MATH 226, HOMEWORK 4, DUE DEC. 11, 2015

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For problems that include programming, please include the code and all outputted figures and tables. Please label these clearly and refer to them appropriately in your answers to the questions.

(1) Derive an explicit linear 2-step method of order 3 and discuss its consistency, stability, and convergence. Recall 2-step Explicit Method has the following form:

$$y_{n+1} = -a_0 y_n - a_1 y_{n-1} + h(b_0 f_n + b_1 f_{n-1})$$

Now if we want it to be consistent under order 3, we have the following 4 equation corresponding to m=0,1,2,3:

$$\begin{aligned} 1 + a_0 + a_1 &= 0 \\ 1 - a_1 - b_0 + b_1 &= 0 \\ 1 + a_1 - 2b_1 &= 0 \\ 1 - a_1 - 3b_1 &= 0 \end{aligned}$$

When we solve the equation, we get:

$$\begin{aligned} a_0 &= 4 \\ a_1 &= -5 \\ b_0 &= 4 \\ b_1 &= 2 \end{aligned}$$

So the 2-step order 3 method we get is $y_{n+1} = -4y_n + 5y_{n-1} + h(4f_n + 2f_{n-1})$ and it is consistent. However with root condition, we know its characteristic function is $\xi^2 + 4\xi - 5 = 0$, which means $\xi_1 = 1, \xi_2 = -5$. Thus $|\xi_2| > 1$, meaning it is not stable and convergent.

(2) Derive the most general two-stage Runge-Kutta method of order 2.
We use Taylor expansion for y_{n+1}: (Note $f_n = f(t_n, y_n)$)

$$\begin{aligned} y_{n+1} &= y_n + hy_n^{'} + \frac{1}{2}h^2 y_n^{''} + o(h^3) \\ &= y_n + hf_n + \frac{1}{2}h^2[\frac{\partial f}{\partial t}_n + \frac{\partial f}{\partial y}_n f_n] + o(h^3) \quad \bigotimes \end{aligned}$$

If we consider the R-K method $y_{n+1} = y_n + h\omega_1 K_1 + h\omega_2 K_2$ with $K_1 = f_n$, $K_2 =$

$f(t_n + hc_2, f_n + h\alpha_{21}K_1) = f_n + hc_2\frac{\partial f}{\partial t}_n + h\alpha_{21}\frac{\partial f}{\partial y}f_n + o(h^3)$ and set $c_1 = 0$

$$\begin{aligned} y_{n+1} &= y_n + h\omega_1 f_n + h\omega_2[f_n + hc_2\frac{\partial f}{\partial t}_n + h\alpha_{21}\frac{\partial f}{\partial y}f_n] + o(h^3) \\ &= y_n + h(\omega_1+\omega_2)f_n + h^2[\omega_2 c_2\frac{\partial f}{\partial t}_n + \omega_2\alpha_{21}\frac{\partial f}{\partial y}_n f_n] + o(h^3) \end{aligned}$$

Let's set $\alpha_{21} = \alpha$ and compare the equation with $\otimes$ above. We will have:

$$\begin{aligned} &\omega_1 + \omega_2 = 1 \\ &\omega_2 c_2 = \frac{1}{2} \\ &\omega_2 \alpha_{21} = \frac{1}{2} \end{aligned}$$

Thus,

$$\begin{aligned} &\omega_1 = 1 - \frac{1}{2\alpha} \\ &\omega_2 = \frac{1}{2\alpha} \\ &c_2 = \alpha \end{aligned}$$

Therefore, we get:$y_{n+1} = y_n + h[(1-\frac{1}{2\alpha})K_1 + \frac{1}{2\alpha}K_2]$, where $K_1 = f_n = f(t_n, y_n)$, $K_2 = f(t_n + \alpha h, y_n + \alpha h K_1)$

(3) Implement the forward Euler method, backward Euler method, Crank-Nicolson method, and Runge-Kutta 4-stage (RK4) method and apply them to solve the following ODE problem, respectively.

$$\begin{cases} y'(t) = 6y(t) - 6y(t)^2, \quad t \in (0, 20] \\ y(0) = 0.5 \end{cases}$$

Using $h = 0.5,\ 0.25,\ 0.1$ to compute the approximate solution of each method. Plot the approximate solutions on the time interval $[0, 20]$ and explain your observations from the numerical experiments.
From the observation we can conclude that:

Table 3-1 Relationship between Method and Convergence under certain h

Method	Forward Euler	Backward Euler	C-N	RK4
$h = 0.50$	F	F	F	TF
$h = 0.25$	T	F	T	T
$h = 0.10$	T	T	T	T

T: converge to right solution F: do not converge TF:converge to wrong solution

Here we notice that for both implicit methods (Backward Euler and C-N)(Figure3.2 and 3.3), they do not converge. It seems to be a contradiction to what we learned in class. However, if we check the iteration, the iteration blows up. So we can conclude that the problem lies in iteration, not in method. Take Backward Euler as an Example: When we try to use iteration, the condition for contraction function is $\Delta y_k = y_{k+1} - y_k \epsilon [1 - \frac{1}{6h}, 1 + \frac{1}{6h}]$, when h becomes small enough, it will converge(Here, y_k is the result of kth iteration). When h is bigger, y_k may diverges or converge locally

For explicit method (RK4 and Forward Euler), they don't blow up when h=0.5 in the domain, but do not converge or do converge incorrectly. The reason should lies in their convergent condition, which means $h < C$ for some constant C.

For all methods, when h becomes smaller, the figure is smoother.