253 Sheng Xu 1205525 HW1 P1

1(a) ① $\frac{V_{k,l+1} - V_{k,l}}{h_t}$

$= \frac{1}{h_t}[V_t h_t] + \frac{1}{h_t} O_\alpha(h_t^2) = V_t + O_1(h_t)$

$\frac{V_{k+1,l} - V_{k-1,l}}{2h_x}$

$= \frac{1}{2h_x}[V_{k,l} + V_x h_x + \frac{1}{2!}h_x^2 + O_\alpha(h_x^3)]$

$- \frac{1}{2h_x}[V_{k,l} - V_x h_x + \frac{1}{2!}h_x^2 + O_\beta(h_x^3)]$

$= V_x + O_1(h_x^2)$

$O_1 = O_\alpha + O_\beta$

$O_1' = a \cdot O_1$ so $E_{xt} = V_t + O_2(h_t) + aV_x + O_1'(h_x^2)$

$= O_2(h_t) + O_1'(h_x^2) = O(h_t, h_x^2)$

Since $V_t + aV_x = 0$

stability ② $V_{k,l+1} = \frac{aht}{2hx} V_{k-1,l} + V_{k,l} - \frac{aht}{2hx} V_{k+1,l}$

So $e_{l+1}^{(j)} \exp(ijkhx) = \lambda e_l^{(j)} \exp[ij(k-1)hx] + e_l^{(j)} \exp[ijkhx] - \lambda e_l^{(j)} \exp[ij(k+1)hx]$

$\lambda = \frac{aht}{2x}$

So $S(j) = \lambda \exp(-ijhx) + 1 - \lambda \exp(ijhx)$

$= 1 - i2\lambda \sin\theta$ $(\theta = jhx)$

So $|S(j)| > 1$ when $\theta \neq k\pi$, $\lambda > 0$, unstable

1 (b) ① $\frac{V_{k,l+1} - V_{k,l}}{h_t}$

$= V_t(x_k, t_l) + \frac{1}{2} V_{tt}(x_k, t_l) h_t + \frac{1}{6} V_{ttt}(x_k, t_l) \cdot h_t^2 + O(h_t^3)$

$\left(\frac{V_{k+1,l+1} - V_{k-1,l+1}}{2h_x} \right)$

$= \frac{1}{2h_x} [V_{k,l} + V_{x\,k,l} h_x + V_{t\,k,l} h_t + \frac{1}{2!} V_{xx\,k,l} h_x^2 + \frac{1}{1!1!} V_{xt\,k,l} h_x h_t$

$+ \frac{1}{2!} V_{tt\,k,l} h_t^2 + \frac{1}{3!} V_{xxx} h_x^3 + \frac{1}{2!} V_{xxt} h_x^2 h_t + \frac{1}{2!} V_{xtt} h_x h_t^2 + \frac{1}{3!} V_{ttt} h_t^3$

$- \frac{1}{2h_x} [V_{k,l} - V_x h_x + V_t h_t + \frac{1}{2!} V_{xx} h_x^2 - \frac{1}{1} V_{xt} h_x h_t$

$+ \frac{1}{2!} V_{tt} h_t^2 - \frac{1}{3!} V_{xxx} h_x^3 + \frac{1}{2!} V_{xxt} h_x^2 h_t - \frac{1}{2!} V_{xtt} h_x h_t^2 + \frac{1}{3!} V_{ttt} h_t^3$

* All h_t^n term → $+ \frac{1}{2h_x} O_\alpha(h_x^4, h_t h_x^3, h_t^2 h_x^2, h_t^3 h_x)$ are cancelled out.

$= \frac{1}{h_x} [V_x h_x + V_{xt} h_x h_t + \frac{1}{3!} V_{xxx} h_x^3 + \frac{1}{2!} V_{xtt} h_x h_t^2]$

$+ O_2(h_x^3, h_t h_x^2, h_t^2 h_x, h_t^3)$

$\frac{V_{k+1,l} - V_{k-1,l}}{2h_x}$

$= \frac{1}{2h_x} [V_{x\,k,l} + V_x h_x + \frac{1}{2} V_{xx} h_x^2 + \frac{1}{3!} V_{xxx} h_x^3]$

$- \frac{1}{2h_x} [V_{x\,k,l} - V_x h_x + \frac{1}{2} V_{xx} h_x^2 - \frac{1}{3!} V_{xxx} h_x^3] + \frac{1}{2h_x} O_\beta(h_x^4)$

$= \frac{1}{h_x} [V_x h_x + \frac{1}{3!} V_{xxx} h_x^3] + O_3(h_x^3)$

Now we combine everything:

$T_{xt} = V_t + \frac{1}{2} V_{tt} h_t + \frac{1}{6} V_{ttt} h_t^2 + O_1(h_t^3)$

$+ aV_x + \frac{a}{2} V_{xt} h_t + \frac{a}{6} V_{xxx} h_x^3 + \frac{a}{4} V_{xtt} h_t^2 +$

$O_2'(h_x^3, h_t h_x^2, h_t^2 h_x, h_t^3) + O_3'(h_x^3)$

$O_2' = \frac{a}{2} \cdot O_2$

$O_3' = \frac{a}{2} O_3$

$= (V_t + aV_x) + \frac{1}{2} h_t (V_{tt} + aV_{xt}) + O_4(h_t^2) + O_5(h_t^2, h_x^2)$

$O_6(h_x^3, h_t h_x^2, h_t^2 h_x, h_t^3)$

We know $V_t + aV_x = 0$. $V_{tt} + aV_{xt} = (V_t + aV_x)_t = O_t = O$

So $T_{xt} = O_4(h_t^2) + O_6(h_x^3, h_t h_x^2, h_t^2 h_x, h_t^3) + O_5(h_t^2, h_x^2)$

$= O(h_t^2, h_x^2)$

253 Sheng Xu HW1 P3

1(b) ② Stability For C-N

$$-\frac{a\Delta t}{4h_x} V_{k-1,l+1} + V_{k,l+1} + \frac{a\Delta t}{4h_x} V_{k+1,l+1} = \frac{a\Delta t}{4h_x} V_{k-1,l} + V_{k,l} - \frac{a\Delta t}{4h_x} V_{k+1,l}$$

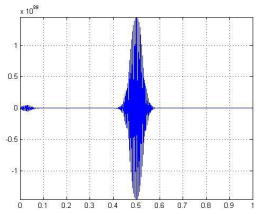
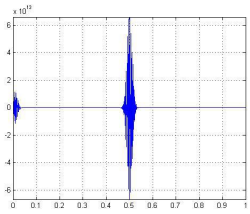
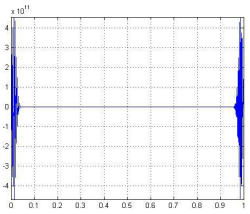
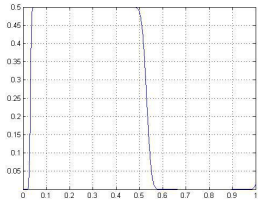
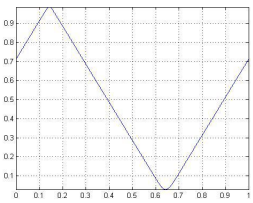
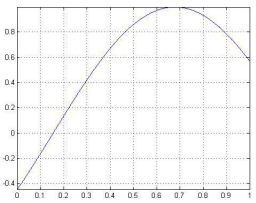
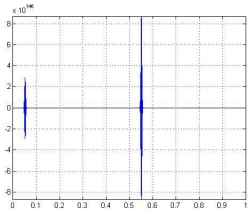
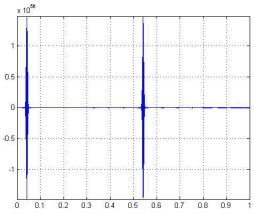
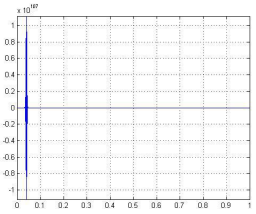
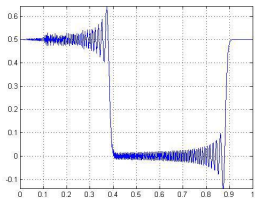
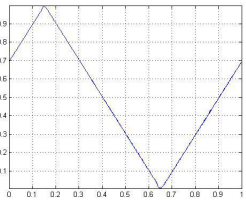
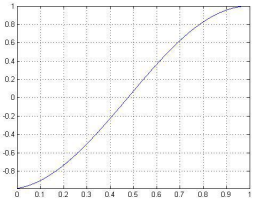
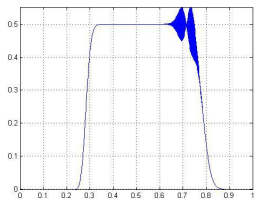
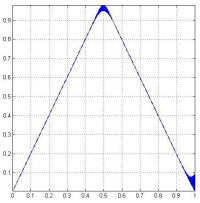
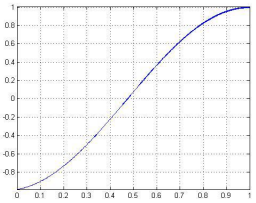
so $\xi(j) = \dfrac{1 - i2\lambda \sin\theta}{1 + i2\lambda \sin\theta}$ $\left(\lambda = \dfrac{a\Delta t}{4h_x} > 0\right)$

so $|g(j)| = 1$ unconditionally stable

Add one step jumped:

$$-\lambda e_{l+1}^{(j)} \exp[ij(k-1)h_x] + e_{l+1}^{(j)} \exp[ijkh_x] + \lambda e_{l+1}^{(j)} \exp[ij(k+1)h_x]$$
$$= \lambda e_l^{(j)} \exp[ij(k-1)h_x] + e_l^{(j)} \exp[ijkh_x] - \lambda e_l^{(j)} \exp[ij(k+1)h_x]$$

Problem 2 Sum_up Sheng Xu

	Discontinuous	C^0	C^∞
FSCT	Unstable 	Unstable 	Unstable 
1st Upwind CFL	Stable with dissipation 	Stable with dissipation 	Stable 
1st Upwind non-CFL	Unstable 	Unstable 	Unstable 
Crank-Nicholson	Stable with Dispersion 	Stable with dissipation 	Stable 
BSCT	Stable with Dispersion 	Stable with Dispersion 	Stable 

Functions	Discontinuous	C^0	C^∞
	F=0 when -1≤ x <-0.5 0≤ x < 0.5 F=0.5 when -0.5≤ x < 0 0.5≤x<1	F=2x+2 when x≤ -0.5 F=-2x when -0.5≤x<0 F=2x when 0≤x<0.5 F=2-2x when 0.5≤ x≤1	cos(3x)

Conclusion:

All stable methods exhibit certain dissipation with C^{∞} functions. FSCT and First Order Upwind method without CFL are never stable.

For Discontinuity and C^1 functions, both implicit methods(C-N and BSCT) exhibits dispersion, making pictures less smooth. BSCT is more effected in C^1 function, while C-N is more effected in Discontinuity function.