Assignment #4

1 (a) $f(x,y) = x^2 - y^2$ subject to $x^2+y^2 = 4$.

Introducing Lagrange's multiplier λ, we get, a new function,

$$L(x,y,\lambda) = x^2 - y^2 - \lambda(x^2+y^2-4)$$

Now, partially differentiating this function, we get,

$$\frac{\partial L}{\partial x} = 2x - 2x\lambda = 0 \quad -(i)$$

$$\frac{\partial L}{\partial y} = -2y - 2y\lambda = 0 \quad -(ii)$$

From (i), we get,

$$x(1-\lambda) = 0$$

So, either, $x=0$ or $(1-\lambda) = 0$ or $\lambda = 1$

From (ii), we get,

$$y(1+\lambda) = 0$$

$$y = 0 \text{ or } \lambda = -1.$$

$x=0$ and $y=0$ cannot be a soln as it doesn't satisfy the constraint $x^2+y^2=4$.

~~When $\lambda=1$, putting $\lambda=1$ in (ii), we get~~

When $x = 0$, $0+y^2 = 4$ or $y = \pm 2$.

$$\left.\begin{array}{l} x=0,\ y=2 \\ x=0,\ y=-2 \end{array}\right\} \text{are solutions.}$$

When $y = 0$, $x = \pm 2$

$$\text{So, } \left.\begin{array}{l} x=2,\ y=0 \\ x=-2,\ y=0 \end{array}\right\} \text{are solutions}$$

Hence, we get 4 set of solutions.

$$\left.\boxed{\begin{matrix} x=0, y=2 \\ x=0, y=-2 \\ x=2, y=0 \\ x=-2, y=0 \end{matrix}}\right\} \text{Ans}$$

All the 4 solutions satisfy the constraint $x^2+y^2=4$.

Also $f(0,2)$ and $f(0,-2) = -4 \Leftarrow$ minima

and $f(2,0)$ and $f(-2,0) = 4 \Leftarrow$ maxima.

Hence, the above 4 sets of x and y are the extrema of the function.

(b) $f(x,y) = x^2y - \log x$ subject to $x+2y=0$

Introducing Lagrange's multiplier, we get a new function,

$$\phi(x,y) = x^2y - \log x - \lambda(x+2y)$$

Now, differentiating partially w.r.t x and y, we get,

$$\frac{\partial\phi(x,y)}{\partial x} = 0 \text{ or } \frac{\partial}{\partial x}\left(x^2y - \log x - \lambda(x+2y)\right) = 0$$

$$\text{or } 2xy - \frac{1}{x} - \lambda = 0$$

$$\text{or } 2x^2y - 1 - \lambda x = 0. \quad \text{—(i)}$$

$$\frac{\partial\phi(x,y)}{\partial y} = 0 \text{ or } \frac{\partial}{\partial y}\left(x^2y - \log x - \lambda(x+2y)\right) = 0$$

$$\text{or } x^2 - 2\lambda = 0$$

$$\text{or } x = \pm\sqrt{2\lambda}$$

For $x^2 = \cancel{\text{}} 2\lambda$, $(x = \pm 2\lambda)$ we can plugin this value of x in eqn (i) and get the below :-

$$4\lambda y - \lambda x - 1 = 0 \quad \text{— (ii)}$$

or $\lambda(4y - \sqrt{2\lambda}) = 1$

or $4\lambda y - \sqrt{2\lambda}\cdot\lambda = 1$

or $y = \dfrac{1 + \sqrt{2\lambda}\cdot\lambda}{4\lambda}$

Now, we replace these values of x & y in the constraint.

$x + 2y = 0$

or $\sqrt{2\lambda} + \cancel{2} \times \left(\dfrac{1 + \sqrt{2\lambda}\cdot\lambda}{{}_2\cancel{4}\lambda}\right) = 0$

or $2\cdot\lambda\cdot\sqrt{2\lambda} + 1 + \sqrt{2\lambda}\cdot\lambda = 0$

or $3\lambda\cdot\sqrt{2\lambda} = -1$

or $\lambda\cdot\sqrt{2\lambda} = -\dfrac{1}{3}$

or $\lambda^2 \times 2\lambda = \dfrac{1}{9}$

or $\lambda^3 = \dfrac{1}{18}$ or $\lambda = \sqrt[3]{\dfrac{1}{18}}$.

$$x = +\sqrt{2\lambda} = +\sqrt{2 \times \left(\frac{1}{18}\right)^{1/3}} = \sqrt{2 \times \frac{1}{3^{2/3} \times 2^{1/3}}}$$

$$= \sqrt{\left(\frac{2}{3}\right)^{2/3}}$$

$$= \left(\frac{2}{3}\right)^{\cancel{2}/3 \times \frac{1}{\cancel{2}}}$$

$$= \left(\frac{2}{3}\right)^{1/3}.$$

$y = \dfrac{1 + x\lambda}{} = +\text{ve}$.

For $x = \left(\frac{2}{3}\right)^{1/3}$, y is always +ve which means

the constraint $x+2y=0$ can't be satisf
for $x=+ve$ and $y=+ve$.

Now, let's calculate y for $x=-\sqrt{2\lambda}$.

$$4\lambda y-\lambda x-1=0 \text{ from equation (ii)}$$

For $x=-\sqrt{2\lambda}$.

$$4\lambda y+\sqrt{2\lambda}\cdot\lambda-1=0$$

$$\text{or } y=\frac{1-\sqrt{2\lambda}\cdot\lambda}{4\lambda}.$$

Now, putting this value in $x+2y=0$

$$-\sqrt{2\lambda}+\cancel{2}\times\left(\frac{1-\sqrt{2\lambda}\cdot\lambda}{\cancel{4}2\lambda}\right)=0$$

$$\text{or } -\sqrt{2\lambda}\cdot 2\lambda+1-\sqrt{2\lambda}\cdot\lambda=0$$

$$\text{or } \cancel{+}3\lambda\cdot\sqrt{2\lambda}=\cancel{+}1$$

$$\text{or } \lambda\cdot\sqrt{2\lambda}=\frac{1}{3}$$

$$\text{or } \lambda^2\times 2\lambda=\frac{1}{9}$$

$$\text{or } \lambda^3=\frac{1}{18}$$

$$\text{or } \lambda=\sqrt[3]{\frac{1}{18}}$$

$$x=-\sqrt{2\lambda}=-\sqrt[3]{\frac{2}{3}}=-\left(\frac{2}{3}\right)^{1/3}$$

$$y=\frac{1+x\lambda}{4\lambda}=\frac{1-\left(\frac{2}{3}\right)^{1/3}\times\left(\frac{1}{18}\right)^{1/3}}{4\times\left(\frac{1}{18}\right)^{1/3}}$$

$$=\frac{1-\left(\frac{\cancel{2}}{3}\right)^{\cancel{1/3}}\times\frac{1}{(3)^{2/3}\times\cancel{(2)^{1/3}}}}{2\times\ldots}$$

$$= \frac{1 - \frac{1}{3}}{\frac{2^2}{3^{2/3} \times 2^{1/3}}}$$

$$= \frac{2}{3} \times \frac{3^{2/3} \times 2^{1/3}}{2^{\cancel{2}}}$$

$$= \frac{2}{3} \cdot \frac{1}{2^{\left(1-\frac{1}{3}\right)} \times 3^{\left(1-\frac{2}{3}\right)}}$$

$$= \frac{1}{(2)^{2/3} \times (3)^{\frac{1}{3}}}$$

$$= \frac{1}{2} \times \frac{2}{(2)^{2/3} \times (3)^{1/3}}$$

$$= \frac{1}{2} \times \left(\frac{2}{3}\right)^{1/3}.$$

So, $x = -\left(\frac{2}{3}\right)^{1/3}$ and $y = \frac{1}{2} \times \left(\frac{2}{3}\right)^{1/3}$ are the solutions. supposedly.

Check :- These solutions satisfy the const.

$x + 2y = 0$

$$-\left(\frac{2}{3}\right)^{1/3} + \cancel{2} \times \frac{1}{\cancel{2}} \times \left(\frac{2}{3}\right)^{1/3} = 0$$

However, $x = -\left(\frac{2}{3}\right)^{1/3}$ which is $-$ve and in $f(x,y)$, $\log(x)$ is not defined for this value of x. (negative values)

So, $\boxed{\text{No soln}}$ exists for this function.

(e) $f(x,y) = x^2+2xy+y^2-2x$ subject to $x^2-y^2=-1$.

Introducing Lagrange's multiplier, the function now becomes,

$$\phi(x,y) = x^2+2xy+y^2-2x-\lambda(x^2-y^2+1)$$

Partially differentiating $\phi(x,y)$ wrt x & y, we get the following :-

$$\frac{\partial\phi(x,y)}{\partial x} = 2x+2y-2-2x\lambda = 0$$

or $x+y-1-x\lambda=0$

or $x(1-\lambda)+y = 1$ — (i)

$$\frac{\partial\phi(x,y)}{\partial y} = 2x+2y+2y\lambda = 0$$

or $x+y+\lambda y = 0$

or $-y(1+\lambda) = x$ — (ii)

Putting this value of x in (i), we get

$-y(1+\lambda)(1-\lambda)+y = 1$

or $-y(1-\lambda^2)+y = 1$

or $\cancel{-y}+y\lambda^2\cancel{+y} = 1$

or $\boxed{\lambda^2 = \frac{1}{y}}$ or $\boxed{y = \frac{1}{\lambda^2}}$

or $\cancel{\lambda = \pm\frac{1}{y}}$

Putting this value of y in eqn (ii), we get :-

$$= -\frac{1}{\lambda^2}(1+\lambda) = -\frac{(1+\lambda)}{\lambda^2}.$$

Putting this value of x in the constraint $x^2 - y^2 = -1$, we get

$$\frac{(1+\lambda)^2}{\lambda^4} - \frac{1}{\lambda^4} = -1$$

or $$\frac{\cancel{1}+\lambda^2+2\lambda-\cancel{1}}{\lambda^4} = -1$$

~~or~~ $$\cancel{\frac{\lambda(\lambda+2)}{\lambda^{4}3}} = \cancel{-1}$$

~~or~~ $$\cancel{\frac{\lambda+2}{\lambda^3}} = \cancel{-1}$$

or $$\lambda^4+\lambda^2+2\lambda = 0$$

or $$\lambda(\lambda^3+\lambda+2) = 0$$

either $\lambda = 0$ or $(\lambda^3+\lambda+2) = 0$

or $\lambda^3+\lambda+2 = 0$ — (iii)

(-1) is a root of the eqn (iii)

or $\lambda^3+\lambda^2-\lambda^2-\lambda+2\lambda+2 = 0$

or $\lambda^2(\lambda+1)-\lambda(\lambda+1)+2(\lambda+1) = 0$

or $(\lambda+1)(\lambda^2-\lambda+2) = 0 \rightarrow$ This equation has complex roots.

or $(\lambda+1)\,\cancel{(\lambda^2-2\lambda+\lambda}$

So, $\lambda = -1$.

For $\lambda = -1$, $y = 1/\lambda^2 = 1$

$x = -\frac{(1+\lambda)}{\lambda^2} = 0$

So, $\boxed{x=0,\ y=1}$ is a soln of this eqn.

It satisfies constraint $x^2-y^2 = 0-1$

$= -1$

$f(x,y) = 0+2+1 = 3.$

So, $\boxed{x=0,\ y=1} \rightarrow$ Ans.