MATH 226, HOMEWORK 1, DUE OCT. 2, 2015

For problems that include programming, please include the code and all outputted figures and tables. Please label these clearly and refer to them appropriately in your answers to the questions.

(1) Consider the Bernstein polynomials, for $f \in C(I)$, $I = [0,1]$, $n = 1, 2, \cdots$,

$$B_n f(x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}$$

(a) Show that the operator $B_n : C(I) \mapsto \mathcal{P}_n(I)$ is linear, where $\mathcal{P}_n(I)$ is the space of polynomial functions of degree at most n on I.

(b) Show that B_n is a positive operator, i.e. if $f(x) \geq 0$ on I, then $B_n f(x) \geq 0$ on I.

(c) Let $f_0 = 1$, $f_1 = x$, and $f_2 = x^2$, show that $B_n f_0 = f_0$, $B_n f_1 = f_1$, and $B_n f_2 = \dfrac{n-1}{n} f_2 + \dfrac{1}{n} f_1$.

(2) Prove the Chebyshev Alternation Theorem.

(3) Derive the error formula for the Hermite interpolation p_{2n+1} on I, i.e.

$$f(x) - p_{2n+1}(x) = \frac{1}{(2n+2)!} f^{(2n+2)}(\xi) \left[(x-x_0)(x-x_1)\cdots(x-x_n)\right]^2,$$

where $\xi \in I$ is a point determined by the $x, x_0, \cdots, x_n$.

(4) Let $I = [-1,1]$ and $\mathcal{P}_n(I)$ be the space of polynomial functions of degree at most n on I. For any $q \in \mathcal{P}_n(I)$, show that $\|q\|_\infty \leq K(n)\|q\|_2$ with $K(n) = \dfrac{n+1}{\sqrt{2}}$.

(5) Consider the function $f(x) = \dfrac{1}{1+x^2}$. Write codes to find the piecewise linear polynomial interpolation and clamped cubic spline to approximate $f(x)$ on $[-5,5]$ with equally distributed points of mesh size h. Plot the approximations and observe how the errors change when the mesh size h decreases.

(6) **(Optional)** If you are only allowed to use no more than 200 points in $[-5,5]$ to approximate $f(x) = \dfrac{1}{1+x^2}$ by the piecewise linear polynomial interpolation, design a strategy to distribute the points and make the error in L_∞-norm as small as possible. Implement and report your error.