Integrable 2d trigonometric $\mathbb{Z}_N$-twisted σ-models

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Integrability in classical field theory

A system is said to be integrable if it possesses an **infinite number of conserved quantities** ($\{\mathcal{H}, \mathcal{Q}_i\} = 0$) that are in **involution** ($\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0$). Sufficient conditions are:

- A flat Lax matrix $\mathcal{L}(x; z)$: $\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] = 0$.
- The Lax satisfies a Maillet bracket: $\{\mathcal{L}_{x\underline{\mathbf{1}}}(x;\mu), \mathcal{L}_{x\underline{\mathbf{2}}}(y;\nu)\} = [\mathcal{R}_{\underline{\mathbf{12}}}(\mu,\nu), \mathcal{L}_{x\underline{\mathbf{1}}}(x;\mu)]\delta_{xy} - [\mathcal{R}_{\underline{\mathbf{21}}}(\nu,\mu), \mathcal{L}_{x\underline{\mathbf{2}}}(y;\nu)]\delta_{xy} - (\mathcal{R}_{\underline{\mathbf{12}}}(\mu,\nu) + \mathcal{R}_{\underline{\mathbf{21}}}(\nu,\mu))\delta'_{xy}$.

4d Chern-Simons

Starting with a **4d Chern-Simons** action on $\mathbb{C}\mathbf{P}^1 \times \Sigma$ with **disorder defects** (zeros of ω)

$$\mathcal{S} = \frac{i}{4\pi} \int_{\mathbb{C}\mathbf{P}^1 \times \Sigma} \omega \wedge \mathrm{CS}_3(A) ,$$

where

- $\mathbb{C}\mathbf{P}^1 = (z, \bar{z})$, a sphere, and $\Sigma = (t, x)$, a 2d space-time manifold,
- $\omega = \varphi(z) dz$, a meromorphic 1-form,
- $\mathrm{CS}_3(A) = \mathrm{tr}(AdA) + \frac{1}{3}\mathrm{tr}(A[A,A])$, a Chern-Simons 3-form,
- $A = A_\mu d\sigma^\mu + A_{\bar{z}} d\bar{z} + A_z dz \rightsquigarrow A_\mu d\sigma^\mu + A_{\bar{z}} d\bar{z}$, a 1-form gauge field.

Redefine A in terms of $\hat{g} \in G^{\mathbb{C}}$ and $\mathcal{L}$ as

$$A = \hat{g}\mathcal{L}\hat{g}^{-1} - d\hat{g}\hat{g}^{-1} = \mathcal{L}^{\hat{g}} ,$$

such that $\mathcal{L} = \mathcal{L}_\mu d\sigma^\mu$ and $\mathcal{L}_{\bar{z}} = 0$. Then the 4d Chern-Simons action can be written as

$$\mathcal{S} = \frac{i}{4\pi} \int_{\mathcal{M}} \omega \wedge \mathrm{CS}_3(\mathcal{L}) + \frac{i}{4\pi} \int_{\mathcal{M}} d\omega \wedge \mathrm{tr}(\hat{g}^{-1} d\hat{g}\, \mathcal{L}) + \frac{i}{4\pi} \int_{\mathcal{M}} \omega \wedge \mathrm{tr}(\hat{g}^{-1} d\hat{g}\, [\hat{g}^{-1} d\hat{g}, \hat{g}^{-1} d\hat{g}]) .$$

The first term is a **bulk** term, the second one is a **boundary** term, and the third is the **Wess-Zumino** term.

Gauge symmetries:

- **External gauge symmetry (e.g.s.)** (on $\mathbb{C}\mathbf{P}^1 \times \Sigma$): $\quad A \to uAu^{-1} - duu^{-1} , \quad \hat{g} \to u\hat{g} .$
- **Internal gauge symmetry (i.g.s.)** (on Σ): $\quad \mathcal{L} \to v^{-1}\mathcal{L}v + v^{-1}dv , \quad \hat{g} \to \hat{g}v .$

Bulk equations of motion

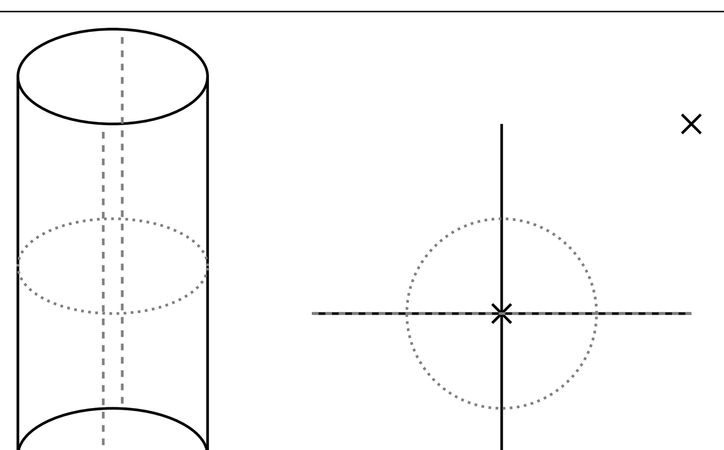
The bulk equations of motion are:

- $\omega \wedge (d_\Sigma \mathcal{L} + \frac{1}{2}[\mathcal{L}, \mathcal{L}]) = 0,$
- $\omega \wedge \partial_{\bar{z}} \mathcal{L} = 0.$

This tells us that:

- $\mathcal{L}$ is flat: $\mathcal{L}(t, x; z)$: $\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] = 0$,
- $\mathcal{L}$ is a meromorphic function of z and is allowed poles at the zeroes of ω: $\mathcal{L}_\pm = \tilde{V}_\pm + \sum_{s \in \mathfrak{z}_\pm} \frac{W_\pm^{(s)}}{z-s}$.

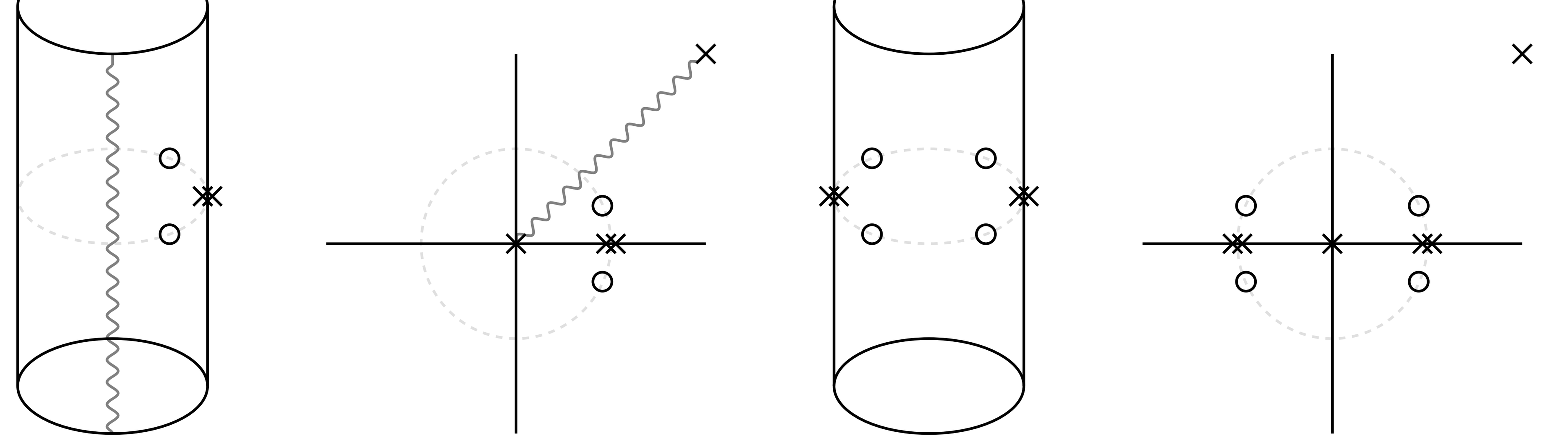
Trigonometric models in 4d Chern-Simons



We consider a setup with simple poles at $z \in \{0, \infty\}$ in the $\mathbb{C}\mathbf{P}^1$. Then we transform to a cylinder by using $z = e^{iw}$, where w, with $\mathrm{Re}(w)$ with $\mathrm{Re}(w) \in (-\pi, \pi]$, is the coordinate on the cylinder. Redefine ω to ω_{trig} such that we split the poles at $\{0, \infty\}$ from the twist function:

$$\omega_{\mathrm{trig}} = \varphi_{\mathrm{trig}}(z) \frac{dz}{z} .$$

Twists of trigonometric models

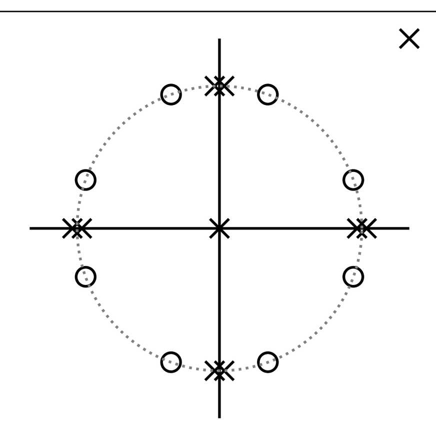


We twist by cutting along the non-compact direction of the cylinder, and create N copies of it, then we glue them together to get the N-fold cover. Therefore, for each field, we will get N copies. We relate these fields by an automorphism σ.

- **Automorphism**: $[\sigma(X), \sigma(Y)] = \sigma([X, Y])$ and $\sigma^N = 1$, where $X, Y \in \mathfrak{g}^{\mathbb{C}}$.
- **Projection operators**:

$$P_0 = \frac{1}{N} \sum_{a \in \mathbb{Z}_N} \sigma^a , \qquad P_a = \frac{1}{N} \sum_{b \in \mathbb{Z}_N} e^{-\frac{2i\pi ab}{N}} \sigma^b .$$

Twist function, defect terms and boundary conditions



$$\omega_{\mathrm{trig}}^{(N)} = -c_0 \hbar \frac{(z^N - e^{N\alpha})(z^N - e^{-N\alpha})}{(z^N - 1)^2} \frac{dz}{z} .$$

Defect terms:

$$-c_0 \hbar \int_\Sigma d^2\sigma \left(\mathrm{tr}(A\delta A)|_{z=0} - \mathrm{tr}(A\delta A)|_{z=\infty} - \frac{4}{N^2} \sinh^2(\frac{N\alpha}{2}) \sum_{a \in \mathbb{Z}_N} e^{\frac{2i\pi a}{N}} \partial_z \,\mathrm{tr}(A\delta A)\big|_{z=e^{\frac{2i\pi a}{N}}} \right) .$$

Boundary conditions:

- Double pole boundary conditions:

$$A\big|_{z=e^{\frac{2i\pi a}{N}}} = 0 , \qquad (\hat{g}, \partial_z \hat{g})\big|_{z=e^{\frac{2i\pi a}{N}}} \xrightarrow{\text{e.g.s.}} (\hat{\sigma}^a(g), 0) , \qquad a \in \mathbb{Z}_N .$$

- η-type boundary conditions:

$$(\tilde{\mathcal{R}} + c_0) A|_{z=0} = (\tilde{\mathcal{R}} - c_0) A|_{z=\infty} , \qquad (\hat{g}|_{z=0}, \hat{g}|_{z=\infty}) \xrightarrow{\text{e.g.s.}} (\tilde{g}, \tilde{g}) \xrightarrow{\text{i.g.s.}} (1, 1) .$$

$$[\tilde{\mathcal{R}}X, \tilde{\mathcal{R}}Y] - \tilde{\mathcal{R}}([\tilde{\mathcal{R}}X, Y] + [X, \tilde{\mathcal{R}}Y]) + c_0^2[X, Y] = 0 , \qquad \mathrm{tr}(X\tilde{\mathcal{R}}Y) + \mathrm{tr}((\tilde{\mathcal{R}}X)Y) = 0 .$$

$\mathbb{Z}_N$-twisted η-model

Using $c_0\eta = \tanh(N\alpha/2)$, $\tilde{j} = \tilde{g}^{-1} d\tilde{g}$, $j = g^{-1}dg$, $J^{(a)} = P_a j$, $\mathcal{R}^0_{\underline{\mathbf{12}}}(\mu, \nu) = \varphi_{\mathrm{trig}} \mathcal{R}_{\underline{\mathbf{12}}}(\mu, \nu)$, and $C_{\underline{\mathbf{12}}} = T_a \otimes T^a$, then:

Lax:

$$\mathcal{L}^{(N)}_{\eta\pm}(z, x) = \left(c_0 + \frac{2c_0}{\frac{1\pm c_0\eta}{1\mp c_0\eta} z^N - 1} + \tilde{\mathcal{R}} \right) \frac{\pm\eta}{1 \pm \eta\tilde{\mathcal{R}}} J^{(0)}_\pm(x) \pm \frac{2c_0\eta}{1 \mp c_0\eta} \frac{1}{\frac{1\pm c_0\eta}{1\mp c_0\eta} z^N - 1} \sum_{a \in \mathbb{Z}_N \setminus \{0\}} z^a J^{(a)}_\pm(x) .$$

Action:

$$\mathcal{S}^{(N)}_\eta(g) = -\frac{4c_0^2\hbar\eta}{1 - c_0^2\eta^2} \int d^2\sigma \;\mathrm{tr}\left(J^{(0)}_+ \frac{1 - c_0^2\eta^2}{1 - \eta\tilde{\mathcal{R}}} J^{(0)}_- + \sum_{a \in \mathbb{Z}_N \setminus \{0\}} \left(1 + c_0\eta\left(1 - \frac{2a}{N}\right)\right) J^{(a)}_+ J^{(N-a)}_- \right) .$$

$\mathcal{R}$-matrix:

$$\mathcal{R}^0_{\underline{\mathbf{12}}}(\mu, \nu) = -\frac{1}{8}\left(\left(\frac{\mu^N + \nu^N}{\mu^N - \nu^N} - \frac{1}{c_0}\tilde{\mathcal{R}}_{\underline{\mathbf{2}}} \right) P_{0\underline{\mathbf{2}}} + \frac{2}{\mu^N - \nu^N} \sum_{a \in \mathbb{Z}_N \setminus \{0\}} \mu^{n-a}\nu^a P_{a\underline{\mathbf{2}}} \right) C_{\underline{\mathbf{12}}}.$$

When σ is an outer automorphism this coincides with the trigonometric $\mathcal{R}$-matrix of the twisted quantum group.

New models

Actions of untwisted and $\mathbb{Z}_2$-twisted η-models are given by:

$$\mathcal{S}^{(1)}_\eta = \int d^2\sigma \;\mathrm{tr}\left(j_+ \frac{1 + \eta^2}{1 - \eta\tilde{\mathcal{R}}} j_- \right) , \qquad \mathcal{S}^{(2)}_\eta = \int d^2\sigma \;\mathrm{tr}\left(j_+ P_0 \frac{1 + \eta^2}{1 - \eta\tilde{\mathcal{R}}} j_- + j_+ P_1 j_- \right) .$$

Examples:

- **Inner automorphism** in $\mathfrak{su}(2)$:

$$\sigma\{T_1, T_2, T_3\} = \{-T_1, -T_2, T_3\} , \qquad \tilde{\mathcal{R}}\{T_1, T_2, T_3\} = \{T_2, -T_1, 0\} .$$

  Symmetry: • untwisted: $\mathrm{U}(1)^2$, • $\mathbb{Z}_2$-twisted: $\mathrm{U}(1)^2$.
  Models are all related to the untwisted models by parameter-dependent field redefinitions, dropping a closed B-field and 2d space-time dualities.

- **Outer automorphism** in $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$:

$$T^\pm_a = \mathrm{diag}(T_a, \pm T_a) , \qquad \sigma(T^\pm_a) = \pm T^\pm_a , \qquad \tilde{\mathcal{R}}\{T^\pm_1, T^\pm_2, T^\pm_3\} = \{T^\pm_2, -T^\pm_1, 0\} .$$

  Symmetry: • untwisted: $\mathrm{U}(1)^4$, • $\mathbb{Z}_2$-twisted: $\mathrm{U}(1)^3$.
  Different global symmetries implies that the models are inequivalent.

Conclusion: More examples are studied in [1]. Although we don't have a complete answer, it is natural to speculate that the inequivalent η-models are enumerated by the outer automorphisms of the Lie algebra.

Generalisations and future directions

- Deformed models and twisted lambda models can be found in [1].
- A key next step would be to analyse them in the Hamiltonian formalism.
- It would also be interesting to understand the $\mathcal{E}$-model formulation.
- Generalisations to other symmetric sigma models.

References

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[2] O. Fukushima, J. Sakamoto, & K. Yoshida, arxiv: 1908.02289.
[3] S Lacroix, arxiv: 2109.14278.
[4] K. Costello & M. Yamazaki, arxiv: 2003.07309.