Problem 3

(a) Consider the scalar function

$$\varphi(x) = \begin{cases} x & \text{when } x \geq 0 \\ +\infty & \text{when } x < 0. \end{cases}$$

Find the expression for the following proximal operator

$$S_\lambda(y) \triangleq \mathrm{prox}_{\lambda\varphi}(y) = \arg\min_{x\in\mathbb{R}} \left\{ \frac{1}{2}(x-y)^2 + \lambda\varphi(x) \right\}, \quad y \in \mathbb{R}.$$

(b) Show the following closed-form expression for the proximal operator

$$\mathrm{prox}_{\lambda\|\cdot\|}(\boldsymbol{y}) = \arg\min_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ \frac{1}{2}\|\boldsymbol{x}-\boldsymbol{y}\|^2 + \lambda\|\boldsymbol{x}\| \right\} = \begin{cases} (\|\boldsymbol{y}\|-\lambda)_+ \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} & \text{when } \boldsymbol{y} \neq 0 \\ \boldsymbol{0} & \text{when } \boldsymbol{y} = 0. \end{cases},$$

where $(x)_+ \triangleq \max(x, 0)$ extracts the positive part of its input.

Hint: Consider the proximal operator of $g(\boldsymbol{x}) = \varphi(\|\boldsymbol{x}\|)$, $\boldsymbol{x} \in \mathbb{R}^n$, where φ is given in (a).

From part (a) we know

$$S_\lambda(y) = (y-\lambda)_+ = \begin{cases} y-\lambda & \text{if } y \geq \lambda \\ 0 & \text{if } y < \lambda. \end{cases}$$

(b) Define $g(\vec{x}) = \varphi(\|\vec{x}\|)$

$\because \|\vec{x}\| \geq 0 \Rightarrow \varphi(\|\vec{x}\|) = \|\vec{x}\| \Rightarrow g(\vec{x}) = \varphi(\|\vec{x}\|) = \|\vec{x}\|.$

$\therefore \mathrm{Prox}_{\lambda\|\cdot\|}(\vec{y}) = \mathrm{Prox}_{\lambda \cdot g}(\vec{y}) \mathrel{.=} \arg\min_{\vec{x}\in\mathbb{R}^n} \left\{ \frac{1}{2}\|\vec{x}-\vec{y}\|^2 + \lambda g(\vec{x}) \right\}.$

Let's consider the following two cases:

① if $\vec{y} = \vec{0}$

$$\mathrm{Prox}_{\lambda\cdot\|\cdot\|}(\vec{y}) = \mathrm{Prox}_{\lambda\cdot\|\cdot\|}(\vec{0}) = \arg\min_{x\in\mathbb{R}^n} \left\{ \frac{1}{2}\|\vec{x}-\vec{0}\|^2 + \lambda\|x\| \right\}.$$

$$= \arg\min_{x\in\mathbb{R}^n} \left\{ \frac{1}{2}\|\vec{x}\|^2 + \lambda\varphi(\|\vec{x}\|) \right\} \mathrel{.=} x^*.$$

$(t = \|\vec{x}\|)$ $\Rightarrow \min_{x\in\mathbb{R}^n} \left\{ \frac{1}{2}\|\vec{x}\|^2 + \lambda\varphi(\|\vec{x}\|) \right\} = \min_{t\in\mathbb{R}} \left\{ \frac{1}{2}t^2 + \lambda\varphi(t) \right\}$

$\Rightarrow t^* = \|x^*\|$

From part(a).

We know $t^* = \arg\min_{t\in\mathbb{R}^n} \left\{ \frac{1}{2}t^2 + \lambda\varphi(t) \right\} = \mathrm{Prox}_{\lambda\varphi}(0) = 0 \Rightarrow x^* = 0.$

② if $\vec{y} \neq 0$,

$$\text{Prox}_{\lambda\|\cdot\|}(\vec{y}) = \underset{x \in \mathbb{R}^n}{\operatorname{argmin}} \left\{ \frac{1}{2} \|\vec{x} - \vec{y}\|^2 + \lambda \varphi(\|\vec{x}\|) \right\}.$$

Let's look at this minimization problem:

$$\min_{\vec{x} \in \mathbb{R}^n} \left\{ \frac{1}{2} \|\vec{x} - \vec{y}\|^2 + \lambda \varphi(\|\vec{x}\|) \right\}.$$

$$= \min_{\vec{x} \in \mathbb{R}^n} \left\{ \frac{1}{2} \|\vec{x}\|^2 - \vec{x}^T \cdot \vec{y} + \frac{1}{2} \|\vec{y}\|^2 + \lambda \varphi(\|\vec{x}\|) \right\}$$

$$= \min_{t \in \mathbb{R}} \left\{ \min_{\vec{x} : \|\vec{x}\| = t} \left\{ \frac{1}{2} t^2 + \lambda \varphi(t) - \vec{x}^T \vec{y} + \frac{1}{2} \|\vec{y}\|^2 \right\}. \right\}.$$

Using Cauchy-Schwarz $\Rightarrow x^* = t^* \frac{\vec{y}}{\|\vec{y}\|}$

with $t^* = \underset{t \in \mathbb{R}}{\operatorname{argmin}} \left\{ \frac{1}{2} t^2 + \lambda \varphi(t) - t\|\vec{y}\| + \frac{1}{2} \|\vec{y}\|^2 \right\}.$

$$= \underset{t \in \mathbb{R}}{\operatorname{argmin}} \left\{ \frac{1}{2} (t - \|\vec{y}\|)^2 + \lambda \varphi(t) \right\}. = \text{Prox}_{\lambda \cdot \varphi}(\|\vec{y}\|)$$

$$= (\|\vec{y}\| - \lambda)_+$$

$$\Rightarrow x^* = t^* \frac{\vec{y}}{\|\vec{y}\|} = (\|\vec{y}\| - \lambda)_+ \frac{\vec{y}}{\|\vec{y}\|}$$

Combining ① ② $\Rightarrow$ $\text{Prox}_{\lambda\|\cdot\|}(\vec{y}) = \begin{cases} 0 & \text{if } y = 0. \\ (\|\vec{y}\| - \lambda)_+ \frac{\vec{y}}{\|\vec{y}\|} & \text{if } y \neq 0 \end{cases}$

Problem 4

Let $g \in \Gamma^0(\mathbb{R}^n)$, and consider $\theta \neq 0$ and $\boldsymbol{z} \in \mathbb{R}^n$.

(a) Show that for $h(\boldsymbol{x}) = g(\theta \boldsymbol{x} + \boldsymbol{z})$, we have

$$\text{prox}_h(\boldsymbol{x}) = \frac{1}{\theta}\left(\text{prox}_{\theta^2 g}(\theta \boldsymbol{x} + \boldsymbol{z}) - \boldsymbol{z}\right).$$

(b) Show that for $r(\boldsymbol{x}) = \theta g(\boldsymbol{x}/\theta)$, we have

$$\text{prox}_r(\boldsymbol{x}) = \theta \text{prox}_{g/\theta}(\boldsymbol{x}/\theta).$$

Solution:

(a). $h(x) = g(\theta x + z)$.

$$\text{Prox}_h(x) = \underset{y \in \mathbb{R}^n}{\text{argmin}} \left\{ \frac{1}{2}\|y - x\|^2 + h(y) \right\}.$$

$$= \underset{y \in \mathbb{R}^n}{\text{argmin}} \left\{ \frac{1}{2}\|y - x\|^2 + g(\theta y + z) \right\}$$

$$= y^*.$$

set $v = \theta y + z \Rightarrow y = \frac{1}{\theta}(v - z)$

$$= \frac{1}{\theta}\left\{ \underbrace{\underset{v \in \mathbb{R}^n}{\text{argmin}} \left\{ \frac{1}{2}\left\|\frac{1}{\theta}(v - z) - x\right\|^2 + g(v) \right\}}_{\triangleq\, v^*} - z \right\}.$$

$$\Rightarrow y^* = \frac{1}{\theta}(v^* - z).$$

$$v^* = \underset{v}{\text{argmin}} \left\{ \frac{1}{2}\left\|\frac{1}{\theta}(v - z) - x\right\|^2 + g(v) \right\}.$$

$$= \underset{v}{\text{argmin}}\ \frac{1}{\theta^2}\left\{ \frac{1}{2}\|(v - z) - \theta x\|^2 + \theta^2 g(v) \right\}.$$

$$= \underset{v}{\text{argmin}} \left\{ \frac{1}{2}\|v - (\theta x + z)\|^2 + \theta^2 g(v) \right\}.$$

$$\Rightarrow v^* = \text{Prox}_{\theta^2 g}(\theta x + z)$$

$$\Rightarrow y^* = \frac{1}{\theta}(v^* - z) = \frac{1}{\theta}\left(\text{Prox}_{\theta^2 g}(\theta x + z) - z\right)$$