# 算法设计 HW06

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**Q1** To prove it is NP-complete, we first prove it is in NP. As a clique S can be a certificate, and we can verfiy whether S is a cllique and whether $|S| = \frac{n}{2}$ in polynomial time. So, it is in NP.

Then, we want to prove it is NP-complete. We can make a reduction from Clique. We have a clique $G_1 = (V, E), k$, so we can make a similar $G' = (V', E')$ in $G$.

If $k \leq \frac{|V|}{2}$, we can add $|V| - 2k$ vertices to $V'$, and add edges between them and all vertices in $V'$. So, the G' has a $|V'|$ of size $2|V| - 2k$. So, if there is a clique of size k in G, so there is at least a clique of size $|V| - k \geq \frac{|V'|}{2}$ in $G'$. And if there is a clique of size $\frac{|V'|}{2}$ in $G'$, so there is also a clique of size $|V| - k - (|V| - 2k) = k$ in $G$.

If $k > \frac{|V|}{2}$, we can add $2k - |V|$ vertices to $V'$, and all of them hava no edges connected to them. So, the G' has a $|V'|$ of size $2k$. So, if there is a clique of size k in G, then there is a clique of size $k > \frac{|V|}{2}$ in $G'$. And if there is a clique of size $k > \frac{|V|}{2}$ in $G'$, then there is a clique of size $k$ in $G$.

Therefore the problem is NP-complete.

**Q2** (a) First when $k = 1$, it is trival.

Then, when $k \geq 2$. We first prove it is in NP. We can use a spanning tree T with maximum degree k, and it is easy to prove that we can use polynomial time to verify whether T is a spanning tree and whether the maximum degree of T is k.

So, we just want to prove it is NP-complete. We can make a reduction from Hamiltonian Path. Given a Hamitonian Path $P \subset G = (V, E)$. We can make a spanning tree $T \subset G' = (V', E')$ with maximum degree at most k by following steps: We add $(k-2)|V|$ new vertices from G to $G'$ and connect each node in G with $k-2$ different new vertices. So, if there is a Hamiltonian Path P in G, then there is a spanning tree with maximum degree k=2 in $G'$.

If the vertices in $G$ connect to k-2 new vertices have degree at most k in $G'$, the vertices in G have degree at most 2 in $G$. So, if there is a spanning tree with maximum degree k in $G'$, then there is a spanning tree with maximum degree at most 2 in G. SO, there is a Hamiltonian Path in G.

So, the problem is NP-complete.

(b)Similarly, it is easy to prove it is in NP.

So, just like (a), we can make a reduction from Hamiltonian Path. Given a Hamiltonian Path $P \subset G = (V, E)$. We can make a spanning tree $T \subset G' = (V', E')$ with maximum degree at most k in $G'$by the same way in (a) when k is fixed.

So, if there is a Hamiltonian Path in G, then there is a spanning tree with maximum degree at most 2 in G. So, we can add k-2 edges from each vertices in G to the new vertices in $G'$, and the maximum degree of the vertices in G is at most k. So, if there is a spanning tree with maximum degree at most k in $G'$.

What's more, if the vertices in G connected to k-2 new vertices have degree at most k in $G'$, then the vertices in G have degree at most 2 in G. So, there is a spanning tree with maximum degree at most k in $G'$. So, there is a spanning tree with maximumdegree at most 2 in G, so there is a Hamiltonian path inn G.

So the problem is NP-complete.

**Q3** Amazing idea!!!!!

First, prove it is in NP, as a color c of each vertice can served as a certificate, and we can verify whether the color is legal in polynomial time.

Then, we can make a reduction from 3-SAT. Given a 3-SAT formula $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$, and the variables are $x_1, x_2, \cdots, x_n$. So, we can construct a graph G=(V, E) in the problem as follws:

We can first create a triangle in G with 3 verticles T, F, A and T stands for true and F stands for false. What's more, it is a triangle, so we can use three colors to color. Then, for each varibles, we can add a triangle with 3 verticles $x_i, \neg x_i, A$ and $x_i$ stands for the i-th variable. And we can use 3 colors to color them. So, the graph is like below:

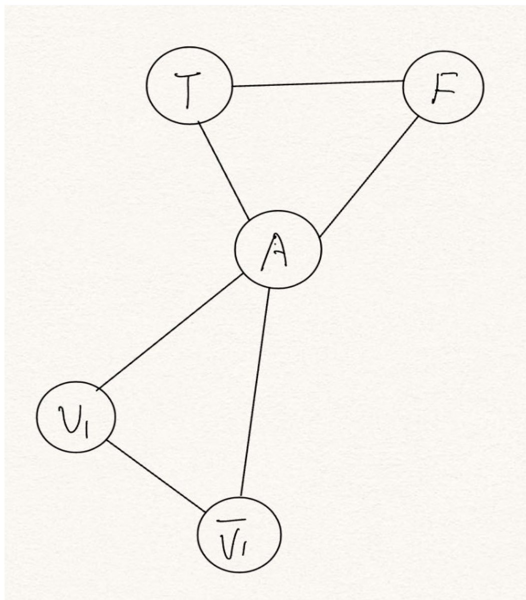


图 1: sample

If G is 3-color, then either $v_i$ or $\neg v_i$ gets true as they are connected to A. and the other gets F. So, we can use the color to replace the value of $v_i$. Also, if the formula is satisfiable, then we can use the value of $v_i$ to color the graph. And the color is 3-color.

So, we next need to draw the clauses. For each clause $\phi_i = a \vee b \vee c$, we can draw a graph by OR-gadget as below:

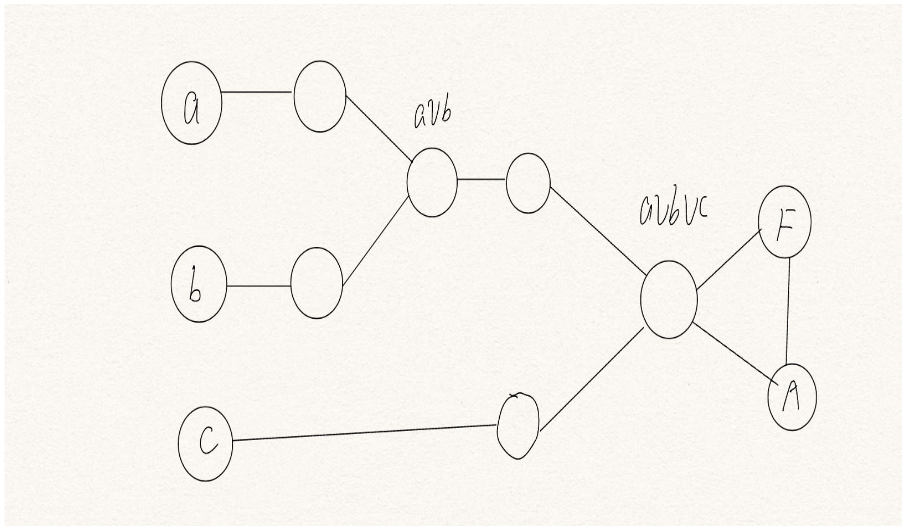


图 2: sample

If a and b are all F in 3-color, then $a \vee b$ must be F, so it is an or gate. So, if a,b,c are all F, the graph cannot be colored by 3-color, which means the formula cannot be satisfied.

If a or b or c has at least one T, WLOG, we assume a is T, then the $a \vee b$ is T, so $a \vee b \vee c$ is T, and the graph is 3-color. So, the formula is satisfied.

As a result, for each $C_i$, we can draw a graph, we can connect the last $a \vee \cdots e$ to the O and F node in the triangle. So, if there is a 3-coloring in the graph, then the formula is satisfied and the value is the color.

On the other hand, if $\phi$ is satisfied, then, we can color $x_i, \neg x_i$ according to the assignment of $x_i$, and the color is unique. Then, we can color the OR-gadget according to the assignment of the variables, and the color is unique. So, the graph is 3-color.

Therefore, the problem is NP-complete.

**Q4** collaborate :蒋松霖

difficulty: 1 and 2 is normal, 3 is very, very, very hard.