Regression analysis_Homework Assignment 5

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2020/11/30

1. Let $f_Y(y;\theta)$ be the distribution of random variable Y under parameter θ and $l = \ln f_Y(y;\theta)$ be the log-likelihood.

a. Show $E_\theta\left[\frac{\partial^2 l}{\partial\theta^2}\right] + E_\theta\left[(\frac{\partial^2 l}{\partial\theta^2})^2\right] = 0$

First of all, $\int f_Y(y;\theta)d\theta = 1$. Under the regularity conditions, than the 1st derivative is

$$\begin{aligned}
0 &= \frac{\partial}{\partial\theta}\int f_Y(y;\theta)dy \\
&= \int \frac{\partial}{\partial\theta} f_Y(y;\theta)dy \\
&= \int \frac{\frac{\partial}{\partial\theta} f_Y(y;\theta)}{f_Y(y;\theta)} f_Y(y;\theta)dy \\
&= \int \left\{\frac{\partial}{\partial\theta}\ln f_Y(y;\theta)\right\} f_Y(y;\theta)dy \\
&= E\left[\frac{\partial}{\partial\theta}\ln f_Y(y;\theta)\right]
\end{aligned}$$

So $E\left[\frac{\partial l}{\partial\theta}\right] = 0$.

The 2nd derivative is

$$\begin{aligned}
0 &= \frac{\partial^2}{\partial\theta^2}\int f_Y(y;\theta)dy \\
&= \int \frac{\partial}{\partial\theta}\left\{\frac{\partial}{\partial\theta}\ln f_Y(y;\theta) f_Y(y;\theta)\right\} dy \\
&= \int \left\{\frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta) f_Y(y;\theta)\right\} + \left\{(\frac{\partial}{\partial\theta}\ln f_Y(y;\theta))^2 f_Y(y;\theta)\right\} dy \\
&= E\left[\frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta)\right] + E\left[(\frac{\partial}{\partial\theta}\ln f_Y(y;\theta))^2\right]
\end{aligned}$$

Since $E\left[\frac{\partial l}{\partial\theta}\right] = 0$, $E\left[(\frac{\partial}{\partial\theta}\ln f_Y(y;\theta) - E\left[\frac{\partial}{\partial\theta}\ln f_Y(y;\theta)\right])^2\right] = Var\left[\frac{\partial}{\partial\theta}\ln f_Y(y;\theta)\right]$. Therefore, one has

$$\begin{aligned}
0 &= E\left[\frac{\partial^2 l}{\partial\theta^2}\right] + E\left[(\frac{\partial l}{\partial\theta})^2\right] \\
&= E\left[\frac{\partial^2 l}{\partial\theta^2}\right] + Var\left[\frac{\partial l}{\partial\theta}\right]
\end{aligned}$$

b. Show $E_\theta\left[\frac{\partial^3 l}{\partial\theta^3}\right] + 3Cov_\theta\left[\frac{\partial^3 l}{\partial\theta^3}, \frac{\partial l}{\partial\theta}\right] + E_\theta\left[(\frac{\partial l}{\partial\theta})^3\right] = 0$

The 3rd derivative is

$$
\begin{aligned}
0 &= \frac{\partial^3}{\partial\theta^3}\int f_Y(y;\theta)dy \\
&= \int \frac{\partial}{\partial\theta}\left\{\frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta)f_Y(y;\theta) + (\frac{\partial}{\partial\theta}\ln f_Y(y;\theta))^2 f_Y(y;\theta)\right\}dy \\
&= \int \left\{\frac{\partial^3}{\partial\theta^3}\ln f_Y(y;\theta)f_Y(y;\theta) + \frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta)\frac{\partial}{\partial\theta}\ln f_Y(y;\theta)f_Y(y;\theta)\right\} + \\
&\quad \left\{2\frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta)\frac{\partial}{\partial\theta}\ln f_Y(y;\theta)f_Y(y;\theta) + (\frac{\partial}{\partial\theta}\ln f_Y(y;\theta))^2\frac{\partial}{\partial\theta}\ln f_Y(y;\theta)f_Y(y;\theta)\right\}dy \\
&= E\left[\frac{\partial^3}{\partial\theta^3}\ln f_Y(y;\theta)\right] + 3E\left[(\frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta))(\frac{\partial}{\partial\theta}\ln f_Y(y;\theta))\right] + E\left[(\frac{\partial}{\partial\theta}\ln f_Y(y;\theta))^3\right]
\end{aligned}
$$

Again, since $E\left[\frac{\partial l}{\partial\theta}\right] = 0$,

$$
\begin{aligned}
&E\left[(\frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta))(\frac{\partial}{\partial\theta}\ln f_Y(y;\theta))\right] \\
=&E\left[(\frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta))(\frac{\partial}{\partial\theta}\ln f_Y(y;\theta))\right] - E\left[\frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta)\right]E\left[\frac{\partial}{\partial\theta}\ln f_Y(y;\theta)\right] \\
=&Cov\left[\frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta), \frac{\partial}{\partial\theta}\ln f_Y(y;\theta)\right]
\end{aligned}
$$

Therefore, one has

$$
\begin{aligned}
0 &= E\left[\frac{\partial^3}{\partial\theta^3}\ln f_Y(y;\theta)\right] + 3Cov\left[\frac{\partial^2}{\partial\theta^2}\ln f_Y(y;\theta), \frac{\partial}{\partial\theta}\ln f_Y(y;\theta)\right] + E\left[(\frac{\partial}{\partial\theta}\ln f_Y(y;\theta))^3\right] \\
&= E\left[\frac{\partial^3 l}{\partial\theta^3}\right] + 3Cov\left[\frac{\partial^2 l}{\partial\theta^2}, \frac{\partial l}{\partial\theta}\right] + E\left[(\frac{\partial l}{\partial\theta})^3\right]
\end{aligned}
$$

2. Check whether Weibull, negative binomial, gamma distribution belong to the exponential family. If so, find the canonical forms.

a. Weibull distribution

Let $Y \sim Weibull(\gamma, \lambda)$, where $\gamma \geq 0$ and $\lambda \geq 0$ are two **unknwon** parameters. The pdf of Y is

$$
f(y|\gamma,\lambda) = \lambda\gamma y^{\gamma-1}e^{-\lambda y^\gamma}1_{\{0,\infty\}}(y)
$$

we can find that a Weibull distribution with **two parameters** can't write as a canonical form.

But if γ is **known**, one can let $Z = Y^\gamma$, the cdf of Z is

$$
P(Z \leq z) = P(Y^\gamma \leq z) = P(Y \leq z^{1/\gamma}) = \int_0^{z^{1/\gamma}} f(y|\gamma,\lambda)dy
$$

and pdf of Z is

$$
\begin{aligned}
f(z|\lambda) &= \frac{d}{dz}P(Z \leq z) \\
&= \frac{1}{\gamma}z^{1/\gamma-1}\lambda\gamma(z^{1/\gamma})^{\gamma-1}e^{-\lambda(z^{1/\gamma})^\gamma}1_{\{0,\infty\}}(z^{1/\gamma}) \\
&= \lambda e^{-\lambda z}1_{\{0,\infty\}}(z)
\end{aligned}
$$

The transformed random variable Z is exactly a exponential distribution with one parameter λ. It can write as the canonical form as follow

$$
\begin{aligned}
f(z|lambda) &= exp\left\{z(-\lambda) + \ln\lambda + \ln(1_{\{0,\infty\}}(z))\right\} \\
&= \exp\left\{\frac{z\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}
\end{aligned}
$$

where $\theta = -\lambda$, $b(\theta) = -\ln(-\theta) = -\ln(\lambda)$, $a(\phi) = \phi = 1$ is a constant, and $c(y,\phi) = \ln(1_{\{0,\infty\}}(z))$. So Weibull distribution with **one parameters** λ can rewrite as a exponential distribution, which belongs to exponential family.

Further more, the expectation and variance of Z are separately

$$
E[y] = \mu = b'(\theta) = -\frac{1}{\theta} = \frac{1}{\lambda}
$$

and

$$
Var[y] = \mu'(\theta) = a(\phi)b''(\theta) = 1(\frac{1}{\theta^2}) = \frac{1}{\lambda^2}
$$

b. Negative binomial distribution

Let $Y \sim NB(r,p)$, where Y denote the number of failures before the rth success, $r > 0$ is a **known** positive integer and $p \in [0,1]$ is a **unknown** parameter. The pdf of Y is

$$
\begin{aligned}
f(y|r,p) &= \binom{r+y-1}{y} p^r (1-p)^y 1_{\{0,1,\ldots\}}(y) \\
&= \exp\left\{y\ln(1-p) + r\ln(p) + \ln\left(\binom{r+y-1}{y} 1_{\{0,1,\ldots\}}(y)\right)\right\} \\
&= \exp\left\{\frac{y(\frac{1}{r}\ln(1-p)) + \ln(p)}{1/r} + \ln\left(\binom{r+y-1}{y} 1_{\{0,1,\ldots\}}(y)\right)\right\} \\
&= \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}
\end{aligned}
$$

where $\theta = \frac{1}{r}\ln(1-p) \Rightarrow p = 1 - e^{r\theta}$, $b(\theta) = -\ln(1-e^{r\theta}) = -\ln(1-p)$, $a(\phi) = \phi = 1/r$ is a constant (since r is known), and $c(y,\phi) = \ln\left(\binom{\phi^{-1}+y-1}{y} 1_{\{0,1,\ldots\}}(y)\right) = \ln\left(\binom{r+y-1}{y} 1_{\{0,1,\ldots\}}(y)\right)$. So the negative binomial distribution belongs to the exponential family.

It follows, as expected, that

$$
E[y] = \mu = b'(\theta) = \frac{re^{r\theta}}{1-e^{r\theta}} = r\frac{1-p}{p}
$$

and

$$
Var[y] = \mu'(\theta) = a(\phi)b''(\theta) = r(\frac{r^2e^{r\theta}}{1-e^{r\theta}} + \frac{r^2(e^{r\theta})^2}{(1-e^{r\theta})^2}) = r(\frac{1-p}{p} + \frac{(1-p)^2}{p^2}) = r\frac{1-p}{p^2}
$$

c. Gamma distribution

Let $Y \sim Gamma(\alpha,\beta)$, with two unknown parameters $\alpha, \beta > 0$. The pdf of Y is

$$
\begin{aligned}
f(y|\alpha,\beta) &= \frac{y^{\alpha-1}e^{-\frac{y}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}1_{(0,\infty)}(y) \\
&= \exp\left\{y(-\frac{1}{\beta}) + \alpha\ln(\frac{1}{\beta}) + (\alpha-1)\ln(y) - \ln(\Gamma(\alpha)) + \ln(1_{(0,\infty)}(y))\right\} \\
&= \exp\left\{\frac{y(-\frac{1}{\alpha\beta}) + \ln(\frac{1}{\alpha\beta})}{1/\alpha} + \big((\alpha-1)\ln(y) - \ln(\Gamma(\alpha)) + \ln(1_{(0,\infty)}(y))\big)\right\} \\
&= \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}
\end{aligned}
$$

where $\theta = -\frac{1}{\alpha\beta}$, $b(\theta) = -\ln(-\theta) = -\ln(\frac{1}{\alpha\beta})$, $a(\phi) = \phi = 1/\alpha$, and $c(y,\phi) = (\phi^{-1}-1)\ln(y) - \ln(\Gamma(\phi^{-1})) + \ln(1_{(0,\infty)}(y)) = (\alpha-1)\ln(y) - \ln(\Gamma(\alpha)) + \ln(1_{(0,\infty)}(y))$.
So gamma distribution belongs to the exponential family.

It follows, as expected, that

$$E[y] = \mu = b'(\theta) = -\frac{1}{\theta} = \alpha\beta$$

and

$$Var[y] = \mu'(\theta) = a(\phi)b''(\theta) = \frac{1}{\alpha}(\frac{1}{\theta^2}) = \frac{1}{\alpha}(\alpha\beta)^2 = \alpha\beta^2$$