Homework Assignment 2
Due date: Oct. 5 請印出來以紙本繳交
#請參考 Regression Analysis (1990) – Theory Methods and Applications
1. **(S.2.2)**
Suppose $y = X\beta + \epsilon$, where $E(\epsilon) = 0, \operatorname{cov}(\epsilon) = \sigma^2 I_n$ the matrix X of dimension $n \times k$ has rank $k \leq n$, and β is a k -vector of regression parameters. Suppose, further, that we wish to predict the $(n+1)$st observation y_{n+1} at $\boldsymbol{x}'_{n+1} = (x_{n+1,1}, \ldots, x_{n+1,k})$; i.e., $y_{n+1} = \boldsymbol{x}'_{n+1}\boldsymbol{\beta} + \epsilon_{n+1}$ where ϵ_{n+1} has the same distribution as the other ϵ_i 's and is independent of them. The predictor based on the least squares estimate of $\boldsymbol{\beta}$ is given by $\hat{y}_{n+1} = \boldsymbol{x}'_{n+1}\boldsymbol{b}$, where $\boldsymbol{b} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$

1. Show that $\hat{y}_{n+1}$ is a linear function of $y_1, \ldots, y_n$ such that $E(\hat{y}_{n+1} - y_{n+1}) = 0$
2. Suppose $\tilde{y}_{n+1} = \boldsymbol{a}'\boldsymbol{y}$ is another predictor of y_{n+1} such that $\mathrm{E}(\tilde{y}_{n+1} - y_{n+1}) = 0$. Show that **a** must satisfy $a'X = x'_{n+1}$
3. Find var $(\hat{y}_{n+1})$ and var $(\tilde{y}_{n+1})$
4. Show that var $(\hat{y}_{n+1}) \leq$ var $(\tilde{y}_{n+1})$

2. **(S.2.3)**
Let $y_i = x'_i\beta + \epsilon_i$ with $i = 1, \ldots, n$ be a regression model where $\mathrm{E}(\epsilon_i) = 0, \operatorname{var}(\epsilon_i) = \sigma^2$ and $\operatorname{cov}(\epsilon_i, \epsilon_j) = 0$ when $i \neq j$. Suppose $e_i = y_i - \hat{y}_i$, where $\hat{y}_i = \boldsymbol{x}'_i\boldsymbol{b}$ and b is the least squares estimator of $\boldsymbol{\beta}$. Let $X' = (x_1, \cdots, x_n)$. Show that the variance of e_i is $\left[1 - x'_i (X'X)^{-1} x_i\right]\sigma^2$
3. **(S.2.4)**
In the model of Exercise 2.3, show that the $\hat{y}_i$ is a linear unbiased estimator of $\boldsymbol{x}'_i\boldsymbol{\beta}$ (that is, $\hat{y}_i$ is a linear function of $y_1, \ldots, y_n$ and $\mathrm{E}(\hat{y}_i) = \boldsymbol{x}'_i\boldsymbol{\beta}$). What is the variance of $\hat{y}_i$? Does there exist any other linear unbiased estimator of $x'_i\beta$ with a smaller variance than the estimator $\hat{y}_i$?
4. **(S.2.6)**
Consider the models $y = X\beta + \epsilon$ and $y^* = X^*\beta + \epsilon^*$ where $\mathrm{E}(\epsilon) = 0, \operatorname{cov}(\epsilon) = \sigma^2 I, y^* = \Gamma y, X^* = \Gamma X, \epsilon^* = \Gamma\epsilon$ and Γ is a known $n \times n$ orthogonal matrix. Show that:

1. $E(\boldsymbol{\epsilon}^*) = 0, \operatorname{cov}(\boldsymbol{\epsilon}^*) = \sigma^2 I$
2. $b = b^*$ and $s^{*2} = s^2$, where b and b^* are the least squares estimates of β and s^2 and s^{*2} are the estimates of σ^2 obtained from the two models.

5. **(S.2.19) (the csv file of Exhibit 2.9 is attached in the mail)**
Exhibit 2.9 gives information on capital, labor and value added for each of three economic sectors: Food and kindred products (20), electrical and electronic machinery, equipment and supplies (36) and transportation equipment (37). The data were supplied by Dr. Philip Israelovich of the Federal Reserve Bank, who also suggested the exercise. For each sector:

1. Consider the model
$$V_t = \alpha K_t^{\beta_1} L_t^{\beta_2} \eta_t$$
where the subscript t indicates year, V_t is value added, K_t is capital, L_t is labor and η_t is an error term, with $\mathrm{E}[\log(\eta_t)] = 0$ and $\operatorname{var}[\log(\eta_t)]$ a constant. Assuming that the errors are independent, and taking logs of both sides of the above model, estimate β_1 and β_2
2. The model given in 1 above is said to be of the Cobb-Douglas form. It is easier to interpret if $\beta_1 + \beta_2 = 1$. Estimate β_1 and β_2 under this constraint.
3. Sometimes the model
$$V_t = \alpha\gamma^t K_t^{\beta_1} L_t^{\beta_2} \eta_t$$
is considered where γ^t is assumed to account for technological development. Estimate β_1 and β_2 for this model.
4. Estimate β_1 and β_2 in the model in 3 , under the constraint $\beta_1 + \beta_2 = 1$.

	Capital			Labor			Real Value Added		
Yr	'20'	'36'	'37'	'20'	'36'	'37'	'20'	'36'	'37'
72	243462	291610	1209188	708014	881231	1259142	6496.96	6713.75	11150.0
73	252402	314728	1330372	699470	960917	1371795	5587.34	7551.68	12853.6
74	246243	278746	1157371	697628	899144	1263084	5521.32	6776.40	10450.8
75	263639	264050	1070860	674830	739485	1118226	5890.64	5554.89	9318.3
76	276938	286152	1233475	685836	791485	1274345	6548.57	6589.67	12097.7
77	290910	286584	1355769	678440	832818	1369877	6744.80	7232.56	12844.8
78	295616	280025	1351667	667951	851178	1451595	6694.19	7417.01	13309.9
79	301929	279806	1326248	675147	848950	1328683	6541.68	7425.69	13402.3
80	307346	258823	1089545	658027	779393	1077207	6587.33	6410.91	8571.0
81	302224	264913	1111942	627551	757462	1056231	6746.77	6263.26	8739.7
82	288805	247491	988165	609204	664834	947502	7278.30	5718.46	8140.0
83	291094	246028	1069651	604601	664249	1057159	7514.78	5936.93	10958.4
84	285601	256971	1191677	601688	717273	1169442	7539.93	6659.30	10838.9
85	292026	248237	1246536	584288	678155	1195255	8332.65	6632.67	10030.5
86	294777	261943	1281262	571454	670927	1171664	8506.37	6651.02	10836.5

EXHIBIT 2.9: Data on Capital, Labor and Value Added for Three Sectors