Tufts University - Department of Mathematics
Math 253 - Homework 2 Solutions

1. Use matlab to visualize the dispersion and dissipation of the wave solutions to $L_i w = 0$ for the three partial differential operators discussed in class:

$$L_1 u = \partial_t u + a\partial_x u$$
$$L_2 u = \partial_t u + a\partial_x u - D\partial_x^2 u, \text{ for } D > 0$$
$$L_3 u = \partial_t u + a\partial_x u - \mu\partial_x^3 u.$$

Note that this does not require you to numerically solve the PDEs! You can simply graph a wave-like solution, $z(x,t) = A_0 e^{i(kx-\omega(k)t)}$, or a superposition of such solutions at various values of t. An important part of this problem is choosing a suitable domain (in both time and space) and values of k, D, and μ so that your visualization clearly shows what you intend. Feel free to test out the "movie" function in Matlab.

For this problem, you will "hand in" your matlab code by uploading (to Trunk) a *single* M-file along with the command to set it running. If you choose to not use Matlab, please discuss your plans with me, so that we can be sure I will be able to grade your work.

Answer: Recall that the following dispersion relations for each of the above PDEs:

$$L_1 u \to \quad \omega = ak \to z(x,t) = A_0 e^{ik(x-at)} \quad \to \text{ No Dissipation nor Dispersion}$$
$$L_2 u \to \quad \omega = ak - iDk^2 \to z(x,t) = A_0 e^{-Dk^2 t} e^{ik(x-at)} \quad \to \text{ Dissipation but no Dispersion}$$
$$L_3 u \to \quad \omega = ak + \mu k^3 \to z(x,t) = A_0 e^{i(kx-(ak+\mu k^3)t)} \quad \to \text{ Dispersion but no Dissipation}$$

The following code runs various tests for testing the dissipation and dispersion relations for each PDE. The results above are confirmed:

```
function [] = DissipationDispersion(PDE)
% This function will create and run the movies for various tests to visualize
% dissipation and dispersion or lack thereof in 3 Differential operators
% given by
% 1. L_1u = u_t + au_x
% 2. L_2u = u_t + au_x-Du_xx    D>0
% 3. L_3u=u_t+au_x-\muu_xxx

if(PDE==1)
    % solution form for 1.
    % w = ak
    wxt = @(x,t,A0, a, k) A0*exp(1i*(k*x-a*k*t))

    % Simple Test for Dissipation
    disp('Testing for Dissipation');
    A0=2;
    a=1;
    k=3;
    x=0:.01:pi;
    tend = 4;
    numt = 100;
    tspace=tend/numt;

    for j=0:numt,
        plot(x,real(wxt(x,j*tspace,A0,a,k)),x,imag(wxt(x,j*tspace,A0,a,k)));
        pause(.1);
    end

    disp('We can see there is no dissipation in the wave over time');
    disp('Press a key to continue')
    waitforbuttonpress;

    % Change the parameter
    disp('Increase the wave number?');
    A0=4;
    a=8;
    k=2;
    x=0:.01:3*pi;
    tend = 4;
    numt = 100;
    tspace=tend/numt;
    for j=0:numt,
```

```
        plot(x,real(wxt(x,j*tspace,A0,a,k)),x,imag(wxt(x,j*tspace,A0,a,k)));
        pause(.1);
    end
    disp('Again, no Dissipation over Time');

    disp('Press a key to continue')
    waitforbuttonpress;
    disp('Test two different waves with different wave numbers')
    A01=1;
    a1=1;
    k1=2;
    x=0:.01:3*pi;
    tend = 5;
    numt = 200;
    tspace=tend/numt;
    A02=1;
    a2=1;
    k2=4;

    for j=0:numt,
        plot(x,real(wxt(x,j*tspace,A01,a1,k1)),x,...
            real(wxt(x,j*tspace,A02,a2,k2)));
        axis([0,3*pi,-2,2]);
        pause(.1);
    end

    disp(['We can see that there is no dispersion here because waves' ...
        ' with different wave numbers still travel at the same speed.']);

    disp('Press a key to continue')
    waitforbuttonpress;
    disp('Superimpose some waves?');
    A01=1;
    a1=1;
    k1=2;
    x=0:.01:3*pi;
    tend = 5;
    numt = 200;
    tspace=tend/numt;

    A02=1;
    a2=1;
    k2=4;

    for j=0:numt,
        plot(x,real(wxt(x,j*tspace,A01,a1,k1)+wxt(x,j*tspace,A02,a2,k2)),x,...
            imag(wxt(x,j*tspace,A01,a1,k1)+wxt(x,j*tspace,A02,a2,k2)));
        axis([0,3*pi,-2,2]);
        pause(.1);
    end

    disp(['We can see that there is also no dispersion in our waves' ...
        ' as our superposition waves remain intact in spite of'...
        ' varying k values.']);

elseif(PDE==2)
    % solution form for 2.
    % w = ak ? iDk^2
    wxt = @(x,t,A0, a, k, D) A0*exp(-D*k^2*t)*exp(1i*k*(x-a*t));

    disp('Simple test for dissipation:');
    A0=2;
    a=3;
    D=1;
    k=1;
    x=0:.01:3*pi;
    tend =5;
    numt = 200;
    tspace=tend/numt;

    for j=0:numt,
        plot(x,real(wxt(x,j*tspace,A0,a,k,D)),x,imag(wxt(x,j*tspace,A0,a,k,D)));
        axis([0,3*pi,-2,2]);
        pause(.1);
    end
    disp(['We clearly observe dissipation in our wave amplitude'...
        ' as time progresses.']);

    disp('Press a key to continue')
    waitforbuttonpress;

    disp('Now try waves with different wave number:');
    A01=2;
    a1=1;
    D1=1;
    k1=1;

    A02=2;
    a2=1;
    D2=1;
    k2=3;

    x=0:.01:3*pi;
    tend =5;
    numt = 200;
    tspace=tend/numt;
```

```
    for j=0:numt,
        plot(x,real(wxt(x,j*tspace,A01,a1,k1,D1)),x,...
            real(wxt(x,j*tspace,A02,a2,k2,D2)));
        axis([0,3*pi,-2,2]);
        pause(.1);
    end
    disp(['We can see that the high frequency waves die out much'...
        ' faster than the low frequency modes.']);

    disp('Press a key to continue')
    waitforbuttonpress;
    disp('Another test?')
    A01=1;
    a1=1;
    k1=.15;
    D1=1;
    x=0:.01:5*pi;
    tend = 5;
    numt = 200;
    tspace=tend/numt;

    A02=1;
    a2=1;
    k2=.5;
    D2=1;

    for j=0:numt,
        plot(x,real(wxt(x,j*tspace,A01,a1,k1,D1)),x,...
            real(wxt(x,j*tspace,A02,a2,k2,D2)));
        axis([0,5*pi,-2,2]);
        pause(.1);
    end
    disp(['We observe that there is no dispersion because the waves'...
        ' with different wave numbers still travel at the same speed.']);

    disp('Press a key to continue')
    waitforbuttonpress;
    disp('Superimpose some waves?');
    A01=1;
    a1=1;
    k1=1;
    D1=1;
    x=0:.01:3*pi;
    tend = 5;
    numt = 200;
    tspace=tend/numt;

    A02=1;
    a2=1;
    k2=2;
    D2=1;

    for j=0:numt,
        plot(x,real(wxt(x,j*tspace,A01,a1,k1,D1)+wxt(x,j*tspace,A02,a2,k2,D2)),x,...
            imag(wxt(x,j*tspace,A01,a1,k1,D1)+wxt(x,j*tspace,A02,a2,k2,D2)));
        axis([0,3*pi,-2,2]);
        pause(.1);
    end

    disp(['Similarly we see no dispersion because our waves with'...
        ' superposition remain intact but with some damping.']);
elseif(PDE==3)
    % solution form for 3.
    % w = ak + \muk^3
    wxt = @(x,t,A0, a, k, mu) A0*exp(1i*(k*x-(a*k+mu*k^3)*t));

    disp('Simple Test.');
    A0=2;
    a=3;
    mu=1;
    k=1;
    x=0:.01:3*pi;
    tend =5;
    numt = 200;
    tspace=tend/numt;

    for j=0:numt,
        plot(x,real(wxt(x,j*tspace,A0,a,k,mu)),x,imag(wxt(x,j*tspace,A0,a,k,mu)));
        axis([0,3*pi,-2,2]);
        pause(.1);
    end
    disp('Clearly we have no dissipation of our wave amplitude');

    disp('Press a key to continue')
    waitforbuttonpress;
    disp('Different wave number test:');
    A01=2;
    a1=3;
    mu1=1;
    k1=1;

    A02=2;
    a2=3;
    mu2=1;
    k2=2;

    x=0:.01:3*pi;
```

```
    tend =5;
    numt = 200;
    tspace=tend/numt;

    for j=0:numt,
        plot(x,real(wxt(x,j*tspace,A01,a1,k1,mu1)),x,...
            real(wxt(x,j*tspace,A02,a2,k2,mu2)));
        axis([0,3*pi,-2,2]);
        pause(.1);
    end

    disp(['Clearly we see dispersion in the waves. The waves with'...
        ' different wave numbers travel at different speeds.']);

    disp('Press a key to continue')
    waitforbuttonpress;
    disp('Superimpose some waves?');
    A01=1;
    a1=1;
    k1=1;
    mu1=1;
    x=0:.01:3*pi;
    tend = 5;
    numt = 200;
    tspace=tend/numt;

    A02=1;
    a2=1;
    k2=2;
    mu2=1;

    for j=0:numt,
        plot(x,real(wxt(x,j*tspace,A01,a1,k1,mu1)+wxt(x,j*tspace,A02,a2,k2,mu2)),x,...
            imag(wxt(x,j*tspace,A01,a1,k1,mu1)+wxt(x,j*tspace,A02,a2,k2,mu2)));
        axis([0,3*pi,-2,2]);
        pause(.1);
    end

    disp(['Dispersion is seen as the waves change shape depending on'...
        ' their frequency']);


end

close all;
```

2. Investigate the numerical dispersion and dissipation for the First-Order Upwind (discussed in class) and Crank-Nicholson discretizations of $u_t + au_x = 0$ for $a > 0$.

   (a) For Crank-Nicholson, analyze the expected dissipation and dispersion by hand, and explain what you expect to see in a numerical study. In particular, graph the expected amount of dissipation and the phase velocity, v_{ph} as a function of k (or of kh_x). Create similar graphs for First-Order Upwind and discuss the results.

   Answer: First recall the results from class on *First-order Upwind.*

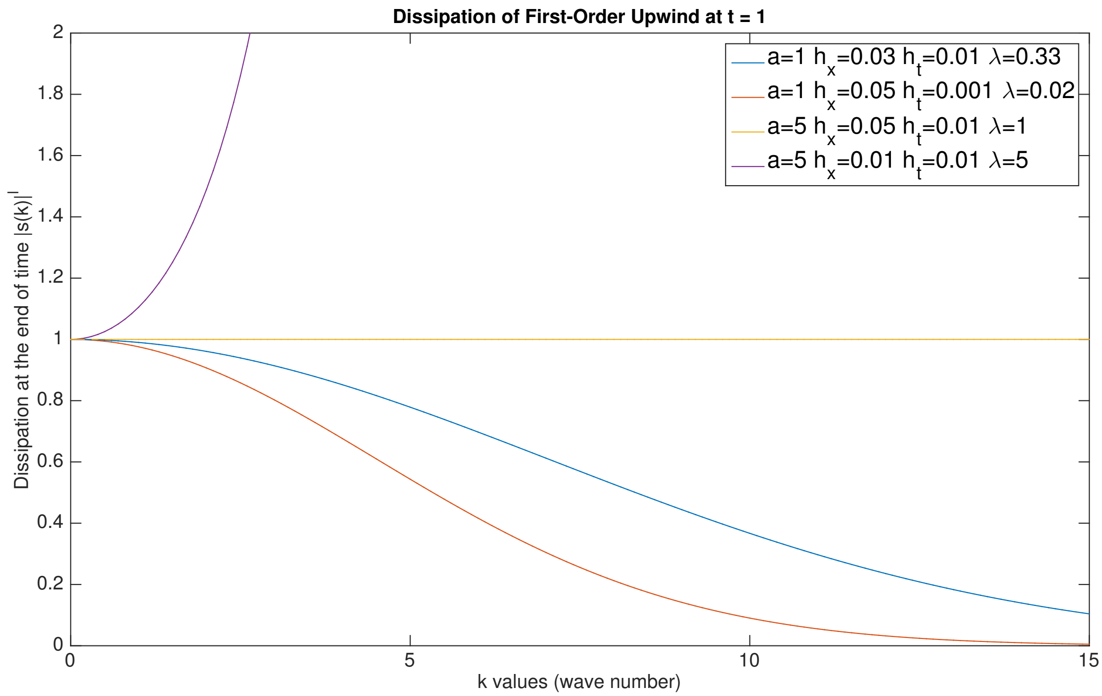
$$s(k) = (1-\lambda) + \lambda e^{-ikh_x}.$$

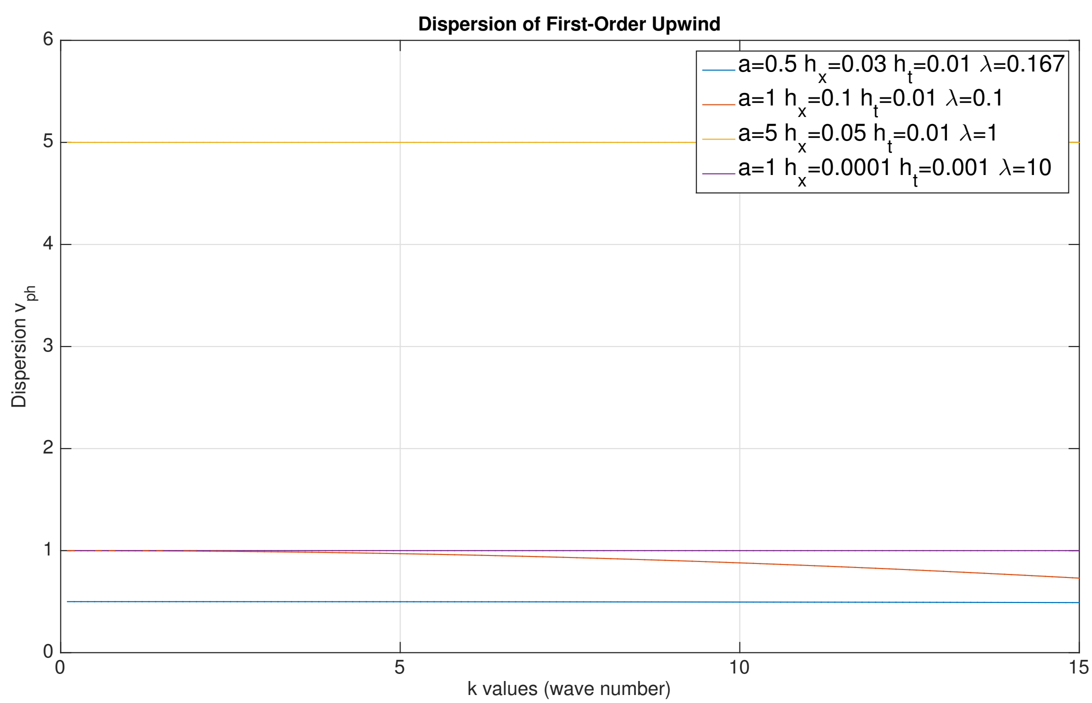
$$|s(k)| = \left(1 - 2\lambda + 2\lambda^2 + 2(\lambda - \lambda^2)\cos(kh_x)\right)^{1/2}$$

   . Letting $s(k) = |s|e^{i\phi} \Rightarrow i\phi = \ln\left(\frac{(1-\lambda))+\lambda e^{-ikh_x}}{|s|}\right)$.

   Then, $v_{ph} = \frac{-\phi(k)}{kh_t} = \frac{i\ln\left(\frac{(1-\lambda))+\lambda e^{-ikh_x}}{|s|}\right)}{kh_t}$. We plot these for a few parameters to confirm the results from class:

   We can see that as λ gets smaller the dissipation increases. When $\lambda = 1$, we have no dissipation as $|s(k)| = 1$. Finally, when we use parameters that do not satisfy the CFL condition, we see the expected numerical instability. Next, we plot the phase field velocity:

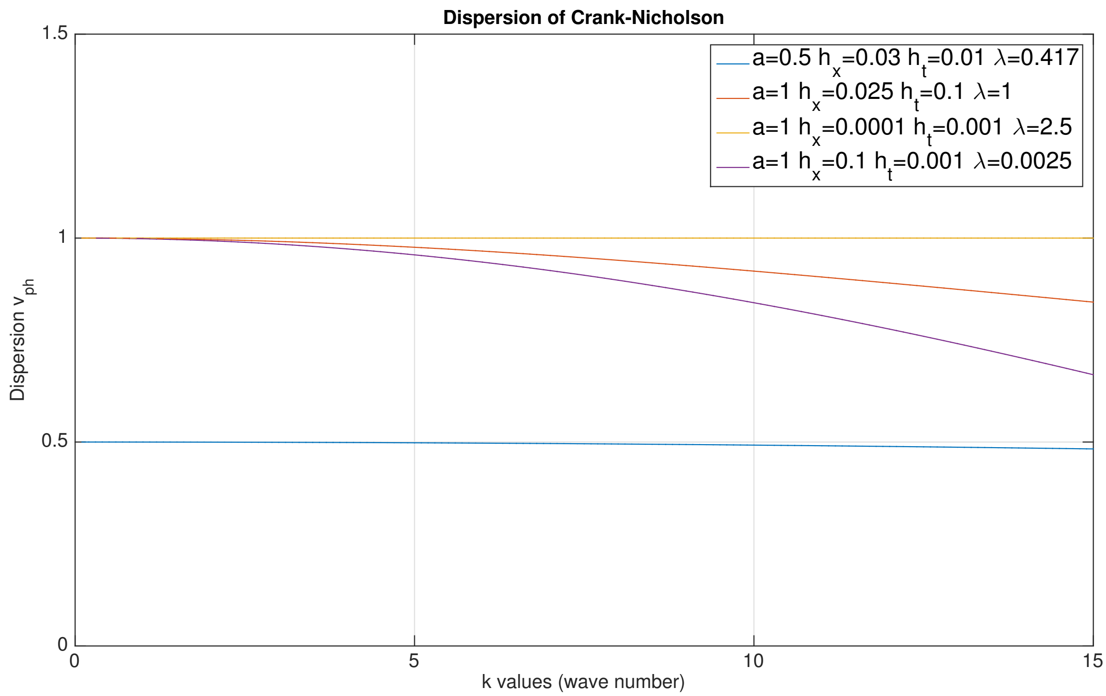




Here we see some slight dispersion for large values of k when h_x is not kept small relative to h_t and a. Additionally, we should have no dispersion when $\lambda = 1$ as this implies $|s(k)| = 1 \Rightarrow v_{ph} = \frac{-\ln(e^{-ikh_x})}{ikh_t} = \frac{ikh_x}{ikh_t} = a\lambda = a$.

Next, we perform a similar analysis for *Crank-Nicholson*. Recall from HW #1 that $s(k) = \frac{1-2\lambda i\sin(kh_x)}{1+2\lambda i\sin(kh_x)} \Rightarrow |s(k)| = 1$, for $\lambda = \frac{ah_t}{4h_x}$. Thus, we expect no numerical dissipation regardless of the values of k, h_t, h_x, and a.

For dispersion, we note that $\phi = -i\ln\left(\frac{s(k)}{|s(k)|}\right) = -i\ln(s(k)) \Rightarrow v_{ph} = \frac{-\phi}{kh_t} = \frac{i\ln\left(\frac{1-2\lambda i\sin(kh_x)}{1+2\lambda i\sin(kh_x)}\right)}{kh_t}$. Thus, there is some dispersion associated with Crank-Nicholson. The following results show some dispersion for various parameters:



As expected for moderate λ there is not much dispersion, but increases as λ goes to zero.

(b) Then, use your codes for these two schemes from HW1 to get numerical results that confirm your expectations. Choose the initial and boundary conditions, $u_0(x)$ and $u_1(t)$, to match a single wave-like solution $e_{j,l} = e_0 e^{i(kjh_x - \omega l h_t)}$ or a superposition of such solutions, and discuss the results. Again, an important part of this problem is the appropriate choice of the parameters, a, k, h_x, and h_t, as well as an appropriate visualization of the results. You may want to use the movie function again, or you can use matlab's "surf" function to display your solution as a surface in space and time. Be sure to label all axes appropriately. **Partial Answer:** Running your code from HW #1, you should see the following results. Recall, here, that $a = 1$. Test the Upwind code with the following data:

$$u_0(x) = \cos(k_1\pi x) + \cos(k_2\pi x) \qquad u_1(t) = \cos(-k_1\pi t) + \cos(-k_2\pi t).$$

k_1	k_2	h_t/h_x	Dissipation	Dispersion
3	3	1	None (true for any k)	None.
10	10	< 1	Significant (especially for high k)	None
3	5	1	None	None
1	7	0.01/0.1	Significant	Significant
1	7	0.001/0.01	Less	Less (Due to small kh_x).

Similar results for Crank-Nicholson are obtained, though, we'd expect no dissipation and some dispersion if kh_x is larger.