P1. Vega of a Digital

$$V(t) = e^{-r(T-t)} N(\phi d_2)$$

where $d_2 = d_1 - \sigma\sqrt{T-t} = \dfrac{\ln \frac{F(t,T)}{K}}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}$

$$\partial_\sigma d_2 = \frac{-\ln \frac{F(t,T)}{K}}{\sigma^2\sqrt{T-t}} - \frac{1}{2}\sqrt{T-t} = \frac{-d_1}{\sigma}$$

$$\nu = \partial_\sigma V = e^{-r(T-t)} N'(\phi d_2)\, \partial_\sigma(\phi d_2)$$

$$= e^{-r(T-t)} N'(\phi d_2) \frac{-\phi}{\sigma} d_1$$

$$= -e^{-r(T-t)} N'(\phi d_2) \frac{\phi d_1}{\sigma}$$

where $\phi = 1$ for call, $\phi = -1$ for put

short digital call:

For $\nu > 0$, $d_1 > 0$, $\dfrac{\ln \frac{F(t,T)}{K}}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} > 0$

$F > K e^{-\sigma^2(T-t)/2}$

$\therefore$ $F > K e^{-\sigma^2(T-t)/2}$, ν is positive

$F < K e^{-\sigma^2(T-t)/2}$, ν is negative

short digital put:

For $\nu > 0$, $d_1 < 0$

$\therefore$ $F > K e^{-\sigma^2(T-t)/2}$, ν is negative

$F < K e^{-\sigma^2(T-t)/2}$, ν is positive.

P2. Theta of digital options:

$$V(t) = e^{-r(T-t)} N(\phi d_2)$$

where $d_2 = \dfrac{\ln \frac{F(t,T)}{K}}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}$

$$\partial_t F = \partial_t e^{(r-q)(T-t)} = (q-r)e^{(r-q)(T-t)} = (q-r)F$$

$$\partial_t d_2 = \frac{\ln \frac{F(t,T)}{K}}{2\sigma(T-t)^{3/2}} + \frac{\frac{K}{F(t,T)}\cdot\frac{1}{K}(q-r)F(t,T)}{\sigma\sqrt{T-t}} + \frac{\sigma}{4}\frac{1}{\sqrt{T-t}}$$

$$= \frac{\ln F(t,T)}{2\sigma(T-t)^{3/2}} + \frac{q-r}{\sigma\sqrt{T-t}} + \frac{\sigma}{4\sqrt{T-t}}$$

$$\Theta = \partial_t V(t) = r e^{-r(T-t)} N(\phi d_2) + e^{-r(T-t)} N'(\phi d_2)\,\phi\,\partial_t d_2$$

$$= e^{-r(T-t)}\left[rN(\phi d_2) + \phi N'(\phi d_2)\left(\frac{\ln\frac{F(t,T)}{K}}{2\sigma(T-t)^{3/2}} + \frac{q-r}{\sigma\sqrt{T-t}} + \frac{\sigma}{4\sqrt{T-t}}\right)\right]$$

sub $\ln\frac{F}{K} = \sigma\sqrt{T-t}\,d_1 - \frac{1}{2}\sigma^2(T-t)$

$$= e^{-r(T-t)}\left[rN(\phi d_2) + \phi N'(\phi d_2)\left(\frac{d_1}{2(T-t)} + \frac{q-r}{\sigma\sqrt{T-t}}\right)\right]$$