253 Sheng Xu 1205525 HW 1 P_1

1 (a) ① $\frac{V_{k,l+1} - V_{k,l}}{h_t}$

$= \frac{1}{h_t}[V_t h_t] + \frac{1}{h_t} O_a(h_t^2) = V_t \pm O_1(h_t)$

$\frac{V_{k+1,l} - V_{k-1,l}}{2h_x}$

$= \frac{1}{2h_x}[\cancel{V_{k,l}} + V_x h_x + \cancel{\frac{1}{2!} h_x^2} + O_\alpha(h_x^3)]$

$- \frac{1}{2h_x}[\cancel{V_{k,l}} - V_x h_x + \cancel{\frac{1}{2!} h_x^2} + O_\beta(h_x^3)]$

$O_1 = O_\alpha + O_\beta$ $\qquad = V_x + O_1(h_x^2)$

$O_1' = a \cdot O_1$ so $E_{xt} = V_t + O_2(h_t) + aV_x + O_1'(h_x^2)$

$= O_2(h_t) + O_1'(h_x^2) = O(h_t, h_x^2)$

since $V_t + aV_x = 0$.

stability ② $V_{k,l+1} = \frac{ah_t}{2h_x} V_{k-1,l} + V_{k,l} - \frac{ah_t}{2h_x} V_{k+1,l}$

So $e_{l+1}^{(j)} \exp(ijkh_x) = \lambda e_l^{(j)} \exp[ij(k-1)h_x] + e_l^{(j)} \exp[ijkh_x] -$

$\lambda e_l^{(j)} \exp[ij(k+1)h_x]$

$\lambda = \frac{ah_t}{2x}$

So $S(j) = \lambda \exp(-ijh_x) + 1 - \lambda \exp(ijh_x)$

$= 1 - i2\lambda \sin\theta$ $\qquad (\theta = jh_x)$

So $|S(j)| > 1$ when $\theta \neq k\pi$, $\lambda > 0$, unstable.

1 (b) ①
$$\frac{V_{k,l+1}-V_{k,l}}{h_t}$$
$$= V_t(x_k,t_l) + \frac{1}{2}V_{tt}(x_k,t_l)\,h_t + \frac{1}{6}V_{ttt}(x_k,t_l)\cdot h_t^2 + O_1(h_t^3)$$

$$\left(\frac{V_{k+1,l+1}-V_{k-1,l+1}}{2h_x}\right)$$
$$= \frac{1}{2h_x}\Big[\cancel{V_{k,l}} + V_{x\,k,l}\,h_x + \cancel{V_{t\,k,l}\,h_t} + \frac{1}{2!}\cancel{V_{xx\,k,l}\,h_x^2} + \frac{1}{1!1!}V_{xt\,k,l}\,h_x h_t$$
$$+\frac{1}{2!}\cancel{V_{tt\,k,l}\,h_t^2} + \frac{1}{3!}V_{xxx}\,h_x^3 + \frac{1}{2!}\cancel{V_{xxt}\,h_x^2 h_t} + \frac{1}{2!}V_{xtt}\,h_x h_t^2 + \frac{1}{3!}\cancel{V_{ttt}\,h_t^3}$$
$$-\frac{1}{2h_x}\Big[\cancel{V_{k,l}} - V_x\,h_x + \cancel{V_t\,h_t} + \frac{1}{2!}\cancel{V_{xx}\,h_x^2} - \frac{1}{1}V_{xt}\,h_x h_t$$
$$+\frac{1}{2!}\cancel{V_{tt}\,h_t^2} - \frac{1}{3!}V_{xxx}\,h_x^3 + \frac{1}{2!}\cancel{V_{xxt}\,h_x^2 h_t} - \frac{1}{2!}V_{xtt}\,h_x h_t^2 + \frac{1}{3!}\cancel{V_{ttt}\,h_t^3}$$

* All h_t^n term are cancelled out.

$$+\boxed{\frac{1}{2h_x}O_4(h_x^4,\ h_t h_x^3,\ h_t^2 h_x^2,\ h_t^3 h_x)}$$
$$= \frac{1}{h_x}\Big[V_x\,h_x + V_{xt}\,h_x h_t + \frac{1}{3!}V_{xxx}\,h_x^3 + \frac{1}{2!}V_{xtt}\,h_x h_t^2\Big]$$
$$+ O_2(h_x^3,\ h_t h_x^2,\ h_t^2 h_x,\ h_t^3)$$

$$\frac{V_{k+1,l}-V_{k-1,l}}{2h_x}$$
$$= \frac{1}{2h_x}\Big[\cancel{V_{k,l}} + V_x\,h_x + \frac{1}{2}\cancel{V_{xx}\,h_x^2} + \frac{1}{3!}V_{xxx}\,h_x^3\Big]$$
$$-\frac{1}{2h_x}\Big[\cancel{V_{k,l}} - V_x\,h_x + \frac{1}{2}\cancel{V_{xx}\,h_x^2} - \frac{1}{3!}V_{xxx}\,h_x^3\Big] + \frac{1}{2h_x}O_\beta(h_x^4)$$
$$= \frac{1}{h_x}\Big[V_x\,h_x + \frac{1}{3!}V_{xxx}\,h_x^3\Big] + O_3(h_x^3)$$

Now we combine everything:

$$T_{xt} = V_t + \frac{1}{2}V_{tt}\,h_t + \frac{1}{6}V_{ttt}\,h_t^2 + O_1(h_t^3)$$
$$+aV_x + \frac{a}{2}V_{xt}\,h_t + \boxed{\frac{a}{6}V_{xxx}\,h_x^2 + \frac{a}{4}V_{xtt}\,h_t^2} +$$
$$\underline{O_2'(h_x^3,\ h_t h_x^2,\ h_t^2 h_x,\ h_t^3) + O_3'(h_x^3)}$$

$O_2' = \frac{a}{2}\cdot O_2$

$O_3' = \frac{a}{2}O_3$

$$= (V_t + aV_x) + \frac{1}{2}h_t(V_{tt} + aV_{xt}) + O_4(h_t^2) + O_5(h_t^2,\ h_x^2)$$
$$O_6(h_x^3,\ h_t h_x^2,\ h_t^2 h_x,\ h_t^3)$$

We know $V_t + aV_x = 0$, $V_{tt} + aV_{xt} = (V_t + aV_x)_t = 0_t = 0$

so $T_{xt} = O_4(h_t^2) + O_6(h_x^3,\ h_t h_x^2,\ h_t^2 h_x,\ h_t^3) + O_5(h_t^2,\ h_x^2)$
$$= O(h_t^2, h_x^2)$$

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1(b) ② Stability For C-N.

$$-\frac{a h_t}{4h_x} V_{k-1,l+1} + V_{k,l+1} + \frac{a h_t}{4h_x} V_{k+1,l+1} = \frac{a h_t}{4h_x} V_{k-1,l} + V_{k,l} - \frac{a h_t}{4h_x} V_{k+1,l}$$

so $\xi(j) = \dfrac{1 - i2\lambda \sin\theta}{1 + i2\lambda \sin\theta}$ $\left(\lambda = \dfrac{a h_t}{4h_x} > 0\right)$

so $|\xi(j)| = 1$ unconditionally stable.

Add one step jumped:

$$-\lambda e_{l+1}^{(j)} \exp[ij(k-1)h_x] + e_{l+1}^{(j)} \exp[ij\,kh_x] + \lambda e_{l+1}^{(j)} \exp[ij(k+1)h_x]$$
$$= \lambda e_l^{(j)} \exp[ij(k-1)h_x] + e_l^{(j)} \exp[ij\,kh_x] - \lambda e_l^{(j)} \exp[ij(k+1)h_x]$$

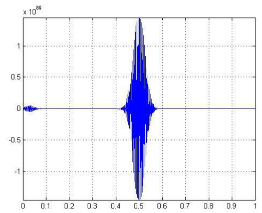
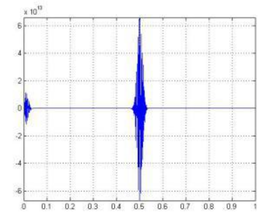
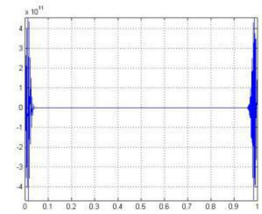
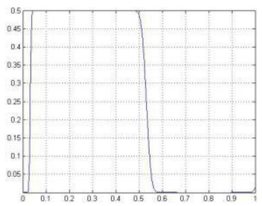
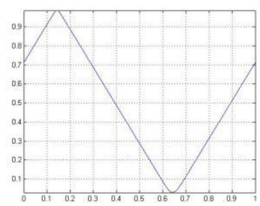
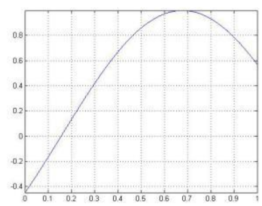
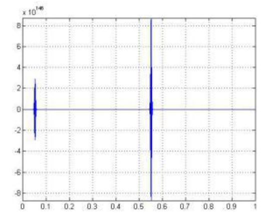
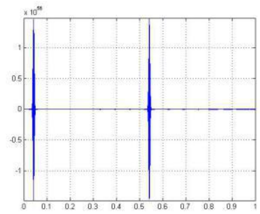
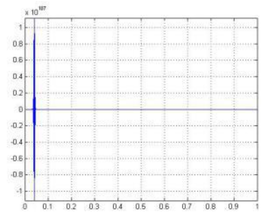
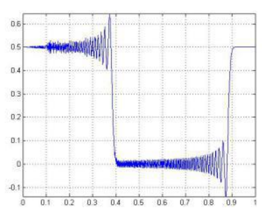
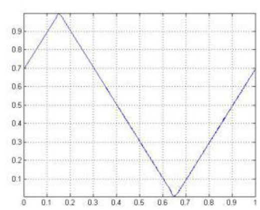
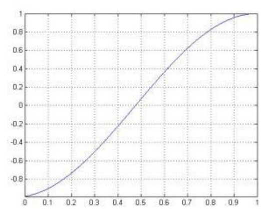
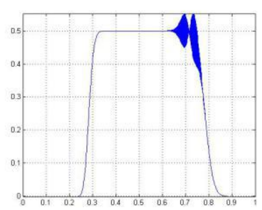
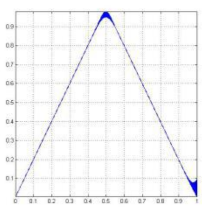
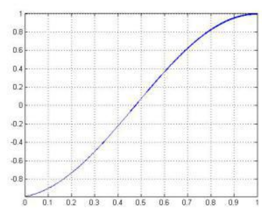
A short note: For Question 1

Both Method are good with Consistency.

The FSCT has truncation error $O(h_t, h_x^2)$, so when $h_t \to 0$, $h_x \to 0$, error goes to 0.

The C-N has truncation error $O(h_t^2, h_x^2)$, so when $h_t \to 0$, $h_x \to 0$, error goes to 0.

Problem 2 Sum_up Sheng Xu

	Discontinuous	C^0	C^∞
FSCT	Unstable 	Unstable 	Unstable 
1st Upwind CFL	Stable with dissipation 	Stable with dissipation 	Stable 
1st Upwind non-CFL	Unstable 	Unstable 	Unstable 
Crank-Nicholson	Stable with Dispersion 	Stable with dissipation 	Stable 
BSCT	Stable with Dispersion 	Stable with Dispersion 	Stable 

Functions	Discontinuous	C^0	C^∞
	F=0 when $-1\le x<-0.5$ $0\le x<0.5$ F=0.5 when $-0.5\le x<0$ $0.5\le x<1$	F=2x+2 when $x\le -0.5$ F=-2x when $-0.5\le x<0$ F=2x when $0\le x<0.5$ F=2-2x when $0.5\le x\le 1$	cos(3x)

Conclusion:

All stable methods exhibit certain dissipation with C^∞ functions. FSCT and First Order Upwind method without CFL are never stable.

For Discontinuity and C^1 functions, both implicit methods(C-N and BSCT) exhibits dispersion, making pictures less smooth. BSCT is more effected in C^1 function, while C-N is more effected in Discontinuity function.