Tufts University - Department of Mathematics
Math 253 - Homework 4

The finite-difference and finite-element methods that we've studied in class both rely upon the smoothness of the solution to prove convergence theorems. While we can apply these techniques to problems with discontinuous solutions, it isn't a good idea to do this without thinking carefully. In this assignment, you will explore another approach called the Finite-Volume Method (FVM). We will apply this for the Burgers' Equation, a PDE with known shock-wave-like solutions.

The Inviscid Burgers' Equation is given by

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = u_t + uu_x = 0 \qquad -\infty < x < \infty, \quad 0 < t \le T$$
$$u(x,0) = u_0(x).$$

We can solve Burgers' Equation analytically using the method of characteristics. Consider the solution $u(x(t),t)$ along a curve $x(t)$ such that $\frac{dx}{dt} = u(x(t),t)$. Then,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} = uu_x + u_t = 0. \tag{1}$$

So, given an initial condition, $u(x,0) = u_0(x)$, Equation (1) has solution $u(x(t),t) = u_0(x_0)$, for points $x(t)$ along the characteristic originating at x_0. This then gives $\frac{dx}{dt} = u_0(x_0)$, yielding $x(t) = x_0 + (u_0(x_0))t$.

So, the characteristics, $x(t)$, are straight lines in the $xt-$plane, with slope given by the initial value at the originating point of the characteristic, $u_0(x_0)$.

1. The first part of the assignment is to investigate some of these solutions.

a) Draw the characteristics in the $xt-$plane for $u_0(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$.

b) Draw the characteristics in the $xt-$plane for $u_0(x) = \begin{cases} 1 & x < 0 \\ 1-x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$.

c) Note, you should find that the characteristics intersect. At what points in the $xt-$plane do the characteristics intersect?

This illustrates an important fact about nonlinear hyperbolic PDEs - continuous initial conditions do not guarantee that continuous solutions exist for all time. When characteristics intersect, we find discontinuous solutions of the PDE that represent shock waves. The focus of this assignment is on the development of a numerical method that can track such a "weak" solution of the PDE.

To properly represent such a solution, we rewrite the conservation law in integral form over an arbitrary interval, $[a,b]$,

$$0 = \int_a^b \left(\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)\right) dx = \frac{d}{dt}\int_a^b u\,dx + \frac{(u(b,t))^2}{2} - \frac{(u(a,t))^2}{2}.$$

Writing $Q(t) = \int_a^b u dx$, we can rewrite this as

$$\begin{aligned} \frac{d}{dt}Q(t) &= \frac{(u(a,t))^2}{2} - \frac{(u(b,t))^2}{2} \\ \text{or } Q(T) - Q(0) &= \int_0^T \left(\frac{(u(a,t))^2}{2} - \frac{(u(b,t))^2}{2} \right) dt, \end{aligned} \tag{2}$$

if we integrate over the time interval $[0, T]$. This second formulation makes sense physically: if $u(x,t)$ represents a mass density, then $Q(t)$ is the total mass in the interval $[a, b]$; Equation (2) says that the change in mass between $t = 0$ and $t = T$ is given by the flux in at $x = a$, $\int_0^T \left(\frac{(u(a,t))^2}{2}\right) dt$, less the flux out at $x = b$, $\int_0^T \left(\frac{(u(b,t))^2}{2}\right) dt$.

In fact, the integral form of the conservation law can be used to predict the speed with which the shock moves.

2. Consider the initial condition $u_0(x) = \begin{cases} u_\ell & x \le 0 \\ u_r & x > 0 \end{cases}$ for $u_\ell > u_r$.

a) Why and where does a shock form at $t = 0$?

b) Consider the region in the $xt-$plane given by $[-1, X] \times [0, T]$ where X is big enough that the $x-$position of the shock is less than X when $t = T$. (You don't need to worry about what X is, just imagine that it's really really big.) Argue, from the principle of conservation of mass, that if the shock moves at a constant speed, s, then $Q(t)$ satisfies

$$Q(T) - Q(0) = sT(u_\ell - u_r).$$

c) Use (2) to compute a second expression for $Q(T) - Q(0)$. From these two expressions, compute s.

We could apply a Finite-Difference Method directly to the solution of Burgers' Equation, but two problems arise. First, conservation of mass is critically important to computing physically realistic solutions, so, we cannot use a dissipative Finite-Difference scheme. However, since we are interested in accurately capturing sharp changes in $u(x,t)$, we cannot use a dispersive scheme, since this will cause unphysical oscillations near the shock. Finally, many standard Finite-Difference schemes give solutions with the wrong shock speed, s.

Instead, we will explore a method known as the local Lax-Friedrichs method. To implement this scheme, we divide the domain $[a, b] \times [0, T]$ into small volumes, $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t_j, t_{j+1}]$, and impose conservation of mass on each volume. Rewriting (2) over the time interval $[t_j, t_{j+1}]$ as

$$Q_i(t_{j+1}) - Q_i(t_j) - \int_{t_j}^{t_{j+1}} \left(\frac{(u(x_{i+\frac{1}{2}}, t))^2}{2} - \frac{(u(x_{i-\frac{1}{2}}, t))^2}{2} \right) dt = 0, \tag{3}$$

where $Q_i(t) = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x,t)dx$, we can define the average value of $u(x,t)$ over cell i at time t and the average value of $f(u) = \frac{u^2}{2}$ over the interface at $x_{i+\frac{1}{2}}$ between t_j and t_{j+1} as

$$\overline{u}_{i,j} = \frac{Q_i(t_j)}{h_x} \text{ and } \overline{f}_{i+\frac{1}{2},j+\frac{1}{2}} = \frac{\int_{t_j}^{t_{j+1}} f(u(x_{i+\frac{1}{2}}, t))dt}{h_t},$$

where $h_x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$ and $h_t = t_{j+1} - t_j$. Using this notation, Equation (3) becomes

$$\frac{\overline{u}_{i,j+1} - \overline{u}_{i,j}}{h_t} + \frac{\overline{f}_{i+\frac{1}{2},j+\frac{1}{2}} - \overline{f}_{i-\frac{1}{2},j+\frac{1}{2}}}{h_x} = 0. \tag{4}$$

Next, we use a "Finite-Difference-like" approximation to (4). Define $v_{i,j}$ to be the numerical approximation to $\overline{u}_{i,j}$ and $f^*(v_{i,j}, v_{i+1,j})$ to approximate $\overline{f}_{i+\frac{1}{2},j+\frac{1}{2}}$. This gives

$$\frac{v_{i,j+1} - v_{i,j}}{h_t} + \frac{f^*(v_{i,j}, v_{i+1,j}) - f^*(v_{i-1,j}, v_{i,j})}{h_x} = 0.$$

There are many ways to choose $f^*(v_{i,j}, v_{i+1,j})$ and each then determines which Finite-Volume scheme we are considering.

For the Inviscid Burgers' Equation, the local Lax-Friedrichs flux function is given by

$$f^*(v_{i,j}, v_{i+1,j}) = \frac{\frac{1}{2}(v_{i,j})^2 + \frac{1}{2}(v_{i+1,j})^2}{2} - \frac{1}{2}\left|\frac{v_{i,j} + v_{i+1,j}}{2}\right|(v_{i+1,j} - v_{i,j}).$$

This scheme is consistent with order 1 in space ($O(h_x)$) and 1 in time ($O(h_t)$) , and is stable if $h_t \leq \min\limits_i \frac{h_x}{v_i}$. If the maximum value of $u_0(x)$ is known, this can provide an absolute bound on h_t for stability. The Lax-Wendroff theorem then says that the numerical solution to this scheme converges to the weak solution of the PDE, with shocks that travel at the correct speed.

3. Implement the Finite-Volume method using the local Lax-Friedrichs flux function for the Inviscid Burgers' Equation. Choose the spatial domain $[a, b]$ to be large enough that your shock never touches the spatial boundaries (and so that the analytical solution has fixed values, u_ℓ and u_r, along these edges) for the time range $[0, T]$ that you simulate.

4. Test your code for the three choices of u_0 given in the exercises above.