P1. Knockout Forward Contract

$$V(T) = (S(T) - K)\,\mathbb{1}_{\{S(T_1) \geq H\}}$$

$$\begin{aligned}
V(t) &= P(t,T)\ \mathbb{E}_t[V(T)] \\
&= e^{-r(T-t)}\mathbb{E}_t[\mathbb{E}_{T_1}[(S(T)-K)\,\mathbb{1}_{\{S(T_1)\geq H\}}]] \quad \text{using tower property} \\
&= e^{-r(T-t)}\mathbb{E}_t[(e^{(r-q)(T-T_1)}S(T_1)-K)\,\mathbb{1}_{\{S(T_1)\geq H\}}] \\
&= e^{-r(T-t)}\left\{e^{(r-q)(T-T_1)}\mathbb{E}_t[S(T_1)\,\mathbb{1}_{\{S(T_1)\geq H\}}] - K\,\mathbb{E}_t[\mathbb{1}_{\{S(T_1)\geq H\}}]\right\} \\
&= e^{-r(T-t)}\left\{e^{(r-q)(T-T_1)}\underbrace{\mathbb{E}_t[(S(T_1)-H)^+]}_{\text{European call}} + (e^{(r-q)(T-T_1)}H-K)\underbrace{\mathbb{E}_t[\mathbb{1}_{\{S(T_1)\geq H\}}]}_{\text{digital option}}\right\}
\end{aligned}$$

$$\begin{aligned}
= e^{-r(T-t)}\Big\{&e^{(r-q)(T-T_1)}\left[e^{-q(T_1-t)}S(t)N(d_1) - e^{-r(T_1-t)}HN(d_2)\right] \\
&+ (e^{(r-q)(T-T_1)}H-K)\,e^{-r(T_1-t)}N(d_2)\Big\}
\end{aligned}$$

where $d_1 = \dfrac{\ln\frac{F(t,T_1)}{K}}{\sigma\sqrt{T_1-t}} + \frac{1}{2}\sigma\sqrt{T_1-t}$ $\quad d_2 = d_1 - \sigma\sqrt{T_1-t}$

# P2. Squared Call and Put

(eq. 10) $dS(t) = (r-q)S(t)dt + \sigma S(t) dW(t)$

(eq. 16) $S(T) = S(t)\, e^{(r-q-\frac{1}{2}\sigma^2)(T-t) + \sigma W(t,T)}$

## I. Squared Call

$$C(T) = \frac{1}{K}(S(T)-K)^2\, \theta(S(T)-K)$$

$$c(t) = e^{-r(T-t)} E_t[\frac{1}{K}(S(T)-K)^2\, \theta(S(T)-K)]$$

sub in (eq. 16)

$$= e^{-r(T-t)} E_t[\frac{1}{K}[S^2(t)\, e^{2(r-q-\frac{1}{2}\sigma^2)(T-t)+2\sigma W(t,T)} - 2S(T)K + K^2]\, \theta(S(T)-K)]$$

$$= e^{-r(T-t)} \frac{1}{K} E_t[(S^2(t)\, e^{2(r-q-\frac{1}{2}\sigma^2)(T-t)+2\sigma W(t,T)} - 2K(S(T)-K) - K^2)\, \theta(S(T)-K)]$$

$$= e^{-r(T-t)} \frac{1}{K}\{\underbrace{e^{2(r-q-\frac{1}{2}\sigma^2)(T-t)} S(t)^2 E_t[e^{2\sigma W(t,T)}\, \theta(S(T)-K)]}_{A}$$

$$\underbrace{-\ 2K}\ \underbrace{E_t[(S(T)-K)\, \theta(S(T)-K)]}_{B\ \text{call}}\ \underbrace{-\ K^2 E_t[\theta(S(T)-K)]}_{C\ \text{digital option}}\} \qquad \text{—— (1)}$$

To compute $E_t[e^{2\sigma W(t,T)}\, \theta(S(T)-K)]$

$$S(T) \geq K \longrightarrow W(t,T) \geq -\frac{1}{\sigma}\ln\frac{S(t)}{K} - \frac{1}{\sigma}(r-q-\frac{1}{2}\sigma^2)(T-t) \quad := x$$

$$E_t[e^{2\sigma W(t,T)}\, \theta(S(T)-K)] = \frac{1}{\sqrt{2\pi(T-t)}}\int_x^\infty e^{-w^2/2(T-t) + 2\sigma w}\, dw \qquad \text{—— (2)}$$

completing square

$$\exp(-w^2/2(T-t) + 2\sigma w)$$

$$= \exp\left(-\frac{1}{2(T-t)}(w^2 - 4(T-t)\sigma w + 4(T-t)^2\sigma^2) + 2(T-t)\sigma^2\right)$$

$$= \exp(2(T-t)\sigma^2)\, \exp[-\frac{1}{2(T-t)}(w-2(T-t)\sigma)^2]$$

let $z = \frac{w}{\sqrt{T-t}} - 2\sigma\sqrt{T-t}$, $\quad dz = \frac{1}{\sqrt{T-t}}dw$

$$= \exp(2(T-t)\sigma^2)\, \exp(-\frac{z^2}{2})$$

$$\therefore \ (2) = \frac{1}{\sqrt{2\pi}} e^{2(T-t)\sigma^2} \int_{-m}^{\infty} e^{-z^2/2} dz$$

$$\therefore \ (A) = e^{2(r-q+\frac{1}{2}\sigma^2)(T-t)} S^2(t) \frac{1}{\sqrt{2\pi}} \int_{-m}^{\infty} e^{-z^2/2} dz$$

where $m = -z(W=x) = -\frac{1}{\sqrt{T-t}}\left(-\frac{1}{\sigma}\ln\frac{S(T)}{K} - \frac{1}{\sigma}(r-q-\frac{1}{2}\sigma^2)(T-t)\right) + 2\sigma\sqrt{T-t}$

$$= \frac{\ln S(T)/K + (r-q)(T-t)}{\sigma\sqrt{T-t}} + \frac{3}{2}\sigma\sqrt{T-t}$$

$$= d_1 + \sigma\sqrt{T-t}$$

$$= e^{2(r-q+\frac{1}{2}\sigma^2)(T-t)} S^2(t) \frac{1}{\sqrt{2\pi}} \int_{-d_1-\sigma\sqrt{T-t}}^{\infty} e^{-z^2/2} dz$$

$$= e^{2(r-q+\frac{1}{2}\sigma^2)(T-t)} S^2(t)\, N(d_1+\sigma\sqrt{T-t})$$

$$(B) = -2K\left[e^{-q(T-t)} S(t) N(d_1) - e^{-r(T-t)} K N(d_2)\right]$$

$$(C) = -K^2 e^{-r(T-t)} N(d_2)$$

$$\therefore \ C(t) = e^{-r(T-t)} \frac{1}{K} \Big\{ e^{2(r-q+\frac{1}{2}\sigma^2)(T-t)} S^2(t) N(d_1+\sigma\sqrt{T-t})$$
$$- 2K\left[e^{-q(T-t)} S(t) N(d_1) - e^{-r(T-t)} K N(d_2)\right]$$
$$- K^2 e^{-r(T-t)} N(d_2) \Big\}$$

## II. Squared Put

$$P(T) = \frac{1}{K}(S(T)-K)^2\, \theta(K-S(T))$$

$$p(t) = e^{-r(T-t)} \mathbb{E}_t[P(T)]$$

$$= e^{-r(T-t)} \mathbb{E}_t\left[\frac{1}{K}\left[S(T)^2 - 2KS(T) + K^2\right]\theta(K-S(T))\right]$$

$$= e^{-r(T-t)} \frac{1}{K} \mathbb{E}_t\left[S^2(t)\, e^{2(r-q-\frac{1}{2}\sigma^2)(T-t)+2\sigma W(t,T)} + 2K(K-S(T)) - K^2\right]\theta(K-S(T))\Big]$$

$$= e^{-r(T-t)} \frac{1}{K} \Big\{ \underbrace{S^2(t)\, e^{2(r-q-\frac{1}{2}\sigma^2)(T-t)}\, \mathbb{E}_t\left[e^{2\sigma W(t,T)}\, \theta(K-S(T))\right]}_{C}$$

$$+ \underbrace{2K\, \mathbb{E}_t\left[(K-S(T))\, \theta(K-S(T))\right]}_{D \quad \text{put}} - \underbrace{K^2\, \mathbb{E}_t\left[\theta(K-S(T))\right]}_{E \quad \text{digital option}}$$

$$K \geq S(T) \rightarrow \quad K \geq S(t)\, e^{(r-q-\sigma^2/2)(T-t)+\sigma W(t,T)}$$

$$W(t,T) \leq \quad -\frac{1}{\sigma} \ln \frac{S(T)}{K} - \frac{1}{\sigma}(r-q-\sigma^2/2)(T-t) := X$$

$$\mathbb{E}_t\left[e^{2\sigma W(t,T)}\, \theta(K-S(T))\right] = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{X} e^{-w^2/2(T-t) + 2\sigma w}\, dw$$

$$= \frac{1}{\sqrt{2\pi}}\, e^{2(T-t)\sigma^2} \int_{-\infty}^{-d_1-\sigma\sqrt{T-t}} e^{-z^2/2}\, dz$$

$$= e^{2(T-t)\sigma^2}\, N(-d_1-\sigma\sqrt{T-t})$$

$$\therefore \quad C = S^2(t)\, e^{2(r-q+\frac{1}{2}\sigma^2)(T-t)}\, N(-d_1-\sigma\sqrt{T-t})$$

$$D = 2K\left[-e^{-q(T-t)} S(t)\, N(-d_1) + e^{-r(T-t)} K\, N(-d_2)\right]$$

$$E = -K^2\left[e^{-r(T-t)}\, N(-d_2)\right]$$

$$\therefore \quad P(t) = e^{-r(T-t)} \frac{1}{K} \Big\{ S^2(t)\, e^{2(r-q+\frac{1}{2}\sigma^2)(T-t)}\, N(-d_1-\sigma\sqrt{T-t})$$

$$+ 2K\left[-e^{-q(T-t)} S(t)\, N(-d_1) + e^{-r(T-t)} K\, N(-d_2)\right]$$

$$- K^2\left[e^{-r(T-t)}\, N(-d_2)\right] \Big\}$$

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