Regression analysis_Homework Assignment 2

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Sen, A., & Srivastava, M. (1990). Regression analysis: theory, methods, and applications.

1. (#S.2.2) Suppose $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\epsilon}$, where $E(\boldsymbol{\varepsilon}) = 0$, $Cov(\boldsymbol{\varepsilon}) = \sigma^2 \boldsymbol{I_n}$ the matrix $\boldsymbol{X}$ of dimension $n \times k$ has rank $k \le n$, and $\boldsymbol{\beta}$ is a k-vector of regression parameters. Suppose, further, that we wish to predict the (n + 1)st observation y_{n+1} at $\boldsymbol{x_{n+1}^\top} = (x_{n+1,1}, \dots, x_{n+1,k})$; ie., $y_{n+1} = \boldsymbol{x_{n+1}^\top \beta} + \varepsilon_{n+1}$ where ε_{n+1} has the same distribution as the other ε_i's and is independent of them. The predictor based on the least squares estimate of $\boldsymbol{\beta}$ is given by $\hat{y}_{n+1} = \boldsymbol{x_{n+1}^\top \hat{\beta}}$, where $\boldsymbol{\hat{\beta}} = \boldsymbol{(X^\top X)^{-1} X^\top y}$.

a. Show that $\hat{y}_{n+1}$ is a linear function of $y_1, \dots, y_n$ such that $E(\hat{y}_{n+1} - y_{n+1}) = 0$.

The least square estimator of $\boldsymbol{\beta}$ is $\boldsymbol{\hat{\beta}} = \boldsymbol{(X^\top X)^{-1} X^\top y}$. One can get that

$$\begin{aligned} \hat{y}_{n+1} &= \boldsymbol{x_{n+1}^\top \hat{\beta}} \\ &= \boldsymbol{x_{n+1}^\top (X^\top X)^{-1} X^\top y} \end{aligned}$$

where $\boldsymbol{x_{n+1}^\top (X^\top X)^{-1} X^\top}$ is a $1 \times n$ vector, therefore $\hat{y}_{n+1}$ is a linear combination of $\{y_1, \dots, y_{n+1}\}$. And the expectation of $\hat{y}_{n+1} - y_{n+1}$ is

$$\begin{aligned} E(\hat{y}_{n+1} - y_{n+1}) &= E(\hat{y}_{n+1}) - E(y_{n+1}) \\ &= E(\boldsymbol{x_{n+1}^\top (X^\top X)^{-1} X^\top y}) - E(\boldsymbol{x_{n+1}^\top \beta} + \boldsymbol{\varepsilon}) \\ &= \boldsymbol{x_{n+1}^\top (X^\top X)^{-1} X^\top} E(\boldsymbol{y}) - \boldsymbol{x_{n+1}^\top \beta} \\ &= \boldsymbol{x_{n+1}^\top (X^\top X)^{-1} X^\top X \beta} - \boldsymbol{x_{n+1}^\top \beta} \\ &= \boldsymbol{x_{n+1}^\top \beta} - \boldsymbol{x_{n+1}^\top \beta} = 0 \end{aligned}$$

So, $\hat{y}_{n+1}$ is an unbiased estimator of y_{n+1}.

b. Suppose $\tilde{y}_{n+1} = \boldsymbol{a^\top y}$ is another predictor of y_{n+1} such that $E(\tilde{y}_{n+1} - y_{n+1}) = 0$. Show that $\boldsymbol{a}$ must satisfy $\boldsymbol{a^\top X} = \boldsymbol{x_{n+1}^\top}$.

Since $\tilde{y}_{n+1}$ should satisfy

$$\begin{aligned} 0 &= E(\tilde{y}_{n+1} - y_{n+1}) \\ &= E(\boldsymbol{a^\top y}) - E(y_{n+1}) \\ &= \boldsymbol{a^\top X \beta} - \boldsymbol{x_{n+1}^\top \beta} \\ &= (\boldsymbol{a^\top X} - \boldsymbol{x_{n+1}^\top})\boldsymbol{\beta} \end{aligned}$$

and $\boldsymbol{\beta}$ is not a zero vector. Therefore, $\tilde{y}_{n+1}$ is an unbiased estimator if it satisfies $\boldsymbol{a^\top X} - \boldsymbol{x_{n+1}^\top} = 0$

c. Find $Var(\hat{y}_{n+1})$ and $Var(\tilde{y}_{n+1})$.

$$\begin{aligned} Var(\hat{y}_{n+1}) &= Var(\boldsymbol{x_{n+1}^\top (X^\top X)^{-1} X^\top y}) \\ &= (\boldsymbol{x_{n+1}^\top (X^\top X)^{-1} X^\top}) Cov(\boldsymbol{y}) (\boldsymbol{x_{n+1}^\top (X^\top X)^{-1} X^\top})^\top \\ &= (\boldsymbol{x_{n+1}^\top (X^\top X)^{-1} X^\top}) (\sigma^2 \boldsymbol{I}) (\boldsymbol{x_{n+1}^\top (X^\top X)^{-1} X^\top})^\top \\ &= \sigma^2 \boldsymbol{x_{n+1}^\top (X^\top X)^{-1} x_{n+1}} \end{aligned}$$

and

$$\begin{aligned} Var(\tilde{y}_{n+1}) &= Var(\boldsymbol{a^\top y}) \\ &= \sigma^2 \boldsymbol{a^\top a} \end{aligned}$$

d. Show that $Var(\hat{y}_{n+1}) \leq Var(\tilde{y}_{n+1})$

If we compare the variance of $\tilde{y}$ and $\hat{y}$

$$\begin{aligned}
Var(\tilde{y}_{n+1}) - Var(\hat{y}_{n+1}) &= \sigma^2 \boldsymbol{a}^\top \boldsymbol{a} - \sigma^2 \boldsymbol{x}_{n+1}^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{x}_{n+1} \\
&= \sigma^2 (\boldsymbol{a}^\top \boldsymbol{a} - \boldsymbol{a}^\top \boldsymbol{X} (\boldsymbol{X}^\top \boldsymbol{X})^{-1} (\boldsymbol{a}^\top \boldsymbol{X})^\top) \qquad (\text{by part b: } \boldsymbol{a}^\top \boldsymbol{X} = \boldsymbol{x}_{n+1}^\top) \\
&= \sigma^2 (\boldsymbol{a}^\top (\boldsymbol{I} - \boldsymbol{X} (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top) \boldsymbol{a}) \\
&:= \sigma^2 (\boldsymbol{a}^\top (\boldsymbol{I} - \boldsymbol{H}) \boldsymbol{a})
\end{aligned}$$

where $\boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top$. We know $(\boldsymbol{I} - \boldsymbol{H})$ is idempotent ($(\boldsymbol{I} - \boldsymbol{H})^2 = (\boldsymbol{I} - \boldsymbol{H})$) and positive semi-definite ($\boldsymbol{b}^\top (\boldsymbol{I} - \boldsymbol{H}) \boldsymbol{b} \geq 0$ for all $\boldsymbol{b} \in \mathbb{R}^n$). Therefore,

$$\begin{aligned}
& Var(\tilde{y}_{n+1}) - Var(\hat{y}_{n+1}) \geq 0 \\
\Rightarrow \quad & Var(\hat{y}_{n+1}) \leq Var(\tilde{y}_{n+1})
\end{aligned}$$

We can conclude that $\hat{y}_{n+1}$ is the best linear unbiased estimator (BLUE) for y_{n+1}, since its variance is smaller than any other linear unbiased estimator (e.g. $\tilde{y}$).

2. (S.2.3) Let $y_i = \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i$ with $i, \ldots, n$ be a regression model where $E(\varepsilon_i) = 0$, $Var(\varepsilon_i) = \sigma^2$ and $Cov(\varepsilon_i, \varepsilon_j) = 0$ when $i \neq j$. Suppose $e_i = y_i - \hat{y}_i$, where $\hat{y}_i = \boldsymbol{x}_i^\top \hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}$ is the least squares estimator of $\boldsymbol{\beta}$. Let $\boldsymbol{X}^\top = (\boldsymbol{x_1}, \ldots, \boldsymbol{x_n})$. Show that the variance of e_i is $[1 - \boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{x}_i] \sigma^2$.

The least square estimator of $\boldsymbol{\beta}$ is $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{y}$, then

$$\begin{aligned}
Var(e_i) &= Var(y_i - \hat{y}_i) = Var(y_i - \boldsymbol{x}_i^\top \hat{\boldsymbol{\beta}}) \\
&= Var(y_i - \boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{y}) \\
&= Var(y_i) + Var(\boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{y}) - 2Cov(y_i, \boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{y}) \\
&= Var(y_i) + Var(\boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{y}) - 2Var(\boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top y_i) \qquad (\text{since } y_i \perp y_j \ \forall i \neq j) \\
&= \sigma^2 - (\boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top) \sigma^2 \boldsymbol{I} (\boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top)^\top \\
&= \sigma^2 [1 - \boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{x}_i]
\end{aligned}$$

3. (S.2.4) In the model of Exercise 2.3, show that the $\hat{y}_i$ is a linear unbiased estimator of $\boldsymbol{x}_i^\top \boldsymbol{\beta}$ (that is, $\hat{y}_i$ is a linear function of $y_1, \ldots, y_n$ and $E(\hat{y}_i) = \boldsymbol{x}_i^\top \boldsymbol{\beta}$). What is the variance of $\hat{y}_i$? Does there exist any other linear unbiased estimator of $\boldsymbol{x}_i^\top \boldsymbol{\beta}$ with a smaller variance than the estimator $\hat{y}_i$?

(1) One can observe that

$$\begin{aligned}
\hat{y}_i &= \boldsymbol{x}_i^\top \hat{\boldsymbol{\beta}} \\
&= \boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{y} \\
&= \sum_{j=1}^{n} w_j y_j
\end{aligned}$$

where $(w_1, \ldots, w_n) = \boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top$. So, $\hat{y}_i$ is a linear function of $\{y_1, \ldots, y_n\}$. Furthermore, its expectation is

$$\begin{aligned}
E(\hat{y}_i) &= \boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{E}(\boldsymbol{y}) \\
&= \boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{X} \boldsymbol{\beta} \\
&= \boldsymbol{x}_i^\top \boldsymbol{\beta}
\end{aligned}$$

Therefore, $\hat{y}_i$ is a linear unbiased estimator of $\boldsymbol{x}_i^\top \boldsymbol{\beta}$.

(2) The variance of $\hat{y}_i$ is

$$\begin{aligned}
Var(\hat{y}_i) &= \boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top Cov(\boldsymbol{y}) (\boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top)^\top \\
&= \sigma^2 \boldsymbol{x}_i^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{x}_i
\end{aligned}$$

(3) Suppose $\tilde{y}_i = \boldsymbol{a}^\top \boldsymbol{y}$, is an linear unbiased estimator for $\boldsymbol{x_i}^\top \boldsymbol{\beta}$, must satisfy

$$\begin{aligned} & \boldsymbol{x_i}^\top \boldsymbol{\beta} = E(\tilde{y}_i) = E(\boldsymbol{a}^\top \boldsymbol{y}) = \boldsymbol{a}^\top \boldsymbol{X}\boldsymbol{\beta} \\ \Rightarrow \quad & \boldsymbol{x_i}^\top = \boldsymbol{a}^\top \boldsymbol{X} \end{aligned}$$

where $\boldsymbol{\beta}$ is not a zeor vector. If we compare variance of $\tilde{y}_i$ and $\hat{y}_i$

$$\begin{aligned} Var(\tilde{y}_i) - Var(\hat{y}_i) &= \sigma^2 \boldsymbol{a}^\top \boldsymbol{a} - \sigma^2 \boldsymbol{x_i}^\top (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{x_i} \\ &= \sigma^2 \boldsymbol{a}^\top (\boldsymbol{I} - \boldsymbol{H}) \boldsymbol{a} \end{aligned}$$

where $\boldsymbol{H} = \boldsymbol{X}(\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top$. We know $(\boldsymbol{I} - \boldsymbol{H})$ is idempotent ($(\boldsymbol{I} - \boldsymbol{H})^2 = (\boldsymbol{I} - \boldsymbol{H})$) and positive semi-definite ($\boldsymbol{b}^\top (\boldsymbol{I} - \boldsymbol{H}) \boldsymbol{b} \geq 0 \text{ for all } \boldsymbol{b} \in \mathbb{R}^n$). Therefore,

$$\begin{aligned} & Var(\tilde{y}_i) - Var(\hat{y}_i) \geq 0 \\ \Rightarrow \quad & Var(\hat{y}_i) \leq Var(\tilde{y}_i) \end{aligned}$$

We can conclude that $\hat{y}_i$ is the best linear unbiased estimator (BLUE) for $\boldsymbol{x_i}^\top \boldsymbol{\beta}$, since its variance is smaller than any other linear unbiased estimator (e.g. $\tilde{y}_i$).

4. (S.2.6) Consider the models $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$ and $\boldsymbol{y}^* = \boldsymbol{X}^*\boldsymbol{\beta} + \boldsymbol{\varepsilon}^*$ where $E(\boldsymbol{\varepsilon}) = 0$, $Cov(\boldsymbol{\varepsilon}) = \sigma^2 \boldsymbol{I}$, $\boldsymbol{y}^* = \boldsymbol{\Gamma}\boldsymbol{y}$, $\boldsymbol{X}^* = \boldsymbol{\Gamma}\boldsymbol{X}$, $\boldsymbol{\varepsilon}^* = \boldsymbol{\Gamma}\boldsymbol{\varepsilon}$ and $\boldsymbol{\Gamma}$ is a known $n \times n$ orthogonal matrix. Show that:

a. $E(\boldsymbol{\varepsilon}^*) = 0$, $Cov(\boldsymbol{\varepsilon}^*) = \sigma^2 \boldsymbol{I}$

If a square matrix $\boldsymbol{\Gamma}$ is said to be an orthogonal, if its columns and rows are orthogonal unit vectors or can express by $\boldsymbol{\Gamma}^\top \boldsymbol{\Gamma} = \boldsymbol{\Gamma}\boldsymbol{\Gamma}^\top = \boldsymbol{I}$.

The expectation of $\boldsymbol{\varepsilon}^*$ is

$$\begin{aligned} E(\boldsymbol{\varepsilon}^*) &= E(\boldsymbol{\Gamma}\boldsymbol{\varepsilon}) \\ &= \boldsymbol{\Gamma}\boldsymbol{E}(\boldsymbol{\varepsilon}) \\ &= \boldsymbol{\Gamma}\boldsymbol{0} = \boldsymbol{0} \end{aligned}$$

and the covariance is

$$\begin{aligned} Cov(\boldsymbol{\varepsilon}^*) &= Cov(\boldsymbol{\Gamma}\boldsymbol{\varepsilon}) \\ &= \boldsymbol{\Gamma}\boldsymbol{Cov}(\boldsymbol{\varepsilon})\boldsymbol{\Gamma}^\top \\ &= \sigma^2 \boldsymbol{\Gamma}\boldsymbol{\Gamma}^\top \\ &= \sigma^2 \boldsymbol{I} \quad (\text{since } \Gamma \text{ is an orthogonal matrix}) \end{aligned}$$

b. $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}}^*$ **and** $s^{*2} = s^2$**, where** $\hat{\boldsymbol{\beta}}$ **and** $\hat{\boldsymbol{\beta}}$**** are the least squares estimates of** $\boldsymbol{\beta}$ **and** s^2 **and** s^{*2} **are the estimates of** σ^2 **obtained from the two models.**

From the classical simple linear regression, we know $\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{y}$ and $s^2 = \frac{\boldsymbol{\varepsilon}^\top \boldsymbol{\varepsilon}}{n-p}$.

Then, minimizing the sum of square errors, the least squares estimator $\hat{\boldsymbol{\beta}}$*:

$$\begin{aligned} \hat{\boldsymbol{\beta}}^* &= (\boldsymbol{X}^{*\top} \boldsymbol{X}^*)^{-1} \boldsymbol{X}^{*\top} \boldsymbol{y}^* \\ &= ((\boldsymbol{\Gamma}\boldsymbol{X})^\top \boldsymbol{\Gamma}\boldsymbol{X})^{-1} (\boldsymbol{\Gamma}\boldsymbol{X})^\top \boldsymbol{\Gamma}\boldsymbol{y} \\ &= (\boldsymbol{X}^\top \boldsymbol{\Gamma}^\top \boldsymbol{\Gamma}\boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{\Gamma}^\top \boldsymbol{\Gamma}\boldsymbol{y} \\ &= (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{y} \end{aligned}$$

is identical to the $\hat{\boldsymbol{\beta}}$. Likewise, the least squares estimator s^{2*}:

$$
\begin{aligned}
s^{2*} &= \frac{(\boldsymbol{\varepsilon}^*)^\top \boldsymbol{\varepsilon}^*}{n-p} \\
&= \frac{(\boldsymbol{\Gamma}\boldsymbol{\varepsilon})^\top \boldsymbol{\Gamma}\boldsymbol{\varepsilon}}{n-p} \\
&= \frac{\boldsymbol{\varepsilon}^\top \boldsymbol{\Gamma}^\top \boldsymbol{\Gamma}\boldsymbol{\varepsilon}}{n-p} \\
&= \frac{\boldsymbol{\varepsilon}^\top \boldsymbol{\varepsilon}}{n-p}
\end{aligned}
$$

is identical to the s^2.

5. (S.2.19) (the csv file of Exhibit 2.9 is attached in the mail) Exhibit 2.9 gives information on capital, labor and value added for each of three economic sectors: Food and kindred products (20), electrical and electronic machinery, equipment and supplies (36) and transportation equipment (37). The data were supplied by Dr. Philip Israelovich of the Federal Reserve Bank, who also suggested the exercise. For each sector:

```
library(tidyverse)
data <- read_csv("exhibit_2.9.csv") %>%
  select(-X1)
data
```

```
# A tibble: 15 x 10
    YEAR Cap.20 Cap.36  Cap.37 Lab.20 Lab.36  Lab.37 Val.20 Val.36 Val.37
   <dbl>  <dbl>  <dbl>   <dbl>  <dbl>  <dbl>   <dbl>  <dbl>  <dbl>  <dbl>
 1    72 243462 291610 1209188 708014 881231 1259142  6497.  6714. 11150
 2    73 252402 314728 1330372 699470 960917 1371795  5587.  7552. 12854.
 3    74 246243 278746 1157371 697628 899144 1263084  5521.  6776. 10451.
 4    75 263639 264050 1070860 674830 739485 1118226  5891.  5555.  9318.
 5    76 276938 286152 1233475 685836 791485 1274345  6549.  6590. 12098.
 6    77 290910 286584 1355769 678440 832818 1369877  6745.  7233. 12845.
 7    78 295616 280025 1351667 667951 851178 1451595  6694.  7417. 13310.
 8    79 301929 279806 1326248 675147 848950 1328683  6542.  7426. 13402.
 9    80 307346 258823 1089545 658027 779393 1077207  6587.  6411.  8571
10    81 302224 264913 1111942 627551 757462 1056231  6747.  6263.  8740.
11    82 288805 247491  988165 609204 664834  947502  7278.  5718.  8140
12    83 291094 246028 1069651 604601 664249 1057159  7515.  5937. 10958.
13    84 285601 256971 1191677 601688 717273 1169442  7540.  6659. 10839.
14    85 292026 248237 1246536 584288 678155 1195255  8333.  6633. 10030.
15    86 294777 261943 1281262 571454 670927 1171664  8506.  6651. 10836.
```

a. Consider the model

$$V_t = \alpha K_t^{\beta_1} L_t^{\beta_2} \eta_t$$

where the subscript t indicates year, V_t is value added, K_t is capital, L_t is labor and η_t is an error term, with $E[log(\eta_t)] = 0$ and $Var[log(\eta_t)]$ a constant. Assuming that the errors are independent, and taking logs of both sides of the above model, estimate β_1 and β_2.

If we take logs of both sides of the above equation, we have

$$log(V_t) = log(\alpha) + \beta_1 log(K_t) + \beta_2 log(L_t) + log(\eta_t)$$

is identical to the classic linear regression model, where $log(K_t)$, $log(L_t)$ are new covariates, $log(V_t)$ is new response variable and $log(\eta_t)$ is random error with $E[log(\eta_t)] = 0$ and $Var[log(\eta_t)] = constant$.

For the convenience, I take log of each variables (except YEAR) at first.

```
logdata <- data %>%
   mutate(across(contains("."), log))
logdata
```

```
# A tibble: 15 x 10
    YEAR Cap.20 Cap.36 Cap.37 Lab.20 Lab.36 Lab.37 Val.20 Val.36 Val.37
   <dbl>  <dbl>  <dbl>  <dbl>  <dbl>  <dbl>  <dbl>  <dbl>  <dbl>  <dbl>
 1    72   12.4   12.6   14.0   13.5   13.7   14.0   8.78   8.81   9.32
 2    73   12.4   12.7   14.1   13.5   13.8   14.1   8.63   8.93   9.46
 3    74   12.4   12.5   14.0   13.5   13.7   14.0   8.62   8.82   9.25
 4    75   12.5   12.5   13.9   13.4   13.5   13.9   8.68   8.62   9.14
 5    76   12.5   12.6   14.0   13.4   13.6   14.1   8.79   8.79   9.40
 6    77   12.6   12.6   14.1   13.4   13.6   14.1   8.82   8.89   9.46
 7    78   12.6   12.5   14.1   13.4   13.7   14.2   8.81   8.91   9.50
 8    79   12.6   12.5   14.1   13.4   13.7   14.1   8.79   8.91   9.50
 9    80   12.6   12.5   13.9   13.4   13.6   13.9   8.79   8.77   9.06
10    81   12.6   12.5   13.9   13.3   13.5   13.9   8.82   8.74   9.08
11    82   12.6   12.4   13.8   13.3   13.4   13.8   8.89   8.65   9.00
12    83   12.6   12.4   13.9   13.3   13.4   13.9   8.92   8.69   9.30
13    84   12.6   12.5   14.0   13.3   13.5   14.0   8.93   8.80   9.29
14    85   12.6   12.4   14.0   13.3   13.4   14.0   9.03   8.80   9.21
15    86   12.6   12.5   14.1   13.3   13.4   14.0   9.05   8.80   9.29
```

Then we apply `lm()` function in R on each economic sector to get β_1 and β_2. The result is as follows

```
lm.20 <- lm(Val.20 ~ 1 + Cap.20 + Lab.20, logdata)
lm.36 <- lm(Val.36 ~ 1 + Cap.36 + Lab.36, logdata)
lm.37 <- lm(Val.37 ~ 1 + Cap.37 + Lab.37, logdata)
lm.list <- list(lm.20, lm.36, lm.37)
```

```
getCoefficient <- function(lm.list, coefNames){
  .coef <- lapply(lm.list, coefficients)
  coef <- do.call(rbind, .coef)
  colnames(coef) <- coefNames
  coef <- as.tibble(coef) %>%
  mutate(sector = c("(20)", "(36)", "(37)"),
         .before = `log(alpha)`)
}
coef <- getCoefficient(lm.list, coefNames = c("log(alpha)", "beta1", "beta2"))
coef %>% mutate(alpha = exp(`log(alpha)`), .before = `log(alpha)`)
```

```
# A tibble: 3 x 5
  sector    alpha `log(alpha)` beta1  beta2
  <chr>     <dbl>        <dbl> <dbl>  <dbl>
1 (20)   1.18e+11         25.5 0.227 -1.46
2 (36)   2.91e- 1        -1.23 0.526  0.254
3 (37)   6.60e- 5        -9.63 0.506  0.845
```

Then we get the estimated model for three sectors:

- For the sector 20: $V_t = 1.18 \times 10^{11} K_t^{0.23} L_t^{-1.46} \eta_t$.
- For the sector 36: $V_t = 0.29 K_t^{0.53} L_t^{0.25} \eta_t$.
- For the sector 37: $V_t = 6.60 \times 10^{-5} K_t^{0.51} L_t^{0.85} \eta_t$.

b. The model given in (a) above is said to be of the Cobb-Douglas form. It is easier to interpret if $\beta_1 + \beta_2 = 1$. Estimate β_1 and β_2 under this constraint.

Since the constraint $\beta_1 + \beta_2 = 1$, we can reparamterize the model in (a) and get

$$\begin{aligned} & log(V_t) = log(\alpha) + \beta_1 log(K_t) + \beta_2 log(L_t) + log(\eta_t) \\ \Rightarrow \quad & log(V_t) = log(\alpha) + \beta_1 log(K_t) + (1 - \beta_1) log(L_t) + log(\eta_t) \\ \Rightarrow \quad & (log(V_t) - log(L_t)) = log(\alpha) + \beta_1 (log(K_t) - log(L_t)) + log(\eta_t) \end{aligned}$$

Then we use `lm()` function to fit this model

```
lm.cnstr.20 <- lm(I(Val.20 - Lab.20) ~ 1 + I(Cap.20 - Lab.20), logdata)
lm.cnstr.36 <- lm(I(Val.36 - Lab.36) ~ 1 + I(Cap.36 - Lab.36), logdata)
lm.cnstr.37 <- lm(I(Val.37 - Lab.37) ~ 1 + I(Cap.37 - Lab.37), logdata)
lm.cnstr.list <- list(lm.cnstr.20, lm.cnstr.36, lm.cnstr.37)
```

```
coef.cnstr <- getCoefficient(lm.cnstr.list, coefNames = c("log(alpha)", "beta
1"))
coef.cnstr %>%
  mutate(alpha = exp(`log(alpha)`), .before = `log(alpha)`) %>%
  mutate(beta2 = 1 - beta1)
```

```
# A tibble: 3 x 5
  sector   alpha `log(alpha)`   beta1   beta2
  <chr>    <dbl>        <dbl>   <dbl>   <dbl>
1 (20)   0.0307         -3.48 1.29    -0.290
2 (36)   0.0220         -3.82 0.900    0.0999
3 (37)   0.00898        -4.71 0.00961  0.990
```

From about result, we get the estimated model for three sectors:

- For the sector 20: $V_t = 0.03 K_t^{1.29} L_t^{-0.29} \eta_t$.
- For the sector 36: $V_t = 0.02 K_t^{0.90} L_t^{0.10} \eta_t$.
- For the sector 37: $V_t = 0.01 K_t^{0.01} L_t^{0.99} \eta_t$.

c. Sometimes the model

$$V_t = \alpha \gamma^t K_t^{\beta_1} L_t^{\beta_2} \eta_t$$

is considered where γ_t is assumed to account for technological development. Estimate β_1 and β_2 for this model.

Again, we take logs of both sides of the above equation, we have

$$log(V_t) = log(\alpha) + log(\gamma) t + \beta_1 log(K_t) + \beta_2 log(L_t) + log(\eta_t)$$

where t(YEAR), $log(K_t)$ and $log(L_t)$ are new covariates, $log(V_t)$ is new response variable, $log(\eta_t)$ is random error and add a new coefficient $log(\gamma)$ corresponding to the covariate t(YEAR). Then we use `lm()` function to fit this model

```
lm2.20 <- lm(Val.20 ~ 1 + YEAR + Cap.20 + Lab.20, logdata)
lm2.36 <- lm(Val.36 ~ 1 + YEAR + Cap.36 + Lab.36, logdata)
lm2.37 <- lm(Val.37 ~ 1 + YEAR + Cap.37 + Lab.37, logdata)
lm2.list <- list(lm2.20, lm2.36, lm2.37)
```

```
coef2 <- getCoefficient(lm2.list, coefNames = c("log(alpha)", "log(eta)", "bet
a1", "beta2"))
coef2 %>%
  mutate(alpha = exp(`log(alpha)`), .before = `log(alpha)`) %>%
  mutate(eta = exp(`log(eta)`), .before = `log(eta)`)
```

```
# A tibble: 3 x 7
  sector   alpha `log(alpha)`   eta `log(eta)`  beta1  beta2
  <chr>    <dbl>        <dbl> <dbl>      <dbl>  <dbl>  <dbl>
1 (20)   3.11e+8         19.6  1.01    0.0110  0.0444 -0.908
2 (36)   2.02e-7        -15.4  1.03    0.0250  0.821   0.882
3 (37)   4.42e-5        -10.0  1.00    0.00458 0.159   1.20
```

From about result, we get the estimated model for three sectors:

- For the sector 20: $V_t = 3.10 \times 10^8 (1.01)^t K_t^{0.04} L_t^{-0.91} \eta_t$.
- For the sector 36: $V_t = 2.02 \times 10^{-7} (1.02)^t K_t^{0.82} L_t^{0.88} \eta_t$.
- For the sector 37: $V_t = 4.42 \times 10^{-5} (1.00)^t K_t^{0.16} L_t^{1.20} \eta_t$.

d. Estimate β_1 and β_2 in the model in (c) , under the constraint $\beta_1 + \beta_2 = 1$.

Since the constraint $\beta_1 + \beta_2 = 1$, similarly to the procedure in (b), reparamterizing the model in (c) can get

$$
\begin{aligned}
& log(V_t) = log(\alpha) + log(\gamma)t + \beta_1 log(K_t) + \beta_2 log(L_t) + log(\eta_t) \\
\Rightarrow \quad & log(V_t) = log(\alpha) + log(\gamma)t + \beta_1 log(K_t) + (1-\beta_1) log(L_t) + log(\eta_t) \\
\Rightarrow \quad & (log(V_t) - log(L_t)) = log(\alpha) + log(\gamma)t + \beta_1 (log(K_t) - log(L_t)) + log(\eta_t)
\end{aligned}
$$

Then we use `lm()` function to fit this model

```
lm2.cnstr.20 <- lm(I(Val.20 - Lab.20) ~ 1 + YEAR + I(Cap.20 - Lab.20), logdat
a)
lm2.cnstr.36 <- lm(I(Val.36 - Lab.36) ~ 1 + YEAR + I(Cap.36 - Lab.36), logdat
a)
lm2.cnstr.37 <- lm(I(Val.37 - Lab.37) ~ 1 + YEAR + I(Cap.37 - Lab.37), logdat
a)
lm2.cnstr.list <- list(lm2.cnstr.20, lm2.cnstr.36, lm2.cnstr.37)
```

```
coef2.cnstr <- getCoefficient(lm2.cnstr.list, coefNames = c("log(alpha)", "log
(eta)", "beta1"))
coef2.cnstr %>%
  mutate(alpha = exp(`log(alpha)`), .before = `log(alpha)`) %>%
  mutate(eta = exp(`log(eta)`), .before = `log(eta)`) %>%
  mutate(beta2 = 1 - beta1)
```

```
# A tibble: 3 x 7
  sector     alpha `log(alpha)`   eta `log(eta)`   beta1 beta2
  <chr>      <dbl>        <dbl> <dbl>      <dbl>   <dbl> <dbl>
1 (20)   0.0000924        -9.29  1.06    0.0546  -0.495  1.49
2 (36)   0.00232          -6.07  1.02    0.0169   0.0345 0.965
3 (37)   0.00641          -5.05  1.00    0.00426 -0.317  1.32
```

From about result, we get the estimated model for three sectors:

- For the sector 20: $V_t = 9.24 \times 10^{-5} (1.06)^t K_t^{-0.49} L_t^{1.49} \eta_t$.
- For the sector 36: $V_t = 2.31 \times 10^{-3} (1.02)^t K_t^{0.03} L_t^{0.97} \eta_t$.
- For the sector 37: $V_t = 6.41 \times 10^{-3} (1.00)^t K_t^{-0.32} L_t^{1.32} \eta_t$.