## Regression analysis\_Homework Assignment 5

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1. Let  $f_Y(y;\theta)$  be the distribution of random variable Y under parameter  $\theta$  and  $l=\ln f_Y(y;\theta)$  be the log-likelihood.

a. Show 
$$E_{ heta}\left[rac{\partial^2 l}{\partial heta^2}
ight]+E_{ heta}\left[(rac{\partial^2 l}{\partial heta^2})^2
ight]=0$$

First of all,  $\int f_Y(y;\theta)d\theta=1$ . Under the regularity conditions, than the 1st derivative is

$$egin{aligned} 0 &= rac{\partial}{\partial heta} \int f_Y(y; heta) dy \ &= \int rac{\partial}{\partial heta} f_Y(y; heta) dy \ &= \int rac{rac{\partial}{\partial heta} f_Y(y; heta)}{f_Y(y; heta)} f_Y(y; heta) dy \ &= \int \left\{ rac{\partial}{\partial heta} \ln f_Y(y; heta) 
ight\} f_Y(y; heta) dy \ &= E \left[ rac{\partial}{\partial heta} \ln f_Y(y; heta) 
ight] \end{aligned}$$

So 
$$E\left[rac{\partial l}{\partial heta}
ight]=0$$
 .

The 2nd derivative is

$$egin{aligned} 0 &= rac{\partial^2}{\partial heta^2} \int f_Y(y; heta) dy \ &= \int rac{\partial}{\partial heta} iggl\{ rac{\partial}{\partial heta} \mathrm{ln} \, f_Y(y; heta) f_Y(y; heta) iggr\} \, dy \ &= \int iggl\{ rac{\partial^2}{\partial heta^2} \mathrm{ln} \, f_Y(y; heta) f_Y(y; heta) iggr\} + iggl\{ (rac{\partial}{\partial heta} \mathrm{ln} \, f_Y(y; heta))^2 f_Y(y; heta) iggr\} \, dy \ &= E \left[ rac{\partial^2}{\partial heta^2} \mathrm{ln} \, f_Y(y; heta) 
ight] + E \left[ (rac{\partial}{\partial heta} \mathrm{ln} \, f_Y(y; heta))^2 
ight] \end{aligned}$$

Since  $E\left[\frac{\partial l}{\partial \theta}\right]=0$ ,  $E\left[(\frac{\partial}{\partial \theta}\ln f_Y(y;\theta)-E\left[\frac{\partial}{\partial \theta}\ln f_Y(y;\theta)\right])^2\right]=Var\left[\frac{\partial}{\partial \theta}\ln f_Y(y;\theta)\right]$ . Therefore, one has

$$egin{aligned} 0 &= E\left[rac{\partial^2 l}{\partial heta^2}
ight] + E\left[(rac{\partial l}{\partial heta})^2
ight] \ &= E\left[rac{\partial^2 l}{\partial heta^2}
ight] + Var\left[rac{\partial l}{\partial heta}
ight] \end{aligned}$$

b. Show 
$$E_{ heta}\left[rac{\partial^3 l}{\partial heta^3}
ight] + 3Cov_{ heta}\left[rac{\partial^3 l}{\partial heta^3},rac{\partial l}{\partial heta}
ight] + E_{ heta}\left[(rac{\partial l}{\partial heta})^3
ight] = 0$$

The 3rd derivative is

$$\begin{split} 0 &= \frac{\partial^3}{\partial \theta^3} \int f_Y(y;\theta) dy \\ &= \int \frac{\partial}{\partial \theta} \left\{ \frac{\partial^2}{\partial \theta^2} \ln f_Y(y;\theta) f_Y(y;\theta) + \left( \frac{\partial}{\partial \theta} \ln f_Y(y;\theta) \right)^2 f_Y(y;\theta) \right\} dy \\ &= \int \left\{ \frac{\partial^3}{\partial \theta^3} \ln f_Y(y;\theta) f_Y(y;\theta) + \frac{\partial^2}{\partial \theta^2} \ln f_Y(y;\theta) \frac{\partial}{\partial \theta} \ln f_Y(y;\theta) f_Y(y;\theta) \right\} + \\ &\left\{ 2 \frac{\partial^2}{\partial \theta^2} \ln f_Y(y;\theta) \frac{\partial}{\partial \theta} \ln f_Y(y;\theta) f_Y(y;\theta) + \left( \frac{\partial}{\partial \theta} \ln f_Y(y;\theta) \right)^2 \frac{\partial}{\partial \theta} \ln f_Y(y;\theta) f_Y(y;\theta) \right\} dy \\ &= E \left[ \frac{\partial^3}{\partial \theta^3} \ln f_Y(y;\theta) \right] + 3E \left[ \left( \frac{\partial^2}{\partial \theta^2} \ln f_Y(y;\theta) \right) \left( \frac{\partial}{\partial \theta} \ln f_Y(y;\theta) \right) \right] + E \left[ \left( \frac{\partial}{\partial \theta} \ln f_Y(y;\theta) \right)^3 \right] \end{split}$$

Again, since  $E\left[rac{\partial l}{\partial heta}
ight]=0$  ,

$$\begin{split} &E\left[(\frac{\partial^{2}}{\partial\theta^{2}}\ln f_{Y}(y;\theta))(\frac{\partial}{\partial\theta}\ln f_{Y}(y;\theta))\right] \\ =&E\left[(\frac{\partial^{2}}{\partial\theta^{2}}\ln f_{Y}(y;\theta))(\frac{\partial}{\partial\theta}\ln f_{Y}(y;\theta))\right] - E\left[\frac{\partial^{2}}{\partial\theta^{2}}\ln f_{Y}(y;\theta)\right] E\left[\frac{\partial}{\partial\theta}\ln f_{Y}(y;\theta)\right] \\ =&Cov\left[\frac{\partial^{2}}{\partial\theta^{2}}\ln f_{Y}(y;\theta),\frac{\partial}{\partial\theta}\ln f_{Y}(y;\theta)\right] \end{split}$$

Therefore, one has

$$egin{aligned} 0 &= E\left[rac{\partial^3}{\partial heta^3} \mathrm{ln}\, f_Y(y; heta)
ight] + 3 Cov\left[rac{\partial^2}{\partial heta^2} \mathrm{ln}\, f_Y(y; heta), rac{\partial}{\partial heta} \mathrm{ln}\, f_Y(y; heta)
ight] + E\left[(rac{\partial}{\partial heta} \mathrm{ln}\, f_Y(y; heta))^3
ight] \ &= E\left[rac{\partial^3 l}{\partial heta^3}
ight] + 3 Cov\left[rac{\partial^2 l}{\partial heta^2}, rac{\partial l}{\partial heta}
ight] + E\left[(rac{\partial l}{\partial heta})^3
ight] \end{aligned}$$

# 2. Check whether Weibull, negative binomial, gamma distribution belong to the exponential family. If so, find the canonical forms.

#### a. Weibull distribution

Let  $Y \sim Weibull(\gamma,\lambda)$ , where  $\gamma \geq 0$  and  $\lambda \geq 0$  are two **unknwon** parameters. The pdf of Y is

$$f(y|\gamma,\lambda) = \lambda \gamma y^{\gamma-1} e^{-\lambda y^{\gamma}} 1_{\{0,\infty\}}(y)$$

we can find that a Weibull distribution with **two parameters** can't write as a canonical form.

But if  $\gamma$  is **known**, one can let  $Z=Y^{\gamma}$  , the cdf of Z is

$$P(Z \leq z) = P(Y^{\gamma} \leq z) = P(Y \leq z^{1/\gamma}) = \int_0^{z^{1/\gamma}} \!\! f(y|\gamma,\lambda) dy$$

and pdf of Z is

$$egin{aligned} f(z|\lambda) &= rac{d}{dz} P(Z \leq z) \ &= rac{1}{\gamma} z^{1/\gamma-1} \lambda \gamma(z^{1/\gamma})^{\gamma-1} e^{-\lambda(z^{1/\gamma})^{\gamma}} 1_{\{0,\infty\}}(z^{1/\gamma}) \ &= \lambda e^{-\lambda z} 1_{\{0,\infty\}}(z) \end{aligned}$$

The transformed random variable Z is exactly a exponential distribution with one parameter  $\lambda$ . It can write as the canonical form as follow

$$egin{aligned} f(z|lambda) &= exp\left\{z(-\lambda) + \ln \lambda + \ln(1_{\{0,\infty\}}(z))
ight\} \ &= \exp\left\{rac{z heta - b( heta)}{a(\phi)} + c(y,\phi)
ight\} \end{aligned}$$

where  $\theta=-\lambda$ ,  $b(\theta)=-\ln(-\theta)=-\ln(\lambda)$ ,  $a(\phi)=\phi=1$  is a constant, and  $c(y,\phi)=\ln(1_{\{0,\infty\}}(z))$ . So Weibull distribution with **one parameters**  $\lambda$  can rewrite as a exponential distribution, which belongs to exponential family.

Further more, the expectation and variance of Z are separately

$$E[y] = \mu = b'( heta) = -rac{1}{ heta} = rac{1}{\lambda}$$

and

$$Var[y] = \mu'( heta) = a(\phi)b''( heta) = 1(rac{1}{ heta^2}) = rac{1}{\lambda^2}$$

### b. Negative binomial distribution

Let  $Y \sim NB(r,p)$ , where Y denote the number of failures before the rth success, r>0 is a **known** positive integer and  $p\in[0,1]$  is a **unknown** parameter. The pdf of Y is

$$egin{aligned} f(y|r,p) &= inom{r+y-1}{y} p^r (1-p)^y 1_{\{0,1,\ldots\}}(y) \ &= \expigg\{y \ln(1-p) + r \ln(p) + \lnigg(inom{r+y-1}{y} 1_{\{0,1,\ldots\}}(y)igg)igg\} \ &= \expigg\{rac{y(rac{1}{r} \ln(1-p)) + \ln(p)}{1/r} + \lnigg(inom{r+y-1}{y} 1_{\{0,1,\ldots\}}(y)igg)igg\} \ &= \expigg\{rac{y heta - b( heta)}{a(\phi)} + c(y,\phi)igg\} \end{aligned}$$

where  $\theta=\frac{1}{r}\ln(1-p)\Rightarrow p=1-e^{r\theta}$ ,  $b(\theta)=-\ln(1-e^{r\theta})=-\ln(1-p)$ ,  $a(\phi)=\phi=1/r$  is a constant (since r is known), and  $c(y,\phi)=\ln\Bigl(\binom{\phi^{-1}+y-1}{y}1_{\{0,1,\ldots\}}(y)\Bigr)=\ln\Bigl(\binom{r+y-1}{y}1_{\{0,1,\ldots\}}(y)\Bigr)$ . So the negative binomial distribution belongs to the exponential family.

It follows, as expected, that

$$E[y] = \mu = b'( heta) = rac{re^{r heta}}{1-e^{r heta}} = rrac{1-p}{p}$$

and

$$Var[y] = \mu'(\theta) = a(\phi)b''(\theta) = r(\frac{r^2e^{r\theta}}{1-e^{r\theta}} + \frac{r^2(e^{r\theta})^2}{(1-e^{r\theta})^2}) = r(\frac{1-p}{p} + \frac{(1-p)^2}{p^2}) = r\frac{1-p}{p^2}$$

#### c. Gamma distribution

Let  $Y \sim Gamma(\alpha, \beta)$ , with two unknown parameters  $\alpha, \beta > 0$ . The pdf of Y is

$$egin{aligned} f(y|lpha,eta) &= rac{y^{lpha-1}e^{-rac{y}{eta}}}{\Gamma(lpha)eta^lpha} 1_{(0,\infty)}(y) \ &= \expigg\{y(-rac{1}{eta}) + lpha \ln(rac{1}{eta}) + (lpha-1)\ln(y) - \ln(\Gamma(lpha)) + \ln(1_{(0,\infty)}(y))igg\} \ &= \expigg\{rac{y(-rac{1}{lphaeta}) + \ln(rac{1}{lphaeta})}{1/lpha} + igl((lpha-1)\ln(y) - \ln(\Gamma(lpha)) + \ln(1_{(0,\infty)}(y))igr)igg\} \ &= \expigg\{rac{y heta - b( heta)}{a(\phi)} + c(y,\phi)igg\} \end{aligned}$$

where  $\theta=-\frac{1}{\alpha\beta}$ ,  $b(\theta)=-\ln(-\theta)=-\ln(\frac{1}{\alpha\beta})$ ,  $a(\phi)=\phi=1/\alpha$ , and  $c(y,\phi)=(\phi^{-1}-1)\ln(y)-\ln(\Gamma(\phi^{-1}))+\ln(1_{(0,\infty)}(y))=(\alpha-1)\ln(y)-\ln(\Gamma(\alpha))+\ln(1_{(0,\infty)}(y))$ . So gamma distribution belongs to the exponential family.

It follows, as expected, that

$$E[y] = \mu = b'( heta) = -rac{1}{ heta} = lpha eta$$

and

$$Var[y] = \mu'( heta) = a(\phi)b''( heta) = rac{1}{lpha}(rac{1}{ heta^2}) = rac{1}{lpha}(lphaeta)^2 = lphaeta^2$$