1

Since $u=\sin((k+0.5)\pi x)$;

$u'=(k+0.5)\pi*\cos((k+0.5)\pi x)$;

So $u(0)=\sin(0)=0$

$U'(1)=(k+0.5)\pi*\cos((k+0.5)\pi)=(k+0.5)\pi*0=0$

$u''=-(k+0.5)^2*\pi^2*\sin((k+0.5)\pi x)$;

So $f=-u''=(k+0.5)^2*\pi^2*\sin((k+0.5)\pi x)$;

2_a:

Notations: $-u''(x) = f(x)$

$u=\sin[(k+0.5)\pi * x]$

$f=(k+0.5)^2 * \pi^2 * u$

$h=1/n$

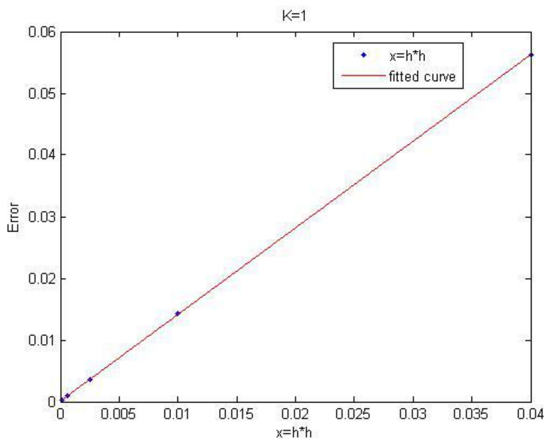
Ritz-Galerkin Approximation of error: (Upper bound)

$E0=0.5*h^2*||f||$;

Errors:

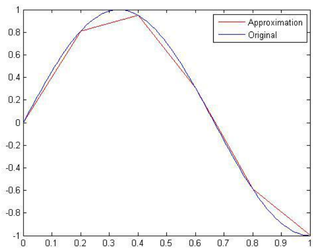
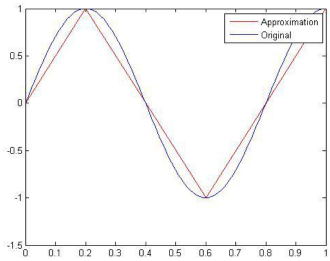
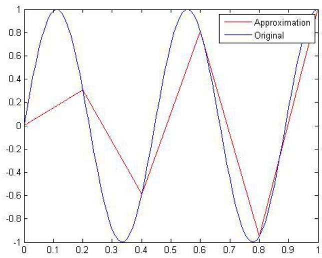
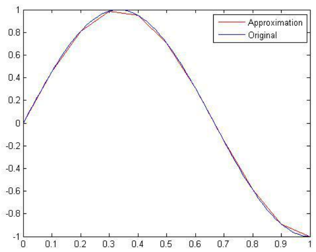
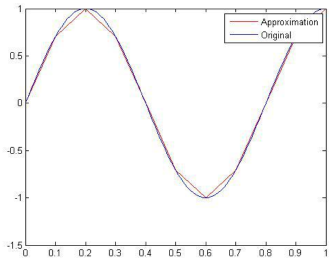
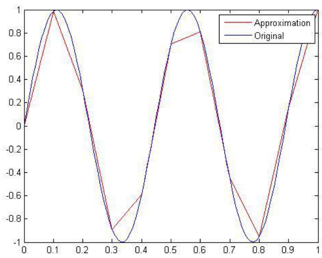
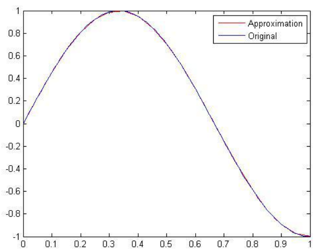
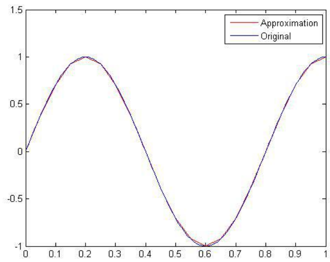
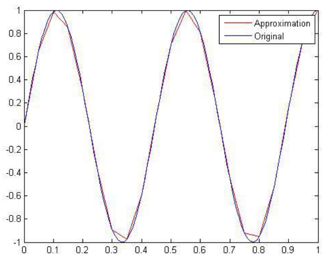
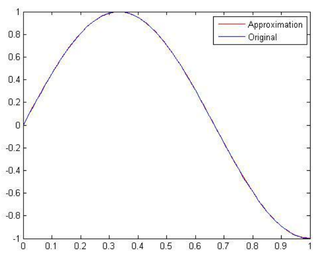
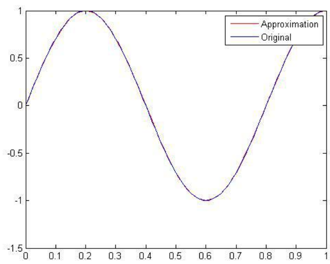
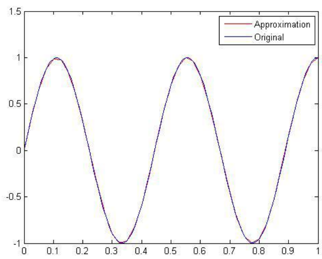
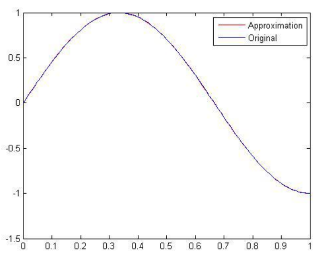
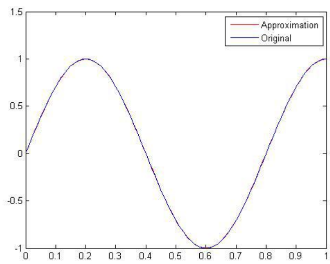
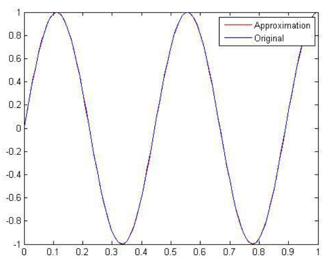
n \ k	1 $\|\|f\|\|=15.7$	2 $\|\|f\|\|=43.6$	4 $\|\|f\|\|=141.3$	8 $\|\|f\|\|=504.2$	16 $\|\|f\|\|=1900.0$	32 $\|\|f\|\|=7371.4$
5	Error: $5.62*10^{-2}$ Upper Bound: 0.314	Error: 0.151 Upper Bound: 0.872	Error: 0.432 Upper Bound: 2.82	Error: 0.950 Upper Bound: 10.1	Error: 0.840 Upper Bound: 38.0	Error: 0.910 Upper Bound: 147
10	Error: $1.43*10^{-2}$ Upper Bound: $7.85*10^{-2}$	Error: $3.93*10^{-2}$ Upper Bound: 0.218	Error: 0.123 Upper Bound: 0.707	Error: 0.393 Upper Bound: 2.52	Error: 0.932 Upper Bound: 9.50	Error: 0.826 Upper Bound: 36.9
20	Error: $3.58*10^{-3}$ Upper Bound: $1.96*10^{-2}$	Error: $9.92*10^{-3}$ Upper Bound: $5.45*10^{-2}$	Error: $3.19*10^{-2}$ Upper Bound: 0.177	Error: 0.111 Upper Bound: 0.630	Error: 0.373 Upper Bound: 2.37	Error: 0.922 Upper Bound: 9.21
40	Error: $8.95*10^{-4}$ Upper Bound: $4.91*10^{-3}$	Error: $2.49*10^{-3}$ Upper Bound: $1.36*10^{-2}$	Error: $8.04*10^{-3}$ Upper Bound: $4.42*10^{-2}$	Error: $2.85*10^{-2}$ Upper Bound: 0.158	Error: 0.104 Upper Bound: 0.594	Error: 0.364 Upper Bound: 2.31
80	Error: $2.24*10^{-4}$ Upper Bound: $1.23*10^{-3}$	Error: $6.22*10^{-4}$ Upper Bound: $3.41*10^{-3}$	Error: $2.01*10^{-3}$ Upper Bound: $1.10*10^{-2}$	Error: $7.17*10^{-3}$ Upper Bound: $3.93*10^{-2}$	Error: $2.69*10^{-2}$ Upper Bound: 0.148	Error: 0.101 Upper Bound: 0.576

The error is going down by approximately 1/4 (as expected), when we half h and $h^2*||f||$ is small enough. The error behaves like h^2. (If we fit error and h^2, we can see they can fit into straight line quite well.)



The error is also growing with the growth of $||f||$. But I can't show the linearity. That probably is due to the function is quite smooth and the error is too far away from the estimate.

Pictures: (We only show k=1, k=2, k=4 here. Please refer to the code for the rest.)

n \ k	1	2	4
5			
10			
20			
40			
80			

2_b:

Notations: -u''(x) = f(x)

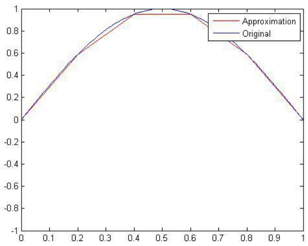
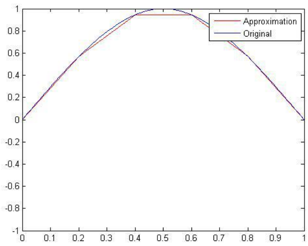
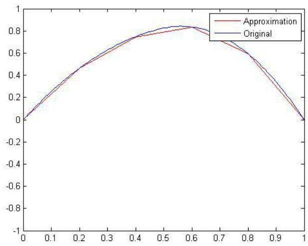
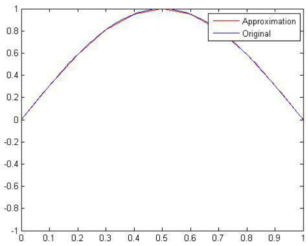
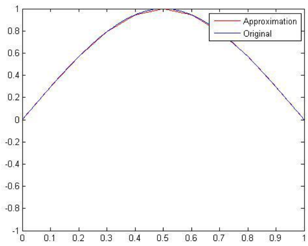
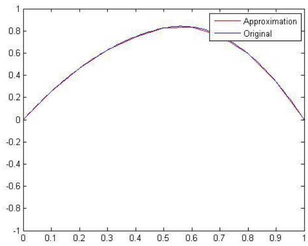
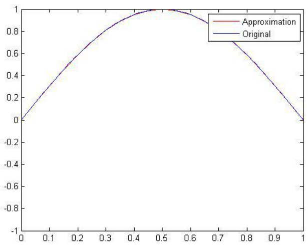
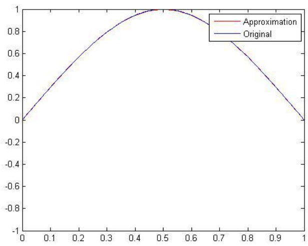
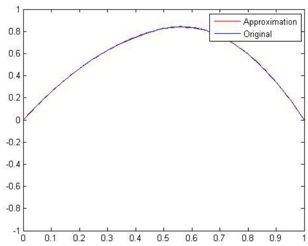
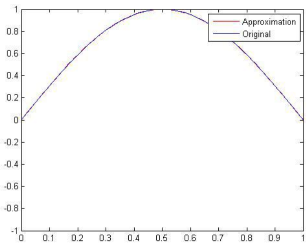
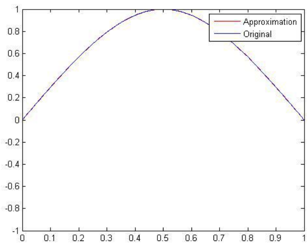
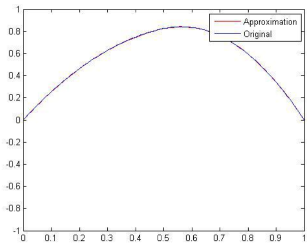
h=1/n

Ratio: error/upper_bound

Function / n	F = sin(π∗x) u = π^2∗sin(π∗x) F is C^∞; ||F||=6.98;	F= 24x (x<0.5) 24(1-x) (x≥0.5) F is C^0 ||F||=6.93 u= -($4x^3$-3x) (x<0.5) -[$4(1-x)^3$+3x-3] (x≥0.5)	F= 4.4 (x≤0.5) 8.8 (x>0.5) F is discont ||F||=6.96 u= -$2.2x^2$+2.75x (x≤0.5) -$4.4x^2$+4.95x-0.45 (x>0.5)
5	Error: $2.53*10^{-2}$ Upper Bound: 0.140 Ratio: 0.181	Error: $2.53*10^{-2}$ Upper Bound: 0.139 Ratio: 0.181	Error: $2.52*10^{-2}$ Upper Bound: 0.139 Ratio: 0.181
10	Error: $6.35*10^{-3}$ Upper Bound: $3.50*10^{-2}$ Ratio: 0.182	Error: $6.29*10^{-3}$ Upper Bound: $3.46*10^{-2}$ Ratio: 0.182	Error: $6.35*10^{-3}$ Upper Bound: $3.48*10^{-2}$ Ratio: 0.182
20	Error: $1.59*10^{-3}$ Upper Bound: $8.72*10^{-3}$ Ratio: 0.182	Error: $1.58*10^{-3}$ Upper Bound: $8.66*10^{-3}$ Ratio: 0.182	Error: $1.59*10^{-3}$ Upper Bound: $8.70*10^{-3}$ Ratio: 0.182
40	Error: $3.98*10^{-4}$ Upper Bound: $2.18*10^{-3}$ Ratio: 0.182	Error: $3.95*10^{-4}$ Upper Bound: $2.17*10^{-3}$ Ratio: 0.182	Error: $3.95*10^{-4}$ Upper Bound: $2.17*10^{-3}$ Ratio: 0.182
80	Error: $9.96*10^{-5}$ Upper Bound: $5.45*10^{-4}$ Ratio: 0.183	Error: $9.88*10^{-5}$ Upper Bound: $5.41*10^{-4}$ Ratio: 0.183	Error: $1..01*10^{-4}$ Upper Bound: $5.44*10^{-4}$ Ratio: 0.187

Here we look into F is C^∞, C^0, discontinuous respectively. Therefore, u is C^∞, C^2, C^1 respectively. It seems that the smoothness doesn't affect much on the error.

Pictures:

Function \ n	$F = \sin(\pi * x)$ $u = \pi^2 * \sin(\pi * x)$ F is C^∞; $\|F\|=6.98$; u is C^∞	F= $24x$ ($x<0.5$) F is C^1 $24(1-x)$ ($x\geq 0.5$) $\|F\|=6.93$ u= $4x^3-3x$ ($x<0.5$) $4(1-x)^3+3x-3$ ($x\geq 0.5$)	F= 4.4 ($x\leq 0.5$) F is discont 8.8 ($x>0.5$) $\|F\|=6.96$ u= $-2.2x^2+2.75x$ ($x\leq 0.5$) $-4.4x^2+4.95x-0.45$ ($x>0.5$)
5			
10			
20			
40			
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