



Real options pricing by the finite element method

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ABSTRACT

Real option pricing problems in investment project evaluation are mostly solved by the simulation-based methods, the lattice methods and by the finite difference method (FDM). Only a few applications of the finite element method (FEM) to these problems have been reported in the literature; although it seems to be an alternative tool for pricing real options.

Unlike the existing finite element-based papers, in this paper we use residual formulation and provide a detailed scheme for practical implementations. The FEM is introduced and developed as a numerical method for real options pricing problems. First of all, a partial differential equation (pde) model is defined, then the problem's domain is discretized by finite elements. The weak formulation of the pde is then obtained, and finally the solution to the real option pricing problem is found by solving an algebraic system. For benchmarking purposes, the FEM is applied to known investment and abandonment option problems found in the literature and the results are compared with those of some traditional methods. These results show a good performance of the FEM and its superiority over the FDM in terms of convergence, and over the simulation-based methods in terms of the optimal exercise policy.

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1. Introduction

It is known that companies make investments to create and to take advantage of the opportunities to expand businesses and to make profits for their own business in competitive and uncertain markets. These opportunities, converted into investment projects, are generally assessed by traditional methods of evaluation like discounted cash flow and Monte Carlo simulation. However, these methods consider only one expected scene for the project's cash flows, so they show some deficiencies when the investment projects are subjected to high volatility and uncertainty during its life. In this case, they are not capable of assessing appropriately the inherent managerial (strategic) flexibility of such projects and price it numerically, and where managerial flexibility means the capability of reviewing the decisions to take the best option present in the project's life in response to the dissipation of the uncertainty with the passage of time. A better approach to assess the managerial flexibility is often made by decision trees, but this approach has its own problem of assigning the right probabilities for different future scenes.

On the other hand, this managerial flexibility of the project can be assessed using a positioning, supporting and flexibility analysis from the strategic theory [1]. Therefore, when assessing investment projects subjected to high volatility and uncertainty like research & development projects it is necessary to use a method which can integrate the financial and the strategic theories to price appropriately the managerial flexibility present in the investment projects as form of options. Myers [2] proposed a method, called the real option method, for assessing economically investment projects which can integrate both the financial and the strategic points of view in assessing the project's investment. The real option method

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captures the value of managerial flexibility to adapt decisions in response to unexpected market developments. It applies financial options theory to quantify the value of management flexibility and leverage uncertainty in a changing world. The method is based on financial options pricing model developed by Black & Scholes in 1973 [3] and extended by Merton [4]. This model is widely used in finance to value different kind of financial instruments. Another widely used method is due to Cox et al. [5]. They have presented a simple approach to the options pricing model developed by Black & Scholes using a lattice method based on a simple discrete time model. A good review of valuation models to the option pricing from its origins to the present can be found in [6].

Until now, real option pricing problems in the investment project evaluation have been solved by the simulation-based methods [7–9], the lattice methods [8,10] and by the FDM method [10–13], delaying the study and the application of others numerical methods like the finite elements method (FEM), which is widely documented and used in others fields of science and engineering for decades. There is not a wide literature about the use of the FEM in real option pricing problems, in most cases the numerical solutions to the pde governing the real option pricing models (based on the Black & Scholes model) are found using the FDM. At the present state of art, some works are related in some way to the FEM applied to the real options pricing problems like the work of Achdou and Tchou [14], who realize a variational analysis of the Black & Scholes equation considering stochastic volatility. Ern et al. [15] used the adaptive FEM to the valuation of European options with local volatility, focussing on adaptive control of errors. Zhang [16] studied the American options valuation through an adaptive FEM using a variational formulation, and Zhu [17] applied the FEM to some exotic options. Topper [18], on the other hand, in his technical note, studied in a generalized way the real options pricing using finite element based on a residual formulation, which is closer to the work done in the present paper.

In this paper, the FEM is introduced as a numerical method that allows finding numerical solutions of the pde governing the real options pricing problems based on the Black & Scholes model. Then, the FEM is applied to two different real option pricing problems. Results obtained are then compared with those obtained by other numerical methods, for benchmarking purposes.

The remainder of this paper is organized as follows. Section 2 presents the real option pricing model defined by a partial differential equation and its boundary conditions. This model is based on the Black & Scholes model written in term of diffusion variables. Section 3 introduces the FEM for the option pricing problems. Section 4 presents the applications of the FEM to the valuation of the investment and abandonment real options. Finally, conclusions are made in Section 5.

2. The real option pricing model

To understand a certain phenomenon of a problem, it is necessary to formulate a model that represents the phenomenon as faithfully as possible. Formulating an exact model is not always possible and it is common to work with a simplified model that achieves a good approximation to the real phenomenon. Basically, there are two important works in financial option pricing that allow one to price real options in the investment projects considering the underlying asset as a real asset, they are the Black & Scholes [3] method and the lattice method [5]. The first one is based on a continuous time model and the second one is based on a discrete time model and both of them are based on the assumptions of riskless arbitrage opportunities absence and on risk-neutral valuation. The present work is focused and based on the Black & Scholes continuous time model for pricing the real options, writing it in terms of diffusion variables to avoid numerical problems when numerical methods are applied [13].

2.1. The backward moving Black & Scholes model

The Black & Scholes model considers that the rate of return of the subjacent real asset follows a generalized Wiener process. This stochastic process determines that the real asset follows a lognormal distribution and its behavior cannot be deterministically determined. To obtain the process followed by the real option price, one can show that the Black & Scholes equation takes the form:

$$\frac{\partial V}{\partial t} + rS \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \quad (1)$$

where S is a real asset value, $0 \leq S < \infty$, V is the (real) option price, r is the risk-free rate, t is the time since the option was issued, $0 \leq t \leq T$, and σ is the real asset volatility [12,19,20]. Eq. (1) is a backward moving equation, i.e. it is solved from the future to the present time.

To complete the model it is necessary to define appropriate time and boundary conditions associated with (1). These conditions depend on the kind of real option (call or put; European or American) being evaluated and on the numerical method to be applied. A call/put option gives the owner the right to buy/sell the underlying real asset for a certain price on a certain date. The specified price is known as the exercise or strike price and will be denoted by K and the specified date is known as the expiration date or maturity date and will be denoted by T . An European option can be exercised only at the expiration date T itself, while an American option can be exercised at any time between the issue date and the expiration date. On the other hand, the FEM requires defining essential and natural boundary conditions to solve the pde model unlike the FDM which requires just essential boundary conditions. An essential boundary condition is defined in terms of the

variable's value at the domain's boundary and a natural boundary condition is defined in terms of the variable's derivative value.

For an European call option the time condition becomes a final condition because its value is known at the maturity date $t = T$ and it is defined as its intrinsic value by:

$$V(S, T) = \max(S - K, 0), \quad \forall S. \quad (2)$$

The boundary conditions of the domain Ω_S can be defined considering that for all time t when the subjacent real asset tends to zero, $S \rightarrow 0$, the call option has no value and when the subjacent real asset tends to infinity, $S \rightarrow \infty$, by put-call parity the call option value tends to its intrinsic value. This last condition can be expressed in terms of its the derivative considering that when the subjacent real asset tends to infinity, $S \rightarrow \infty$, the derivative of the call option value with respect to the real asset tends to unity. Then, the essential boundary condition for $S = 0$ and the natural boundary conditions for $S = S_\infty$ can be expressed as follows:

$$V(0, t) = 0 \quad (3)$$

$$\frac{\partial V}{\partial S}(S_\infty, t) = 1, \quad \forall t. \quad (4)$$

For an European put option the final condition at the maturity date $t = T$ is defined as:

$$V(S, T) = \max(K - S, 0), \quad \forall S. \quad (5)$$

The essential and natural boundary conditions of the domain Ω_S can be defined considering that for all time t when the subjacent real asset tends to zero, $S \rightarrow 0$, the put option tends to today's strike price, $Ke^{-r(T-t)}$, and when the subjacent real asset, $S \rightarrow \infty$, the put option has no value and its derivative with respect to the real asset tends to zero. Therefore, the essential boundary condition for $S = 0$ and the natural boundary conditions for $S = S_\infty$ can be expressed as follows:

$$V(0, t) = Ke^{-r(T-t)} \quad (6)$$

$$\frac{\partial V}{\partial S}(S_\infty, t) = 0, \quad \forall t. \quad (7)$$

The boundary conditions for (1) are now fully defined. We refer [20] for more details on conditions (2)–(7).

2.2. The diffusion forward moving Black & Scholes model

Numerical instability may occur when solving Eq. (1) numerically due to the fact that it is a pde with variables coefficients and due to the existence of the convective term [13]. This can be overcome by the following transformation of variables:

$$\begin{aligned} x &= \ln \frac{S}{K}, \\ \tau &= \frac{\sigma^2}{2}(T - t), \\ v(x, \tau) &= \frac{1}{K}V(S, t)e^{\frac{1}{2}(k_1-1)x + \frac{1}{4}(k_1+1)^2\tau}, \end{aligned} \quad (8)$$

where the constant term is defined as follows:

$$k_1 = \frac{r}{\sigma^2/2}. \quad (9)$$

Under the above transformation, it can be shown that pde (1) reduces to the following diffusion partial differential equation [20]:

$$\frac{\partial v}{\partial \tau} - \frac{\partial^2 v}{\partial x^2} = 0, \quad (10)$$

which is valid for $-\infty < x < +\infty$ and $0 \leq \tau \leq \sigma^2 T/2$; the variables, v , x and τ defined by (8) are the diffusion variables. The parabolic equation (10) is known as the one-dimensional heat conduction equation and it can be interpreted as the distribution of temperature $v(x, \tau)$ for a beam at time τ . An advantage of using (10) to value real options is that it is a forward moving model which is easier to solve numerically. The time and boundary conditions (2)–(4) and (5)–(7) respectively for call and put options must be re-written using the transformation of variables given in (8). Thus, for an European call option the final condition at the maturity date $t = T$ defined as its intrinsic value can be rewritten as follows:

$$v(x, 0) = \max(e^{\frac{1}{2}(k_1+1)x} - e^{\frac{1}{2}(k_1-1)x}, 0), \quad \forall x, \quad (11)$$

and the essential boundary condition for $S = 0$ and the natural boundary conditions for $S = S_\infty$ can be rewritten as follows:

$$v(x_{-\infty}, \tau) = 0, \quad (12)$$

$$\frac{\partial v}{\partial x}(x_\infty, \tau) = \frac{1}{2}((k_1 + 1)e^{x_\infty} - (k_1 - 1)e^{-k_1\tau})e^{\frac{1}{2}(k_1-1)x_\infty + \frac{1}{4}(k_1+1)^2\tau}, \quad \forall \tau, \quad (13)$$

where $x_{-\infty}$ and x_∞ are the lower and upper bounds for x , respectively. The time and the boundary conditions [20] for an European put option are as follows:

$$v(x, 0) = \max(e^{\frac{1}{2}(k_1-1)x} - e^{\frac{1}{2}(k_1+1)x}, 0), \quad \forall x, \quad (14)$$

$$v(x_{-\infty}, \tau) = e^{\frac{1}{2}(k_1-1)x_{-\infty} + \frac{1}{4}(k_1+1)^2\tau} \quad (15)$$

$$\frac{\partial v}{\partial x}(x_\infty, \tau) = 0, \quad \forall \tau. \quad (16)$$

Again, we refer [20] for more details on the conditions (11)–(16).

Having defined the diffusion forward moving Black & Scholes model with appropriate time and boundary conditions to value real European options, we now describe how (10) can be used to value American options. However, unlike an European option, any solution method for (10) must ascertain that the following features be retained.

- (i) Since an American option can be exercised at any time until its expiration date, the question of the optimal exercise policy arises. Hence, the exercise policy must maximize the real option value.
- (ii) The option value, $V(S, t)$, cannot be less than the payment, $g(S)$, obtained if the option is exercised at the time t , i.e. $V(S, t) \geq g(S)$. Otherwise, it can be demonstrated that arbitrage opportunities would exist [12].

Two approaches can be used. The first one considers the resolution of the complementary linear problem obtained from the free-boundary formulation of the American option pricing problem through the use of variational techniques [16,6], where the free-boundary is the boundary that determines the separation between the optimal and non-optimal exercise region in the problem domain, $\Omega_{t,S} = \{(t, S) | 0 \leq t \leq T, 0 \leq S < +\infty\}$. The second approach is based on Bellman's dynamic programming principle [12]. Based on this principle, it is possible to value an American option through any backward moving in time method like the lattice method and numerical methods like the FEM and FDM applied to the pde model (10) with appropriate boundary conditions. It is also possible to value an option through forward moving in time methods like the least squares Monte Carlo simulation (LSM), see for example the Refs. [7–9]. In this paper, we have adopted the second approach with the diffusion forward moving Black & Scholes model through the application of the FEM.

3. Finite element applied to real option pricing

Until now, the approximated methods used to value options have been based on the lattice method [8,10] and on the FDM [10–13]. In this section, we present the FEM as an approximate method that allows solving the pde model governing the real option pricing problems. The FEM discretizes the continuous domain of the problem by means of a series of simple geometric forms called finite elements, for which the governing relations on the entire continuous domain are valid on each element. Under this assumption, the approximate solution in the entire continuous domain of the problem can be obtained by means of trial functions also called the form functions. The FEM transforms the differential equation into an algebraic system of equations which can then be solved easily by known numerical methods. We have used the well-known Galerkin weighted residue method to find the finite element solution of (10).

3.1. Discretization of variables in the FEM

We consider the case of valuing an European call option $V(S, t)$ through the diffusion forward moving Black & Scholes model defined previously by (10), (11)–(13).

To solve this problem through the FEM it is necessary to discretize appropriately the time-space domain of the problem,

$$\Omega_{\tau,x} = \left\{ (\tau, x) \mid 0 \leq \tau \leq \frac{\sigma^2 T}{2}, x_{-\infty} \leq x \leq x_{+\infty} \right\}.$$

For this purpose, the Kantorovitch's discretization is used [21]. This kind of discretization is a partial discretization by finite elements, because just the space domain Ω_x is discretized by finite element whereas the time domain Ω_τ is discretized by finite difference, see [15,16]. Using Kantorovitch's discretization the approximate solution is obtained at each time τ for the space domain Ω_x . Let us consider the following discretization of the continuous domain $\Omega_{\tau,x}$:

$$\tau_i = \tau_0 + i\Delta\tau, \quad i = 0, 1, \dots, n, \quad (17)$$

$$x_j = x_{-\infty} + (j - 1)\Delta x, \quad j = 1, 2, \dots, m + 1. \quad (18)$$

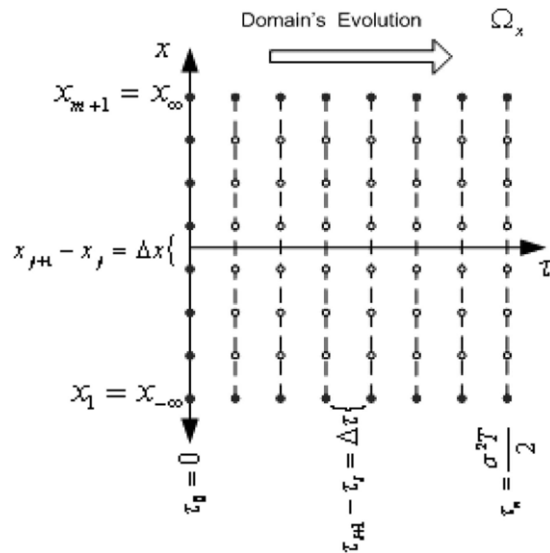


Fig. 1. Discretization of the pde model.

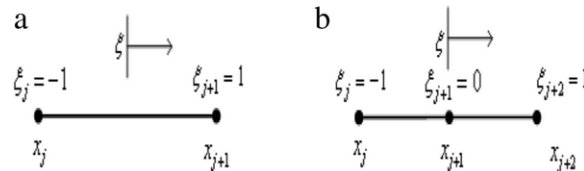


Fig. 2. Two and three nodes finite elements.

Using the above discretization we obtain a regular mesh of $(n+1)(m+1)$ of discrete points (τ_i, x_j) for independent variables of the model. In this way, the domain's intervals $[0, \sigma^2 T/2]$ and $[x_{-\infty}, x_{+\infty}]$ are divided in constant length subintervals $\Delta\tau$ and Δx defined by

$$\Delta\tau = \frac{\tau_n - \tau_0}{n} = \frac{\delta^2 T}{2n}, \quad (19)$$

$$\Delta x = \frac{x_{m+1} - x_1}{m} = \frac{x_{\infty} - x_{-\infty}}{m}. \quad (20)$$

Fig. 1 shows the one dimensional space discretization Ω_x and its evolution through the scaled time τ . The finite element can be defined by two adjacent discrete points x_j and x_{j+1} and or by three adjacent discrete points x_j , x_{j+1} and x_{j+2} as shown in Fig. 2(a) and (b), respectively. The number of finite elements in Ω_x is m and $m/2$ respectively in the first and the second case, for $m+1 > 3$, m is an even number. In Fig. 2, ξ denotes the local coordinate which can takes integer values only. It is used to define all variables at each finite element. At each j finite element it is necessary to define the approximate solution v and the space variable x as a combination of its values at the finite element's nodes. For this purpose the form functions $N_k(\xi)$, which is defined in terms of the local coordinate ξ , are used. The Lagrangian form functions [21] are commonly used. These are linear or quadratic type depending on the required approximation. The linear form functions are required for the finite element with two nodes, and the quadratic function is needed for the elements with three nodes; this is explained using Figs. 3 and 4. It can be seen in these figures that each form function $N_j(\xi)$ is related to a finite element's node j , which takes the value of unity at j and zero at all other nodes. The approximate solution v and the space variable x , when restricted to a typical finite element, can be written as a linear combination of node values using Lagrangian linear form functions as:

$$v = \sum_{k=j}^{j_k} N_k(\xi) q_k(\tau) = \frac{1-\xi}{2} q_j + \frac{1+\xi}{2} q_{j+1}, \quad (21)$$

$$x = \sum_{k=j}^{j_k} N_k(\xi) x_k = \frac{1-\xi}{2} x_j + \frac{1+\xi}{2} x_{j+1}, \quad (22)$$

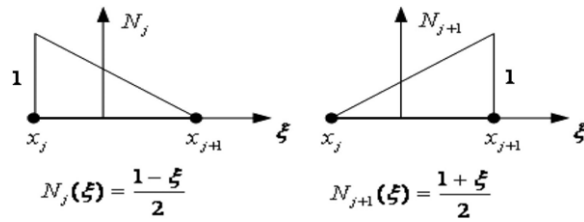


Fig. 3. Lagrangian linear form functions.

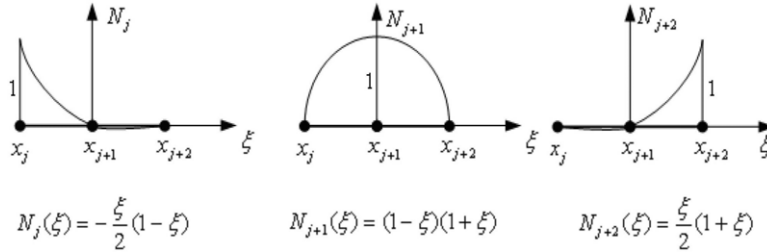


Fig. 4. Lagrangian quadratic form functions.

where j_k denotes the number of nodes in the finite element. The solution v and the space variable x can be written as a quadratic combination of node values using Lagrangian quadratic form functions as:

$$v = \sum_{k=j}^{j_k} N_k(\xi) q_k(\tau) = -\xi \frac{1-\xi}{2} q_j + (1-\xi)(1+\xi) q_{j+1} + \xi \frac{1+\xi}{2} q_{j+2}, \quad (23)$$

$$x = \sum_{k=j}^{j_k} N_k(\xi) x_k = -\xi \frac{1-\xi}{2} x_j + (1-\xi)(1+\xi) x_{j+1} + \xi \frac{1+\xi}{2} x_{j+2}. \quad (24)$$

When using the same form functions to define the approximate solution v and the space variable x an isoparametric formulation is obtained from which the values q_j need to be found to know the approximate solution of the real option pricing problem.

3.2. Integral formulation of the partial differential equation

Once the domain has been discretized it is necessary to find the value of time dependent parameters $q_j(\tau)$, called the generalized variables, which allow one to write the approximate solution at each finite element using the Eqs. (21) or (23). To get these generalized variables, it is necessary to define the weak or integral formulation of the differential equation of the real option pricing problem which is obtained by the Galerkin weighted residue method.

The Galerkin weighted residue method obtains the generalized variables $q_j(\tau)$ by minimizing the residue, say R . The residue, R , is the result of the exact solution being replaced by the approximate solution in $L(v) - f = 0$, where L is the differential operator and f a function of independent variables. Hence, $R = L(v) - f$, where v is an approximate solution. The residue R is minimized to zero by weighting it with the so-called weight functions w_k . In the Galerkin method, the form functions are used as the weight functions e.g., $w_k = N_k(\xi)$. This results in the equation

$$\int_{\Omega} w_k R d\Omega = \int_{\Omega} w_k (L(v) - f) d\Omega = \int_{\Omega} N_k(\xi) \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial \tau} \right) d\Omega = 0 \quad (25)$$

to be solved, where k represents all nodes in Ω_x . Using $N^T = (N_j(\xi), N_{j+1}(\xi), \dots, N_K(\xi))$, the approximate solution $v = N \cdot q$, $x = N \cdot X$, $X^T = (x_j, x_{j+1}, \dots, x_K)$ and $q^T = (q_j, q_{j+1}, \dots, q_K)$, the expression (25) can be written as:

$$\int_{x_j}^{x_K} N^T \frac{\partial^2 (N \cdot q)}{\partial x^2} dx - \int_{x_j}^{x_K} N^T \frac{\partial (N \cdot q)}{\partial \tau} dx = 0. \quad (26)$$

Changing to local variables for each finite element and integrating by part the first integral of (26) one has

$$N^T \frac{\partial N}{\partial x} q \Big|_{-1}^1 - \int_{-1}^1 \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial x} q d\xi - \int_{-1}^1 N^T N \frac{\partial q}{\partial \tau} d\xi = 0, \quad (27)$$

where $J = \frac{\partial x}{\partial \xi}$. It can be shown using (22) and (24), respectively, that J equals $\Delta x/2$ for the two node finite element and Δx for the three node finite element. The term $N^T \frac{\partial N}{\partial x} q|_{-1}^1$ represents the natural boundary conditions evaluated at the extreme nodes of the element and defined by the column arrays

$$F^e = \left[-\frac{\partial v_j}{\partial x}, \frac{\partial v_{j+1}}{\partial x} \right]^T \quad (28)$$

$$F^e = \left[-\frac{\partial v_j}{\partial x}, 0, \frac{\partial v_{j+1}}{\partial x} \right]^T \quad (29)$$

for two and three node finite elements, respectively.

Eq. (27) demands that the form function used be a C_0 class function. This means that Eq. (27) requires a continuous form function in the domain, this restriction is accomplished by the Lagrangian form functions used in this work. Rearranging and introducing the array of form function's derivatives $B = \frac{\partial N}{\partial x} = \frac{\partial N}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial N}{\partial \xi} J^{-1} = \left(\frac{\partial N_j}{\partial \xi}, \frac{\partial N_{j+1}}{\partial \xi}, \dots, \frac{\partial N_K}{\partial \xi} \right) J^{-1}$ the Eq. (27) can be rewritten for each finite element as follows:

$$\left(\int_{-1}^1 N^T N J d\xi \right) \dot{q} + \left(\int_{-1}^1 B^T B J d\xi \right) q = F^e, \quad (30)$$

which is an algebraic system. Eq. (30) can be re-written as

$$C^e \dot{q} + K^e q = F^e \quad (31)$$

where \dot{q} denotes the temporal derivative of the generalized variable, $\frac{\partial q}{\partial \tau}$, and where $C^e = \left(\int_{-1}^1 N^T N J d\xi \right)$ and $K^e = \left(\int_{-1}^1 B^T B J d\xi \right)$. These expressions can be expressed for a two node finite element as:

$$C^e = \frac{\Delta x}{3} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}, \quad K^e = \frac{1}{\Delta x} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad (32)$$

and for a three node finite element as:

$$C^e = \frac{\Delta x}{15} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \quad K^e = \frac{1}{6\Delta x} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}. \quad (33)$$

3.3. The assembly procedure

The system of equations presented by (31) is valid just for a single finite element. Therefore, it is necessary to find the general system of equations that defines the approximate solution in the entire domain Ω at a certain time τ . The system of equations for the entire finite element system is represented by

$$C\dot{q} + Kq = F, \quad (34)$$

where $K = \sum_e K^e$, $C = \sum_e C^e$ and $F = \sum_e F^e$, the sum is taken over all finite elements. We describe (34) by the following example. Let us discretize Ω by two linear finite elements (e^1 and e^2), each of two nodes. K in (34) is given by

$$K = K^{e^1} + K^{e^2} = \frac{1}{\Delta x} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{\Delta x} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{1}{\Delta x} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad (35)$$

where the element K_{22} is obtained from the intermediate node that connects the two finite elements. The same procedure is applied to obtain C and F , see [20] for more details on this.

3.4. Discretization of the time variable

To solve the above algebraic system of Eq. (34) it is necessary to approximate the temporal derivative of the generalized variable $\dot{q} = \partial q / \partial \tau$ at τ_i . For this purpose, a backward finite difference approximation is used, i.e. \dot{q} at τ_i is approximated by

$$\dot{q}^i = \frac{q^i - q^{i-1}}{\Delta \tau}, \quad (36)$$

where $\Delta \tau = \tau_i - \tau_{i-1}$. Substituting (36) in (34) we get

$$\left[\frac{1}{\Delta \tau} C + K \right] q^i = \left[\frac{1}{\Delta \tau} C \right] q^{i-1} + F, \quad i = 1, 2, \dots, n, \quad (37)$$

where n is the number of nodes in the time domain Ω_t . The above system of equations is of the form $Aq = b$, which can be solved to obtain q^i at τ_i in terms of its previous value q^{i-1} at τ_{i-1} .

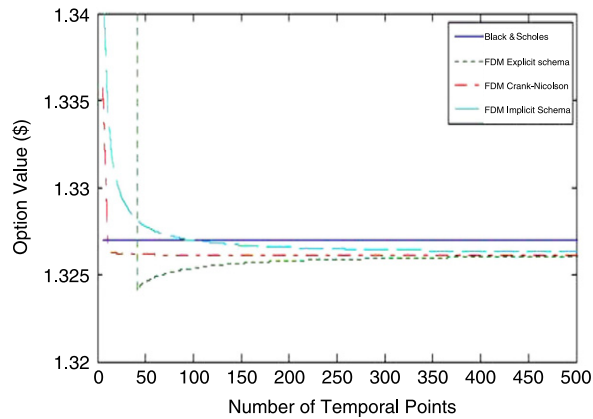


Fig. 5. Investment option value obtained with FDM.

3.5. Application of time and boundary conditions

We now present the initial condition at $\tau_0 = 0$ and boundary conditions for (37) and show that the system can be reduced further. These conditions are given by

$$q^0 = \max(e^{\frac{1}{2}(k_1+1)x - \frac{1}{2}(k_1-1)x}, 0), \quad (38)$$

$$q_1^i = 0, \quad i = 1, 2, \dots, n, \quad (39)$$

$$F_{m+1} = \frac{1}{2}((k_1 + 1)e^{x_\infty} - (k_1 - 1)e^{-k_1\tau})e^{(\frac{1}{2}(k_1-1)x_\infty + \frac{1}{4}(k_1+1)^2\tau)}. \quad (40)$$

The condition (38) is the initial condition and the conditions (39)–(40) are the essential and natural boundary conditions for an European call option.

The condition $q_1^i = 0$ in (39) transforms the system (37) with $m - 1$ equations and $m - 1$ unknowns, see [20].

Now, once the algebraic system (37) is solved for each time τ_i , the approximate solution $V(S, t)$ in terms of the original variables can be obtained using inverse transformation of variable via (8).

4. Solving European and American options pricing problems

To study numerically the performance of the FEM presented here, it is necessary to apply it to known real option pricing problems found in the literature. The results obtained can then be compared with the known results for these problems. We implement the FEM to solve an European and an American real option pricing problems. The FEM was implemented on a personal computer using the Matlab programming language.

4.1. European real option pricing problem

Let us consider the real call option pricing problem presented in [9]. The problem considers an investment project valuation whose present value of the cash flows are dependent on the product sale price S through the linear relationship $V(S) = 10S$, with S following a stochastic process. The project does not need an initial investment at $t_0 = 0$ year, but it requires an investment of $I_1 = \$10$ millions at $t_1 = 1$ year, which, if invested, generates cash flows with a present value $V(S)$ at t_1 . Depending upon the present value $V(S)$ at t_1 the option holder can decide whether to invest or not to invest at time t_1 without an abandonment value. The investment option or the project's value can be calculated as the value of an European call option written over the present value of the cash flows $V(S)$ with an exercise price of I_1 and an expiration date t_1 as $C_e = \max(V(S) - I_1, 0)$, where C_e stands for the value of the European call option. The analytical solution of the problem can be obtained using the Black & Scholes formula considering an annual risk-free rate of $r = 10\%$ and an annual volatility of the present value of the cash flows of $\sigma = 20\%$. Then, for a sale price of $S = \$1$ or equivalently a present value of cash flows $V(S) = \$10$ the analytical option value is $C_e = \$1.327$. This can be used as the benchmark value for numerical comparisons. For comparison purposes, the FDM is applied and compared to the results obtained analytically and numerically with the FEM.

For the FDM, we use three different schemes: the explicit scheme, the Crank–Nicolson scheme, and the implicit scheme. For more details of these schemes, we refer [12]. Fig. 5 shows the investment option value obtained using the three schemes for a present value of the cash flows of $V(S) = \$10$ and considering $N_s = 103$ spatial points and varying the amount of temporal points N_t on the x-axis. Fig. 5 shows that all schemes of the FDM converge at the same option value of $C_e = \$1.326$

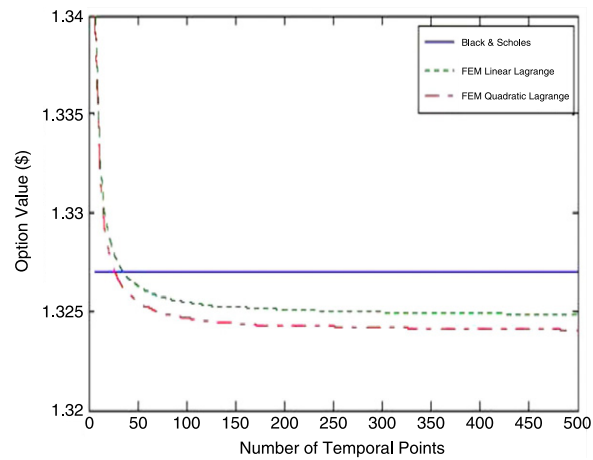


Fig. 6. Investment option value obtained with FEM.

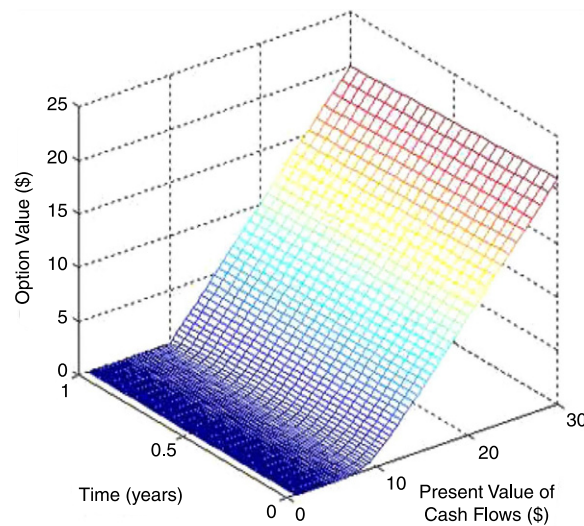


Fig. 7. Graphical representation of $V(S)$, C_e against time.

Table 1

Investment option values for various $V(S)$.

$V(S)$	8.75	9.05	9.356	9.67	10	10.4	10.689	11.05	11.42	11.81
C_e	0.57	0.72	0.898	1.1	1.328	1.58	1.864	2.17	2.5	2.85

with increasing N_t . This value is closer to the analytical option value obtained from the Black & Scholes formula. From Fig. 5, it is also clear that the Crank–Nicolson scheme converges faster than other schemes. Moreover, for $N_t < 50$, the explicit scheme diverges due to the fact that this is a conditionally convergent scheme for the FDM.

We now obtain the results of the FEM using linear and quadratic form functions. The investment option value, obtained for a present value of the cash flows of $V(S) = \$10$ and considering $N_s = 103$ spatial points and varying the number of temporal points N_t on the x-axis. Results are summarized in Fig. 6. Fig. 6 shows that the real option value obtained by the FEM converges to $C_e = \$1.3240$ and $C_e = \$1.3248$ for linear and quadratic form function, respectively. Both the values are close $C_e = \$1.327$ and $C_e = \$1.326$, the value obtained analytically from the Black & Scholes formula and with the FDM, respectively. Fig. 6 shows that, unlike the explicit scheme of the FDM, no divergence of the option value has occurred even for smaller N_t .

Fig. 7 shows the results mesh obtained from the valuation of the investment option as an European call option using the FEM with linear Lagrangian form function and considering a number of $N_s = 103$ spatial points and $N_t = 25$ temporal point. For this results mesh the Table 1 shows some values of the investment real option C_e obtained for the issued time $t_0 = 0$ in the proximity of $V(S) = \$10$, which is presented in bold.

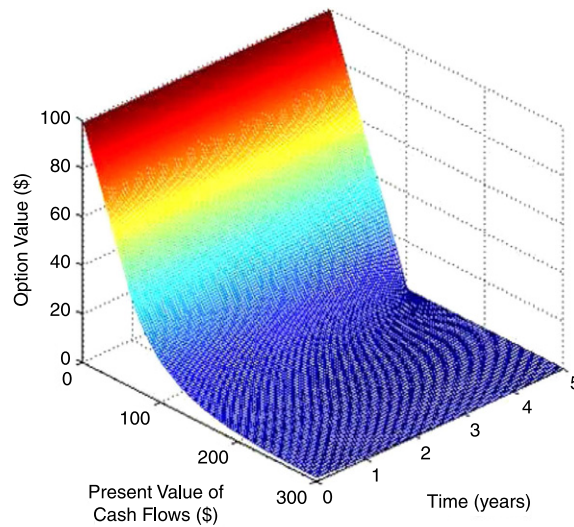


Fig. 8. Results mesh for the abandonment option pricing problem using FEM.

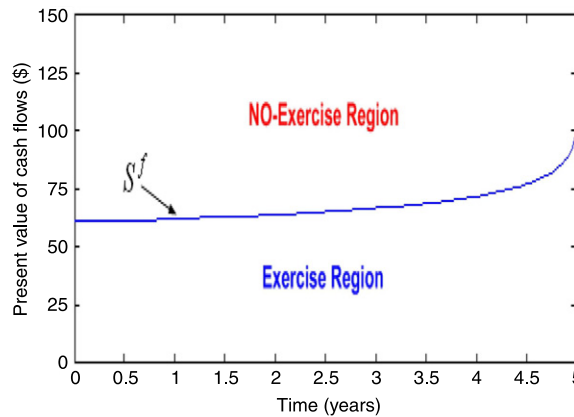


Fig. 9. Exercise boundary for the abandonment option.

4.2. American real option pricing problem

We now consider the abandonment real option pricing problem presented in Chapter 7 of the book by Mun [22].

In this work the abandonment real option present in an investment project can be priced as an American put option written over the present value of the project cash flows that can depend on some sale price $V(S)$ and exercise price I equal to the rescue value of the project.

The problem considers a pharmaceutical company developing a new medicament. Due to various types and levels of inherent uncertainties such as the market demand, the success in animal and human testing and the legal approval, the management considers strategically the abandonment option with a rescue value. This abandonment option can be evaluated as an American put option. The option defined as the present value of the cash flows V . The present value is $V = \$150$ millions, the exercise price equals the rescue value of the project estimated as $I = \$100$ millions, and expiration date of $T = 5$ years. For simplicity, we consider the rescue value as a constant; although it can be treated as a time dependent function, see [23]. The annual risk-free rate and the annual volatility of the present value of the cash flows are estimated as $r = 10\%$ and $\sigma = 20\%$ respectively. The problem does not have an analytical solution. Mun [22], using the binomial method with 1000 binomial trials, obtained the option value of $P_a = \$7.0878$ millions, where P_a denotes the value of the American put option. We apply the FEM to the problem using linear form function with $N_S = 305$ and $N_t = 101$. The relationship between $V(S)$ and P_a at various time points is presented graphically with Fig. 8. The values of P_a obtained for the issued time t_0 as a function of the present value of cash flow V are also reported in Table 2. It can be seen in Table 2 that for a present value of $V = \$150$ millions the abandonment option has a value of $P_a = \$7.10$ millions (in bold), which is close to the solution obtained by Mun [22]. Being an American put option, it is important to see the optimal and non-optimal exercise regions within the entire time domain. We have presented this with Fig. 9, where the x-axis represents the time in years and the y-axis represents the present value of the cash flows; S^f separates the optimal and non-optimal regions.

Table 2Abandonment option values for various $V(S)$.

$V(S)$	141.8	144.48	147.21	150	152.83	155.72	158.67	161.67	164.72
P_a	8.06	7.70	7.35	7.02	6.69	6.38	6.07	5.78	5.5

5. Conclusions

In this paper the finite element method is introduced as a new numerical method for solving real option pricing problems. The finite element method allows valuing both European and American options as well as both call and put options.

Results obtained by the finite element methods are comparative but the method has some advantages over other methods. First, an important feature of the FEM is that the optimal exercise policy for American options is obtained as part of the solution due to the fact that it is applied to a backward moving differential model and it does not need to determine the expected value to continue keeping the option through some kind of regression like simulation-based methods. Second, the FEM performs better with respect to the convergence than the explicit FDM when small number of time points are used. The FEM converges to the solution when the explicit scheme of FDM diverges. Further research is underway for tackling more complex and larger real option pricing problems using the FEM.

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