Finance Research Letters 10 (2013) 17–26

Contents lists available at SciVerse ScienceDirect

Finance Research Letters

journal homepage: www.elsevier.com/locate/frl

Superconvergence of the finite element solutions of the Black–Scholes equation

A. Golbabai [a], L.V. Ballestra [b,*], D. Ahmadian [a]

[a] *School of Mathematics, Iran University of Science and Technology, Narmak, 16844 Tehran, Iran*
[b] *Dipartimento di Economia, Seconda Università di Napoli, Corso Gran Priorato di Malta, 81043 Capua, Italy*

ARTICLE INFO

Article history:
Received 28 June 2012
Accepted 11 September 2012
Available online 19 September 2012

JEL classification:
C02
C1
C63
G11
G12

Keywords:
Finite element method
Black–Schoels equation
Vanilla option
Barrier options
Gauss–Lobatto
Superconvergence

ABSTRACT

We investigate the performances of the finite element method in solving the Black–Scholes option pricing model. Such an analysis highlights that, if the finite element method is carried out properly, then the solutions obtained are superconvergent at the boundaries of the finite elements. In particular, this is shown to happen for quadratic and cubic finite elements, and for the pricing of European vanilla and barrier options. To the best of our knowledge, lattice-based approximations of the Black–Scholes model that exhibit nodal superconvergence have never been observed so far, and are somehow unexpected, as the solutions of the associated partial differential problems have various kinds of irregularities.

1. Introduction

A very popular approach to derivative pricing is the use of mathematical models based on partial differential equations. In particular, among the most commonly employed models, there is the famous Black–Scholes (BS) model (Black and Scholes, 1973), which provides exact closed-form solutions for various kinds of financial derivatives.

* Corresponding author.
E-mail addresses: golbabai@iust.ac.ir (A. Golbabai), lucavincenzo.ballestra@unina2.it (L.V. Ballestra), d_ahmadian@iust.ac.ir (D. Ahmadian).

1544-6123/$ - see front matter © 2012 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.frl.2012.09.002

However, in the everyday financial practice it is also frequent the use of partial differential models that are very similar to the BS model but for which exact analytical solutions are not available. Among such small deviations from the original BS paradigm there are, for example, the non-linear BS models (Barles and Soner, 1998; Company et al., 2008), models with non-local volatility (Dupire, 1994), models with time-dependent barriers (Ballestra et al., xxxx), models with two stochastic factors (Wilmott, 2006), and interest rate models (Daglish, 2010; Hull and White, 1990). These latter models must be solved using numerical techniques, among which the most widespread is the finite difference method (FDM), see (Clarke and Parrot, 1999; Company et al., 2008; Hull, 2000; Oosterlee et al., xxxx).

Nevertheless, a more modern approach to solve partial differential equations of BS-type is given by the finite element method (FEM), which has been applied, for instance, by Allegretto et al. (2001), Ballestra and Sgarra (2010), Chacur et al. (2011), Hughes (1987), int Hout and Volders (2009), Markolefas (2008), Sapariuc et al. (2004) and Topper (2005). The standard theory of the FEM predicts convergence in the L^2 norm as $O(h^{r+1})$, where h is the mesh width and r is the degree of the piecewise polynomial basis employed. However, some authors (see, for example, Arnold and Douglas, 1979; Arnold et al., 1981; Bakker, 1980; Bakker, 1982; Bakker, 1984; Bramble et al., 1977; Chen, 1979; Thomée, 1980) have found that in certain situations there may exist points in the computational domain at which the convergence rate of the FEM is higher than $O(h^{r+1})$. In particular, for quadratic partial differential equations in one space variable, Bakker (1980) has shown that, if the FEM is performed with a uniform mesh, if the variational integrals are computed either exactly or by Gauss–Lobatto quadrature, and if the initial solution is collocated at Gauss–Lobatto nodes, then at the boundaries of the finite elements the convergence rate is $O(h^{2r})$. This rate is clearly higher than h^{r+1} when r is greater than one.

Nevertheless, in Bakker (1980), such a superconvergence result has been proven under certain regularity assumptions. In particular, the initial condition of the quadratic problem being solved is required to belong to the Sobolev space H^{r+1}, that is to the space of functions whose first $r + 1$ distributional derivatives are square integrable functions. Now, in models of BS-type, such an assumption is generally not satisfied (at least for r greater than one), as the first-order derivatives of the initial data have usually a jump discontinuity at the strike prices.

Therefore, it appears to be interesting to study if the FEM achieves nodal superconvergence also when it is used for problems of BS-type, which is done in the present paper. In particular, we focus our attention on the cases of quadratic and cubic FEMs ($r = 2$ and $r = 3$), and on the problems of pricing European vanilla and barrier options under the traditional BS model. Note that we are considering problems with exact closed-form solutions, as this allows us to perform an accurate evaluation of the approximation error, (the errors obtained are often of order 10^{-8} or even smaller, see Section 5).

Furthermore, the FEM is carried out according to the directions given in Bakker (1980), i.e., the finite element subdivision has uniform spacing, the variational integrals are evaluated by Gauss–Lobatto quadrature, and the initial solution is collocated at Gauss–Lobatto nodes. In addition, the computational mesh is chosen such that one of the finite element boundaries is positioned on the strike price. This strategy has already proven crucial to improve the convergence rates of lattice-based numerical methods for the BS model (see, for example, Pooley et al., 2003; Ritchken, 1995).

Furthermore, in the case of vanilla options, the FEM is tested on both the original BS equation and a modified BS equation that is obtained from the original BS equation after applying the change of variables proposed in Oosterlee et al. (xxxx). Such a change of variables is often employed in the numerical approximation of partial differential equations of BS-type, as it helps to reduce the losses of accuracy due to the non-smoothness of the options' payoffs (see for example Ballestra and Sgarra, 2010; Clarke and Parrot, 1999).

The numerical experiments performed reveal that the FEM is superconvergent, even though the initial data of the BS problems considered are non-smooth, and the superconvergence rates obtained are in accordance with those that have been theoretically established in Bakker (1980) under the assumption of regular initial data. This result is striking, especially for what concerns the barrier option problem, where at the initial time the solution, besides being non-differentiable in the price variable, is not even continuous in time.

We believe that the present paper gives an interesting contribution to the literature, as we provide a clear numerical evidence of the fact that superconvergent discretizations of the BS model can actually be obtained. Note that, to the best of our knowledge, the possibility of having lattice-based numerical approximations of the BS model that exhibit nodal superconvergence has never been considered so far, presumably due to the presence of the non-smooth initial data. Furthermore, in this paper we also show that the FEM, if carried out properly, yield numerical solutions of the BS model that, at certain points of the computational domain, are extremely accurate (also when polynomial basis of relatively low degree are employed).

The remainder of the paper is organized as follows: in Section 2 the BS model for European vanilla and barrier options is introduced; in Section 3 the change of variables proposed in Oosterlee et al. (xxxx) is applied to the BS equation; in Section 4 the (discrete) variational formulation of the FEM is developed; in Section 5 the results of the numerical experiments are presented and discussed; finally, in Section 6 some conclusions are drown.

2. The Black–Scholes model

According to the Black–Scholes (BS) model, the price of an option's underlying asset, which we denote by S, follows (under the risk-neutral measure, see Hull, 2000) a geometric Brownian motion:

$$dS = rS\,dt + \sigma S\,dW, \tag{1}$$

where r and σ are the (constant) interest rate and volatility, respectively, and W is a Wiener standard process (Hull, 2000).

Let us consider an European option on the above asset, with maturity T. Moreover, let $t \leqslant T$ denote a generic instant of time, let us define the time to maturity $\tau = T - t$, and let $V(S,\tau)$ denote the price of the above option on the above asset. It can be shown that V satisfies the following partial differential equation:

$$\frac{\partial V(S,\tau)}{\partial \tau} - rS\,\frac{\partial V(S,\tau)}{\partial S} - \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V(S,\tau)}{\partial S^2} + rV(S,\tau) = 0. \tag{2}$$

2.1. Vanilla Call option

Let us consider a vanilla Call option with strike price E. In this case, Eq. (2) must be solved with the initial condition:

$$V(S,0) = \Pi(S), \tag{3}$$

where

$$\Pi(S) = \max(S - E, 0), \tag{4}$$

and with the boundary conditions:

$$V(0,\tau) = 0,\ \lim_{S\to+\infty} V(S,\tau) = S - Ee^{-r\tau}. \tag{5}$$

Problems (2), (4), and (5) have an exact closed-form solution (see, for example, Black and Scholes, 1973; Wilmott, 2006).

2.2. Up-and-out barrier Call option

In the present paper, we focus our attention on the case of an up-and-out barrier Call option, i.e., an option that expires when the asset price reaches the barrier level. However, there are several other types of barrier options (see for example Hull, 2000), to which the FEM could be applied as well.

Let us consider an up-and-out barrier Call option with strike price E and (constant) barrier B. In this case, Eq. (2) must be solved with the initial condition:

$$V(S,0) = \Pi(S), \tag{6}$$

where Π is given by (4), and with boundary conditions:

$$V(0,\tau) = 0,\ V(B,\tau) = 0. \tag{7}$$

Problems (2), (6), and (7) have an exact closed-form solution (see, for example, Wilmott, 2006).

3. Change of variables

In the case of the vanilla Call option the FEM is tested on both the BS Eq. (2) and a modified equation that is obtained from (2) after applying the change of variables proposed in Oosterlee et al. (xxxx). Such a change of variables is actually a grid-stretching approach which is frequently employed when solving partial differential problems of BS-type (see for example Ballestra and Sgarra, 2010; Clarke and Parrot, 1999), as it helps to reduce the losses of accuracy due to the non-smooth initial data. Note that in the case of the barrier option the aforementioned change of variables is not applied, as it gives a computational mesh which is too coarse in the region close to the barrier (see below), where the solution experiences strong spatial gradients.

First of all, the spatial domain $[0,+\infty)$ is truncated to $[0,S_{\max}]$, where $S_{\max}$ will be chosen large enough such that the error due to the truncation is negligible (see Section 5). Then let us consider the change of variables:

$$y(S) = \frac{\sinh^{-1}(\xi(S-E)) - c_1}{c_2 - c_1}, \tag{8}$$

where

$$c_1 = \sinh^{-1}(-\xi E), \;\; c_2 = \sinh^{-1}(\xi(S-E)), \tag{9}$$

and ξ is a constant parameter, that will be specified later. Relation (8) can be easily inverted:

$$S(y) = \frac{1}{\xi}\sinh(c_2 y + c_1(1-y)) + E. \tag{10}$$

The change of variables (8) maps the S-domain $[0,S_{\max}]$ to the y-domain $[0,1]$. In particular, equally spaced points in $[0,1]$ are transformed by (10) to points in $[0,S_{\max}]$ that cluster around the strike price E. Therefore, in the domain $[0,1]$ it is possible to choose a uniform mesh of finite elements (which is in line with what prescribed by Bakker (1982)), and to this mesh will correspond in the physical domain $[0,S_{\max}]$ a mesh of finite elements which are concentrated in a region around E, where the initial solution is non-smooth. Note that the amount of mesh refinement is directly proportional to the parameter ξ. Let us define:

$$u(y,\tau) = V(S(y),\tau). \tag{11}$$

By replacing the domain $[0,+\infty)$ with the domain $[0,S_{\max}]$, and by using the change of variables (10) and (11), the BS Eq. (2), the initial condition (3), and the boundary conditions (5) are rewritten as follows:

$$\frac{\partial u(y,\tau)}{\partial \tau} = z_1(y)\frac{\partial^2 u(y,\tau)}{\partial y^2} + z_2(y)\frac{\partial u(y,\tau)}{\partial y} - ru(y,\tau), \tag{12}$$

$$u(y,0) = \Pi(S(y)), \tag{13}$$

$$u(0,\tau) = 0, \;\; u(1,\tau) = S_{\max} - Ee^{-r\tau}, \tag{14}$$

where

$$z_1(y) = \frac{1}{2}\sigma^2\left(\frac{S(y)}{S'(y)}\right)^2, \quad z_2(y) = r\frac{S(y)}{S'(y)} - \frac{1}{2}\sigma^2\frac{S^2(y)S''(y)}{(S'(y))^3}. \tag{15}$$

4. The FEM

In this section it is shown how to perform the superconvergent FEM discretization. For the sake of brevity, we will only consider the transformed BS problems (12)–(14), as the application of the FEM to the original BS equation or to the barrier option problem is substantially analogous.

First of all, in $[0,1]$ let us consider $N+1$ equally spaced points $y_1, y_2, \ldots, y_{N+1}$, such that $y_j = (j-1)h, j = 1, 2, \ldots, N+1$, where $h = \frac{1}{N}$. Then let us define the finite elements $I_1, I_2, \ldots, I_N$ as $I_j = [y_j, y_{j+1}], j = 1, 2, \ldots, N$.

Following a procedure that is very common and usually yields optimal convergence rates (see for example Pooley et al., 2003), one of the nodes $y_1, y_2, \ldots, y_{N+1}$ is made coincide with the option's strike price E. This will be obtained by choosing suitable values of ξ, $S_{\max}$, and N (see Section 5).

Let r denote an integer greater than one, and let $N_1 = N\ r$. According to the directions given in Bakker (1980), let us consider the $N_1 + 1$ nodes $\eta_1, \eta_2, \ldots, \eta_{N_1+1}$ that are obtained by putting together the $r + 1$ Gauss–Lobatto nodes associated to each of the finite elements $I_1, I_2, \ldots, I_N$. That is $\eta_1, \eta_2, \ldots, \eta_{r+1}$ are the $r + 1$ Gauss–Lobatto nodes in the interval I_1, $\eta_{r+1}, \eta_{r+2}, \ldots, \eta_{2r+1}$ are the $r + 1$ Gauss–Lobatto nodes in the interval I_2, and so on. The reader who was unfamiliar with Gauss–Lobatto interpolation is referred, for instance, to Quarteroni and Valli (1994). Here we simply observe that Gauss–Lobatto nodes include also the endpoints of the interval to which they belong, and thus the nodes $\eta_1, \eta_{r+1}, \eta_{2r+1}, \ldots, \eta_{Nr+1}$ coincide with the finite element boundaries $y_1, y_2, \ldots, y_{N+1}$.

We want to approximate the solution of problems (12)–(14) by a linear combination of piecewise continuous polynomial functions of degree smaller or equal to r:

$$u(y,\tau) = \sum_{j=1}^{N_1+1} w_j(y) u_j(\tau), \tag{16}$$

where for $j = 1, 2, \ldots, N_1 + 1$, $u_j(\tau)$ is a function to be determined and $w_j(y)$ denotes the piecewise polynomial function of degree smaller or equal to r and such that $w_j(\eta_j) = 1$ and $w_j(\eta_i) = 0$, $i = 1, 2, \ldots, N_1 + 1$, $i \neq j$.

We multiply Eq. (12) by $w_i(y)$, $i = 2, 3, \ldots, N_1$, and integrate over the domain [0,1]:

$$\int_0^1 \left(\frac{\partial u(y,\tau)}{\partial \tau} - z_1(y) \frac{\partial^2 u(y,\tau)}{\partial y^2} - z_2(y) \frac{\partial u(y,\tau)}{\partial y} + ru(y,\tau) \right) w_i(y) dy = 0, \quad i = 2,\ 3,\ \ldots, N_1. \tag{17}$$

Using integration by parts Eq. (17) is rewritten as follows:

$$\int_0^1 \left[\frac{\partial u(y,\tau)}{\partial \tau} + \left(\frac{dz_1(y)}{dy} - z_2(y) \right) \frac{\partial u(y,\tau)}{\partial y} + ru(y,\tau) \right] w_i(y) + z_1(y) \frac{\partial u(y,\tau)}{\partial y} \frac{\partial w_i(y)}{\partial y} dy$$
$$= 0, \quad i = 2,\ 3,\ \ldots,\ N_1, \tag{18}$$

Substitution of (16) in (18) yields:

$$\sum_{j=1}^{N_1+1} m_{i,j} \frac{du_j(\tau)}{d\tau} + (a_{i,j} + d_{i,j} + k_{i,j}) u_j(\tau) = 0, \quad i = 2,\ 3,\ \ldots,\ N_1, \tag{19}$$

where

$$m_{i,j} = \int_0^1 w_i(y) w_j(y) dy, \quad i = 2,\ 3,\ \ldots,\ N_1, \quad j = 1,\ 2,\ \ldots,\ N_1 + 1, \tag{20}$$

$$a_{i,j} = \int_0^1 \left(\frac{dz_1(y)}{dy} - z_2(y) \right) w_i(y) \frac{dw_j(y)}{dy} dy, \quad i = 2,\ 3,\ \ldots,\ N_1, \quad j = 1,\ 2,\ \ldots,\ N_1 + 1, \tag{21}$$

$$k_{i,j} = \int_0^1 z_1(y) \frac{dw_i(y)}{dy} \frac{dw_j(y)}{dy} dy, \quad d_{i,j} = rm_{i,j}, \quad i = 2,\ 3,\ \ldots,\ N_1, \quad j = 1,\ 2,\ \ldots,\ N_1 + 1. \tag{22}$$

Moreover, using (16) in (14), we obtain:

$$u_1(\tau) = 0, \quad u_{N_1+1}(\tau) = S_{\max} - Ee^{-r\tau}. \tag{23}$$

Then, let us also define:

$$m_{i,j} = 0, \quad a_{i,j} = 0, \quad k_{i,j} = 0, \quad i = 1, \quad N_1 + 1, \quad j = 1,\ 2,\ \ldots,\ N_1 + 1, \tag{24}$$

$$d_{1,1} = 1, \quad d_{1,j} = 0, \quad d_{N_1+1,N_1+1} = 1, \quad d_{N_1+1,j} = 0, \quad j = 2,\ 3,\ \ldots,\ N_1 + 1, \tag{25}$$

and let M, A, K and D denote the $(N_1 + 1) \times (N_1 + 1)$ matrices with entries $m_{i,j}$, $a_{i,j}$, $k_{i,j}$ and $d_{i,j}$, respectively, $i = 1, 2, \ldots, N_1 + 1$, $j = 1, 2, \ldots, N_1 + 1$. Finally, let us consider the vectors $U(\tau) = [u_1(\tau), u_2(\tau), \ldots, u_{N_1+1}(\tau)]^T$

and $b(\tau) = [0,0,\ldots,0,S_{\max} - Ee^{-r\tau}]^T$ (b has N_1 components equal to zero). Relations (20)–(25) can be rewritten in the more compact form:

$$M\frac{dU(\tau)}{d\tau} + (A + K + D)U(\tau) = b(\tau_k). \tag{26}$$

Thus, we have found a system of ordinary differential equations in the unknown vector function $U(\tau)$. This system must be solved with the initial condition:

$$U(0) = [\Pi(S(\eta_1)), \Pi(S(\eta_2)), \ldots, \Pi(S(\eta_{N_1+1}))]^T, \tag{27}$$

which is obtained by imposing (13). Note that relation (27) amounts to collocating the initial datum $\Pi(S(y))$ at the Gauss–Lobatto nodes $\eta_1,\ \eta_2,\ \ldots,\ \eta_{N_1+1}$, which is in line with the directions given in Bakker (1980)).

Problem (26) and (27) is discretized in time using the Crank–Nicholson finite difference method with Rannacher time-stepping (Rannacher, 1984). Precisely, the first two time iterations are performed by the implicit Euler scheme, whereas the subsequent ones are performed by the Crank–Nicholson scheme, see also (Pooley et al., 2003). Nevertheless, we make notice that any other time discretization approach could be employed as well.

On the FEM superconvergence. In the next section we are going to present numerical results showing that the FEM described above is $O(h^{2r})$ accurate at the boundaries of the finite elements, even though the initial data of the problems considered are non-smooth.

In this paper we do not provide a rigorous mathematical proof of the superconvergence of the finite element approximation of the BS equation, as it would be a very difficult task (due to the presence of the non-smooth initial data), and goes far beyond our purposes. Nevertheless, we can explain the very practical reasons why the strategy proposed by Bakker in Bakker (1980), if combined with a mesh aligned with the option's strike price, allows us to achieve an optimal convergence rate.

First of all let us observe that, according to the procedure described above, the initial solution of the problem, i.e. the option's payoff, is interpolated at the Gauss–Lobatto points. Now, such an approach, together with the use of a mesh aligned with the strike price, is crucial to efficiently handle the non-smoothness of the initial datum. In fact, interpolating the option's payoff at the $r + 1$ Gauss–Lobatto nodes is approximately equivalent to projecting the option's payoff to the space of piecewise continuous polynomials of degree r (see, e.g., Bakker, 1980; Quarteroni and Valli, 1994). Therefore, thanks to the fact that one of the mesh nodes coincides with the strike price, the difference between the option's payoff and its aforementioned projection turns out to be negligible (if the change of variables (8) is not applied, the option's payoff and its projection coincide, as the function (4) is already in the space of piecewise continuous polynomials of degree r).

Moreover, as recommended by Bakker (1980), the FEM integrals are evaluated by Gauss–Lobatto integration. This is another key point to achieving superconvergence, as the matrices in (26) must be evaluated with a sufficiently small error: if, in place of the Gaussian quadrature, we employed the standard Newton Cotes formulae, then the matrices in (26) would be at most $O(h^{r+2})$ accurate (see Quarteroni and Valli, 1994), and the $O(h^{2r})$ convergence rate could not be obtained (see also Section 5.3).

Finally, always following Bakker (1980), the mesh is chosen uninform. The advantage of a uniform grid spacing can be heuristically explained as follows. In (26) the matrices A, K, D are banded matrices with band width equal to $2r + 1$ (as follows from (20), (21), (22), (24) and (25) and from the shape of the function w_i, $i = 1, 2, \ldots, N_1$). Therefore, from the practical standpoint, the FEM can be thought of as a discretization scheme based on stencils of $2r + 1$ nodes (i.e. the stencil of nodes $\eta_1, \eta_2, \ldots, \eta_{2r+1}$, the stencil of nodes $\eta_{r+1}, \eta_{r+2}, \ldots, \eta_{3r+1}$, and so on). Now, if the mesh is uniform, each one of these stencils turns out to be symmetric with respect to each one of the FEM boundary nodes $y_2, y_3, \ldots, y_N$. As a consequence, thinking to the fact that, in approximation theory, symmetric discretization stencils usually yield the highest levels of accuracy, we can understand, on a heuristic basis, the importance of using a uniform mesh in order to obtain the superconvergence rate.

5. Numerical results

We restrict our attention to the cases of piecewise quadratic and piecewise cubic FEMs ($r = 2$ and $r = 3$), which are the most commonly employed among the polynomial finite elements of order greater than one. For the vanilla option, the parameters and data are chosen as follows: $\sigma = 0.2$, $r = 0.05$, $T = 0.5$, $E = 10$. Instead, for the barrier option, we set: $\sigma = 0.2$, $r = 0.05$, $T = 0.5$, $E = 100$, $B = 120$.

In all the numerical simulations the number of time-steps used to solve the system of ordinary differential Eqs. (26) and (27) is chosen large enough such that the error due to the time discretization is negligible with respect to the spatial error (which is the error we are interested in).

Following Bakker (1980), we want to test the convergence of the FEM at the boundaries of the finite elements. Thus, the error of the FEM (at $\tau = T$) is measured as follows:

$$Error = \max_{l=1,2,\dots,N+1} |u_{(l-1)r+1}(T) - u(y_l, T)|, \tag{28}$$

where the exact solution $u(y,T)$ is calculated using the analytical formulae available, for example, in Wilmott (2006). The convergence of the FEM is tested by varying the number of finite elements employed. In particular, following a very common approach, the value of N is progressively doubled, and the convergence rates are estimated by computing the ratios of the errors obtained with N and $2N$ finite elements.

5.1. The quadratic FEM

To make the strike price E coincide with one of the finite element boundaries we proceed as follows. In the case of the vanilla option, we simply set $S_{\max} = 2E$ and choose N odd (the reader may easily verify that this is enough to obtain $y_{\frac{N}{2}} = E$). Instead, in the case of the barrier option, where $S_{\max}$ must be set equal to B, the mesh is aligned with the strike price by choosing suitable values of N (see Table 2).

The results obtained using the quadratic FEM are reported in Tables 1 and 2. Note that in Table 3, for the sake of brevity, we write $\xi = 0$ to refer to the case in which the grid-stretching is not employed, i.e. the FEM is directly applied to the original BS equation.

We may observe that, for both the vanilla and the barrier options, the error ratio tends to be very close to $2^4 = 16$ as N increases. This clearly indicates that at the boundaries of the finite elements the quadratic FEM superconverges as $O(h^4)$, as established by Bakker (1980) under the assumption of regular solutions.

5.2. The cubic FEM

To make the strike price E coincide with one of the finite element boundaries we proceed as follows. In the case of the vanilla option, we can no longer set $S_{\max} = 2E$, as we did for the quadratic FEM, because we would have a truncation error that is larger than the error due to the finite element approximation. Therefore, for various choices of the grid-stretching parameter ξ, suitable values of $S_{\max}$ that give $y_j = E$ for some positive integer j (smaller than $N + 1$) are obtained using a bisection algorithm.

Table 1
Superconvergence of the quadratic FEM, vanilla option.

N	$\xi = 0.1$		$\xi = 0.5$		$\xi = 1$		$\xi = 0$	
	Error	Ratio	Error	Ratio	Error	Ratio	Error	Ratio
18	3.09e−004		2.55e−004		6.67e−004		5.70e−004	
36	2.14e−005	14.39	1.61e−005	15.80	4.18e−005	15.96	3.53e−005	16.30
72	1.40e−006	15.33	1.00e−006	16.01	2.61e−006	16.01	2.18e−006	16.16
144	8.72e−008	16.06	6.91e−008	14.60	1.63e−007	15.97	1.37e−007	15.94

Table 2
Superconvergence of the quadratic FEM, European barrier option.

N	Error	Ratio
16	2.74e−003	
32	1.97e−004	13.57
64	1.19e−005	16.55
128	7.54e−007	15.78

Table 3
Superconvergence of the cubic FEM, vanilla option.

N	$\xi = 0.1$		$\xi = 0.5$		$\xi = 1$		$\xi = 0$	
	Error	Ratio	Error	Ratio	Error	Ratio	Error	Ratio
16	3.79e−004		2.62e−005		7.71e−005		9.47e−005	
32	6.04e−006	62.74	3.68e−007	71.15	1.36e−006	56.59	3.44e−007	27.52
64	7.30e−008	82.73	8.10e−009	45.48	2.98e−008	45.63	3.52e−008	97.58
128	1.02e−009	72.27	1.37e−010	59.12	5.00e−010	59.59	5.72e−010	61.60

Table 4
Superconvergence of the cubic FEM, European barrier option.

N	Error	Ratio
16	9.47e−005	
32	5.69e−007	119.62
64	7.71e−009	73.82
128	1.25e−010	61.57

Table 5
Fourth-order convergence of the cubic FEM with equally spaced nodes and Newton–Cotes quadrature, vanilla option.

N	$\xi = 0.1$		$\xi = 0.5$		$\xi = 1$		$\xi = 0$	
	Error	Ratio	Error	Ratio	Error	Ratio	Error	Ratio
16	5.71e−003		1.24e−003		3.48e−003		4.24e−003	
32	3.09e−004	17.51	8.05e−005	15.49	4.28e−004	8.12	1.94e−004	21.86
64	1.63e−005	18.94	5.01e−006	16.04	2.53e−005	16.94	1.06e−005	18.18
128	1.02e−006	15.96	3.15e−007	15.92	1.60e−006	15.76	6.65e−007	16.04

In particular we use: $S_{max} = 32$ if $\xi = 0$, $S_{max} = 79.999999997819685$ if $\xi = 0.1$, $S_{max} = 57.163233586136059$ if $\xi = 0.5$ and $S_{max} = 83.983744432305912$ if $\xi = 1$. Instead, in the case of the barrier option, the mesh is aligned with the strike price by choosing appropriate values of N (see Table 4). The results obtained are reported in Tables 3 and 4.

We note that the ratio of the errors seems to tend to $2^6 = 64$. Actually this trend is not perfectly clear, presumably because the errors obtained using the cubic FEM are extremely small (they are often of order 10^{-8} and sometimes even smaller), and thus are affected by numerical noise (due, for instance, to the numerical inversion of the matrix of the system of ordinary differential Eq. (27)). However, the error of the FEM decreases at a rate that is much higher than the $O(h^4)$ rate predicted by the standard theory, which clearly reveals a superconvergence effect at the boundaries of the finite elements.

5.3. Comparison with other FEM approaches

In Section 4, following the directions given by Bakker (1980), the nodes $\eta_1, \eta_2, \ldots, \eta_{N_1+1}$ have been obtained by putting together the $r + 1$ Gauss–Lobatto points associated to each finite element. Note that, as explained at the end of Section 3, the use of Gauss–Lobatto points and of Gauss–Lobatto integration yields a remarkable advantage as far as the FEM superconvergence is concerned.

Nevertheless, in standard FEM discretizations, the nodes $\eta_1, \eta_2, \ldots, \eta_{N_1+1}$, instead of being Gauss–Lobatto points, are normally chosen equally spaced, and the FEM integrals are evaluated by Newton–Cotes quadrature (Quarteroni and Valli, 1994). Now, in the special case of quadratic elements ($r = 2$), equally spaced nodes and Gauss–Lobatto nodes coincide, so that, actually, the FEM strategy proposed in Bakker (1980) and the standard FEM turn out to be equivalent. Instead, if $r \geqslant 3$, equally spaced nodes and Gauss–Lobatto nodes are different. Therefore, for comparison purposes, in the case of cubic finite elements ($r = 3$), we have also tried to use equally spaced nodes (in place of the Gauss–Lobatto nodes), and to evaluate the variational integrals by Newton–Cotes quadrature. The results obtained (those concerning the vanilla option are reported in the paper, see Table 5) reveal that such a standard implementation of the FEM is fourth-order accurate only (which is in perfect agreement with the usual FEM theory). Thus, if the nodes $\eta_1, \eta_2, \ldots, \eta_{N_1+1}$ are chosen equally spaced, as happens in conventional FEM discretizations, the $O(h^{2r})$ convergence rate is not obtained (unless $r = 2$).

So far, according to the approach in Bakker (1980), we have taken into account the error of the FEM only at the boundaries of the finite elements (see relation (28)). However, for testing purposes, and following (Chen, 1979), we have checked the accuracy of the FEM also by considering the maximum norm of the error over *all* the nodes $\eta_1, \eta_2, \ldots, \eta_{N_1+1}$. By doing that, we have found that the convergence rate of the FEM is $O(h^4)$ in the case of quadratic elements, and is $O(h^5)$ in the case of cubic elements (tables are not reported to save space). This is in accordance with a theoretical result for elliptic partial differential equations established by Chen (1979), and indicates that at the Gauss–Lobatto nodes laying in the interior of the finite elements the convergence rate is $O(h^{r+2})$. That is at the interior Gauss–Lobatto nodes the FEM is still superconvergent, but, for $r \geqslant 3$, the convergence rate obtained is lower than the $O(h^{2r})$ rate experienced at the boundaries of the finite elements.

6. Conclusions

We have investigated the convergence of the FEM in solving the Black–Scholes model for option pricing. We have found that, if the FEM is carried out following directions given by Bakker (1980) and the mesh is aligned with the options' payoff, then the computed solutions exhibit nodal superconvergence, even though the initial data are non-smooth. Moreover, and the convergence rates obtained are in accordance with those that have been theoretically established in Bakker (1980) and Chen (1979) under the assumption of regular solutions. In the present manuscript, the reasons why the approach in Bakker (1980) is crucial to achieve superconvergence are clearly highlighted, also by performing a numerical comparison with the standard FEM approach.

We believe that this paper gives an interesting contribution to the already existing literature, as we provide a clear numerical evidence of the fact that superconvergent numerical approximations of the BS model can actually be obtained. In addition, we have shown how the FEM, if carried out properly, allows us to obtain very accurate results using polynomial basis of relatively low degree. In our opinion, this makes the FEM particularly suitable to solve partial differential problems of BS-type.

References

Allegretto, W., Lin, Y., Yang, H., 2001. Finite element error estimates for a nonlocal problem in American option valuation. SIAM Journal on Numerical Analysis 39, 834–857.

Arnold, D.N., Douglas Jr., J., 1979. Superconvergence of the Galerkin approximation of a quasilinear parabolic equation in a single space variable. Calcolo 16, 345–369.

Arnold, D.N., Douglas Jr., J., Thomée, Vidar, 1981. Superconvergence of a finite element approximation to the solution of a Sobolev equation in a single space variable. Mathematics of Computation 36, 53–63.

Bakker, M., 1980. On the numerical solution of parabolic equations in a single space variable by the continuous time Galerkin method. SIAM Journal on Numerical Analysis 17, 162–177.

Bakker, M., 1982. Galerkin methods for even-order parabolic equations in one space variable. SIAM Journal on Numerical Analysis 19, 571–587.

Bakker, M., 1984. One-dimensional Galerkin methods and superconvergence at interior nodal points. SIAM Journal on Numerical Analysis 21, 101–110.

Ballestra, L.V., Sgarra, C., 2010. The evaluation of American options in a stochastic volatility model with jumps: an efficient finite element approach. Computers and Mathematics with Applications 6, 1571–1590.

Ballestra, L.V., Pacelli, G., xxxx. A Very Fast and Accurate Boundary Element Method for Options with Moving Barrier and Time-Dependentr Rebate. Working paper. <http://www.dss.univpm.it/files/dss/pubbl/quaderni/qdd030.pdf>.

Barles, G., Soner, H.M., 1998. Option pricing with transaction costs and a nonlinear Black–Scholes equation. Finance and Stochastics 2, 369–397.

Black, F., Scholes, M., 1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637–659.

Bramble, J.H., Schatz, A.H., Thomée, V., Wahlbin, L.B., 1977. Some convergence estimates for semidiscrete Galerkin type approximations for parabolic equations. SIAM Journal on Numerical Analysis 14, 218–241.

Chacur, A.A., Ali, M.M., Salazar, J.G., 2011. Real options pricing by the finite element method. Computers and Mathematics with Applications 61, 2863–2873.

Chen, C.M., 1979. Optimal points of approximation solution for Galerkin method for two-point boundary value problem. Numerical Mathematics, A Journal of Chinese Universities 1, 73–79.

Clarke, N., Parrot, K., 1999. Multigrid for American option pricing with stochastic volatility. Applied Mathematical Finance 6, 177–179.

Company, R., Navarro, E., Pintos, J.R., Ponsoda, E., 2008. Numerical solution of linear and nonlinear Black–Scholes option pricing equations. Computers and Mathematics with Applications 56, 813–821.

Daglish, T., 2010. Lattice methods for no-arbitrage pricing of interest rate securities. Journal of Derivatives 18, 7–19.

Dupire, B., 1994. Pricing with a smile. Risk 7, 18–20.

Hughes, T.J.R., 1987. The Finite Element Method, Linear Static and Dynamic Finite Element Analysis. Prentice-Hall, Englewood Cliffs, NJ.

Hull, J.C., 2000. Options, Futures and Other Derivatives. Prentice Hall, Englewood Cliffs, NJ.

Hull, J., White, A., 1990. Pricing interest-rate-derivative securities. Review of Financial Studies 3, 573–592.

int Hout, K.J., Volders, K., 2009. Stability of central finite difference schemes on non-uniform grids for the Black–Scholes equation. Applied Numerical Mathematics 59, 2593–2609.

Markolefas, S., 2008. Standard Galerkin formulation with high order Lagrange finite elements for option markets pricing. Applied Mathematics and Computation 195, 707–720.

Oosterlee, C.W., Leentvaar, C.C.W., Huang, X., xxxx. Accurate American option pricing by grid stretching and high order finite differences, Working paper. <http://ta.twi.tudelft.nl/users/vuik/numanal/huang_paper.pdf>.

Pooley, D.M., Vetzal, K.R., Forsyth, P.A., 2003. Convergence remedies for non-smooth payoffs in option pricing. Journal of Computational Finance 6, 25–40.
Quarteroni, A., Valli, A., 1994. Numerical Approximation of Partial Differential Equations. Springer-Verlag, Heidelberg.
Rannacher, R., 1984. Finite element solution of diffusion problems with irregular data. Numerische Mathematik 43, 309–327.
Ritchken, P., 1995. On pricing barrier options. Journal of Derivatives 3, 19–28.
Sapariuc, I., Marcozzi, M.D., Flaherty, J.E., 2004. A numerical analysis of variational valuation techniques for derivative securities. Applied Mathematics and Computation 159, 171–198.
Thomée, V., 1980. Negative norm estimates and superconvergence in Galerkin methods for parabolic problems. Mathematics of Computation 34, 93–113.
Topper, J., 2005. Financial Engineering with Finite Elements. John Wiley & Sons, New York.
Wilmott, P., 2006. Paul Wilmott on Quantitative Finance. John Wiley & Sons, New York.