2 (a1). First order upwind:

$S(k) = 1-\lambda + \lambda \cdot \exp(-ikh_x)$. $\lambda = \frac{a h_t}{h_x}$

So For $|S(k)|$ we have Figure 2-a1-1 when $\lambda \le 1$ (CFL)

2-a1-2 when $\lambda > 1$

For V_{ph}

$$V_{ph} = i\ln\frac{S(k)}{|S(k)|} / (k h_t).$$

Here we only discuss V_{ph} when $\lambda \le 1$. 2-a1-3

We notice if $\lambda = 1$, No dissipation and no dispersion

2. (a2) $s(k) = \frac{1-2i\lambda\sin\theta}{1+2i\lambda\sin\theta} = \frac{1-4\lambda^2\sin^2\theta}{1+4\lambda^2\sin^2\theta} - i\,\frac{4\lambda\sin\theta}{1+4\lambda^2\sin^2\theta}$

$(\theta = k h x)$

Here $|s(k)| = 1$ so there is no dissipation. (Fig. 2-a2-1)

For $s(k) = e^{i\phi}$

so $\phi = i \ln \frac{1-2i\lambda\sin\theta}{1+2i\lambda\sin\theta} \Big/ (kht)$

On the other hand;

$s(k) = e^{i\phi} = \cos\phi + i\sin\phi$

so $\phi = -\arcsin\left(\frac{4\lambda\sin khx}{1+4\lambda^2\sin^2 khx}\right)$

so $V_{ph} = \frac{\arcsin\left[\frac{4\lambda\sin(khx)}{1+4\lambda^2\sin^2(khx)}\right]}{k\,ht}$ $\quad \lambda = \frac{a\,ht}{hx}$

For plot please see Fig. 2-a2-2, 2-a2-3, 2-a2-4.

If we set $a=1$ (we can always re-sale x ~~t~~ to get that),

For small hx, when $ht \to 0$ $(\lambda \to 0)$ $V_{ph} \to 1$ Fig 2-a2-2

For small ht, when $hx \to 0$ $V_{ph} \to 1$ Fig 2-a2-3

For small λ, when $hx \to 0$, $V_{ph} \to 1$ Fig 2-a2-4

2-a1 For First Order Upwind:

For dissipation, one can set a=1, since we can re-scale x and t to achieve that. That means V_{ph} is only related to h_t and h_x.

We first set h_t=0.01 to see what's going on if CFL is satisfied. As h_x goes to h_t, |s(k)|goes to 1. (Figure 2-a1-1). This is true since S(k)=exp(-ikh$_x$) when λ=1.

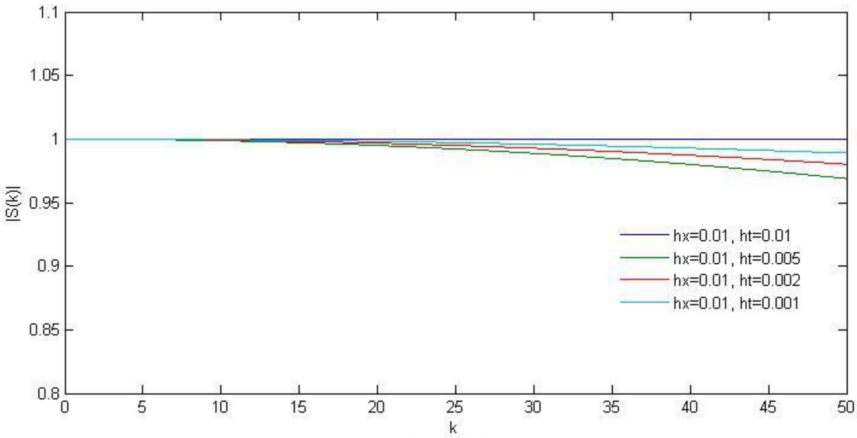


Figure 2-a1-1

We first set h_t=0.01 to see what's going on if CFL is not satisfied. As h_x goes bigger, |s(k)| blows up. (Figure 2-a1-2). This will bring unstability. (L=30.)

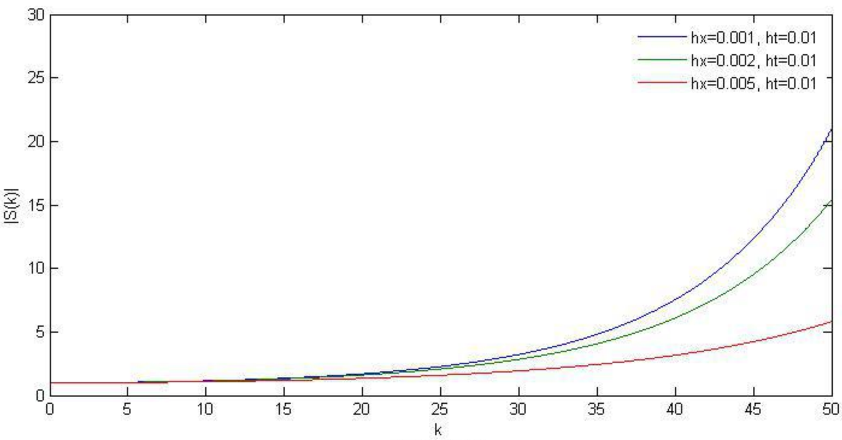


Figure 2-a1-2

For dispersion, if λ→1 or λ→0, V_{ph}→1. And λ→1 has a bigger effect in eliminating dispersion.

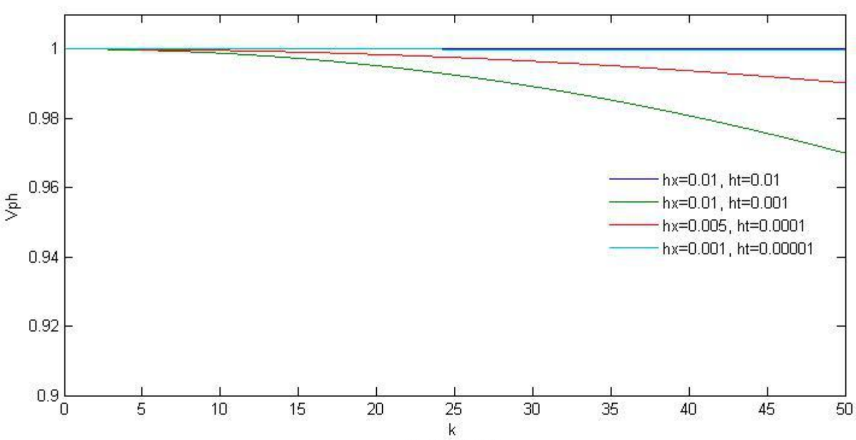


Figure 2-a1-3

2-a2 For Crank-Nicholson:

For dissipation, $|s(k)|=1$. So there is no dissipation in C-N. (Figure 2-a2-1)

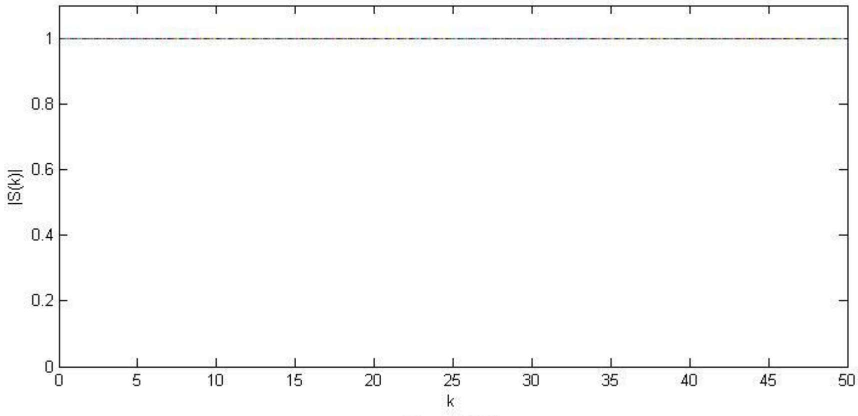


Figure 2-a2-1

For dispersion, again one can set a=1, since we can re-scale x and t to achieve that. That means V_{ph} is only related to h_t and h_x.

We first set h_t=0.01 to see what's going on. As h_x goes to 0, V_{ph} goes to 1. (Figure 2-a2-2)

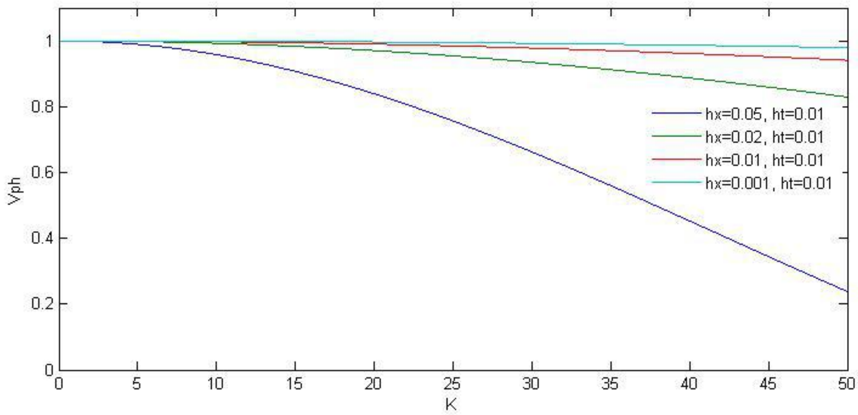


Figure 2-a2-2

We than set h_x=0.01 to see what's going on. As h_t goes to 0, V_{ph} goes to 1. (Figure 2-a2-3)

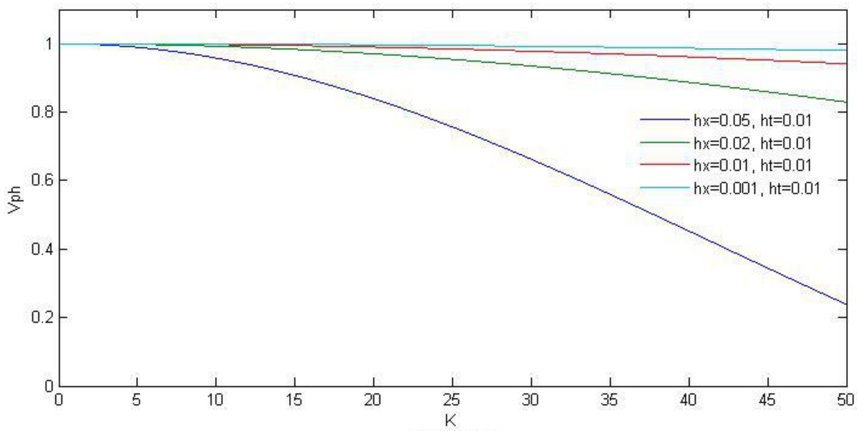


Figure 2-2a-3

Finally, we set λ=0.25(h_t=h_x) to see what's going on. As h_x goes to 0, V_{ph} goes to 1. (Figure 2-a2-4)

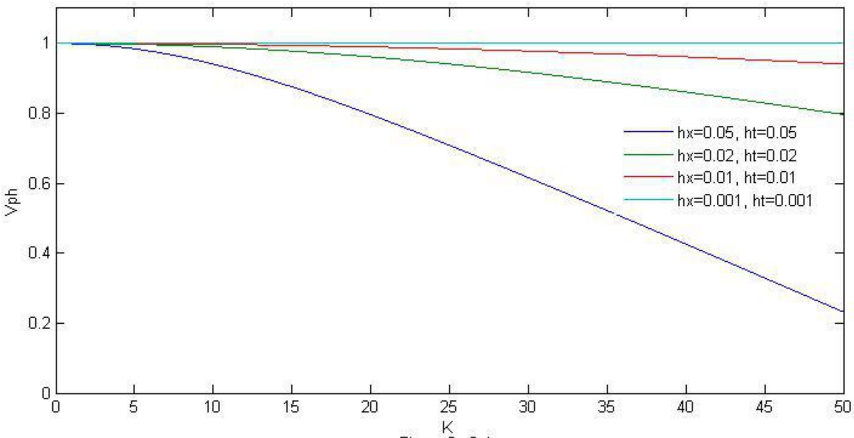
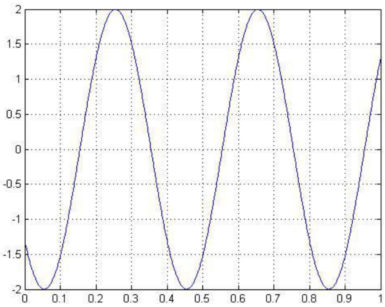
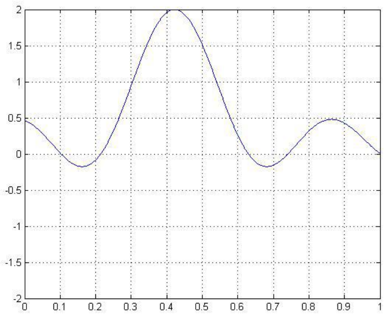
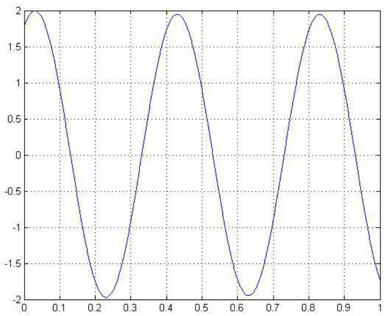
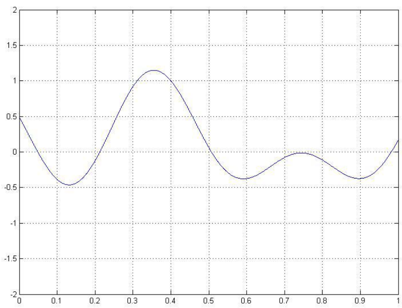
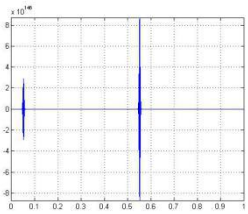
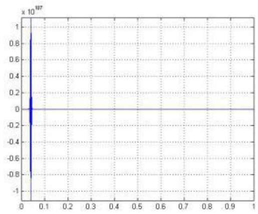
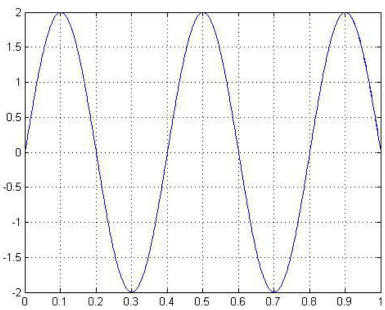
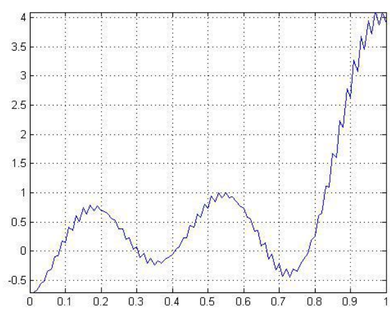


Figure 2-a2-4

Our conclusion here is for C-N, the dispersion always exists. To control it, we can try to make both h_x and h_t smaller. However, for functions decomposed into Fourier series with distinct small K_m and large K_n(i.e $K_n \to \infty$), we cannot avoid dispersion.

2-b Observation of Dissipation and Dispersion

	Dissipation	Dispersion
1st Upwind CFL λ=1	 2Cos(5πx) No dissipation	 0.5(cos(15x)+cos(x)+cos(5x)+cos(10x)) No Dispersion
1st Upwind CFL λ=0.5	 2Cos(5πx) Dissipation: The max is smaller than 2 on the right.	 0.5(cos(5πx)+cos(0.5πx)-cos(πx)-cos(3πx)); Dispersion: Small Wiggles Observed
1st Upwind non-CFL λ=10	Unstable  2Cos(5πx)	Unstable  0.5(cos(15x)+cos(x)+cos(5x)+cos(10x))
Crank-Nicholson	 2Cos(5πx) No dissipation	 0.5(cos(15x)+cos(x)+cos(5x)+cos(10x)) Dispersion