CS 512

Li Xu
A20300818

A. Let $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$, find

1. $2\vec{A} - \vec{B} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$

2. $\|\vec{A}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

Let $\vec{R} = (1, 0, 0)$ be ~~x axis~~ x axis.

$\cos\theta = \frac{\vec{A} \cdot \vec{R}}{\|A\|\|R\|} = \frac{1\times1 + 2\times0 + 3\times0}{\sqrt{14} \times 1} = \frac{1}{\sqrt{14}} = \frac{\sqrt{14}}{14}$

$\Rightarrow \theta = \arccos\theta \frac{1}{\sqrt{14}}$

3. Unit vector of $\vec{A}$: $\hat{A} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{14}}{14} \\ \frac{\sqrt{14}}{7} \\ \frac{3\sqrt{14}}{14} \end{bmatrix}$

4. $\vec{A} = 1i + 2j + 3k$

$\therefore \cos\alpha = 1$

$\cos\beta = 2$

$\cos\gamma = 3$

5. $\vec{A} \cdot \vec{B} = (i + 2j + 3k) \cdot (4i + 5j + 6k)$

$= (1\times4) + (2\times5) + (3\times6)$

$= 32 = \vec{B} \cdot \vec{A}$

6. $\theta = \arccos\left(\frac{\vec{A} \cdot \vec{B}}{\|\vec{A}\| \|\vec{B}\|}\right) = \arccos\frac{32}{\sqrt{14}\sqrt{77}} = \arccos\frac{32}{\sqrt{1078}}$

7 a $\vec{A} \cdot \vec{A}^{\perp} = 0$

$\therefore$ $\vec{A}^{\perp} = (x, y, z)$

$x + 2y + 3z = 0$

$\vec{A}^{\perp}$ can be $(1, 2, -1)$

$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $B \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ $C \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

8. $\vec{A} \times \vec{B} = (a_2b_3 - a_3b_2, \ a_3b_1 - a_1b_3, \ a_1b_2 - a_2b_1)$

$= (2\times6 - 3\times5, \ 3\times4 - 1\times6, \ 1\times5 - 2\times4)$

$= (-3, 6, -3)$

$\vec{B} \times \vec{A} = (b_2a_3 - b_3a_2, \ b_3a_1 - b_1a_3, \ b_1a_2 - b_2a_1)$

$= (5\times3 - 6\times2, \ 6\times1 - 4\times3, \ 4\times2 - 5\times1)$

$= (3, -6, 3)$

9. $\begin{cases} x + 2y + 3z = 0 \\ 4x + 5y + 6z = 0 \end{cases} \Rightarrow (1, -2, 1)$

10. Determinant $= \begin{vmatrix} 1 & 4 & -1 \\ 2 & 5 & 1 \\ 3 & 6 & 3 \end{vmatrix}$

$= 15 + 12 - 12 + 15 - 24 - 6$

$= 0$

$\therefore$ A, B, C are not linearly independent.

B、 $\vec{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{bmatrix}$ $\vec{B} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -4 \\ 3 & -2 & 1 \end{bmatrix}$ $\vec{C} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{bmatrix}$

1、 $2\vec{A} - \vec{B} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & -4 & 6 \\ 0 & 10 & -2 \end{bmatrix} + \begin{bmatrix} -1 & -2 & -1 \\ -2 & -1 & 4 \\ -3 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 5 \\ 6 & -5 & 10 \\ -3 & 12 & -1 \end{bmatrix}$

2、 $\vec{A}\vec{B} = \begin{bmatrix} 1\times1+2\times2+3\times3 & 1\times2+2\times1+3\times(-2) & 1\times1+2\times(-4)+3\times1 \\ 4\times1+(-2)\times2+3\times3 & 4\times2+(-2)\times1+3\times(-2) & 4\times1+(-2)\times(-4)+(3\times1) \\ 0\times1+5\times2+(-1)\times3 & 0\times2+5\times1+(-1)\times(-2) & 0\times1+3\times(-4)+(-1)\times1 \end{bmatrix}$

$= \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}$

$\vec{B}\vec{A} = \begin{bmatrix} 9 & 3 & 8 \\ 6 & -18 & 13 \\ -5 & 15 & 2 \end{bmatrix}$

3、 $(\vec{A}\vec{B})^T = \begin{bmatrix} 14 & -2 & -4 \\ 9 & 0 & 15 \\ 7 & 7 & -21 \end{bmatrix}^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ 14 & 15 & -21 \end{bmatrix}$

$\vec{B}^T\vec{A}^T = (\vec{A}\vec{B})^T = \begin{bmatrix} 14 & 9 & 7 \\ -2 & 0 & 7 \\ 14 & 15 & -21 \end{bmatrix}$

4、 $|\vec{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 0 & 5 & -1 \end{vmatrix} \cancel{= 2+0+20+6-8=15}$

$= 2+0+60-0-15+8 = 55$

$|\vec{C}| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 1 & 3 \end{vmatrix} = 15-12+12+15-24-6$

$= 0$

5. B's row vectors can form a orthogonal set.
Because each row multiply to another equals to 0

6. $\vec{A}^{-1} = \begin{bmatrix} \frac{-13}{55} & \frac{17}{55} & \frac{12}{55} \\ \frac{4}{55} & \frac{1}{55} & \frac{9}{55} \\ \frac{4}{11} & \frac{1}{11} & \frac{-2}{11} \end{bmatrix}$

$\vec{B}^{-1} = \begin{bmatrix} \frac{1}{6} & \frac{2}{21} & \frac{3}{14} \\ \frac{1}{3} & \frac{1}{21} & \frac{-1}{7} \\ \frac{1}{6} & \frac{-4}{21} & \frac{1}{14} \end{bmatrix}$

c. $\vec{A} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ $\quad \vec{B} = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$

1. $\det(\vec{A} - \lambda\vec{I}) = 0$

$= \left| \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$

$= \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(2-\lambda) - 6 = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$(\lambda+1)(\lambda-4) = 0$

$\lambda = -1, 4$ are eigenvalues.

~~when~~ $\cancel{\lambda = -1} \quad \cancel{\begin{bmatrix} 1+1 & 2 \\ 3 & 2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

$\cancel{\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$ $\cancel{x_1 + 2x_2 = 0}$

$\cancel{4x_1 + 2x_2 = 0}$ $\cancel{\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$
$3x_1 + 3x_2 = 0$

when $\lambda = -1$

$$\begin{bmatrix} 1+1 & 2 \\ 3 & 2+1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2x_1 + 2x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ is eigenvector}$$

$\lambda = 4$,

$$\begin{bmatrix} 1-4 & 2 \\ 3 & 2-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{cases} -3x_1 + 2x_2 = 0 \\ 3x_1 - 2x_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

2. $V = \begin{bmatrix} -1 & \frac{2}{3} \\ 1 & 1 \end{bmatrix}$, $V^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} \end{bmatrix}$

$$V^{-1} A V = \begin{bmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -1 & \frac{2}{3} \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$$

3. $\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} = -\frac{2}{3} + 1 = \frac{1}{3}$

4. $\det(\vec{B} - \lambda \vec{I}) = 0$

$$\left| \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(5-\lambda) - 4 = 0$$

$$\lambda = 1, 6$$

when $\lambda = 1$, $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow \begin{cases} x_1 - 2x_2 = 0 \\ -2x_1 + 4x_2 = 0 \end{cases} \Rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

5.

when $\lambda = 6$, $\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{cases} -4x_1 - 2x_2 = 0 \\ -2x_1 - x_2 = 0. \end{cases}$$

$$\Rightarrow \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -2 + 2 = 0.$$

5. The eigenvectors are an orthogonal set, because their dot product equals to 0

D. $f(x) = x^2 + 3$, $g(x, y) = x^2 + y^2$

1. $f'(x) = 2x$

   $f''(x) = 2$

2. $\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(y^2) = 2x + 0 = 2x$

   $\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2) = \frac{\partial}{\partial y}(x^2) + \frac{\partial}{\partial y}(y^2) = 0 + 2y = 2y$

3. $\nabla g(x, y) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = (2x, 2y)$

4. $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$