A Finite-Volume Approach to Black-Scholes Formula

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Black-Scholes Formula

Black-Scholes(BS) Model[1] is one of the most commonly employed model based on partial difference equations, which provides an exact closed form solutions for financial derivatives. The equation is stated as:

BS Formula

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{1}$$

where S is a real asset value, $0 \leq S \leq \infty$, V is the (real) option price, r is the risk-free rate, t is the time since the option was issued, $0 \leq t \leq T$, and σ is the real asset volatility. Eq. (1) is a backward moving equation, i.e. it is solved from the future to the present time.

Black-Scholes Formula

For an European option the time condition becomes a final condition because its value is known at the maturity date t = T and it is defined as its intrinsic value by:

BS Call

$$V(S, T) = max(S - K, 0), \forall S. \tag{2}$$

BS Put

$$V(S, T) = max(K - S, 0), \forall S. \tag{3}$$

FVM

The finite volume method (FVM) is a method for representing and evaluating partial differential equations[2, 3]. Similar to the finite difference method or finite element method, values are calculated at discrete places on a meshed geometry. "Finite volume" refers to the small volume surrounding each node point on a mesh. In the finite volume method, volume integrals in a partial differential equation that contain a divergence term are converted to surface integrals, using the divergence theorem. These terms are then evaluated as fluxes at the surfaces of each finite volume. Because the flux entering a given volume is identical to that leaving the adjacent volume, these methods are conservative. Another advantage of the finite volume method is that it is easily formulated to allow for unstructured meshes. [4]

Inverse Time

In section 1, we introduced the Black-Schole Model and describe the parameters. It is known that we know a strike at time $t = T$ and want to calculate the price at time $0 \leq t < T$. To deal with the problem, we define $\tau = T - t$ and transform the formula into:

Inverse Time

$$\frac{\partial V}{\partial \tau} = rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV \tag{4}$$

where S is a real asset value, $0 \leq S \leq \infty$, V is the (real) option price, r is the risk-free rate, τ is the time before the option is exercised, $0 \leq \tau \leq T$, and σ is the real asset volatility.

Integration

We than integrate the function (4) on a divided domain $A_i \times T_i = [S_{i-\frac{1}{2}}, S_{i+\frac{1}{2}}] \times [\tau_j, \tau_{j+1}]$, where $S_{i-\frac{1}{2}} = \frac{1}{2}(S_{i-1} + S_i)$, $S_{i+\frac{1}{2}} = \frac{1}{2}(S_i + S_{i+1})$. Let $\|T_i\| = \Delta\tau = \tau_{j+1} - \tau_j$, $\|A_i\| = \|S_{i-\frac{1}{2}} - S_{i+\frac{1}{2}}\|$, $V_n^i = V(S_i, \tau_n)$. This will give us the formula:

Integration

$$\int_{A_i} V_i^{n+1} - V_i^n = \Delta\tau\left(\int_{A_i} rS\frac{\partial V}{\partial S} + \int_{A_i} \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \|A_i\| rV\right) \quad (5)$$

Integration-LHS

LHS:

Integration

$$\int_{A_i} V_i^{n+1} - V_i^n = \|A_i\|(V_i^{n+1} - V_i^n) \tag{6}$$

Integration-RHS

RHS:

Integration-RHS

$$\int_{A_i} rS\frac{\partial V}{\partial S} \approx rS_i[V_{i+\frac{1}{2}} - V_{i-\frac{1}{2}}] \approx \frac{1}{2}rS_i[V_{i+1} - V_{i-1}] \tag{7}$$

Integration-RHS

$$\int_{A_i} \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} \approx \frac{1}{2}\sigma^2 S_i^2[(V_S)_{i+\frac{1}{2}} - (V_S)_{i-\frac{1}{2}}] \approx \frac{1}{2}\sigma^2 S_i^2(\frac{V_{i+1} - V_i}{S_{i+1} - S_i} - \frac{V_i - V_{i-1}}{S_i - S_{i-1}}) \tag{8}$$

Results

Now let's assemble every thing together and it will give an implicit formula of V^n and V^{n+1}.

Results

$$\begin{aligned}\|A_i\|(V_i^{n+1} - V_i^n) \approx & \Delta\tau(\frac{1}{2}rS_i[V_{i+1} - V_{i-1}] \\ & + \frac{1}{2}\sigma^2 S_i^2(\frac{V_{i+1} - V_i}{S_{i+1} - S_i} - \frac{V_i - V_{i-1}}{S_i - S_{i-1}}) \\ & - \|A_i\|rV)\end{aligned} \tag{9}$$

Results: Call Option

Settings:

$$\sigma = 0.4; r = 0.05; K = 100; T = 1;$$

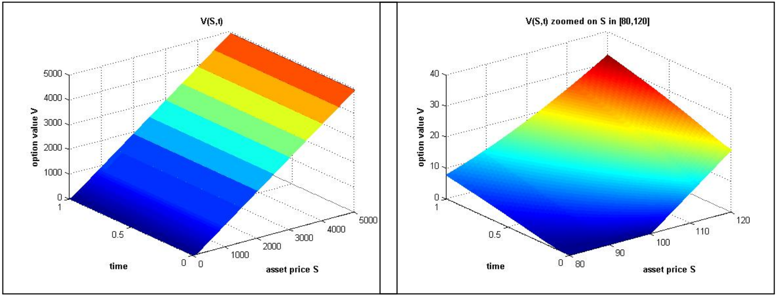

Figure: European Call Option: S $\times$ T= [0,5000] $\times$ [0,1]

Figure: European Call Option: S $\times$ T= [80,120] $\times$ [0,1]

Results: Call Option

Settings:

$$\sigma = 0.4; r = 0.05; K = 100; T = 1;$$

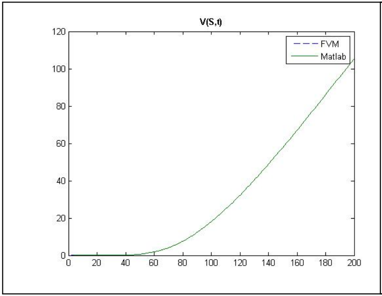

Figure: European Call Option: S $\times$ T= [0,200] $\times$ [0,1]

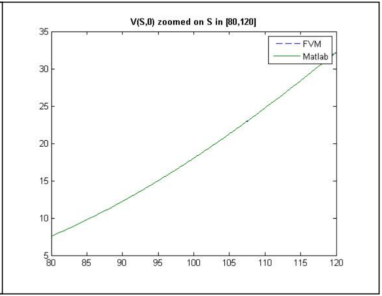

Figure: European Call Option: S $\times$ T= [80,120] $\times$ [0,1]

Results: Put Option

Settings:

$$\sigma = 0.4; r = 0.05; K = 100; T = 1;$$

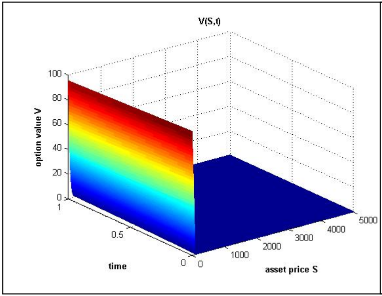

Figure: European Put Option: S $\times$ T= [0,5000] $\times$ [0,1]

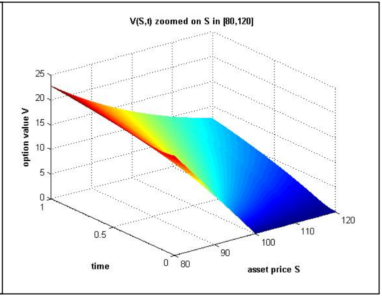

Figure: European Put Option: S $\times$ T= [80,120] $\times$ [0,1]

Results: Put Option

Settings:

$$\sigma = 0.4; r = 0.05; K = 100; T = 1;$$

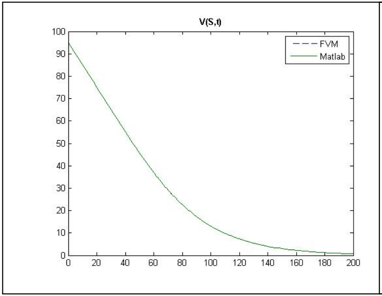

Figure: European Put Option Compared with Matlab: S $\times$ T= [0,200] $\times$ 1

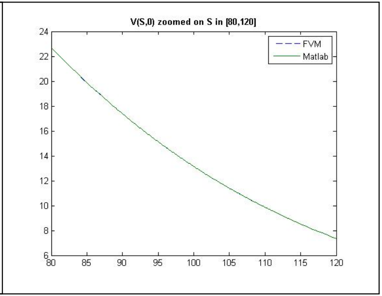

Figure: European Put Option Compared with Matlab: S $\times$ T= [80,120] $\times$ 1

Conclusions

In this project, we investigated the application of finite-volume method to the Black-Scholes Model and provide a detail scheme for pratical implementations. A modified finite-volume methhod is introduced as a numerical simulation of European Option Pricing. Detailed derivation process of the method is developed and numerical results is provided. The numerical results are compared with Matlab built in solver and show a good performance of the method.

Future Work

Error Analysis

Compute Faster: Jacobi or Gauss-Seidel

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