Twists of trigonometric sigma models

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Outline

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Classical integrability in field theories

1. Infinite number of conserved quantities $\mathcal{Q}_i$, i.e. $\{\mathcal{H}, \mathcal{Q}_i\} = 0$.
   - Lax connection $\mathcal{L}(z, t, x)$.
   - Flatness condition $\partial_+ \mathcal{L}_- - \partial_- \mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-] = 0$.
   - Monodromy matrix $\mathcal{M}(z, t) = \mathrm{P}\overleftarrow{\exp}\left(-\int_{-\infty}^{\infty} dx\, \mathcal{L}_x(z, t, x)\right)$.
2. Conserved quantities are in involution, i.e. $\{\mathcal{Q}_i, \mathcal{Q}_j\} = 0$.
   - Maillet bracket: $\{\mathcal{L}_{x\underline{\mathbf{1}}}(\mu, x), \mathcal{L}_{x\underline{\mathbf{2}}}(\nu, y)\} =$
     $[\mathcal{R}_{\underline{\mathbf{12}}}(\mu, \nu), \mathcal{L}_{x\underline{\mathbf{1}}}(\mu, x)]\delta_{xy} - [\mathcal{R}_{\underline{\mathbf{21}}}(\nu, \mu), \mathcal{L}_{x\underline{\mathbf{2}}}(\nu, y)]\delta_{xy} - (\mathcal{R}_{\underline{\mathbf{12}}}(\mu, \nu) + \mathcal{R}_{\underline{\mathbf{21}}}(\nu, \mu))\delta'_{xy}$.

4D Chern-Simon theories

4d Chern-Simon action:

$$\mathcal{S}[A] = \frac{i}{4\pi}\int_{\mathbb{C}\mathrm{P}^1\times\Sigma} \omega\wedge \mathrm{CS}_3(A)\ , \qquad \mathrm{CS}_3(A) = \mathrm{tr}(AdA) + \frac{1}{3}\mathrm{tr}(A[A,A])\ , \qquad \omega = \varphi(z)dz\ .$$

Lax connection:

$$A = \hat{g}\mathcal{L}\hat{g}^{-1} - d\hat{g}\hat{g}^{-1} = \mathcal{L}^{\hat{g}}\ , \qquad \hat{g}\in \mathrm{G}^{\mathbb{C}}\ , \qquad \mathcal{L} = \mathcal{L}_\mu d\sigma^\mu\ , \qquad \mathcal{L}_{\bar{z}} = 0\ .$$

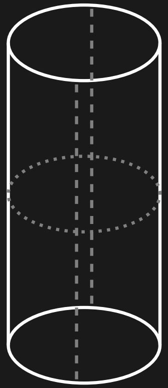
Bulk equations of motion:

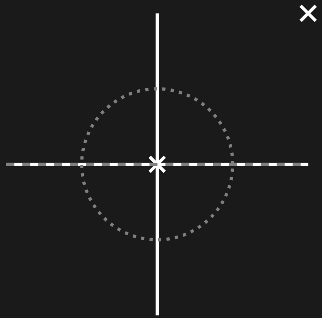
$$\omega\wedge\left(d_\Sigma\mathcal{L} + \frac{1}{2}[\mathcal{L},\mathcal{L}]\right) = 0\ , \qquad \omega\wedge\partial_{\bar{z}}\mathcal{L} = 0\ .$$

Gauge symmetries:

$$A\to uAu^{-1} - duu^{-1}\ , \qquad \hat{g}\to u\hat{g}\ , \qquad \mathcal{L}\to v^{-1}\mathcal{L}v + v^{-1}dv\ , \qquad \hat{g}\to\hat{g}v\ .$$

Trigonometric models in 4d Chern-Simons



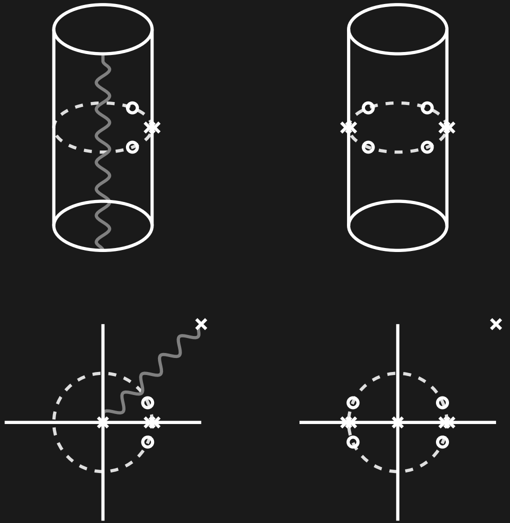


$$z = e^{iw} \ ,$$

where w, with $\mathrm{Re}(w)$ with $\mathrm{Re}(w) \in (-\pi, \pi]$, is the coordinate on the cylinder and z is the coordinate on $\mathbb{C}\mathrm{P}^1$.

$$\omega_{\text{trig}} = c_0 \varphi_{\text{trig}}(z) \frac{dz}{z} \ ,$$

Twists of trigonometric models



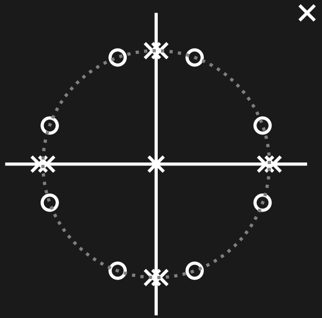
Automorphism:

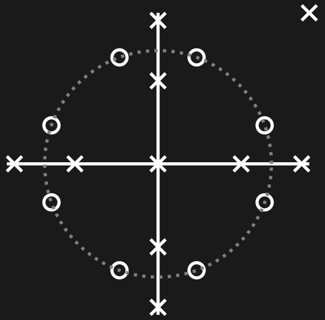
$$[\sigma(X), \sigma(Y)] = \sigma([X, Y]) \ , \ \sigma^N = 1 \ ,$$

where $X, Y \in \mathfrak{g}^{\mathbb{C}}$. Projection operators:

$$P_0 = \frac{1}{N} \sum_{a \in \mathbb{Z}_N} \sigma^a \ , \qquad P_a = \frac{1}{N} \sum_{b \in \mathbb{Z}_N} e^{-\frac{2i\pi ab}{N}} \sigma^b \ .$$

Twist function





$$\omega_{\mathrm{trig}}^{(N)} = -c_0 \hbar \frac{(z^N - e^{N\alpha})(z^N - e^{-N\alpha})}{(z^N - 1)^2} \frac{dz}{z} \, , \qquad \omega_{\mathrm{trig}}^{(N)} = -c_0 \hbar \frac{(z^N - e^{N\alpha})(z^N - e^{-N\alpha})}{(z^N - e^{N\beta})(z^N - e^{-N\beta})} \frac{dz}{z} \, .$$

Poles, zeros, and Lax connection

Meromorphic 1-form for $\mathbb{Z}_N$-twisted trigonometric model:

$$\omega^{(N)}_{\text{trig}} = -c_0 \hbar \frac{(z^N - e^{N\alpha})(z^N - e^{-N\alpha})}{(z^N-1)^2} \frac{dz}{z} \; .$$

Poles:

$$\mathfrak{p}_0 = \{0, \infty\} \; , \qquad \mathfrak{p}_2 = \left\{ e^{\frac{2i\pi a}{N}} \; : \; a \in \mathbb{Z}_N \right\} \; .$$

Zeros:

$$\mathfrak{z} = \mathfrak{z}_+ \cup \mathfrak{z}_- \; , \qquad \mathfrak{z}_\pm = \left\{ e^{\mp\alpha + \frac{2i\pi a}{N}} \; : \; a \in \mathbb{Z}_N \right\} \; .$$

General formula for Lax connection:

$$\mathcal{L}_\pm = \tilde{V}_\pm + \sum_{s \in \mathfrak{z}_\pm} \frac{W^{(s)}_\pm}{z - s} \; .$$

Defect terms and boundary conditions

Defect terms:

$$-c_0\hbar\int_\Sigma d^2\sigma\,\Big(\mathrm{tr}(A\delta A)|_{z=0}-\mathrm{tr}(A\delta A)|_{z=\infty}$$
$$-\frac{4}{N^2}\sinh^2(\frac{N\alpha}{2})\sum_{a\in\mathbb{Z}_N}e^{\frac{2i\pi a}{N}}\partial_z\mathrm{tr}(A\delta A)\big|_{z=e^{\frac{2i\pi a}{N}}}\Big)\ ,$$

Boundary conditions:

- double pole boundary conditions:

$$A\big|_{z=e^{\frac{2i\pi a}{N}}}=0\ ,\qquad \partial_z\hat{g}\big|_{z=e^{\frac{2i\pi a}{N}}}=0\ ,\qquad \hat{g}\big|_{z=e^{\frac{2i\pi a}{N}}}=\hat{\sigma}^a(g)\in\hat{\sigma}^a(\mathrm{G})\ ,\qquad a\in\mathbb{Z}_N\ .$$

- λ-type boundary conditions:

$$A|_{z=0}=A|_{z=\infty}\ ,\qquad \hat{g}|_{z=0}=\tilde{g}\in\mathrm{G}_0\ ,\qquad \hat{g}|_{z=\infty}=1\ .$$

- η-type boundary conditions:

$$(\tilde{\mathcal{R}}+c_0)A|_{z=0}=(\tilde{\mathcal{R}}-c_0)A|_{z=\infty}\ ,\qquad \hat{g}|_{z=0}=\hat{g}|_{z=\infty}=\tilde{g}\qquad\rightarrow\qquad \hat{g}|_{z=0}=\hat{g}|_{z=\infty}=1\ .$$
$$[\tilde{\mathcal{R}}X,\tilde{\mathcal{R}}Y]-\tilde{\mathcal{R}}([\tilde{\mathcal{R}}X,Y]+[X,\tilde{\mathcal{R}}Y])+c_0^2[X,Y]=0\ ,\qquad \mathrm{tr}(X\tilde{\mathcal{R}}Y)+\mathrm{tr}((\tilde{\mathcal{R}}X)Y)=0\ .$$

Actions

Algebra fields:

$$\tilde{\jmath} = \tilde{g}^{-1} d\tilde{g} \ , \qquad j = g^{-1} dg \ , \qquad J^{(a)} = P_a j \ .$$

Action of $\mathbb{Z}_N$-twisted η-model:

$$\mathcal{S}_\eta^{(N)}(g) = -\frac{4c_0^2 \hbar \eta}{1-c_0^2\eta^2} \int d^2\sigma \, \mathrm{tr}\Big(J_+^{(0)} \frac{1-c_0^2\eta^2}{1-\eta\tilde{\mathcal{R}}} J_-^{(0)} + \sum_{a\in\mathbb{Z}_N\setminus\{0\}} \Big(1+c_0\eta\Big(1-\frac{2a}{N}\Big)\Big) J_+^{(a)} J_-^{(N-a)} \Big) \ .$$

Action of $\mathbb{Z}_N$-twisted λ-model:

$$\begin{aligned} \mathcal{S}_\lambda^{(N)}(g,\tilde{g}) = -2\hbar \int d^2\sigma \, \Big(& \mathrm{tr}\Big(((1-\mathrm{Ad}_{\tilde{g}}^{-1}) J_+^{(0)} - \tilde{\jmath}_+) \frac{1}{1-\lambda^{-1}\mathrm{Ad}_{\tilde{g}}} ((1-\mathrm{Ad}_{\tilde{g}}^{-1}) J_-^{(0)} - \tilde{\jmath}_-) \Big) \\ & + \mathrm{tr}\Big(\sum_{a\in\mathbb{Z}_N\setminus\{0\}} (\lambda-1)\Big(1+\frac{a}{N}(\lambda^{-1}-1)\Big) J_+^{(a)} J_-^{(N-a)} \Big) \Big) \\ & + 2\hbar \int d^2\sigma \, \mathrm{tr}\Big(\frac{1}{2}\tilde{\jmath}_+\tilde{\jmath}_- - J_+^{(0)}\tilde{\jmath}_- + \tilde{\jmath}_+ \mathrm{Ad}_{\tilde{g}}^{-1} J_-^{(0)} + J_+^{(0)}(1-\mathrm{Ad}_{\tilde{g}}^{-1}) J_-^{(0)} \Big) - \hbar \mathcal{S}_{\mathrm{WZ}}(\tilde{g}) \ . \end{aligned}$$

Automorphisms

Actions of untwisted and $\mathbb{Z}_2$-twisted η-models:

$$\mathcal{S}^{(1)}_\eta = \int d^2\sigma\, \mathrm{tr}\Big(j_+ \frac{1+\eta^2}{1-\eta\tilde{\mathcal{R}}} j_-\Big) \;, \qquad \mathcal{S}^{(2)}_\eta = \int d^2\sigma\, \mathrm{tr}\Big(j_+ P_0 \frac{1+\eta^2}{1-\eta\tilde{\mathcal{R}}} j_- + j_+ P_1 j_-\Big) \;,$$

Inner automorphism in $\mathfrak{su}(2)$: $\sigma = \mathrm{Ad}_{\exp(\frac{\pi}{2}T_3)}$,

$$\sigma\{T_1, T_2, T_3\} = \{-T_1, -T_2, T_3\} \;, \qquad \tilde{\mathcal{R}}\{T_1, T_2, T_3\} = \{T_2, -T_1, 0\} \;.$$

Outer automorphism in $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$:

$$T^\pm_a = \mathrm{diag}(T_a, \pm T_a) \;, \qquad \sigma(T^\pm_a) = \pm T^\pm_a \;, \qquad \tilde{\mathcal{R}}\{T^\pm_1, T^\pm_2, T^\pm_3\} = \{T^\pm_2, -T^\pm_1, 0\} \;.$$

Conclusions and future directions

- Conjecture inequivalence for outer automorphisms: Although we don't have a complete answer to this question, it is natural to speculate that the inequivalent η-models might be enumerated by the outer automorphisms of the Lie algebra.
- Hamiltonian formalism: Poisson bracket of the Lax matrix and extracting the conserved charges, and investigating their underlying affine Gaudin models.
- Understand the $\mathcal{E}$-model formulation, which unifies models with the same twist function in a single first-order 2d model on a doubled space.
- Generalisations to other symmetric sigma models.
- Construct coupled models.

Questions