Tufts University - Department of Mathematics
Math 253 Homework 1

1. Let $a > 0$ and consider the one-way wave equation, $u_t + au_x = 0$. For both the Forward-Central,

$$\frac{v_{k,l+1} - v_{k,l}}{h_t} + a\frac{v_{k+1,l} - v_{k-1,l}}{2h_x} = 0,$$

and Crank-Nicholson,

$$\frac{v_{k,l+1} - v_{k,l}}{h_t} + \frac{a}{2}\left(\frac{v_{k+1,l+1} - v_{k-1,l+1}}{2h_x} + \frac{v_{k+1,l} - v_{k-1,l}}{2h_x}\right) = 0,$$

finite-difference schemes, investigate their consistency and stabilty. In particular, show that the truncation error for each scheme goes to zero as $h_t, h_x \to 0$ and perform a von Neumann stability analysis for each scheme.

2. Consider the one-way wave equation, $u_t + u_x = 0$, with $a = 1$ on a finite interval in space and time, $[0,1] \times [0,1]$. For this domain, we need to specify both an initial condition, $u(x,0) = u_0(x)$ for $0 \leq x \leq 1$, and a boundary condition, $u(0,t) = u_1(t)$ for $0 \leq t \leq 1$. To avoid contradiction, note that these two conditions should match at $(0,0)$: $u_0(0) = u_1(0)$.

Implement the Forward-Central, First-Order Upwind, Crank-Nicholson, and Backward-Central,

$$\frac{v_{k,l+1} - v_{k,l}}{h_t} + a\frac{v_{k+1,l+1} - v_{k-1,l+1}}{2h_x} = 0,$$

finite-difference schemes for this problem. Note that the Backward-Central scheme is unconditionally von Neumann stable (or feel free to prove this). Also note that for the Crank-Nicholson and Backward-Central schemes, you will need to solve a linear system to compute the numerical solution at time $l+1$ from that at time l. You can use Matlab's '\' command to do this.

Investigate the convergence of these schemes for various choices of initial and bounday conditions and choices of grid spacings, h_t and h_x. In particular, investigate cases that do and do not conform to the CFL condition for the First-Order Upwind scheme, $h_t \leq \frac{h_x}{a}$, or for those found in Problem 1. Also, be aware that the expressions for the truncation error for these schemes are based on assumptions of the smoothness of u. Investigate how the smoothness of the exact solution (reflected in the smoothness of $u_0(x)$ and $u_1(t)$) affects the performance of the numerical schemes.