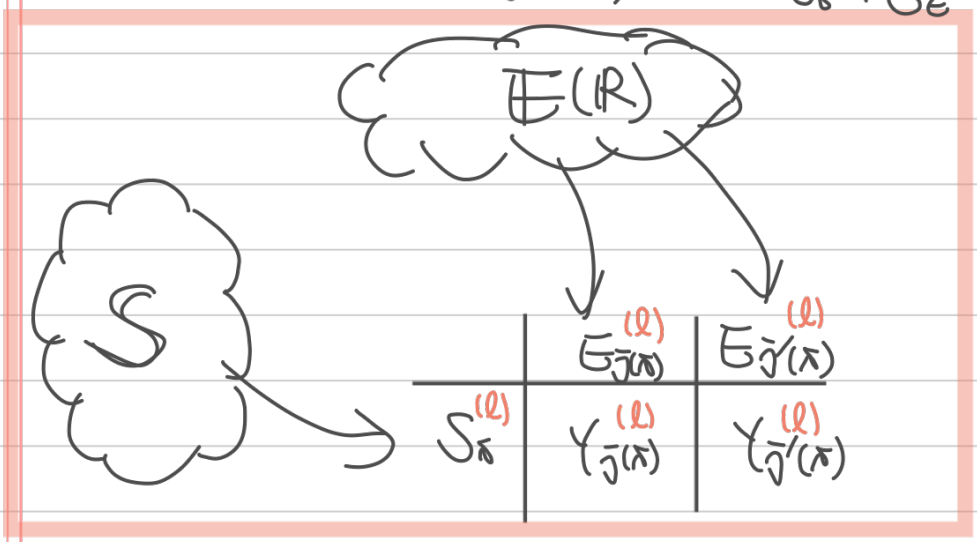
Case 2: One-way ANOVA

* Model: $Y_{j(i)} = \mu + S_i + E_{j(i)}$

- $S_i \sim N(0, \sigma_S^2)$
- $E_{j(i)} \sim N(0, \sigma_E^2)$

$\Big) \perp\!\!\!\perp$

* $ICC(2) = Cor(Y_{j(i)}, Y_{j'(i)}) = \dfrac{\sigma_S^2}{\sigma_S^2 + \sigma_E^2}$



$\Downarrow$

$l = 1, \dots, N^{Rep}$

	$E_{1(1)}$ $E_{2(1)}$	$E_{1(2)}$ $E_{2(2)}$	$E_{1(3)}$ $E_{2(3)}$	...
S_1				
S_2				
S_3				
$\vdots$				

$\Rightarrow$

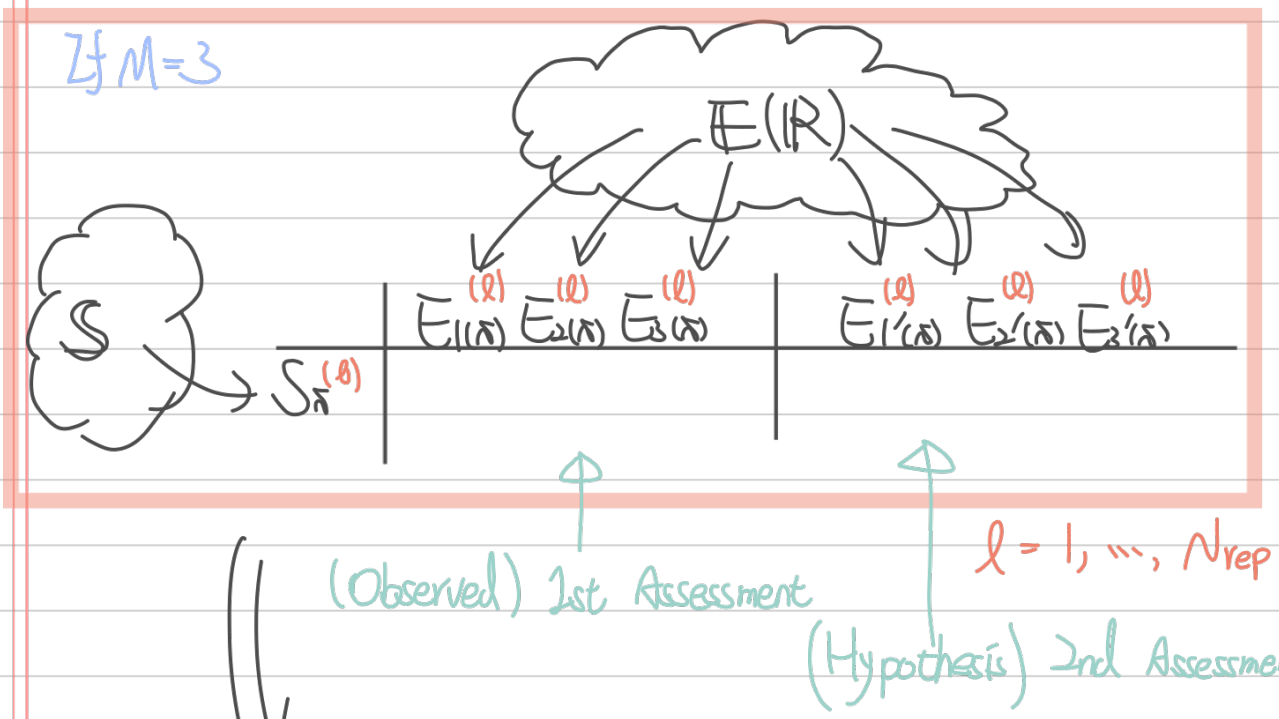
	$E_{j(i)}$	$E_{j'(i)}$
S_1		
S_2		
S_3		
$\vdots$	$\vdots$	$\vdots$

$\Rightarrow Cor(\ ,\ )$

$$* \quad ICC(k=M) = Cor\left(\overline{Y}_{\bullet(s)}, \overline{Y}_{\bullet'(s)}\right) = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_\varepsilon^2/M}$$

- $\overline{Y}_{\bullet(s)} = \frac{1}{M}\sum_{j=1}^{M} Y_{j(s)}$

If $M=3$



	$E_{1(1)}$ $E_{2(1)}$ $E_{3(1)}$	$E_{1'(1)}$ $E_{2'(1)}$ $E_{3'(1)}$	$E_{1(2)}$ $E_{2(2)}$ $E_{3(2)}$	$E_{1'(2)}$ $E_{2'(2)}$ $E_{3'(2)}$	...
S_1	$\times$ $\times$ $\times$	$\times$ $\times$ $\times$ $\times$			
S_2			$\times$ $\times$ $\times$	$\times$ $\times$ $\times$	
S_3					
$\vdots$					

	$\overline{E}_{\bullet(s)}$	$\overline{E}_{\bullet'(s)}$
S_1		
S_2		
S_3		
$\vdots$	$\vdots$	

$\Rightarrow Cor(\quad)$

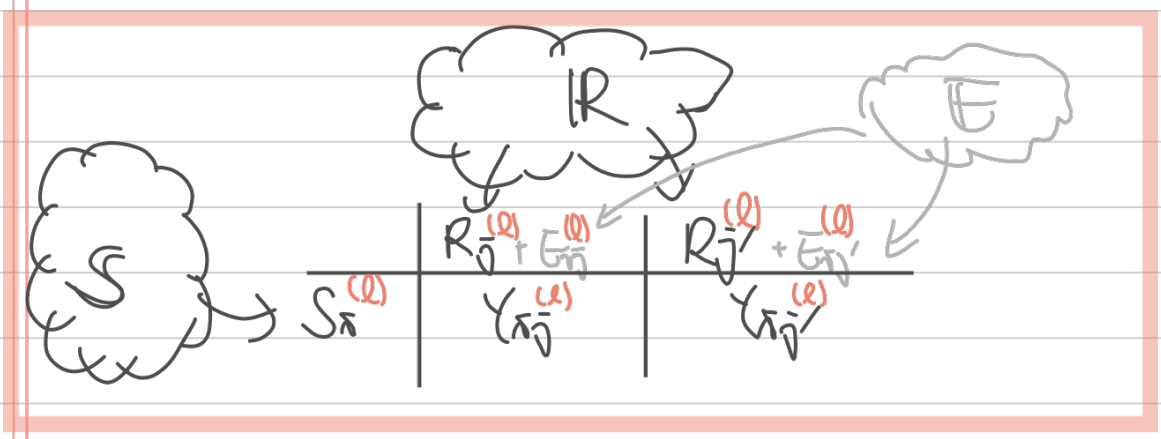
Case 2: Two-way Random-Effect Model

* Model: $Y_{ij} = \mu + S_i + R_j + E_{ij}$

  - $S_i \sim N(0, \sigma_S^2)$
  - $R_j \sim N(0, \sigma_R^2)$
  - $E_{ij} \sim N(0, \sigma_E^2)$

  $\perp\!\!\!\perp$

* $ICC(A,1) = Cor(Y_{ij}, Y_{ij'}) = \dfrac{\sigma_S^2}{\sigma_S^2 + \sigma_R^2 + \sigma_E^2}$



$\Downarrow$

$l = 1, \dots, N_{rep}$

	R_1	R_1'	R_2	R_2'	R_3	R_3'	...
S_1							
S_2							
S_3							
⋮							

$\Rightarrow Cor(\ ,\ )$

different subjects use different pairs of raters

Problem What is the difference between Cases 1 & 2?

I suppose it should be something like the data matrix we got

	R_1	R_2	R_3	R_4
S_1				
S_2				
S_3				

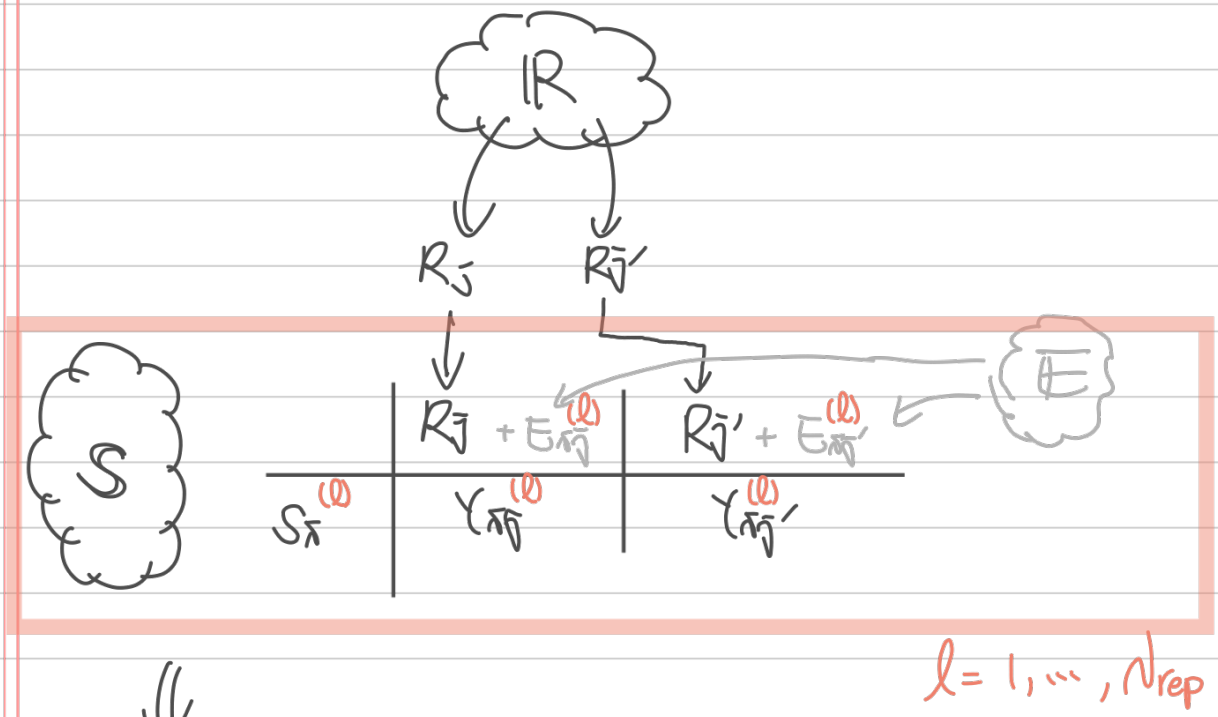
$\Rightarrow Cor(\ ,\ )$

subjects are evaluated by the same set of raters

$$* \quad ICC(A,k) = Cor(\bar{Y}_{\bar{i}\cdot}, Y_{\bar{i}\cdot'}) = \frac{\sigma_S^2}{\sigma_S^2 + \sigma_R^2/M + \sigma_E^2/M}$$

$\boxed{\text{Skip}}$

$$* \quad ICC(C,1) = Cor(Y_{\bar{i}\bar{j}}, Y_{\bar{i}\bar{j}'} \mid \bar{j}, \bar{j}') = \frac{\sigma_S^2}{\sigma_S^2 + \sigma_E^2}$$



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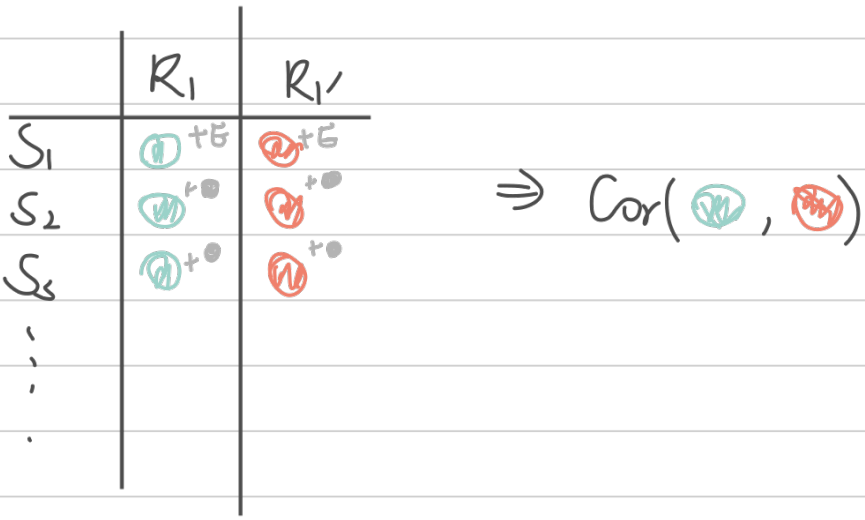


$$* \quad ICC(C,k) = Cor(\bar{Y}_{\bar{i}\cdot}, \bar{Y}_{\bar{i}\cdot'} \mid \forall \bar{j}, \forall \bar{j}') = \frac{\sigma_S^2}{\sigma_S^2 + \sigma_E^2}$$

$\boxed{\text{Skip}}$

Case 3 : Two-way Mixed-Effect Model

* Model = $Y_{ij} = \mu + S_i + R_j + E_{ij}$

  - $S_i \sim N(0, \sigma_S^2)$
  - $\sum_{j=1}^{M} R_j = 0$, $\theta_R^2 \triangleq \frac{\sum R_j^2}{M-1}$
  - $E_{ij} \sim N(0, \sigma_E^2)$

* $ICC(A,1) = \frac{\sigma_S^2}{\sigma_S^2 + \theta_R^2 + \sigma_E^2}$ (McGraw & Wang, 1996)

$Cor(Y_{ij}, Y_{ij'}) \overset{?}{=} \frac{\sigma_S^2}{\sigma_S^2 + \sigma_E^2} = ICC(C,1)$

$Cor(Y_{IJ}, Y_{IJ'}) \overset{?}{\neq} \frac{\sigma_S^2 - \frac{1}{M-1}\theta_R^2}{\sigma_S^2 + \theta_R^2 + \sigma_E^2}$

$\boxed{\text{Question}}$ $Var(Y_{ij}) = Var(\mu + S_i + R_j + E_{ij}) = \sigma_S^2 + \sigma_R^2$

$= E[(Y_{ij} - E[Y_{ij}])^2] = E[(S_i + E_{ij})^2] = \sigma_S^2 + \sigma_E^2$

$\mu + S_i + R_j + E_{ij}$ $\quad$ $\mu + 0 + R_j + 0$

Random effect model : $E[MS_R] = \sigma_E^2 + N\sigma_R^2$

Fixed effect model : $E[MS_R] = \sigma_E^2 + N \underbrace{\boxed{\frac{1}{M-1}\sum_{j=1}^{M} R_j^2}}_{\theta_R^2}$

$\frac{1}{M}\sum_{j=1}^{M}(\bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot})^2$ $\quad$ $= \sigma_E^2 + N\theta_R^2$

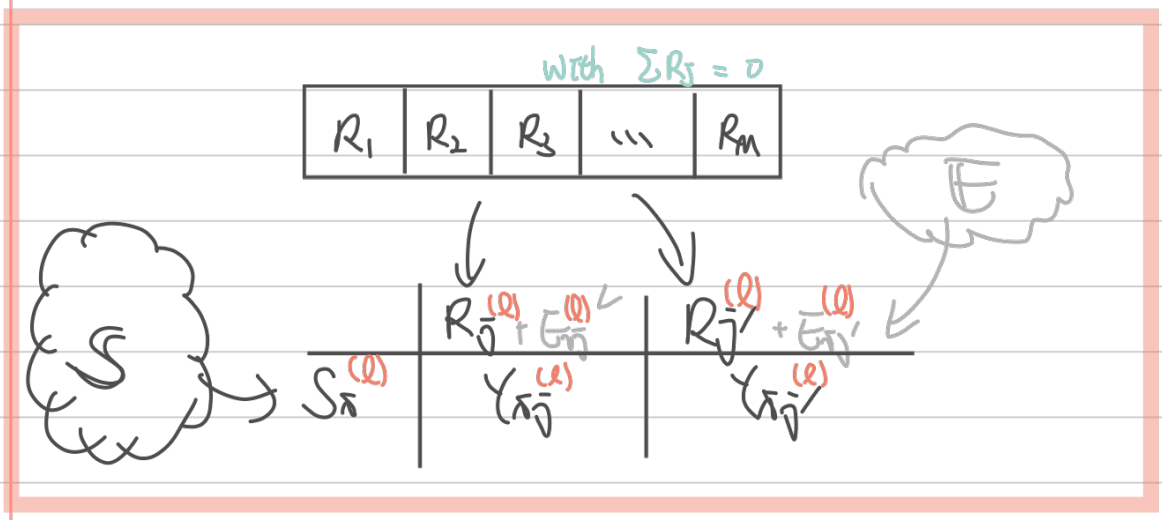
If I allow, say $\mathbb{R} = \{R_1, R_2, \dots, R_M\}$, a finite set exist. Let R_j is sampled from this set $\mathbb{R}$ with the probability $= \frac{1}{M}$

Could I say $Var(R_j) = \theta_R^2$ and $Cov(R_j, R_{j'}) = -\frac{M}{M-1}\theta_R^2$??

$$0 = Var\left(\sum R_j\right) = \sum_{j=1}^{M} Var(R_j) + \sum\sum_{j \neq j'} Cov(R_j, R_{j'})$$

$$\Rightarrow \quad 0 = M\theta_R^2 + M(M-1)\, Cov(R_j, R_{j'})$$

$$\Rightarrow \quad Cov(R_j, R_{j'}) = -\frac{\theta_R^2}{M-1}$$



$l = 1, \dots, N_{rep}$

But, here, we still randomly sample pairs of raters for each subject, so

$$cor(\cdot, \cdot) = \frac{\sigma_S^2 - \frac{1}{M-1}\theta_R^2}{\sigma_S^2 + \theta_R^2 + \sigma_E^2} \neq ICC(A,1)$$

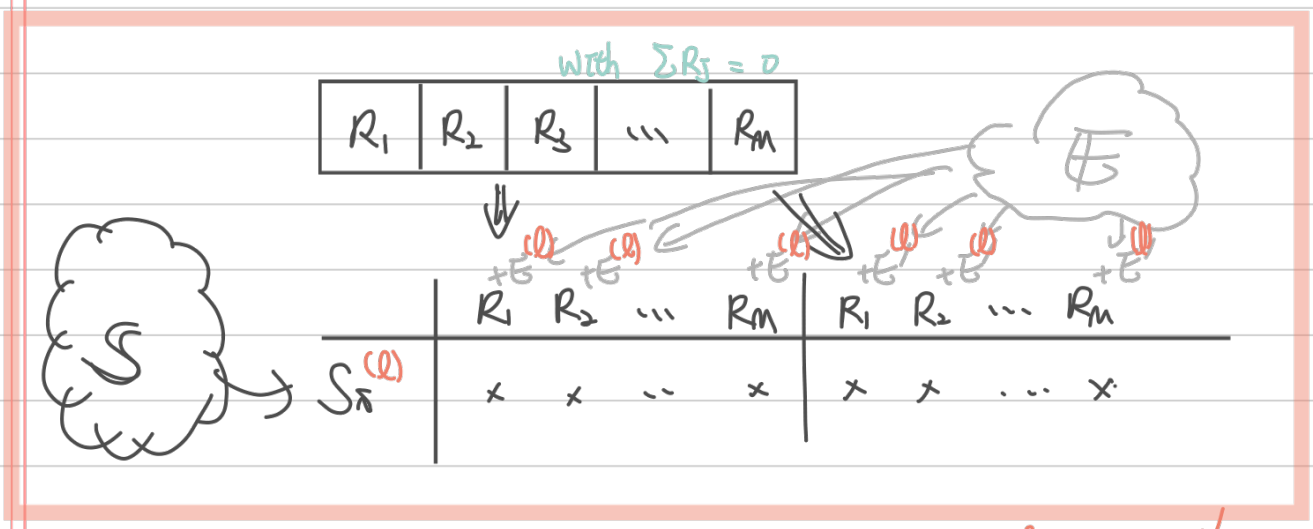
$$\neq ICC(C,1)$$

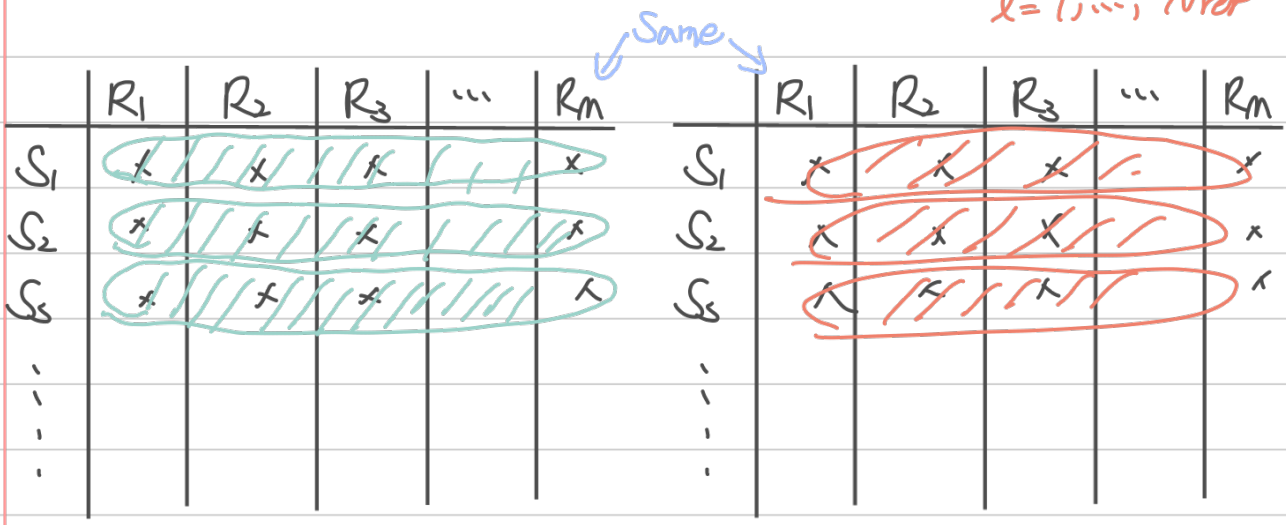
$$* \quad ICC(A,k) = \frac{\sigma_S^2}{\sigma_S^2 + \sigma_R^2/m + \sigma_E^2/m} \qquad (M\&W, 1996)$$

$$\parallel$$

$$Cor(\bar{Y}_{i\cdot}, \bar{Y}_{i\cdot'}) = Cor\Big(\mu + S_i + \cancel{\tfrac{1}{m}\sum R_j}^{\,0} + \tfrac{1}{m}\sum E_{ij},\ \mu + S_i + \cancel{\tfrac{1}{m}\sum R_{j'}}^{\,0} + \tfrac{1}{m}\sum E_{ij}\Big)$$

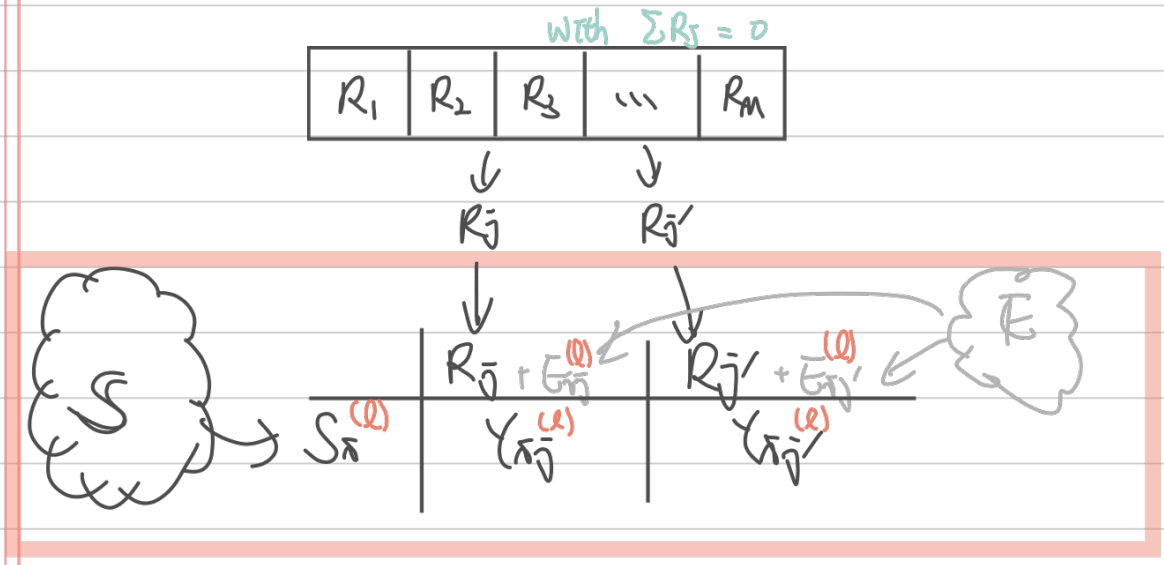
$$= \frac{\sigma_S^2}{\sigma_S^2 + \sigma_E^2/m} = ICC(C,k)$$





$$\Rightarrow Cor(\ ,\ ) = ICC(C, k=m)$$

$$* \quad ICC(C,1) = Cor(Y_{\bar{s}\bar{j}}, Y_{\bar{s}\bar{j}'} \mid \bar{j}, \bar{j}')$$



$l = 1, \dots, N_{rep}$

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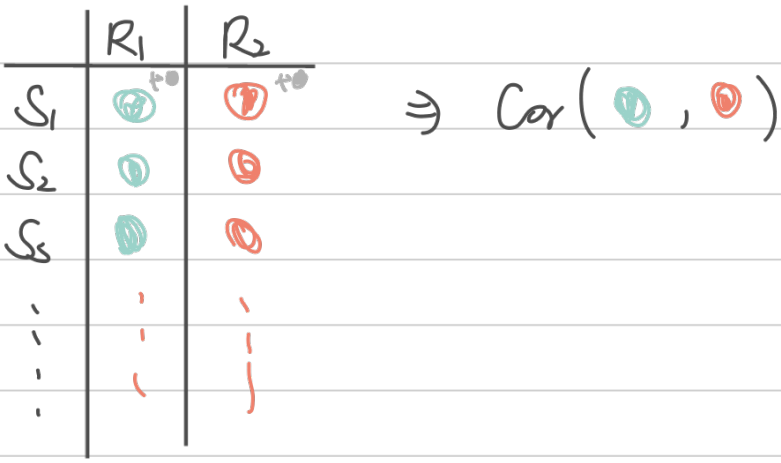
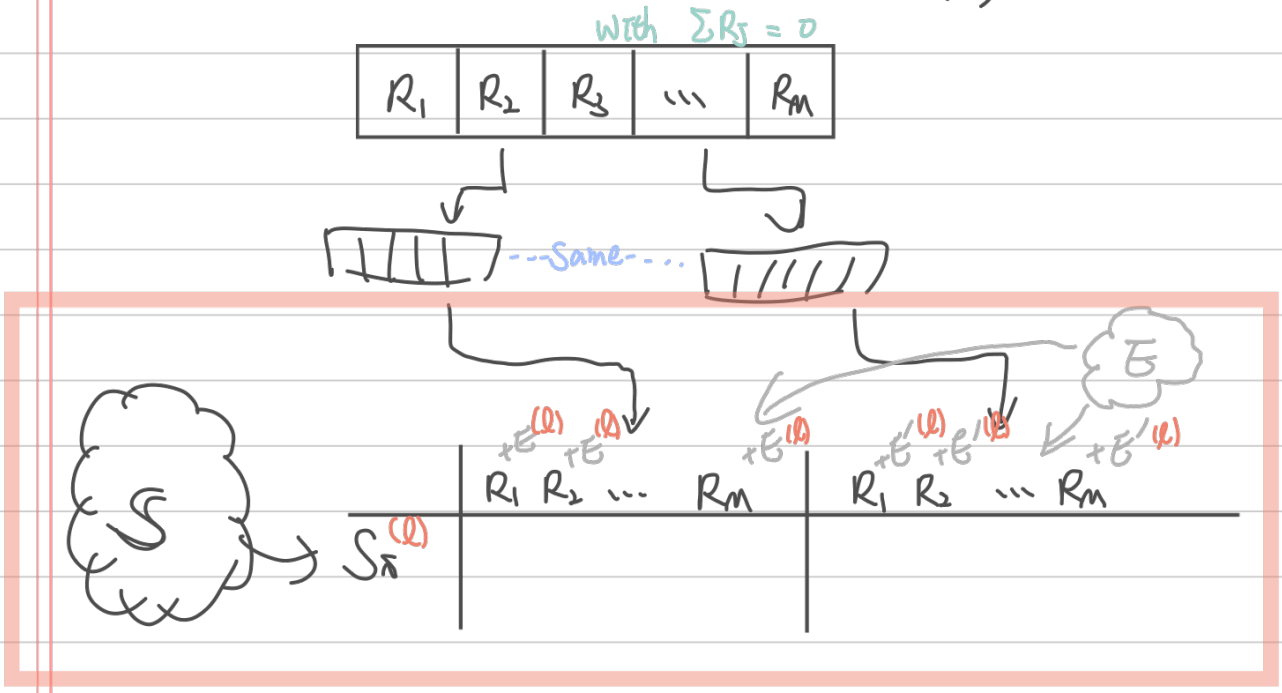


$$* \quad ICC(C,k) = Cor(\bar{Y}_{\bar{s}\bullet}, \bar{Y}_{\bar{s}\bullet'} \mid \forall \bar{j}, \forall \bar{j}')$$



$l = 1, 2, \dots, N_{rep}$

Same

	R_1	R_2	R_3	...	R_m
S_1	x	x	x		x
S_2	x	x	x		x
S_3	x	x	x		x
⋮					

	R_1	R_2	R_3	...	R_m
S_1	x	x	x		x
S_2	x	x	x		x
S_3	x	x	x		x
⋮					

$$\Rightarrow \mathrm{Cor}(\ ,\ ) = \mathrm{ICC}(C, k=m)$$