



Innovative Applications of O.R.

## Generalised soft binomial American real option pricing model (fuzzy–stochastic approach)

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### ARTICLE INFO

#### Article history:

Received 13 May 2009

Accepted 27 May 2010

Available online 31 May 2010

#### Keywords:

Finance

Pricing

Investment analysis

Fuzzy sets

Real options

Binomial model

### ABSTRACT

The stochastic discrete binomial models and continuous models are usually applied in option valuation. Valuation of the real American options is solved usually by the numerical procedures. Therefore, binomial model is suitable approach for appraising the options of American type. However, there is not in several situations especially in real option methodology application at disposal input data of required quality. Two aspects of input data uncertainty should be distinguished; risk (stochastic) and vagueness (fuzzy). Traditionally, input data are in a form of real (crisp) numbers or crisp-stochastic distribution function. Therefore, hybrid models, combination of risk and vagueness could be useful approach in option valuation. Generalised hybrid fuzzy–stochastic binomial American real option model under fuzzy numbers (T-numbers) and Decomposition principle is proposed and described. Input data (up index, down index, growth rate, initial underlying asset price, exercise price and risk-free rate) are in a form of fuzzy numbers and result, possibility-expected option value is also determined vaguely as a fuzzy set. Illustrative example of equity valuation as an American real call option is presented.

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### 1. Introduction

For valuation of financial derivatives both the discrete (binomial, trinomial) models and the analytical continuous ones are often used. For American option, exotic and real options with complicated payoff function and non-simple underlying stochastic processes including their dependences the numerical valuation models on the binomial basis are very useful and usually used. Real options topic is under consideration of academic and managerial community for several years, and it is the basic novelty of corporate finance. In energy and electricity sector real options are often applied as well. Basic papers and books concerning the real options applications represent for instance Black and Scholes (1973), Brennan and Schwartz (1985), McDonald and Siegel (1986), Kulatilaka (1993), Dixit and Pindyck (1994), Sick (1995), Smith and Nau (1995), Trigeorgis (1998), Brennan and Trigeorgis (1999), Bellalah (2001), Ronn (2002), Smit and Trigeorgis (2004), Howel et al. (2001), Brandao and Dyer (2005), Zmeskal (2008).

In the case of an option methodology application the estimation of an option value is determined by input data precision, especially concerning of validity, quality and availability of input data. There are also problems with data frequency and stochastic error.

Typical feature of financial environment is uncertainty. This term is understood mostly as risk uncertainty (probability, stochastic) and is modelled by stochastic apparatus. However, the term uncertainty has the second aspect, the vagueness (sometimes called imprecision, non-preciseness, ambiguity and softness) which is often neglected and could be modelled by a fuzzy methodology. In this respect it is apparent that the general term the uncertainty includes two aspects: risk (stochastic) and vagueness (fuzzy) ones. Even if terms introduced are not used uniquely, these terms will be used solely in the paper.

One of suitable approaches for solving the hybrid problems is to apply a fuzzy–stochastic methodology and create fuzzy–stochastic models. There are several references combining the fuzzy and stochastic processes and their aspects (see e.g. Dubois and Prade, 1980; Puri and Ralescu, 1986; Kruse and Meyer, 1987; Kacprzyk and Fedrizzi, 1988; Luhandjula, 1996; Wang and Qiao, 1993; Viertl, 1996; Ramík and Vlach, 2001; Sakawa et al., 2003; Luhandjula, 2006; Van Hop, 2007).

There occur in financial modelling both types of uncertainty and thus models are often depicted as hybrid models. We can generally distinguish and see two basic approaches in dealing with soft (fuzzy–stochastic) financial modelling. The first one concerns using a fuzzy

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<sup>1</sup> Previous drafts of the paper were presented on Euro Working Group on Financial Modelling meetings held in Brescia, Yogyakarta and Rotterdam.

measure in contrast with a probability measure where a sub-additive property is not fulfilled. The second approach is based on assumption that input data (parameters, distribution functions) is possible to introduce only vaguely. A survey of vaguely formulated problems in finance and accounting decision-making is in Siegel et al. (1995), and financial engineering in Riberio et al. (1999). Other financial applications examples, except option valuation models, are Lai and Hwang (1993), Inuiguchi and Ramik (2000), Tanaka et al. (2000), Cherubini and Lunga (2001), Carlsson et al. (2002), Zmeskal (2005), Koissi and Shapiro (2006), Xu and Kaymak (2008).

Special attention in hybrid (fuzzy–stochastic) financial modelling is paid to option valuations, because of providing abundant and a lot of applications. Researchers has dealt with relatively new topic of fuzzy–stochastic option valuation models which are presented, e.g., in Zmeskal (2001), Simonelli (2001), Yoshida (2002), Carlsson and Fuller (2003), Yoshida (2003), Muzzioli and Torricelli (2004), Wu (2005), Cheng et al. (2005), Yoshida et al. (2005), Liyan Han and Wenli Chen (2006), Zhang et al. (2006), Wu (2007), Muzzioli and Reynaerts (2007) and Thiagarajaha et al. (2007), Guerra et al. (2007), Chrysafis and Papadopoulos (2009).

We have declared that real option models are solved mostly by discrete American option methodology. Therefore hybrid (fuzzy–stochastic) binomial option models will be basically derived. We can find a few papers dealing with fuzzy binomial models methodology approaches. There is supposed a fuzzy volatility (see Yoshida, 2002; Muzzioli and Torricelli, 2004; Yoshida et al., 2005; Muzzioli and Reynaerts, 2007) or simultaneously fuzzy volatility and fuzzy risk-free rate (see Cheng et al., 2005).

The intention and goal of the paper is to propose and verify generalised hybrid (fuzzy–stochastic) binomial American real option model. We suppose in comparing with previous approaches published that all input parameters are given vaguely, fuzzy volatility (up index, down index), risk-free rate, growth rate, initial underlying asset price, exercise price. The reason for considering all input data defined as fuzzy is that in any situations and especially real option application (corporate finance, real assets) is sometimes difficult to give input data precisely, e.g. value of assets, debt, cost, risk-free rate, etc. The generalised fuzzy–stochastic methodology applied is based on replication strategy and follows especially an approach applying decomposition principle (resolution identity).

The paper is organised as follows: There is in the first section fuzzy–stochastic methodology described, next section includes description of a traditional crisp-stochastic binomial American real option model based on replication strategy, generalised fuzzy–stochastic binomial American option model is described in subsequent section, last section is devoted to application of proposed generalised fuzzy–stochastic American real option binomial model in valuation of company equity.

## 2. Fuzzy–stochastic methodology description under normal fuzzy sets

There are several approaches how to construct hybrid fuzzy–stochastic models. Following basic fuzzy–stochastic elements are very useful from an application point of view: (i) fuzzy set, (ii) normal and T-number sets, (iii) fuzzy-random variable, (iv)  $\varepsilon$ -cut, (v) extension principle, (vi) decomposition principle, (vii) fuzzy-probability function.

**Definition 1.** A fuzzy set (depicted with tilde) is commonly defined by a membership function ( $\mu$ ) as representation from  $E^n$  (Euclid  $n$ -dimensional space,  $n > 1$ ) to a set of  $E^1$  especially to the interval of  $[0; 1]$ ,  $\tilde{s} \equiv \mu_{\tilde{s}}(x)$ , where  $\tilde{s}$  is fuzzy set,  $x$  is vector and  $x \in X \subset E^n$ ,  $\mu_{\tilde{s}}(x)$  is membership function.

It is evident that many fuzzy sets could be created. The most common type of the fuzzy set meeting the specified preconditions of normality, convexity and continuity with upper semi-continuous membership function is very well known normal fuzzy set (see e.g. Dubois and Prade, 1980; Ramik and Vlach, 2001). The set of normal fuzzy sets are depicted  $F_M(E^n)$ . One of the most widely applied normal fuzzy set types is the  $T$ -number.

**Definition 2.** A fuzzy set meeting preconditions of normality, convexity, continuity with upper semi-continuous membership function and closeness and being defined as quadruple  $\tilde{s} = (s^L, s^U, s^\alpha, s^\beta)$ , where  $\phi(x)$  is a non-decreasing function and  $\psi(x)$  is a non-increasing function, as follows,

$$\tilde{s} \equiv \mu_{\tilde{s}}(x) = \begin{cases} 0 & \text{for } x \leq s^L - s^\alpha; & \phi(x) & \text{for } s^L - s^\alpha < x < s^L \\ 1 & \text{for } s^L \leq x \leq s^U; & \psi(x) & \text{for } s^U < x < s^U + s^\beta \\ 0 & \text{for } x \geq s^U + s^\beta \end{cases},$$

and is called the  $T$ -number. Let us denote the set of  $T$ -numbers by  $F_T(E)$ .

**Definition 3.** The  $\varepsilon$ -cut of the fuzzy set  $\tilde{s}$ , depicted  $\tilde{s}^\varepsilon$ , is defined as follows.  $\tilde{s}^\varepsilon = \{x \in E^n; \mu_{\tilde{s}}(x) \geq \varepsilon\} = [-s^\varepsilon, +s^\varepsilon]$ , where

$$-s^\varepsilon = \inf\{x \in E^n; \mu_{\tilde{s}}(x) \geq \varepsilon\}, \quad +s^\varepsilon = \sup\{x \in E^n; \mu_{\tilde{s}}(x) \geq \varepsilon\}.$$

The crucial category in the fuzzy–stochastic modelling is the fuzzy-random variable.

**Definition 4.** It is said,  $\tilde{s}: \Omega \rightarrow F_T(E)$  is the fuzzy-random variable (depicted with tilde and line), if for every  $w \in \Omega$  and  $\varepsilon \in ]0, 1]$ ,  $\tilde{s}_w^\varepsilon = \{x: x \in E^n, \tilde{s}_w \geq \varepsilon\} = [-s_w^\varepsilon, +s_w^\varepsilon]$ , is random interval (random  $\varepsilon$ -cut),  $-s_w^\varepsilon$ ,  $+s_w^\varepsilon$  are two random variables (or finite measurable functions). Let us denote the set of the fuzzy-random variables  $FR(\Omega, P)$  where  $P: \Omega \rightarrow [0, 1]$  and thus  $\tilde{s} \in FR(\Omega, P)$ .

From definition implies that,

- (a)  $\tilde{s} = \bigcup_{\varepsilon} \tilde{s}^\varepsilon$ , because  $\forall w \in \Omega$ ,  $\tilde{s}_w = \bigcup_{\varepsilon} \tilde{s}_w^\varepsilon$ , where  $\tilde{s}_w$  is fuzzy set and  $\tilde{s}_w^\varepsilon$  is random interval,
- (b)  $\tilde{s}$  is the fuzzy-random variable if  $\tilde{s}_w^\varepsilon$  is a random interval, it means  $\tilde{s} = \bigcup_{\varepsilon, w} \tilde{s}_w^\varepsilon$  for every  $w \in \Omega$  and  $\varepsilon \in ]0, 1]$ .

Very useful and powerful instrument that might be used for calculating a function of fuzzy sets is the extension principle (see Zadeh, 1965).

**Definition 5.** The extension principle is derived by the sup min composition between fuzzy sets  $\tilde{r}_1 \cdots \tilde{r}_n$  and  $\tilde{s} = f(\tilde{r}_1 \cdots \tilde{r}_n)$  as follows. Let  $f: E^n \rightarrow E^1$ , then the membership function of a function of fuzzy set  $\tilde{s} = f(\tilde{r}_1 \cdots \tilde{r}_n)$  is defined by

$$\mu_{\tilde{s}}(y) \equiv \tilde{s} = \sup_{y=f(x_1 \cdots x_n)} \min[\mu_{\tilde{r}_1}(x_1) \cdots \mu_{\tilde{r}_n}(x_n)], \quad x_i, y \in E^1.$$

In general conditions an analytic solution according to extension principle is not available. Assuming a fuzzy set is of fuzzy number type (the T-number type as well) there is possible to solve function of fuzzy numbers  $\tilde{s} = f(\tilde{r}_1 \cdots \tilde{r}_n)$  in accordance with the extension principle by decomposition principle as the approximate procedure of  $\varepsilon$ -cuts.

**Definition 6.** Decomposition principle (resolution identity) is defined as follows:

$$\mu_{\tilde{s}}(y) = \sup_{\varepsilon} \{ \varepsilon \cdot I_{\tilde{s}^\varepsilon}; y \in \tilde{s}^\varepsilon \} \quad \text{for any } y \in E^n \quad \text{and} \quad \varepsilon \in [0; 1],$$

where  $\tilde{s}^\varepsilon = [-s^\varepsilon, +s^\varepsilon]$  is  $\varepsilon$ -cut,  $y = f(x)$ ,  $-s^\varepsilon(x) = \min_{x \in \tilde{r}^\varepsilon \subset E^n} f(x)$ ,  $+s^\varepsilon(x) = \max_{x \in \tilde{r}^\varepsilon \subset E^n} f(x)$ . Here  $I_{\tilde{s}^\varepsilon}$  is characterisation function,  $I_{\tilde{s}^\varepsilon} = \begin{cases} 1 & \text{if } y \in [-s^\varepsilon, +s^\varepsilon] \\ 0 & \text{if } y \notin [-s^\varepsilon, +s^\varepsilon] \end{cases}$ .

Advantage of procedure can be seen in generalized application possibility, especially for convex, normal fuzzy numbers. Disadvantage consists in a computation difficulty.

The following definition is useful in accordance with Definitions 4 and 6 for evaluating and computing the functions of fuzzy-random variables and a constructing of the fuzzy-probabilities and the fuzzy-expected values.

**Definition 7.** Let us suppose, that  $\tilde{s} \in FR(\Omega, P)$ ,  $s_w^\varepsilon = [-s_w^\varepsilon, +s_w^\varepsilon]$ ,  $\varepsilon \in [0; 1]$ ,  $x \in E^n$ .

Let  $-g_w^\varepsilon$  (resp.  $+g_w^\varepsilon$ ) be a function of  $-s_w^\varepsilon$  (resp.  $+s_w^\varepsilon$ ). Then we call  $\tilde{g}(y) = \sup_{\varepsilon} \{ \varepsilon \cdot I_{\tilde{g}^\varepsilon}; y \in \tilde{g}^\varepsilon \}$  for any  $y \in E^n$  and  $\varepsilon \in [0; 1]$ , the fuzzy-probability function or fuzzy-expected function of  $\tilde{s}$ .

It is apparent that applying the Definition 6 the function of fuzzy numbers  $\tilde{s} = f(\tilde{r}_1 \cdots \tilde{r}_n)$  could be transformed and solved as several mathematical programming problems for  $\varepsilon$  in this way.

**Problem P1.**

$$\begin{aligned} & \max s = +s^\varepsilon, \quad \text{or} \quad \min s = -s^\varepsilon, \\ & \text{s.t.} \quad x_i \in [-x_i^\varepsilon, +x_i^\varepsilon] \quad \text{for } i \in \{1, 2, \dots, n\}, \quad \text{and} \quad \varepsilon \in [0; 1], \\ & \text{where} \quad s = f(x_1 \cdots x_n). \end{aligned}$$

**Definition 8.** Application of the Decomposition principle for function of fuzzy numbers allows expressing selected fuzzy operations  $\tilde{s} \tilde{r}$  among fuzzy numbers directly, as follows:

$$\tilde{w} = \tilde{s} \tilde{r} = \bigcup_{\varepsilon} \varepsilon(w^\varepsilon) = \bigcup_{\varepsilon} \varepsilon(s^\varepsilon * r^\varepsilon).$$

Fuzzy addition:  $s^\varepsilon + r^\varepsilon = [-s^\varepsilon + -r^\varepsilon, +s^\varepsilon + +r^\varepsilon]$ ,

Fuzzy subtract:  $s^\varepsilon - r^\varepsilon = [-s^\varepsilon - +r^\varepsilon, +s^\varepsilon - -r^\varepsilon]$ ,

Fuzzy scalar product:  $k \cdot s^\varepsilon = [k \cdot -s^\varepsilon, k \cdot +s^\varepsilon]$  for  $k \geq 0$ ,  $k \cdot s^\varepsilon = [k \cdot +s^\varepsilon, k \cdot -s^\varepsilon]$  for  $k < 0$ .

Fuzzy multiplication:  $s^\varepsilon \cdot r^\varepsilon = [-s^\varepsilon \cdot -r^\varepsilon, +s^\varepsilon \cdot +r^\varepsilon]$  for  $\tilde{s} > 0$ ,  $\tilde{r} > 0$ ,

$$s^\varepsilon \cdot r^\varepsilon = [-s^\varepsilon \cdot +r^\varepsilon, +s^\varepsilon \cdot -r^\varepsilon] \quad \text{for } \tilde{s} < 0, \quad \tilde{r} > 0, \quad s^\varepsilon \cdot r^\varepsilon = [+s^\varepsilon \cdot +r^\varepsilon, -s^\varepsilon \cdot -r^\varepsilon] \quad \text{for } \tilde{s} < 0, \quad \tilde{r} < 0.$$

Fuzzy division:  $s^\varepsilon : r^\varepsilon = [-s^\varepsilon : +r^\varepsilon, +s^\varepsilon : -r^\varepsilon]$  for  $\tilde{s} > 0$ ,  $\tilde{r} > 0$ ,  $s^\varepsilon : r^\varepsilon = [+s^\varepsilon : +r^\varepsilon, -s^\varepsilon : -r^\varepsilon]$  for  $\tilde{s} < 0$ ,  $\tilde{r} > 0$ ,  $s^\varepsilon : r^\varepsilon = [+s^\varepsilon : -r^\varepsilon, -s^\varepsilon : +r^\varepsilon]$  for  $\tilde{s} < 0$ ,  $\tilde{r} < 0$ .

Fuzzy max:  $\max(s^\varepsilon) = [\max -s^\varepsilon, \max +s^\varepsilon]$ .

Here  $\tilde{s} > 0$  is positive fuzzy number, positive  $\tilde{s} : \{x; \text{ for which } \mu_{\tilde{s}} \geq 0\}$  and simultaneously  $x \in E^+$  (set of positive numbers), negative  $\tilde{s} : \{x; \text{ for which } \mu_{\tilde{s}} \geq 0\}$  and simultaneously  $x \in E^-$  (set of negative numbers).

We are able to show, that result of fuzzy operation of addition, subtraction, and scalar product for linear T-numbers is also a linear T-number. For other operations it is not valid, but sometimes the approximation is applied such a way, that the result is also a linear T-number.

### 3. Binomial American option model on crisp-stochastic approach basis

One of the basic approaches of derivatives valuation under complete market is replication strategy. Having derived the replication strategy, we suppose a compact (effective) market, asset-bearing the incomes (dividends, coupons, etc.) proportional to an asset price. The replication strategy will be applied for the discrete binomial model and one risk (random) factor. The model is of discrete version and for the sake of simplicity an intra-interval continuous compounding is applied.

Replication strategy is based on creation a portfolio from underlying asset  $S$  and risk-free asset  $B$  so, for every situation to be replicated derivative value, it means a derivative value equals a portfolio value. Portfolio value in appraising a moment  $t$  is  $\Pi_t \equiv a \cdot S_t + B_t = f_t$ , portfolio value in a moment  $t + dt$  for growing price is  $\Pi_{t+dt} \equiv a \cdot S_{t+dt}^u + B_t \cdot e^{r \cdot dt} = f_{t+dt}^u$ , the portfolio value in a moment  $t + dt$  for declining price,  $\Pi_{t+dt} \equiv a \cdot S_{t+dt}^d + B_t \cdot e^{r \cdot dt} = f_{t+dt}^d$ . Here  $\Pi_t$  is portfolio value,  $S$  is underlying asset value,  $a$  is an amount of underlying asset,  $B$  is risk-free asset

value,  $f$  is derivative value,  $r$  is risk-free rate,  $u(d)$  are indexes of up (down) movement of underlying asset,  $S_{t+dt}^u(S_{t+dt}^d)$  are their prices in up-movements (down-movements).

By solution of three equations for variables  $a, B, f_t$ , we can get a general formula for derivative price,

$$f_t = e^{-r \cdot dt} \cdot \left\{ f_{t+dt}^u \cdot \left[ \frac{e^{r \cdot dt} \cdot S_t - S_{t+dt}^d}{S_{t+dt}^u - S_{t+dt}^d} \right] + f_{t+dt}^d \cdot \left[ \frac{S_{t+dt}^u - e^{r \cdot dt} \cdot S_t}{S_{t+dt}^u - S_{t+dt}^d} \right] \right\}.$$

This is the general formula for derivative price valuation by the replication strategy, which should be written as follows:

$$f_t = e^{-r \cdot dt} \cdot [f_{t+dt}^u \cdot (p) + f_{t+dt}^d \cdot (1-p)], \quad \text{or} \quad f_t = e^{-r \cdot dt} \cdot E(f_{t+dt}).$$

Here  $p = \frac{e^{r \cdot dt} \cdot S_t - S_{t+dt}^d}{S_{t+dt}^u - S_{t+dt}^d}$  implies the risk-neutral probability of up-movement and  $E(f_{t+dt})$  is the risk-neutral expected value. This probability can be considered neither a market growth nor a subjective probability. The derivative price is equal to the present value of risk-neutral expected value of subsequent period, which coincides with generalised martingale principle (see Harrison and Kreps, 1979).

There are several ways how to calibrate the generalised model (see e.g. Cox et al. (1979); Jarrow and Rudd, 1983; Boyle, 1988; Boyle et al., 1989; Madan et al., 1989; Kamrad and Ritchken, 1991; Trigeorgis, 1991; Kulatilaka, 1993; Smith and Nau, 1995; Luenberger, 1998). Applying the approach of Cox et al. (1979) we can express the underlying asset price, under the proportional continuous income  $c$ , due to Geometric Brown's process as follows,  $S_{t+dt}^u = S_t \cdot e^{u+c \cdot dt}$ ;  $S_{t+dt}^d = S_t \cdot e^{d+c \cdot dt}$ . Because of  $e^u = e^{\sigma \cdot \sqrt{dt}}$ ,  $e^d = e^{-\sigma \cdot \sqrt{dt}}$ , then after substitution and after re-arranging we get a particular risk-neutral probability formula  $p = \frac{e^{(r-c) \cdot dt} - e^d}{e^u - e^d}$  and  $S_{t+dt}^u = S_t \cdot e^u$ ;  $S_{t+dt}^d = S_t \cdot e^d$ . These formulas can be after substituting for  $U = e^u = e^{\sigma \cdot \sqrt{dt}}$ ,  $D = e^d = e^{-\sigma \cdot \sqrt{dt}}$ ,  $e^{(r-c) \cdot dt} = \frac{1+R}{1+C}$  denoted as follows  $p = \frac{\frac{1+R}{1+C} - D}{U - D}$ ,  $S_{t+dt}^u = S_t \cdot U$ ,  $S_{t+dt}^d = S_t \cdot D$ .

### 3.1. Valuation procedure of an American option

Option pricing using binomial model can be divided into following steps.

- Modelling an evolution of underlying asset in accordance with the observed volatility as follows,  $S_{t+dt}^u = S_t \cdot U$ ;  $S_{t+dt}^d = S_t \cdot D$ .
- Computation of intrinsic value (payoff function)  $g$ . For example, in the case of a call option  $g_t^u = \max(S_t^u - X; 0)$ , and a put option  $g_t^d = \max(X - S_t^d; 0)$ , where  $X$  is exercise price.
- At maturity day  $T$ , option price is equal to intrinsic value,  $f_T^u = g_T^u$  or  $f_T^d = g_T^d$ .
- Having working backwards, from the end to the beginning of the binomial tree, the American option can be exercised whenever during pre-specified period and its price can be written  $f_t = \max(g_t; F_t)$ , where  $F_t = (1+R)^{-dt} \cdot [f_{t+dt}^u \cdot (p) + f_{t+dt}^d \cdot (1-p)]$  and  $p = \frac{\frac{1+R}{1+C} - D}{U - D}$ . This equation is Bellman dynamic programming optimal equation.
- Price of option  $f_0$  is at the beginning of whole period.

## 4. Generalised fuzzy-stochastic binomial American option model

We suppose a binomial model based on replication strategy with fuzzy variables, initial underlying asset  $\tilde{S}$ , exercise price  $\tilde{X}$ , fuzzy indexes  $\tilde{U}$ ,  $\tilde{D}$ , risk-free rate  $\tilde{R}$  and growth (dividend) rate  $\tilde{C}$ , consequently fuzzy-probabilities  $\tilde{p}$ ,  $\tilde{q}$  are derived and computed. Fuzzy sets are of fuzzy number type, see Definition 2, Decomposition principle due to Definition 6 is applied, and operations between fuzzy numbers are used according to Definition 8 as well. Backward procedure in coincidence with the replication strategy is applied as follows,

$$\tilde{f}_t = \max(\tilde{g}_t; \tilde{F}_t), \text{ here payoff function is } \tilde{g}_t = \max(\tilde{S}_t \sim \tilde{X}; 0), \\ \tilde{F}_t = (1+\tilde{R})^{-dt} \cdot [\tilde{f}_{t+dt}^u \cdot \tilde{p} + \tilde{f}_{t+dt}^d \cdot (1-\tilde{p})] \text{ and } \tilde{p} = [(1+\tilde{R}) \cdot \tilde{S}_t \sim \tilde{S}_{t+dt}^d] \cdot [\tilde{S}_{t+dt}^u \sim \tilde{S}_{t+dt}^d].$$

### 4.1. Fuzzy-stochastic procedure of a binomial American option

Application of the soft approach is based on Decomposition principle. Therefore, following procedure is represented by  $\varepsilon$ -cuts and includes following steps.

- Modelling an evolution of soft underlying asset in accordance with observed fuzzy volatility and fuzzy initial underlying asset price applying fuzzy operations as follows,  $-S_{t+dt}^{eu} = -S_t^e \cdot -U^e$ ;  $+S_{t+dt}^{eu} = +S_t^e \cdot +U^e$ ;  $-S_{t+dt}^{ed} = -S_t^e \cdot -D^e$ ;  $+S_{t+dt}^{ed} = +S_t^e \cdot +D^e$ .
- Computation of intrinsic value (payoff function) depicted  $g$ . For example, for call option,  $-g_t^{ue} = \max(-S_t^{ue} - +X^e; 0)$ ,  $+g_t^{ue} = \max(+S_t^{ue} - -X^e; 0)$ ,  $-g_t^{de} = \max(-S_t^{de} - +X^e; 0)$ ,  $+g_t^{de} = \max(+S_t^{de} - -X^e; 0)$  and where  $X$  is exercise price.
- At maturity day  $T$  option price is equal to intrinsic value,  $-f_T^{ue} = -g_T^{ue}$ ,  $+f_T^{ue} = +g_T^{ue}$ , or  $-f_T^{de} = -g_T^{de}$ ,  $+f_T^{de} = +g_T^{de}$ .
- Having working backwards, from the end to the beginning of the binomial tree, the American option can be exercised whenever during pre-specified period and its price can be written  $-f_t^e = \max(-g_t^e, -F_t^e)$ ,  $+f_t^e = \max(+g_t^e, +F_t^e)$ , where variables  $+F_t^e$ ,  $-F_t^e$  are computed by following mathematical programming problem.

### Problem P2

$$\begin{aligned} \max F_t = +F_t^e, \quad \text{or} \quad \min F_t = -F_t^e, \quad \text{for } \varepsilon \in [0; 1], \\ \text{s.t. } p \in [0; 1], \quad q \in [0; 1], \quad p + q = 1, \quad \left( \frac{1+R}{1+C} \right) \cdot S_t - S_{t+dt}^d \geq 0, \quad S_{t+dt}^u - S_{t+dt}^d \geq 0, \\ S_t \in [-S_t^e, +S_t^e], \quad S_{t+dt}^d \in [-S_{t+dt}^{de}, +S_{t+dt}^{de}], \quad S_{t+dt}^u \in [-S_{t+dt}^{ue}, +S_{t+dt}^{ue}], \quad f_{t+dt}^d \in [-f_{t+dt}^{de}, +f_{t+dt}^{de}], \\ f_{t+dt}^u \in [-f_{t+dt}^{ue}, +f_{t+dt}^{ue}], \quad R \in [-R^e, +R^e], \quad C \in [-C^e, +C^e], \end{aligned}$$

where

$$F_t = (1 + R)^{-dt} \cdot [f_{t+dt}^u \cdot (p) + f_{t+dt}^d \cdot (q)], p = \left[ \frac{\left( \frac{1+R}{1+C} \right) \cdot S_t - S_{t+dt}^d}{S_{t+dt}^u - S_{t+dt}^d} \right].$$

- Soft value of an option at the beginning of the period thus at the initial node is  $\tilde{f}_0^e = [-f_0^e, +f_0^e]$ .

### 5. Example of equity value determination as American call option (fuzzy–stochastic approach)

We consider for the sake of simplicity as real option application a calculation of company equity value without dividend payment under soft conditions. Problem is formulated as an American option solved by a binomial option model applying fuzzy–stochastic methodology. We are using the methodology and procedure described in Chapter 4. Option value calculation is following,  $\tilde{f}_t = (\tilde{g}_t; \tilde{F}_t)$ , here payoff function is  $\tilde{g}_t = \max(\tilde{S}_t - \tilde{X}; 0)$ ,  $\tilde{F}_t = (1 + \tilde{R})^{-dt} \cdot [\tilde{f}_{t+dt}^u \cdot \tilde{p} + \tilde{f}_{t+dt}^d \cdot (1 - \tilde{p})]$ , and  $\tilde{p} = [(1 + \tilde{R}) \cdot \tilde{S}_t - \tilde{S}_{t+dt}^d] / [\tilde{S}_{t+dt}^u - \tilde{S}_{t+dt}^d]$ .

Fuzzy input data are in a form of T-numbers  $\tilde{x} = [x^L, x^U, x^Z, x^B]$  and represented by this way  $\tilde{x} = [-x^0, -x^1, +x^1, +x^0]$ , where  $-x^0 = x^L - x^Z$ ,  $-x^1 = x^L$ ,  $+x^1 = x^U$ ,  $+x^0 = x^U + x^B$ . In the applied example we suppose that following variables are given vaguely: company initial (underlying) asset value  $\tilde{S}$ , nominal debt value (exercise price)  $\tilde{X}$ , up-movement index  $\tilde{U}$ , down-movement index  $\tilde{D}$  and risk-free rate  $\tilde{R}$ . Applying real option method, it is sometimes difficult to know input asset value, debt value as crisp values and determination by fuzzy numbers is more realistic. Similarly, risk-free rate is useful considered as fuzzy and volatility as well. We are considering for analysing vagueness three variants: variant A with all fuzzy input data, variant B supposing crisp risk-free rate and other fuzzy input data, variant C with all input crisp data. Table 1 shows the input fuzzy data and crisp data.

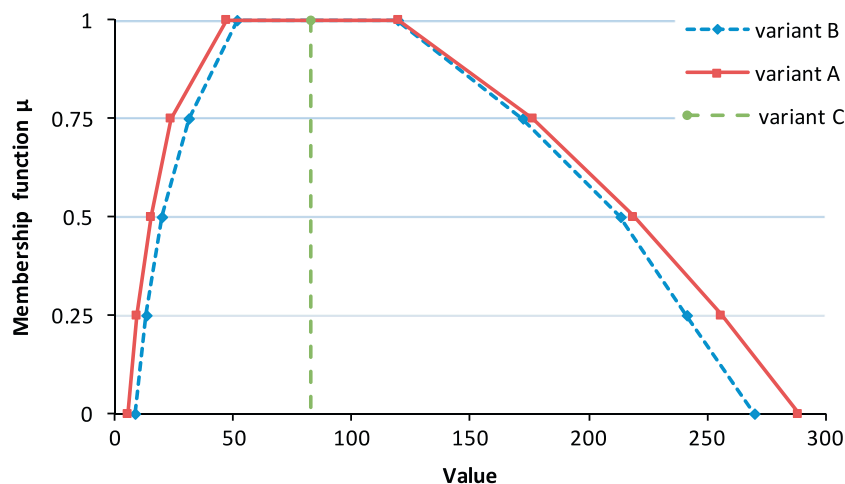
Computation procedure of the fuzzy underlying asset value  $\tilde{S}$ , and fuzzy option value  $\tilde{f}_t$  for limiting  $\varepsilon$ -cuts ( $\varepsilon = 1, \varepsilon = 0$ ) is shown in Appendix A. The results in  $\varepsilon$ -cut form for variant A, variant B and variant C is presented in Table 2 and Graph A1.

**Table 1**  
Input data of fuzzy–stochastic equity option valuation models.

Number type	Symbol	Actual asset value $\tilde{S}$	Debt $\tilde{X}$	Up index $\tilde{U}$	Down index $\tilde{D}$	Risk-free rate $\tilde{R}$
T-number	$-x^0$	990	1095	1.08	0.892857	0.03
	$-x^1$	1000	1098	1.09	0.909091	0.04
	$+x^1$	1000	1100	1.1	0.917431	0.05
	$+x^0$	1010	1105	1.12	0.925926	0.06
Crisp number	$x$	1000	1100	1.1	0.917431	0.05

**Table 2**  
Calculated equity (option) values results in  $\varepsilon$ -cuts.

$\varepsilon$ -cut	Variant A		Variant B		Variant C
	$-\tilde{f}_0^A$	$+\tilde{f}_0^A$	$-\tilde{f}_0^B$	$+\tilde{f}_0^B$	
1	47.00321	119.2532	51.27713	119.2532	82.51072
0.75	23.03663	176.3915	29.05214	172.0812	0
0.5	15.33285	218.9135	18.68931	218.8741	0
0.25	9.033978	255.829	11.26894	251.6295	0
0	5.124421	288.4715	6.879291	282.9507	0



**Graph A1.** Graphical representation of resulting equity value.

It is apparent that a company equity (option) values presented in Table 1 are  $\tilde{f}_0^A = [5.124421; 47.00321; 119.2532; 288.4715]$ ,  $\tilde{f}_0^B = [6.879291; 51.27713; 119.2532; 282.9507]$  and  $f_0^C = 82.51072$ . The results should be considered to be a soft American call option value respecting input data uncertainty, both risk and vagueness.

The result can be used for financial analysis, sensitivity analysis and for decision-making and interpreted in several ways. Firstly, fuzzy result can give information about value and decision-making space depending on input data preciousness. Secondly, for stated aspiration level, given by  $\varepsilon$ -cut representation, a financial analyst can found relevant value interval. Thirdly, applying some of known defuzzification methods (e.g. centre of gravity area, first of maxima, last of maxima, mean of maxima, centre of maxima (median), centroid method, bisector method) we can gain one crisp value for decision-making.

## 6. Conclusion

The purpose of the paper was to propose and verify generalised binomial American option valuation model under vagueness conditions on the fuzzy–stochastic basis. Generalised hybrid soft binomial option model on replication strategy basis was explained, methodology described and illustrative example presented.

We have supposed that all input parameters are given vaguely in a form of fuzzy numbers: initial underlying asset value, exercise price, up-movement index, down-movement index, risk-free rate, growth (dividend) rate. It was shown, that under fuzzy numbers and fuzzy-random variables applying decomposition (resolution) principle, problem should be transformed and solved by several mathematical programming problems, see Problem P1 or Problem P2. Three input data variants were computed and results were in a form of fuzzy number as well. Possibilities of interpretation and application in decision-making were outlined and discussed.

An option and real option application is one of crucial topic in financial modelling and decision-making. Conditions and input data availability determine undoubtedly approaches and models applied. There is in some situations usage of hybrid fuzzy–stochastic models and thus development and verifying these ones useful. We can see it especially in real options applications, because the input data are real company assets, debts, cash flow, cost and are possible to give sometimes only vaguely. Certainly, we cannot expect better and more precise estimation. However, model can better coincide with soft input data and decision-making conditions. Vagueness of results reflects vagueness of objectives and input data of model. Therefore, successful application of the particular models depends on decision-making circumstances, financial analyst objectives and input data availability and preciousness. The described generalised approach and model

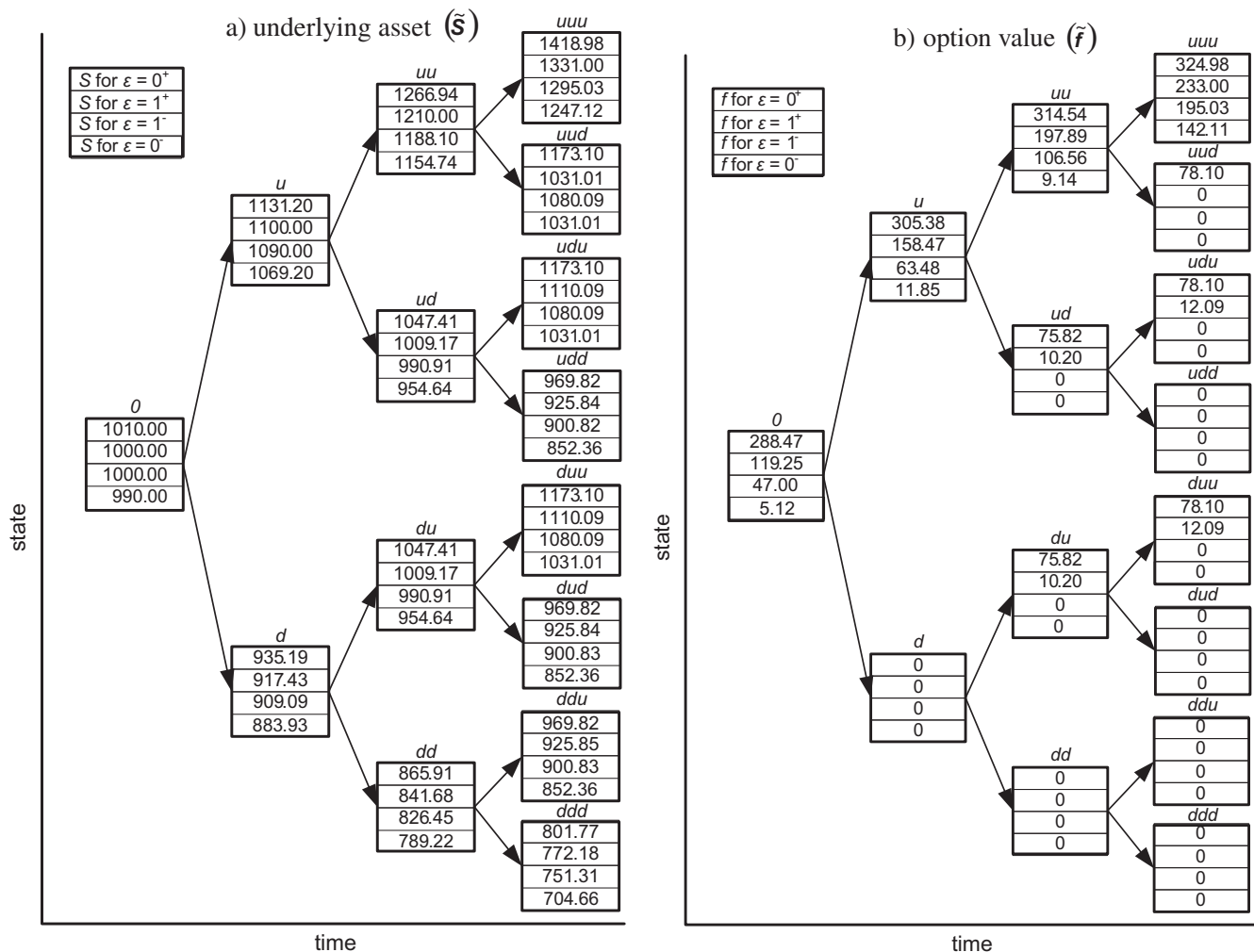


Fig. A1. Calculation procedure of fuzzy–stochastic binomial model for  $\varepsilon$ -cuts,  $\varepsilon = 1, \varepsilon = 0$



might thus be considered to be a generalised sensitivity analysis instrument of real option value determination under contingent claim conditions and non-precise input data.

## Acknowledgements

The paper is supported by the research project GACR 402/08/1234. This support is gratefully acknowledged.

## Appendix A

See Fig. A1.

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