Rayleigh-scattering calculations for the terrestrial atmosphere

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Rayleigh-scattering cross sections and volume-scattering coefficients are computed for standard air; they incorporate the variation of the depolarization factor with wavelength. Rayleigh optical depths are then calculated for the 1962 U.S. Standard Atmosphere and for five supplementary models. Analytic formulas are derived for each of the parameters listed. The new optical depths can be 1.3% lower to 3% higher at midvisible wavelengths and up to 10% higher in the UV region compared with previous calculations, in which a constant or incorrect depolarization factor was used. The dispersion of the depolarization factor is also shown to affect the Rayleigh phase function slightly, by approximately 1% in the forward, backscattered, and 90° scattering-angle directions.

Key words: Rayleigh scattering, Rayleigh optical depth, Rayleigh cross section.

1. Introduction

An accurate estimate of the amount of Rayleigh scattering in the terrestrial atmosphere is required in many applications. The parameters that characterize this type of scattering are well defined (see, for example, McCartney[1]) and are summarized here.

The total Rayleigh-scattering cross section per molecule, σ, is given by the following formula:

$$\sigma(\lambda) = \frac{24\pi^3(n_s^2 - 1)^2}{\lambda^4 N_s^2(n_s^2 + 2)^2}\left(\frac{6 + 3\rho_n}{6 - 7\rho_n}\right), \qquad (1)$$

where λ is the wavelength (in centimeters), n_s is the refractive index for standard air at λ, N_s is the molecular number density (2.54743×10^{19} cm^{-3}) for standard air, and ρ_n is the depolarization factor—a term that accounts for the anisotropy of the air molecule and that varies with wavelength. "Standard air is defined as dry air containing 0.03% CO_2 by volume at normal pressure 760 mm Hg (= 1013.25 mb) and having an air temperature of 15 °C." (see Penndorf[2]). The cross section σ is typically given in units of squared centimeters.

The amount of scattering for a volume of gas, characterized by the total Rayleigh volume-scattering coefficient β is given by the product of the total Rayleigh cross section per molecule, σ, as defined in Eq. (1), and the molecular number density N at a given pressure and temperature, or altitude, z:

$$\beta(\lambda, z) = N(z)\sigma(\lambda) \qquad (2)$$

where β is in units of inverse centimeters.

The Rayleigh optical depth τ at altitude z_0 is then given as the integral of the total volume-scattering coefficient from z_0 to the top of the atmosphere:

$$\tau(\lambda, z_0) = \int_{z_0}^{\infty} \beta(\lambda, z)\mathrm{d}z. \qquad (3)$$

The angular distribution of the scattered light is described by the Rayleigh phase function P_{ray} and is defined below.

A variety of tabulations or formulations of the Rayleigh-scattering parameters given above can be found in the literature. Typically it is taken for granted that these parameters are well known, therefore it hardly seems necessary to recompute them. However, as pointed out by Teillet[3], the Rayleigh optical depth as determined by various sources can vary by as much as 3–4% depending on the values of the index of refraction and the depolarization factor that were used. Many previous calculations are based on old values for these two parameters, and only one source that includes the dispersion, or variation, of the depolarization factor with wave-

length has been found. In addition, confusion arose a few years ago when an incorrect value for the depolarization factor was used in computations of the Rayleigh optical depth.[4] A brief historical review may help to clarify the situation.

Penndorf[2] first published tabular values of the refractive index of standard air and the Rayleigh-scattering coefficients for the 0.2–20.0-µm wavelength range. Specifically, he calculated the Rayleigh-scattering cross section, the Rayleigh mass-scattering coefficient, and the Rayleigh volume-scattering coefficient as a function of wavelength, through the use of Edlen's[5] formula for the index of refraction and a constant value with wavelength of 0.035 for the depolarization factor. He also calculated surface Rayleigh optical depths when an isothermal atmosphere was assumed. Eltermann[6,7] extended Penndorf's[2] research by tabulating values of the Rayleigh volume-scattering coefficient and the Rayleigh optical depth as a function of wavelength (0.27–4.0 µm) and altitude (0–50 km) with the U.S. Standard Atmosphere (1962) data.[8] He also used Edlen's[5] formula for the refractive index of air and a constant value of 0.035 for the depolarization factor.

Many tabulations or formulations of the Rayleigh-scattering parameters found in the literature can be traced back to Penndorf[2] or Eltermann.[6,7] For example, McCartney[1] simply lists Penndorf's[2] Rayleigh-scattering-coefficient and refractive-index values, the series of LOWTRAN atmospheric transmittance–radiance models (Kneizys et al.[9–11]) all use some form of least-squares fit to Penndorf's[2] Rayleigh volume-scattering coefficients in their calculations, whereas Margraff and Griggs[12] use a least-squares fit to Eltermann's[6] optical depth data.

Based on updated depolarization data, Hoyt[13] tabulated values of the Rayleigh optical depth as a function of wavelength (0.3–1.5 µm) for six standard atmospheres (U.S. Standard Atmosphere supplement, 1966[14]) through the use of Edlen's[5] formula for the index of refraction and a constant value of 0.0139 for the depolarization factor. Frölich and Shaw,[15] who also used updated depolarizaton data, later tabulated values of the Rayleigh optical depth as a function of wavelength (0.26–1.5 µm) for five standard atmospheres[14] into which model water vapor and ozone profiles were added (McClatchey et al.[16]). They combined Edlen's[17] refractive-index formula, which takes into account the effect of water vapor, and Peck and Reeder's[18] four-term formula for the index of refraction, and used a constant value of 0.0095 for the depolarization factor. They also determined an analytic formula for the Rayleigh optical depth from a least-squares fit of their data as a function of wavelength. However, as pointed out in a series of articles by Young,[4,19–21] the values of the depolarization factor used by Hoyt[13] and Frölich and Shaw[15] were based on measurements that excluded the effects of the Raman-scattered lines and therefore did not completely represent the depolarization factor for Rayleigh scattering, This led to optical-depth

values that were a few percent too low. Based on the latest measurements, Young[19] recommended a value of 0.0279 for the depolarization factor.

In all the tabulations or formulations listed above, a constant value for the depolarization factor with wavelength was assumed. This is a fairly good approximation at midvisible wavelengths but breaks down toward near-UV wavelengths. In Fig. 1 it can be seen that this factor varies by approximately 60% from the near IR to the UV spectral region, introducing a corresponding variation with wavelength of approximately 3% in the Rayleigh-scattering coefficients. Bates[22] tabulated values of the depolarization factor and the Rayleigh-scattering cross section as a function of wavelength that took into account the dispersion of the depolarization factor (see Table 1). Nicolet[23] then derived an analytic expression from a fit to Bates' cross-section values.[22] No source for the Rayleigh volume-scattering coefficient or the Rayleigh optical depth that includes this dispersion has been found. Nor has any source that includes the effect of the dispersion of molecular anisotropy on the Rayleigh phase function been found.

It seems appropriate, therefore, that the Rayleigh-scattering parameters (i.e., the Rayleigh-scattering cross section, volume-scattering coefficient, and optical depth) should be updated by the use of the latest estimates of the index of refraction and the depolarization factor of air and by inclusion of the dispersion of the depolarization factor with wavelength. In this paper tabulated values of these Rayleigh-scattering parameters are presented, and least-squares-derived analytic formulas for each parameter, which can be easily incorporated into computer codes, are given as a function of wavelength. Comparisons of the values given here with the most common previous calculations of the Rayleigh-scattering parameters are also presented. Section 2 describes the calculation of the Rayleigh-scattering cross section, and Section 3 describes the calculation of the Rayleigh volume-scattering coefficient. Section 4 discusses the effect of the dispersion of the molecular anisot-
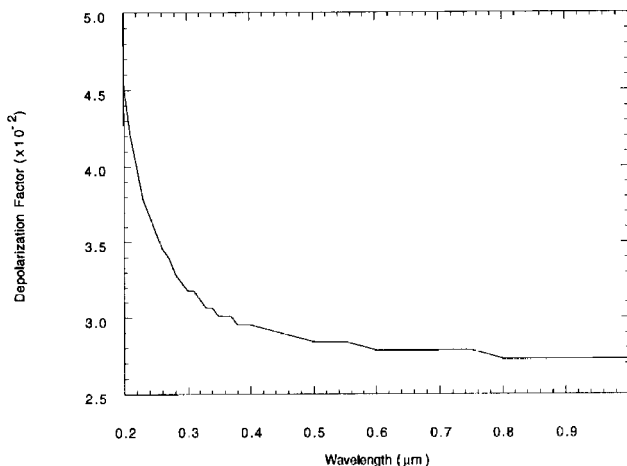

Fig. 1. Dispersion of the depolarization factor with wavelength (adapted from Bates[22]).

Wavelength (μm)	F_k(air)	ρ_n	γ
0.200	1.080	4.545 ($\times 10^{-2}$)	2.326 ($\times 10^{-2}$)
0.205	1.077	4.384	2.241
0.210	1.074	4.221	2.156
0.215	1.072	4.113	2.100
0.220	1.070	4.004	2.043
0.225	1.068	3.895	1.986
0.230	1.066	3.785	1.930
0.240	1.064	3.675	1.872
0.250	1.062	3.565	1.815
0.260	1.060	3.455	1.758
0.270	1.059	3.400	1.729
0.280	1.057	3.289	1.672
0.290	1.056	3.233	1.643
0.300	1.055	3.178	1.614
0.310	1.055	3.178	1.614
0.320	1.054	3.122	1.586
0.330	1.053	3.066	1.557
0.340	1.053	3.066	1.557
0.350	1.052	3.010	1.528
0.360	1.052	3.010	1.528
0.370	1.052	3.010	1.528
0.380	1.051	2.955	1.499
0.390	1.051	2.955	1.499
0.400	1.051	2.955	1.499
0.450	1.050	2.899	1.471
0.500	1.049	2.842	1.442
0.550	1.049	2.842	1.442
0.600	1.048	2.786	1.413
0.650	1.048	2.786	1.413
0.700	1.048	2.786	1.413
0.750	1.048	2.786	1.413
0.800	1.047	2.730	1.384
0.850	1.047	2.730	1.384
0.900	1.047	2.730	1.384
0.950	1.047	2.730	1.384
1.000	1.047	2.730	1.384

ropy on the Rayleigh phase function, and Section 5 describes the calculation of the Rayleigh optical depth for the 1962 U.S. Standard Atmosphere and for five supplementary models that take into account both seasonal and latitudinal variations.

2. Total Rayleigh-Scattering Cross-Section Calculation

The amount of scattering from a single air molecule is characterized by its scattering cross section. Table 2 lists the total Rayleigh-scattering cross section per molecule for standard air as a function of wavelength computed with Eq. (1).

The refractive index n_s for standard air ($T = 15\ °C$) was calculated with the equations of Peck and Reeder.[18] For wavelengths greater than 0.23 μm, the four-parameter formula was used:

$$(n_s - 1) \times 10^8$$

$$= \frac{5{,}791{,}817}{238.0185 - (1/\lambda)^2} + \frac{167{,}909}{57.362 - (1/\lambda)^2}, \quad (4)$$

where the wavelength is given in micrometers. For wavelengths less than or equal to 0.23 μm, the five-parameter formula was used:

$$(n_s - 1) \times 10^8 = 8060.51 + \frac{2{,}480{,}990}{132.274 - (1/\lambda)^2}$$

$$+ \frac{17{,}455.7}{39.32957 - (1/\lambda)^2}. \quad (5)$$

Equation (4) yields a slightly better fit to the laboratory-measured refractive-index values in the IR than do the equations of Edlen,[5,17] and the five-parameter formula [Eq. (5)] is required for a description of the sparse measurements below 0.23 μm (Peck and Reeder[18]).

A comparison of the refractive-index term from Eq. (1), $(n_s^2 - 1)^2/(n_s^2 + 2)^2$, calculated through the use of the equations of Peck and Reeder[18] and Edlen[5], shows a difference of less than 0.01% between the two. A comparison of this same term calculated through the use of the equations of Peck and Reeder[18] and of Edlen,[17] in which the effect of water vapor is included, again shows a small difference (less than 0.3%), where a typical water vapor density value of 7 g/m^3 was used in the calculation (McClatchey et al.[24]). Therefore, whether the index of refraction is calculated through the use of the equations of Peck and Reeder[18] or of Edlen,[5] or whether water vapor is included (Edlen[17]) will not significantly affect the Rayleigh-scattering cross section. Also, Bates[22] states that the difference between the refractive-index values used in his calculations and those of Peck and Reeder[18] is less than 0.1% for wavelengths greater than 0.195 μm. Although the differences in the values of the refractive-index term calculated with the formulations listed above are small, Teillet[3] has pointed out that some formulations may differ by up to 4% from those given here.

The King correction factor F_k, which accounts for the anisotropy of air molecules, is defined by the second factor in Eq. (1):

$$F_k = \left(\frac{6 + 3\rho_n}{6 - 7\rho_n}\right). \quad (6)$$

The dispersion of this factor with wavelength (see Table 1), as given by Bates,[22] was used to calculate the cross-section values in Table 2. Bates[22] calculated the King correction factor from 0.2 to 0.3 μm (0.005 μm), 0.3 to 0.4 μm (0.01 μm), and 0.4 to 1.0 μm (0.05 μm). These factors are listed in Table 1 along with values for the depolarization factor ρ_n. Linear interpolation of these values was used to calculate the depolarization term at the wavelengths given in Table 2. For comparison, Penndorf[2] used a King correction value of 1.061, Young[19] suggested 1.048, Hoyt[13] used 1.0235, and Frölich and Shaw[15] used 1.0160. As can be seen in Table 1, Penndorf's[2] King correction value is more appropriate to wavelengths near 0.26 μm, and Young's[19] value is valid for wavelengths greater than approximately 0.5 μm. Both Hoyt's[13]

Table 2. Total Rayleigh-Scattering Cross Section σ and Total Rayleigh Volume-Scattering Coefficient β_s for Standard Air ($P_S = 1013.25$ mbars, $T_S = 288.15$ K)

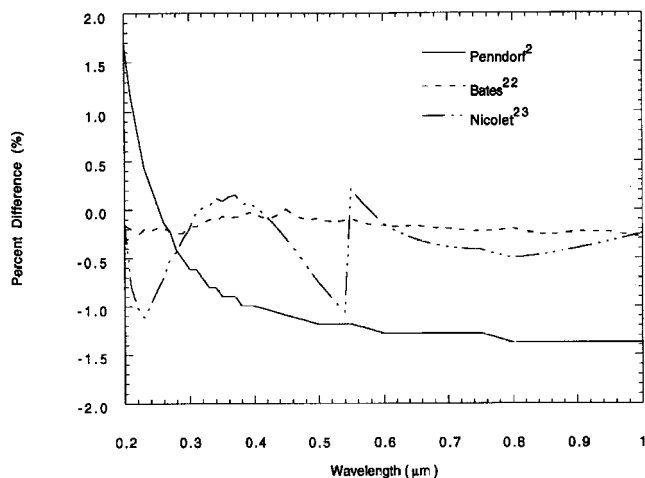| Wavelength (μm)	Rayleigh-Scattering Cross Section σ (cm^2)	Rayleigh Volume-Scattering Coefficient β_s (km^{-1})
0.20	3.612 ($\times 10^{-25}$)	9.202 ($\times 10^{-1}$)
0.21	2.836	7.225
0.22	2.269	5.781
0.23	1.841	4.691
0.24	1.515	3.859
0.25	1.259	3.208
0.26	1.056	2.690
0.27	8.939 ($\times 10^{-26}$)	2.277
0.28	7.614	1.940
0.29	6.534	1.665
0.30	5.642	1.437
0.31	4.903	1.249
0.32	4.279	1.090
0.33	3.752	9.557 ($\times 10^{-2}$)
0.34	3.307	8.425
0.35	2.924	7.450
0.36	2.598	6.619
0.37	2.317	5.901
0.38	2.071	5.275
0.39	1.858	4.734
0.40	1.673	4.261
0.41	1.509	3.845
0.42	1.366	3.479
0.43	1.239	3.156
0.44	1.127	2.870
0.45	1.027	2.616
0.46	9.378 ($\times 10^{-27}$)	2.389
0.47	8.583	2.187
0.48	7.871	2.005
0.49	7.232	1.842
0.50	6.656	1.696
0.51	6.138	1.564
0.52	5.669	1.444
0.53	5.245	1.336
0.54	4.860	1.238
0.55	4.509	1.149
0.56	4.189	1.067
0.57	3.897	9.927 ($\times 10^{-3}$)
0.58	3.630	9.247
0.59	3.385	8.624
0.60	3.161	8.053
0.61	2.956	7.530
0.62	2.767	7.049
0.63	2.593	6.605
0.64	2.432	6.196
0.65	2.284	5.819
0.66	2.147	5.470
0.67	2.020	5.146
0.68	1.903	4.847
0.69	1.793	4.568
0.70	1.692	4.310
0.71	1.597	4.069
0.72	1.510	3.846
0.73	1.428	3.637
0.74	1.351	3.442
0.75	1.280	3.261
0.76	1.213	3.090
0.77	1.150	2.931
0.78	1.092	2.781
0.79	1.037	2.641
0.80	9.854 ($\times 10^{-28}$)	2.510
0.90	6.129	1.561

Table 2. (continued)

Wavelength (μm)	Rayleigh-Scattering Cross Section σ (cm^2)	Rayleigh Volume-Scattering Coefficient β_s (km^{-1})
1.00	4.010	1.022
1.10	2.734	6.964 ($\times 10^{-4}$)
1.20	1.927	4.909
1.30	1.398	3.560
1.40	1.038	2.644
1.50	7.872 ($\times 10^{-29}$)	2.005
1.60	6.077	1.548
1.70	4.766	1.214
1.80	3.790	9.656 ($\times 10^{-5}$)
1.90	3.052	7.775
2.00	2.485	6.331
2.20	1.697	4.322
2.40	1.197	3.050
2.60	8.691 ($\times 10^{-30}$)	2.214
2.80	6.460	1.646
3.00	4.901	1.249
3.50	2.645	6.737 ($\times 10^{-6}$)
4.00	1.550	3.948

and Frölich and Shaw's[15] values are too low, as previously pointed out by Young.[4]

Note that Eq. (1) differs slightly from that given by Penndorf[2] and others because the simplifying approximation that n_s is very nearly equal to unity has not been made. Although this approximation only introduces an error of the order of 0.05%, there is no reason to include it in these calculations. It is also important to note that according to Lorenz–Lorentz theory the quantity $(n_s^2 - 1)/(n_s^2 + 2)$ is proportional to the molecular number density N_s and that although Eq. (1) describes the cross section for a single air molecule, and is therefore independent of the temperature and the pressure of the gas, one must be consistent in the values for n_s and N_s.[1]

In Fig. 2 the Rayleigh cross-section values com-

Fig. 2. Comparison of the total Rayleigh-scattering cross section as computed in this paper (see Table 2) with those calculated by Penndorf,[2] Bates,[22] and Nicolet.[23] The percent difference is defined as $100[x_B(\lambda) - x_i(\lambda)]/x_B(\lambda)$, where x_B is the cross section computed here at wavelength λ and x_i is the cross section at λ of Penndorf,[2] Bates,[22] or Nicolet.[23]

puted in this paper are compared with those calculated by Penndorf,[2] Bates,[22] and Nicolet.[23] The percent difference is defined as

$$\text{Percent Difference} = 100 \left[\frac{x_B(\lambda) - x_i(\lambda)}{x_B(\lambda)} \right], \quad (7)$$

where x_B represents the cross section computed here at wavelength λ and x_i is the cross section at λ of Penndorf,[2] Bates,[22] or Nicolet.[23] The values calculated in this paper are from 1.7% higher to 1.4% lower than those of Penndorf,[2] with the best agreement occurring near 0.26 μm, where the depolarization values are similar. Figure 2 also shows good agreement with Bates,[22] although the values computed here are consistently lower than those from Bates, by approximately 0.2%. This difference can be attributed to the differences in the indices of refraction and the no-approximation form of the cross-section formula that was used in this paper. However, the cross-section values given in Table 2 are still slightly closer to Bates' values[22] than those calculated with the analytic formula of Nicolet.[23]

A least-squares fit of the Rayleigh cross-section data given in Table 2 yields the following analytic formula:

$$\sigma = A\lambda^{-(B+C\lambda+D/\lambda)}, \quad (8)$$

where λ is in micrometers; $A = 3.01577 \times 10^{-28}$, $B = 3.55212$, $C = 1.35579$, and $D = 0.11563$ for the wavelength range 0.2–0.5 μm; and $A = 4.01061 \times 10^{-28}$, $B = 3.99668$, $C = 1.10298 \times 10^{-3}$, and $D = 2.71393 \times 10^{-2}$ for wavelengths greater than 0.5 μm (see Table 3). Equation (8) fits the data to within 0.4% for wavelengths less than 0.25 μm, to better than 0.2% in the 0.25–0.5-μm range, and to better than 0.1% for wavelengths greater than 0.5 μm.

3. Total Rayleigh Volume-Scattering-Coefficient Calculation

As stated above, the amount of scattering for a unit volume of air is characterized by the total Rayleigh volume-scattering coefficient β, which is given by the product of the total Rayleigh cross section per molecule, σ, as defined in Eq. (1), and the molecular number density N at a given pressure and temperature, or altitude, z [see Eq. (2)].

Table 2 lists the total Rayleigh volume-scattering coefficient as a function of wavelength computed with Eq. (2). The calculations were done for standard conditions and put in units of inverse kilometers so that

$$\beta_s(\lambda) = N_s \sigma(\lambda)(10^5 \text{ cm/km}). \quad (9)$$

The analytic formula given in Eq. (8) can be used to describe these data simply by multiplying the A coefficients given above by $N_s(10^5 \text{ cm/km})$: from 0.2–0.5 μm, $A = 7.68246 \times 10^{-4}$, and for wavelengths greater than 0.5 μm, $A = 10.21675 \times 10^{-4}$ (see Table 3). The B, C, and D coefficients will be the same as those given in Table 3. Because the only difference is a multiplicative constant, these equations will fit the volume-scattering-coefficient data of Table 2 to the same accuracy as the cross-section fits in Section 2, that is, to within 0.4% for wavelengths less than 0.25 μm, to better than 0.2% in the 0.25–0.5-μm range, and to better than 0.1% for wavelengths greater than 0.5 μm. Similarly, a comparison of the volume-scattering coefficients given in Table 2 with those given by Penndorf[2] or Eltermann[7] shows the same differences of from 1.7% to −1.4%, as shown in the cross-section comparisons (see Fig. 2) when the volume-scattering coefficients are given at the same molecular number densities.

Because β scales with the molecular number density, to correct the Rayleigh volume-scattering coefficients of Table 2 to any pressure P and temperature T is straightforward[1]:

$$\beta = \beta_s \frac{N}{N_s} = \beta_s \frac{P}{P_s} \frac{T_s}{T}, \quad (10)$$

where P_s and T_s represent the reference pressure and temperature at which β_s was calculated, respectively. In Table 2, $P_s = 1013.25$ mbars and $T_s = 288.15$ K (15
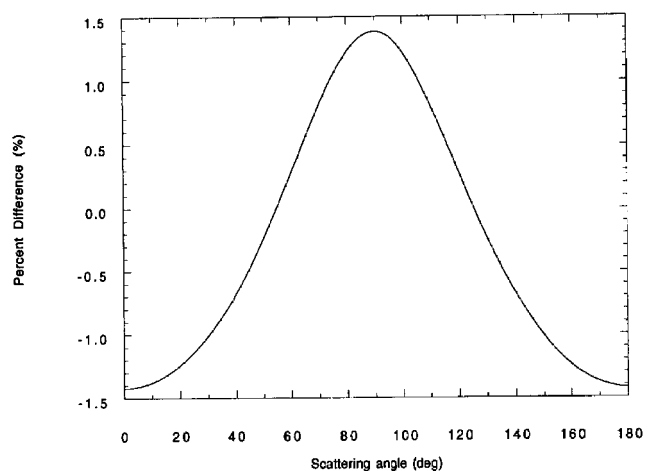
Fig. 3. Comparison of the Rayleigh phase function at 0.5 μm computed with and without taking into account the effect of molecular anisotropy. The percent difference is defined as in Fig. 2, where x_B now represents the phase function computed with molecular anisotropy included and x_i represents the phase function computed without molecular anisotropy taken into account.

Table 3. Constants Used in Analytic Formulas for the Total Rayleigh-Scattering Cross Section and Total Rayleigh Volume-Scattering Coefficient (for Standard Air): σ or $\beta = A\lambda^{-(B+C\lambda+D/\lambda)}$

	λ	
Coefficients	0.2–0.5 μm	>0.5 μm
A (cross section)	3.01577×10^{-28}	4.01061×10^{-28}
A (volume scattering)	7.68246×10^{-4}	10.21675×10^{-4}
B	3.55212	3.99668
C	1.35579	1.10298×10^{-3}
D	0.11563	2.71393×10^{-2}

Table 4. Rayleigh Optical Depths at 0-km Altitude for Six Different Atmosphere Models

Wavelength (μm)	Tropical (15° N)	Midlatitude Summer (45° N, July)	Midlatitude Winter (45° N, January)	Subarctic Summer (60° N, July)	Subarctic Winter (60° N, January)	1962 U.S. Standard
0.20	7.819	7.807	7.826	7.761	7.783	7.788
0.21	6.139	6.129	6.145	6.093	6.111	6.114
0.22	4.912	4.904	4.917	4.876	4.890	4.892
0.23	3.986	3.979	3.990	3.956	3.968	3.970
0.24	3.279	3.274	3.282	3.255	3.264	3.266
0.25	2.725	2.721	2.728	2.705	2.713	2.714
0.26	2.286	2.282	2.288	2.269	2.276	2.277
0.27	1.935	1.932	1.937	1.920	1.926	1.927
0.28	1.648	1.645	1.650	1.636	1.640	1.641
0.29	1.414	1.412	1.416	1.404	1.408	1.409
0.30	1.221	1.219	1.222	1.212	1.216	1.216
0.31	1.061	1.060	1.062	1.053	1.056	1.057
0.32	$9.262\,(\times 10^{-1})$	$9.247\,(\times 10^{-1})$	$9.271\,(\times 10^{-1})$	$9.193\,(\times 10^{-1})$	$9.220\,(\times 10^{-1})$	$9.225\,(\times 10^{-1})$
0.33	8.121	8.108	8.129	8.061	8.084	8.088
0.34	7.158	7.147	7.165	7.105	7.126	7.130
0.35	6.330	6.320	6.336	6.283	6.301	6.304
0.36	5.624	5.615	5.629	5.582	5.598	5.601
0.37	5.014	5.006	5.019	4.977	4.991	4.994
0.38	4.482	4.475	4.486	4.449	4.461	4.464
0.39	4.022	4.016	4.026	3.992	4.004	4.006
0.40	3.620	3.615	3.624	3.594	3.604	3.606
0.41	3.267	3.262	3.270	3.243	3.252	3.254
0.42	2.956	2.952	2.959	2.935	2.943	2.945
0.43	2.682	2.678	2.684	2.662	2.670	2.671
0.44	2.439	2.435	2.441	2.421	2.428	2.429
0.45	2.223	2.219	2.225	2.206	2.212	2.214
0.46	2.030	2.027	2.032	2.015	2.021	2.022
0.47	1.858	1.855	1.860	1.844	1.849	1.850
0.48	1.704	1.701	1.705	1.691	1.696	1.697
0.49	1.565	1.563	1.567	1.554	1.558	1.559
0.50	1.441	1.438	1.442	1.430	1.434	1.435
0.51	1.329	1.326	1.330	1.319	1.322	1.323
0.52	1.227	1.225	1.228	1.218	1.222	1.222
0.53	1.135	1.133	1.136	1.127	1.130	1.131
0.54	1.052	1.050	1.053	1.044	1.047	1.048
0.55	$9.760\,(\times 10^{-2})$	$9.745\,(\times 10^{-2})$	$9.769\,(\times 10^{-2})$	$9.688\,(\times 10^{-2})$	$9.761\,(\times 10^{-2})$	$9.721\,(\times 10^{-2})$
0.56	9.067	9.053	9.076	9.000	9.026	9.031
0.57	8.435	8.422	8.443	8.373	8.396	8.401
0.58	7.857	7.845	7.865	7.799	7.821	7.826
0.59	7.328	7.316	7.335	7.274	7.294	7.298
0.60	6.842	6.832	6.849	6.792	6.811	6.815
0.61	6.398	6.388	6.404	6.351	6.369	6.372
0.62	5.989	5.980	5.995	5.945	5.962	5.965
0.63	5.612	5.604	5.618	5.571	5.587	5.590
0.64	5.265	5.257	5.270	5.226	5.241	5.244
0.65	4.944	4.936	4.949	4.908	4.922	4.924
0.66	4.647	4.640	4.652	4.613	4.626	4.629
0.67	4.373	4.366	4.377	4.340	4.353	4.355
0.68	4.118	4.112	4.122	4.088	4.099	4.102
0.69	3.882	3.876	3.885	3.853	3.864	3.866
0.70	3.662	3.656	3.666	3.635	3.645	3.647
0.71	3.458	3.452	3.461	3.432	3.442	3.444
0.72	3.268	3.262	3.271	3.243	3.253	3.254
0.73	3.090	3.085	3.093	3.067	3.076	3.078
0.74	2.925	2.920	2.928	2.903	2.912	2.913
0.75	2.770	2.766	2.773	2.750	2.758	2.759
0.76	2.626	2.621	2.628	2.606	2.614	2.615
0.77	2.490	2.486	2.492	2.472	2.479	2.480
0.78	2.363	2.359	2.365	2.346	2.352	2.354
0.79	2.244	2.241	2.246	2.228	2.234	2.235
0.80	2.133	2.129	2.135	2.117	2.123	2.124
0.90	1.326	1.324	1.328	1.317	1.320	1.321
1.00	$8.680\,(\times 10^{-3})$	$8.666\,(\times 10^{-3})$	$8.688\,(\times 10^{-3})$	$8.616\,(\times 10^{-3})$	$8.640\,(\times 10^{-3})$	$8.645\,(\times 10^{-3})$

Table 4.   (continued)

Wavelength (µm)	Tropical (15° N)	Midlatitude Summer (45° N, July)	Midlatitude Winter (45° N, January)	Subarctic Summer (60° N, July)	Subarctic Winter (60° N, January)	1962 U.S. Standard
1.10	5.917	5.908	5.923	5.873	5.890	5.893
1.20	4.171	4.165	4.175	4.141	4.152	4.155
1.30	3.025	3.020	3.028	3.003	3.011	3.013
1.40	2.247	2.243	2.249	2.230	2.237	2.238
1.50	1.704	1.701	1.705	1.691	1.696	1.697
1.60	1.315	1.313	1.317	1.306	1.309	1.310
1.70	1.032	1.030	1.033	1.024	1.027	1.027
1.80	$8.204\,(\times 10^{-4})$	$8.191\,(\times 10^{-4})$	$8.212\,(\times 10^{-4})$	$8.143\,(\times 10^{-4})$	$8.167\,(\times 10^{-4})$	$8.171\,(\times 10^{-4})$
1.90	6.606	6.596	6.613	6.557	6.576	6.580
2.00	5.379	5.371	5.384	5.339	5.355	5.358
2.20	3.672	3.666	3.676	3.645	3.655	3.658
2.40	2.592	2.588	2.594	2.573	2.580	2.581
2.60	1.881	1.878	1.883	1.867	1.873	1.874
2.80	1.398	1.396	1.400	1.388	1.392	1.393
3.00	1.061	1.059	1.062	1.053	1.056	1.057
3.50	$5.724\,(\times 10^{-5})$	$5.715\,(\times 10^{-5})$	$5.730\,(\times 10^{-5})$	$5.682\,(\times 10^{-5})$	$5.698\,(\times 10^{-5})$	$5.702\,(\times 10^{-5})$
4.00	3.355	3.350	3.358	3.330	3.340	3.341

°C) for standard air.   In Eq. (10) temperatures must be given in degrees Kelvin, whereas pressures need only be in consistent units.

4.   Rayleigh Phase Function Calculation

For unpolarized, incident radiation, the angular distribution of the light scattered by air is given by the Rayleigh phase function P_{ray}, which is typically written as

$$P_{ray}(\theta) = \tfrac{3}{4}(1 + \cos^2 \theta), \qquad (11)$$

where θ represents the scattering angle.   However, because molecular anistropy also affects the angular distribution of Rayleigh-scattered light, a more accurate formula is given by Chandrasekhar[25]:

$$P_{ray}(\theta) = \frac{3}{4(1 + 2\gamma)}\left[(1 + 3\gamma) + (1 - \gamma)\cos^2 \theta\right], \quad (12)$$

where γ is defined by[25]

$$\gamma = \frac{\rho_n}{2 - \rho_n} \qquad (13)$$

and ρ_n is the depolarization factor.   In Fig. 3 the Rayleigh phase function for the full range of scattering angles at 0.5 µm calculated with [see Eq. (12)] and without [see Eq. (11)], with the effect of molecular anisotropy taken into account, are compared.   The greatest differences, approximately 1.5%, arise in the forward and the backscattered directions and near the 90° scattering angle.

Because ρ_n varies with wavelength, as discussed above, the effect of the anisotropy on the phase function [see Eq. (12)] will also vary slightly with wavelength.   Table 1 lists the values of γ as a function of wavelength derived from the King correction factors of Bates.[22]   As with the depolarization factor, it can be seen from Table 1 that the γ term

varies by approximately 60% from the near IR to the UV spectral region.   In the forward or the backscattered directions this variation in γ would introduce a change of approximately 1% in the Rayleigh phase function computed with Eq. (12).   A rigorous calculation of the Rayleigh phase function should therefore include the dispersion with wavelength of γ as given in Table 1.

To include the dispersion with wavelength of the depolarization factor consistently in the calculation of the angular volume-scattering coefficient $\beta(\theta, \lambda, z)$ is straightforward[1]:

$$\beta(\theta, \lambda, z) = \frac{\beta(\lambda, z)}{4\pi}P_{ray}(\theta, \lambda), \qquad (14)$$

where $\beta(\lambda, z)$ is calculated by the use of Table 2 and Eq. (10) and $P_{ray}(\theta, \lambda)$ is calculated from Eq. (12) with γ as given in Table 1.   A similar expression holds for the angular-scattering cross section, where σ is substituted for β in Eq. (14).

5.   Rayleigh Optical-Depth Calculation

As shown in Eq. (3), the Rayleigh optical depth τ at altitude z_0 is given as the integral of the total volume-scattering coefficient from z_0 to the top of the atmosphere. Using Eq. (10), Eq. (3) can be rewritten as

$$\tau(\lambda, z_0) = \int_{z_0}^{\infty} \beta_s(\lambda)\frac{P(z)}{P_s}\frac{T_s}{T(z)}\,\mathrm{d}z. \qquad (15)$$

For a given temperature and pressure profile Eq. (15) was integrated from 0 to 100 km in 1-km altitude increments through the use of the trapezoidal rule. Table 4 lists the surface ($z_0 = 0$) optical depths calculated in this way for the 1962 U.S. Standard Atmosphere[8] and five supplementary models[14] that take into account both seasonal and latitudinal variations: Tropical (15° N), Midlatitude Summer (45° N, July),

Midlatitude Winter (45° N, January), Subarctic Summer (60° N, July), and Subarctic Winter (60° N, January).

Figure 4 compares the Rayleigh optical depths computed in this paper for the 1962 U.S. Standard Atmosphere[8] case with those calculated by Elterman[7] and by Frölich and Shaw.[15] The optical depths calculated here differ from those of Elterman[7] in exactly the same way as the cross sections calculated above differed from those of Penndorf[2] (see Fig. 2). This is to be expected because Elterman[7] used Penndorf's[2] cross-section values. Figure 4 also shows the comparisons of the Rayleigh optical depths from this paper with the optical depths calculated by Frölich and Shaw[15] and with their values multiplied by 1.031, a correction factor suggested by Young[19] to compensate for the error in their depolarization term. Although this correction drastically improves the data from Frölich and Shaw near 0.7 μm, the error increases at longer and shorter wavelengths because the dispersion of the anisotropy was not taken into account. In addition, Frölich and Shaw's[15] data only extend from 0.26 to 1.5 μm. Using their analytic formulas beyond these wavelengths leads to errors of 2–25%, even for the corrected case.

Equation (15) can be shown to be equivalent to

$$\tau(\lambda, z_0) = \sigma(\lambda)N_{\text{col},z_0}, \tag{16}$$

where N_{col,z_0} is the column density at altitude z_0. Least-squares fits of the data in Table 4 will therefore yield expressions of the form of Eq. (8), where the A coefficients are multiplied by the column densities for the given atmospheric profile. The B, C, and D coefficients will be the same as those given in Table 3. Table 5 lists the A coefficients derived for each model atmosphere along with the surface pressure and the temperature of each model. Again, because the only difference is a multiplicative constant, these equations will fit the optical-depth data of Table 4 to the same accuracy as the cross-section fits, that is, to within 0.4% for wavelengths less than 0.25 μm, to better than 0.2% in the 0.25–0.5-μm range, and to better than 0.1% for wavelengths greater than 0.5 μm.

Equation (16) shows that the Rayleigh optical depth scales with the column density, therefore the optical depth at any given pressure can be approximated from the data in Tables 4 and 5 when the most appropriate model atmosphere is chosen and the following approximation is used:

$$\tau(\lambda, z) = \tau(\lambda, z_0) \frac{N_{\text{col},z}}{N_{\text{col},z_0}} \approx \tau(\lambda, z_0) \frac{P(z)}{P(z_0)}. \tag{17}$$

6. Summary

This paper has presented a consistent set of newly calculated values for the total Rayleigh-scattering cross section, the total Rayleigh volume-scattering coefficient, and the Rayleigh optical depth as a function of wavelength (0.2–0.4 μm), through the use of the best estimates of the refractive index[18] and the depolarization factor[22] of air. Most important, the dispersion of the depolarization factor with wavelength was included in the calculations so that the values given here are valid throughout the wavelength range stated.

In addition, all the Rayleigh-scattering parameters listed above were fitted with least-squares-derived analytic formulas as a function of wavelength that can be easily incorporated into computer codes (see Tables 3 and 5). These formulas fit the data to within 0.4% for wavelengths less than 0.25 μm, to better than 0.2% in the 0.25–0.5-μm range, and to better than 0.1% for wavelengths greater than 0.5 μm.

A comparison with some of the most common previous calculations of the Rayleigh-scattering parameters showed the improvement in the values given in this paper. The cross sections and the
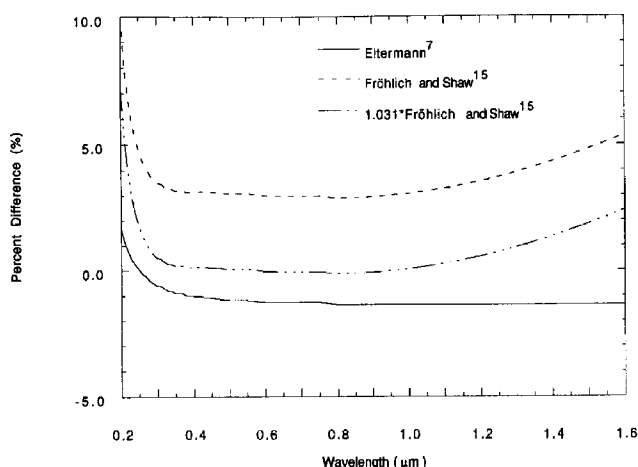
Table 5. Constants Used in the Analytic Formula for the Optical Depth Along with the Surface Pressure and Temperature of Each Model Atmosphere Used in Table 4[a]

Model	A 0.2–0.5 μm	λ > 0.5 μm	$P(z_0 = 0)$ (mbars)	$T(z_0 = 0)$ (K)
Tropical	6.52965×10^{-3}	8.68094×10^{-3}	1013	300
Midlatitude Summer	6.51949×10^{-3}	8.66735×10^{-3}	1013	294
Midlatitude Winter	6.53602×10^{-3}	8.68941×10^{-3}	1018	272.2
Subarctic Summer	6.48153×10^{-3}	8.61695×10^{-3}	1010	287
Subarctic Winter	6.49997×10^{-3}	8.64145×10^{-3}	1013	257.1
1962 U.S. Standard	6.50362×10^{-3}	8.64627×10^{-3}	1013	288.1

[a]Form of the analytic formula is as given in Table 3. The B, C, and D coefficients are given in Table 3.

Fig. 4. Comparison of the Rayleigh optical depth at 0-km altitude for the 1962 U.S. Standard Atmosphere[8] as computed in this paper (see Table 4) with those optical depths calculated by Elterman,[7] Frölich and Shaw,[15] along with Frölich and Shaw's value[15] multiplied by a 1.031 correction factor. The percent difference is defined as in Fig. 2, where x now represents optical depth.

volume-scattering coefficients of Penndorf[2] and Elter-mann[7] are from 1.7% lower to 1.4% higher than those calculated here, depending on the wavelength, with the best agreement occurring where the depolarization values are similar, near 0.26 µm (see Fig. 2). The same differences hold when the optical depths of Eltermann[7] are compared with those calculated in this paper for the 1962 U.S. Standard Atmosphere model[8] (see Fig. 4). The uncorrected optical depths of Frölich and Shaw[15] are too low by approximately 3% at visible wavelengths to 10–20% too low at longer and shorter wavelengths (see Fig. 4). Although the corrected optical depths of Frölich and Shaw[15] show much better agreement at visible wavelengths, large errors still exist at longer and shorter wavelengths (see Fig. 4).

Finally, the importance of including the dispersion of the molecular anisotropy in the Rayleigh phase function calculation was illustrated (see Fig. 3). The largest errors made when molecular anistropy, and the dispersion of the molecular anisotropy, are ignored occur in the forward and the backscattered directions and at a 90° scattering angle. These errors are of the order of 1% to 1.5%.

With the inclusion of the variation of the depolarization factor with wavelength, the tabular values of the Rayleigh-scattering parameters listed in this paper and the least-squares-derived analytic formulas given provide a more accurate estimate of the amount of Rayleigh scattering in the terrestrial atmosphere throughout the stated wavelength range than previous calculations, in which a constant depolarization factor was assumed.

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