ESE415 Optimization

Lecture 20 : Sensitivity vectors

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Our goal today

$$(P) \qquad \text{minimize} \;\; f(\boldsymbol{x}) \;\text{subject to}\; \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} \;\text{where}\; \boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}^n.$$

$$f^* = \inf_{\boldsymbol{x} \in \mathcal{G}} \{f(\boldsymbol{x})\} \quad \text{where} \quad \mathcal{G} = \{\boldsymbol{x} \in \mathcal{X} : \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}\}$$

$$\boldsymbol{g}(\boldsymbol{x}) : \begin{cases} g_1(\boldsymbol{x}) \leq 0 \\ g_2(\boldsymbol{x}) \leq 0 \\ g_3(\boldsymbol{x}) \leq 0 \end{cases}$$

Sensitivity Vector

$$(P(\boldsymbol{z})) \qquad \text{minimize} \;\; f(\boldsymbol{x}) \;\text{subject to}\; \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{z} \;\text{where}\; \boldsymbol{x} \in \mathcal{X} \subseteq \mathbb{R}^n,$$

$$f_{\mathrm{p}}^*(\boldsymbol{z}) = \inf_{\boldsymbol{x} \in \mathcal{G}(\boldsymbol{z})} \{f(\boldsymbol{x})\} \quad \text{where} \quad \mathcal{G}(\boldsymbol{z}) = \{\boldsymbol{x} \in \mathcal{X} : \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{z}\}.$$

$$\boldsymbol{g}(\boldsymbol{x}) : \begin{cases} g_1(\boldsymbol{x}) \leq z_1 \\ g_2(\boldsymbol{x}) \leq z_2 \\ g_3(\boldsymbol{x}) \leq z_3 \end{cases}$$

A function

Definition of the domain of the function f_P^*(z)

Definition 1. We define the *domain* of the function $f_p^*(\boldsymbol{z})$ as the set

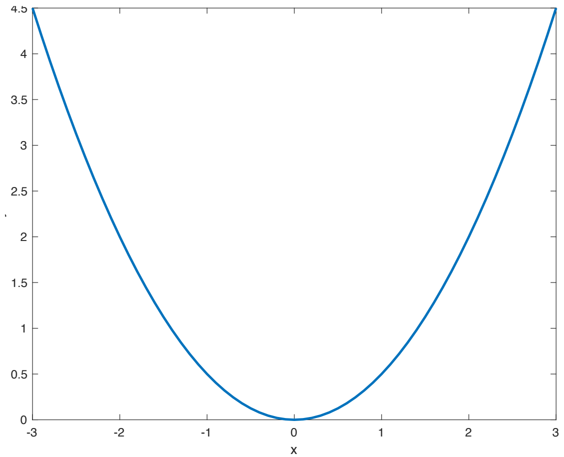
$$\text{dom}\{f_p^*\} \triangleq \{\boldsymbol{z} \in \mathbb{R}^m : \mathcal{G}(\boldsymbol{z}) \neq \varnothing\}.$$

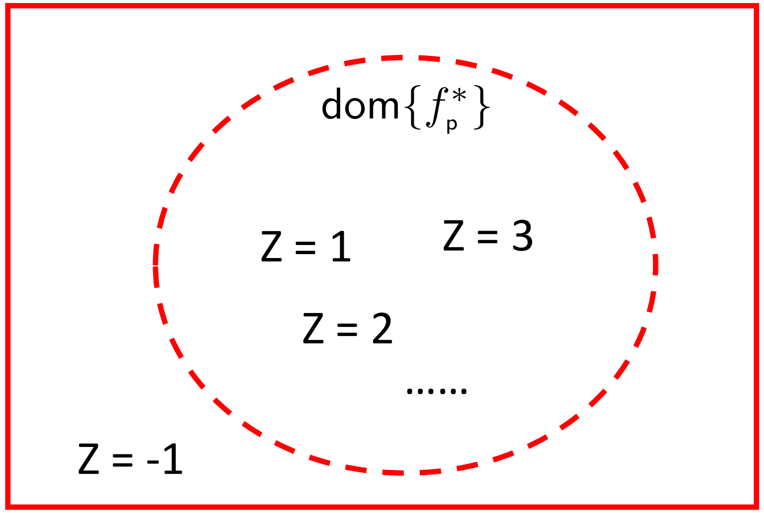
This means that the domain is the set of all vectors for which the feasibility region $\mathcal{G}(\boldsymbol{z})$ is not empty.

A set that contains all **reasonable constrains z**

$$\mathcal{G}(\boldsymbol{z}) = \{\boldsymbol{x} \in \mathcal{X} : \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{z}\}$$

$\text{g}(\text{x}) = 0.5\boldsymbol{x}^2$





1. $f_p^*(\boldsymbol{z}) = -\infty$
2. f_p^* is a finite real number

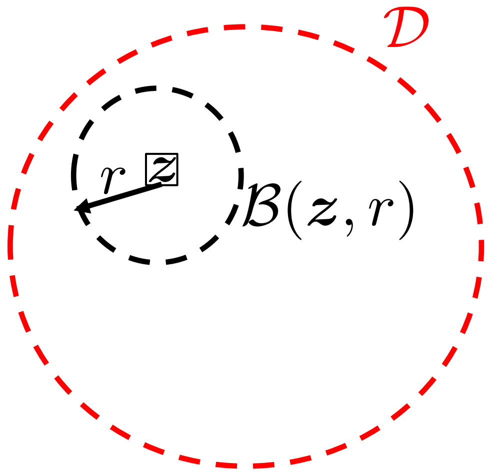
Definition of interior point

Before stating our first theorem, we remind the definition of an interior point.

Definition 2. A point $\boldsymbol{z} \in \mathbb{R}^m$ is an *interior point* of $\mathcal{D}$ if

$$\exists r > 0 \quad \text{s.t.} \quad \mathcal{B}(\boldsymbol{z}, r) \subset \mathcal{D},$$

which means that there is an open ball that is fully contained in $\mathcal{D}$.



(a) Establish the convexity of function f_P^* and its domain

Our first result establishes the convexity of the function f_{p}^* and of its domain

Theorem 1. We have the following two results:

a) If $(P(\boldsymbol{z}))$ is the perturbation of a convex program (P) by $\boldsymbol{z} \in \mathbb{R}^m$, then $f_{\mathrm{p}}^*(\boldsymbol{z})$ is a convex function and $\mathrm{dom}\{f_{\mathrm{p}}^*\} \subseteq \mathbb{R}^m$ is a convex set.

b) If (P) is superconsistent, then $\mathbf{0}$ is an interior point of $\mathrm{dom}\{f_{\mathrm{p}}^*\}$.

- (a) Reminder of the definitions:

(1)

$$f_{\mathrm{p}}^*(\boldsymbol{z}) = \inf_{\boldsymbol{x} \in \mathcal{G}(\boldsymbol{z})} \{f(\boldsymbol{x})\} \quad \text{where} \quad \mathcal{G}(\boldsymbol{z}) = \{\boldsymbol{x} \in \mathcal{X} : \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{z}\}.$$

(2)

Definition 1. We define the *domain* of the function $f_{\mathrm{p}}^*(\boldsymbol{z})$ as the set

$$\mathrm{dom}\{f_{\mathrm{p}}^*\} \triangleq \{\boldsymbol{z} \in \mathbb{R}^m : \mathcal{G}(\boldsymbol{z}) \neq \varnothing\}.$$

This means that the domain is the set of all vectors for which the feasibility region $\mathcal{G}(\boldsymbol{z})$ is not empty.

(b) Proof that **0** is a interior point of domain of f_P^*

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Theorem 1. We have the following two results:

a) If $(P(\boldsymbol{z}))$ is the perturbation of a convex program (P) by $\boldsymbol{z} \in \mathbb{R}^m$, then $f_\mathrm{p}^*(\boldsymbol{z})$ is a convex function and $\mathrm{dom}\{f_\mathrm{p}^*\} \subseteq \mathbb{R}^m$ is a convex set.

b) If (P) is superconsistent, then $\mathbf{0}$ is an interior point of $\mathrm{dom}\{f_\mathrm{p}^*\}$.

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(2)

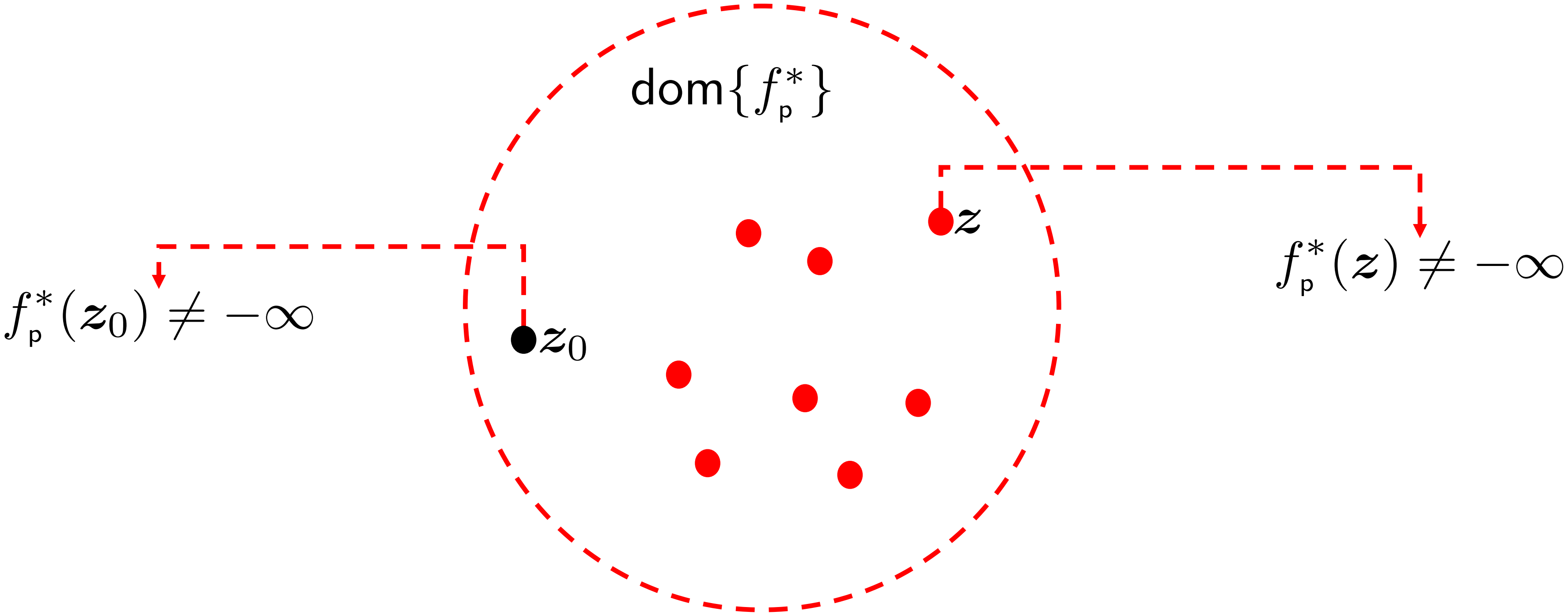
Definition 2. A point $\boldsymbol{z} \in \mathbb{R}^m$ is an *interior point* of $\mathcal{D}$ if

$$\exists r > 0 \quad \text{s.t.} \quad \mathcal{B}(\boldsymbol{z}, r) \subset \mathcal{D},$$

which means that there is an open ball that is fully contained in $\mathcal{D}$.

One constrain finite, everywhere in the domain finite

Theorem 2. If f_{p}^* is finite at any interior point z_0 of $\mathrm{dom}\{f_{\mathrm{p}}^*\}$, then f_{p}^* is finite on the entire domain.



(1) Introduction and the existence of the sensitivity vector

Theorem 3. If (P) is a superconsistent convex program such that $f^* = f_{\mathrm{p}}^*(\mathbf{0})$ is finite, then f_{p}^* is finite on the entire $\mathrm{dom}\{f_{\mathrm{p}}^*\}$. Additionally,

$$\exists \boldsymbol{\nu} \in \mathbb{R}_+^m \quad \text{such that} \quad f_{\mathrm{p}}^*(\boldsymbol{z}) \geq f_{\mathrm{p}}^*(\mathbf{0}) - \boldsymbol{\nu}^{\mathsf{T}}\boldsymbol{z}, \quad \forall \boldsymbol{z} \in \mathrm{dom}\{f_{\mathrm{p}}^*\}.$$

(1)

Theorem 1. We have the following two results:

a) If $(P(\boldsymbol{z}))$ is the perturbation of a convex program (P) by $\boldsymbol{z} \in \mathbb{R}^m$, then $f_{\mathrm{p}}^*(\boldsymbol{z})$ is a convex function and $\mathrm{dom}\{f_{\mathrm{p}}^*\} \subseteq \mathbb{R}^m$ is a convex set.

b) If (P) is superconsistent, then $\mathbf{0}$ is an interior point of $\mathrm{dom}\{f_{\mathrm{p}}^*\}$.

Theorem 2. If f_{p}^* is finite at any interior point $\boldsymbol{z}_0$ of $\mathrm{dom}\{f_{\mathrm{p}}^*\}$, then f_{p}^* is finite on the entire domain.

(2) Introduction and the existence of the sensitivity vector

Theorem 3. If (P) is a superconsistent convex program such that $f^* = f_\mathrm{p}^*(\mathbf{0})$ is finite, then f_p^* is finite on the entire $\mathrm{dom}\{f_\mathrm{p}^*\}$. Additionally,

$$\exists \boldsymbol{\nu} \in \mathbb{R}_+^m \quad \text{such that} \quad f_\mathrm{p}^*(\boldsymbol{z}) \geq f_\mathrm{p}^*(\mathbf{0}) - \boldsymbol{\nu}^\mathsf{T}\boldsymbol{z}, \quad \forall \boldsymbol{z} \in \mathrm{dom}\{f_\mathrm{p}^*\}.$$

(2)

Definition 3. We define a *sensitivity vector* of a convex program (P) with a finite f^* as a vector

$$\boldsymbol{\nu} \in \mathbb{R}_+^m \quad \text{such that} \quad f_\mathrm{p}^*(\boldsymbol{z}) \geq f^* - \boldsymbol{\nu}^\mathsf{T}\boldsymbol{z}, \quad \forall \boldsymbol{z} \in \mathrm{dom}\{f_\mathrm{p}^*\}.$$

Remark. Superconsistent convex programs with finite infima *always* have sensitivity vectors.

Example shows the meaning of of sensitivity vectors

Example. Suppose that for a convex program with three constraints

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) \\ &\text{subject to } g_1(\boldsymbol{x}) \le 0,\ g_2(\boldsymbol{x}) \le 0,\ g_3(\boldsymbol{x}) \le 0, \end{aligned}$$

we find a sensitivity vector $\boldsymbol{\nu} = (100, 1, 0)$. Then for $\boldsymbol{z} = (z_1, z_2, z_3) \in \mathrm{dom}\{f_{\mathrm{p}}^*\}$, we have

$$f_{\mathrm{p}}^*(\boldsymbol{z}) \ge f_{\mathrm{p}}^*(\boldsymbol{0}) - \boldsymbol{\nu}^{\mathsf{T}}\boldsymbol{z} = f^* - 100z_1 - z_2. \quad (1)$$

$$\boldsymbol{g}(\boldsymbol{x}) : \begin{cases} g_1(\boldsymbol{x}) \le z_1 \\ g_2(\boldsymbol{x}) \le z_2 \\ g_3(\boldsymbol{x}) \le z_3 \end{cases} \quad \boldsymbol{z}$$

$$\boldsymbol{z} = [0, 0, 0]^T$$

$$\boldsymbol{g}(\boldsymbol{x}) : \begin{cases} g_1(\boldsymbol{x}) \le 0 \\ g_2(\boldsymbol{x}) \le 0 \\ g_3(\boldsymbol{x}) \le 0 \end{cases}$$

$$f^*$$

$$\boldsymbol{z} = [1, 0, 0]^T$$

$$\boldsymbol{g}(\boldsymbol{x}) : \begin{cases} g_1(\boldsymbol{x}) \le 1 \\ g_2(\boldsymbol{x}) \le 0 \\ g_3(\boldsymbol{x}) \le 0 \end{cases}$$

$$f_{\mathrm{p}}^*(1, 0, 0) \ge f^* - 100$$

$$\boldsymbol{z} = [0, 1, 0]^T$$

$$\boldsymbol{g}(\boldsymbol{x}) : \begin{cases} g_1(\boldsymbol{x}) \le 0 \\ g_2(\boldsymbol{x}) \le 1 \\ g_3(\boldsymbol{x}) \le 0 \end{cases}$$

$$f_{\mathrm{p}}^*(1, 0, 0) \ge f^* - 1$$

$$\boldsymbol{z} = [0, 0, 1]^T$$

$$\boldsymbol{g}(\boldsymbol{x}) : \begin{cases} g_1(\boldsymbol{x}) \le 0 \\ g_2(\boldsymbol{x}) \le 0 \\ g_3(\boldsymbol{x}) \le 1 \end{cases}$$

$$f_{\mathrm{p}}^*(1, 0, 0) \ge f^*$$

Example illustrates that f_{p}^* may not be differentiable at $\boldsymbol{z} = \mathbf{0}$.

Remark. Note that if f_{p}^* is differentiable at $\boldsymbol{z} = \mathbf{0}$, then the sensitivity vector can be taken to be $\boldsymbol{\nu} = -\nabla f_{\mathrm{p}}^*(\mathbf{0})$. However, as we shall see f_{p}^* may fail to be differentiable at $\boldsymbol{z} = \mathbf{0}$.

Example. Consider the following convex program (P)

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) = \sqrt{x_1^2 + x_2^2} \\ &\text{subject to } g(\boldsymbol{x}) = x_1 + x_2 \leq 0. \end{aligned}$$

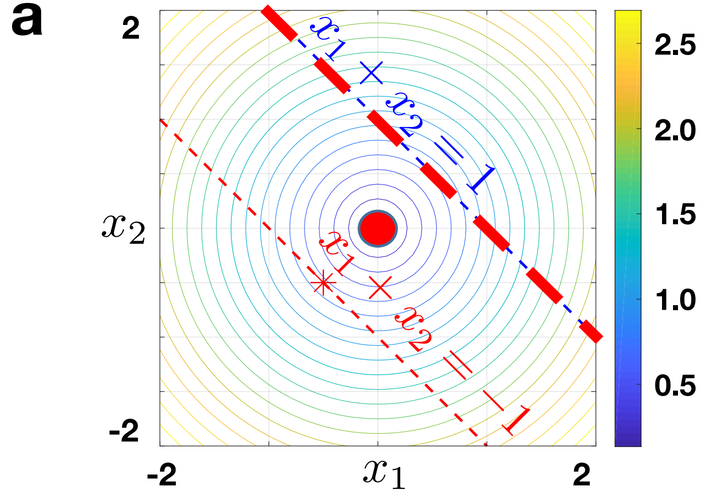
The perturbation $(P(z))$ of (P) is given by

$$\begin{aligned} &\text{minimize} \quad f(\boldsymbol{x}) = \sqrt{x_1^2 + x_2^2} \\ &\text{subject to } g(\boldsymbol{x}) = x_1 + x_2 \leq z. \end{aligned}$$

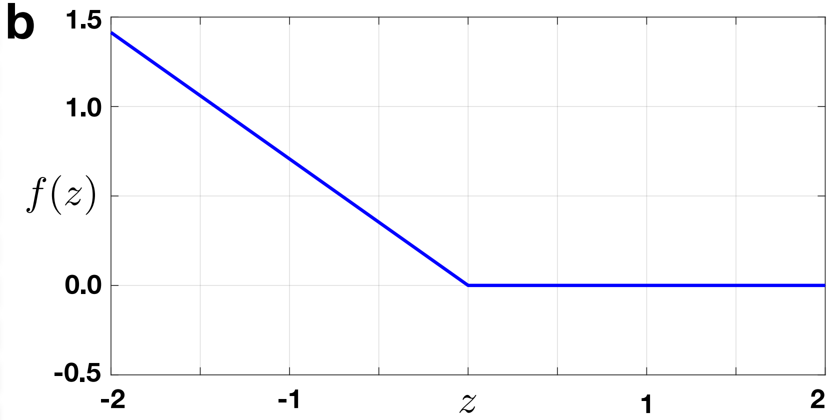
When $z \geq 0$, $(P(z))$ is minimized at $\boldsymbol{x} = (0,0)$. On the other hand for $z < 0$, the minimizer is $\boldsymbol{x} = (z/2, z/2)$. Therefore, we have that

$$f_{\mathrm{p}}^*(z) = \begin{cases} 0 & \text{when } z \geq 0 \\ -\dfrac{z}{\sqrt{2}} & \text{when } z < 0 \end{cases},$$

which is not differentiable at $z = 0$.



(a)Visual of the function $f(\boldsymbol{x}) = \sqrt{x_1^2 + x_2^2}$.



(b) Visual illustration of $f_{\mathrm{p}}^*(z)$ for the problem (P).

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Thanks