CHAPTER 3: LINEAR PROGRAMMING: SENSITIVITY ANALYSIS AND INTERPRETATION OF SOLUTION

INTRODUCTION TO SENSITIVITY ANALYSIS

Sensitivity analysis (or post-optimality analysis) is used to determine how the optimal solution is affected by changes, within specified ranges, in:

- the objective function coefficients
- the right-hand side (RHS) values

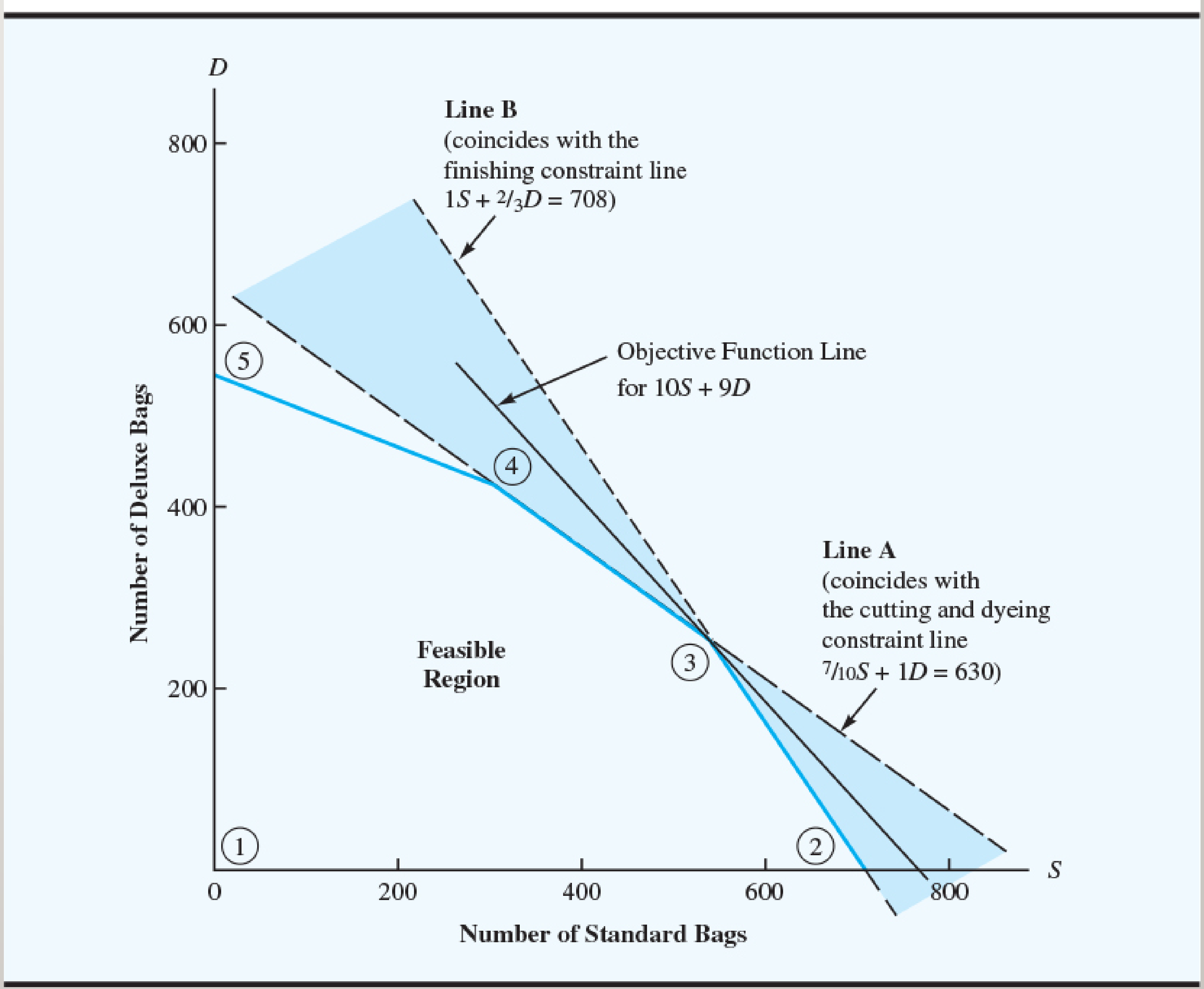
- Sensitivity analysis is important to a manager who must operate in a dynamic environment with imprecise estimates of the coefficients.
- Sensitivity analysis allows a manager to ask certain what-if questions about the problem.

OBJECTIVE FUNCTION COEFFICIENTS

- Let us consider how changes in the objective function coefficients might affect the optimal solution.
- The **range of optimality** for each coefficient provides the range of values over which the current solution will remain optimal.
- Managers should focus on those objective coefficients that have a narrow range of optimality and coefficients near the endpoints of the range.

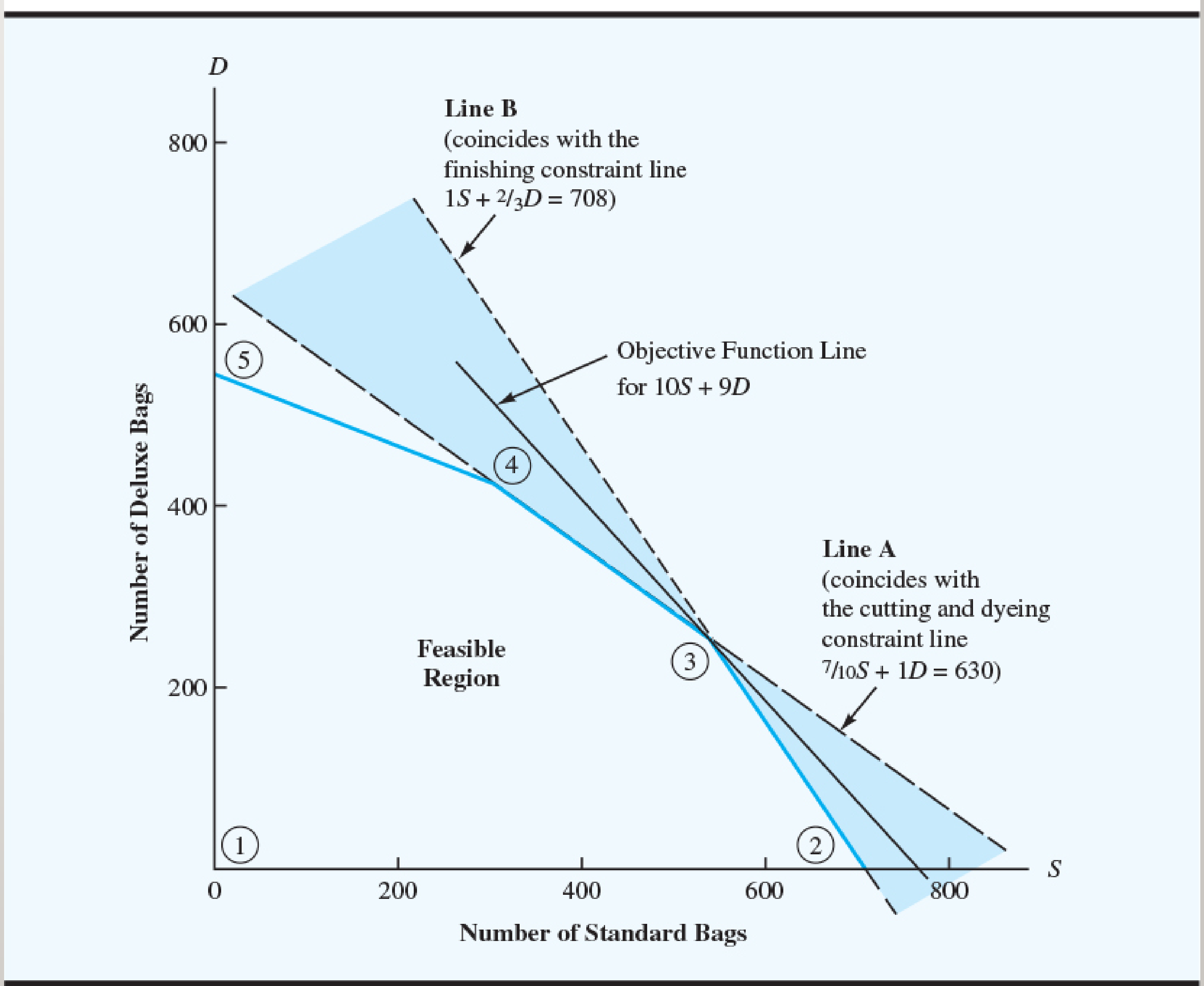
RANGE OF OPTIMALITY (1 OF 7)

Graphically, the limits of a range of optimality are found by changing the slope of the objective function line within the limits of the slopes of the binding constraint lines.



RANGE OF OPTIMALITY (2 OF 7)

As long as the slope of the objective function line is between the slope of line A (which coincides with the cutting and dyeing constraint line) and the slope of line B (which coincides with the finishing constraint line), extreme point 3 with
S = 540 and
D = 252 will be optimal.



RANGE OF OPTIMALITY (3 OF 7)

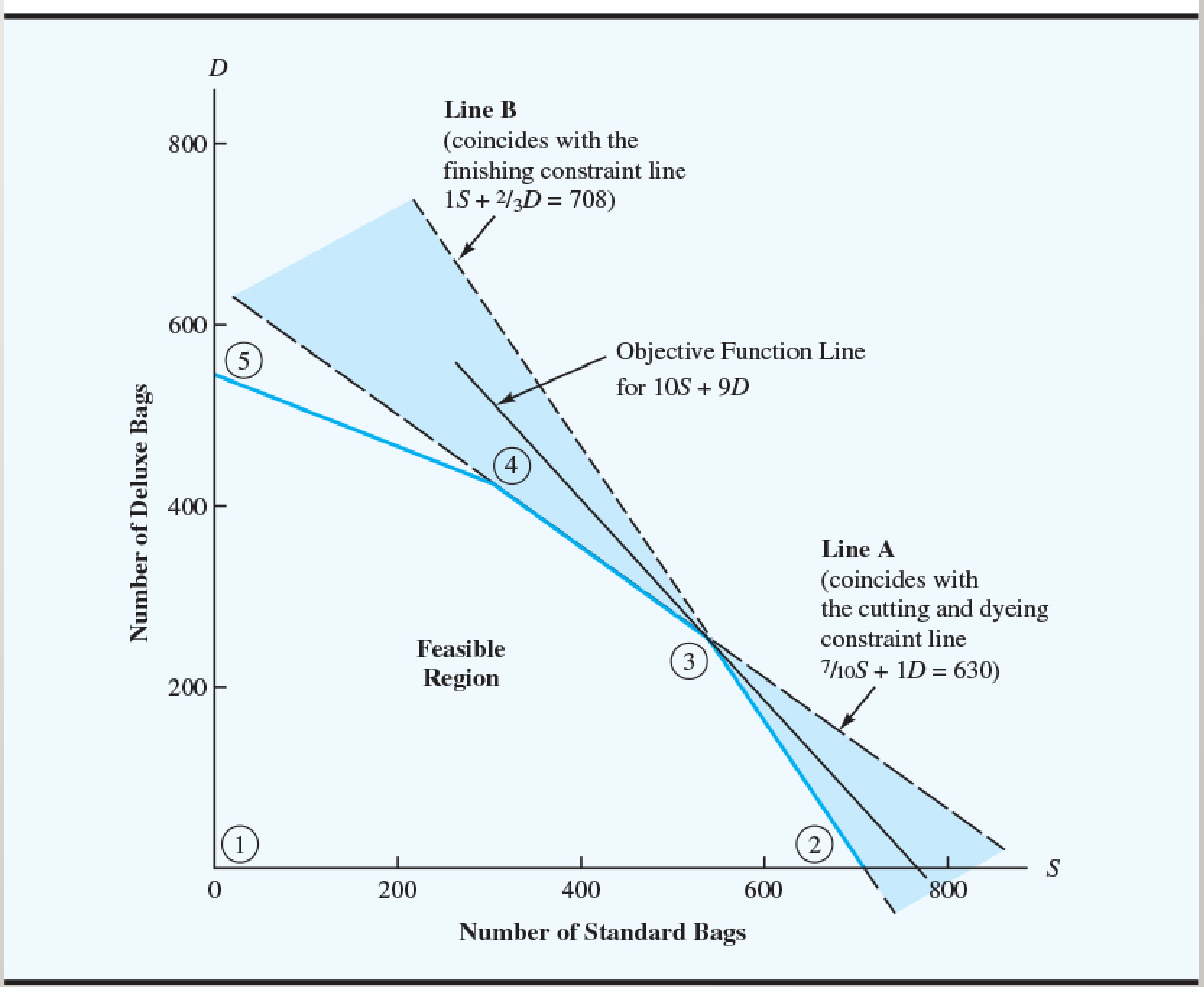
Extreme point 3 will be the optimal solution as long as

Slope of line B ≤ slope of the objective function line ≤ slope of line A.

In slope-intercept form.

Line A: $$D = -\frac{7}{10}S + 630$$

Line B: $$D = -\frac{2}{3}S + 1062$$



RANGE OF OPTIMALITY (4 OF 7)

Let us now consider the general form of the slope of the objective function line. Let C_S denote the profit of a standard bag, C_D denote the profit of a deluxe bag, and P denote the value of the objective function.
Using this notation, the objective function line can be written as

$$P = C_s S + C_D D$$

In slope-intercept form: $$D = -\frac{C_S}{C_D} S + \frac{P}{C_D}$$

Extreme point 3 will be optimal as long as $$-\frac{3}{2} \leq -\frac{C_S}{C_D} \leq -\frac{7}{10}.$$

RANGE OF OPTIMALITY (5 OF 7)

To compute the range of optimality for the standard-bag profit contribution, we hold the profit contribution for the deluxe bag fixed at its initial value

$$C_D = 9.$$

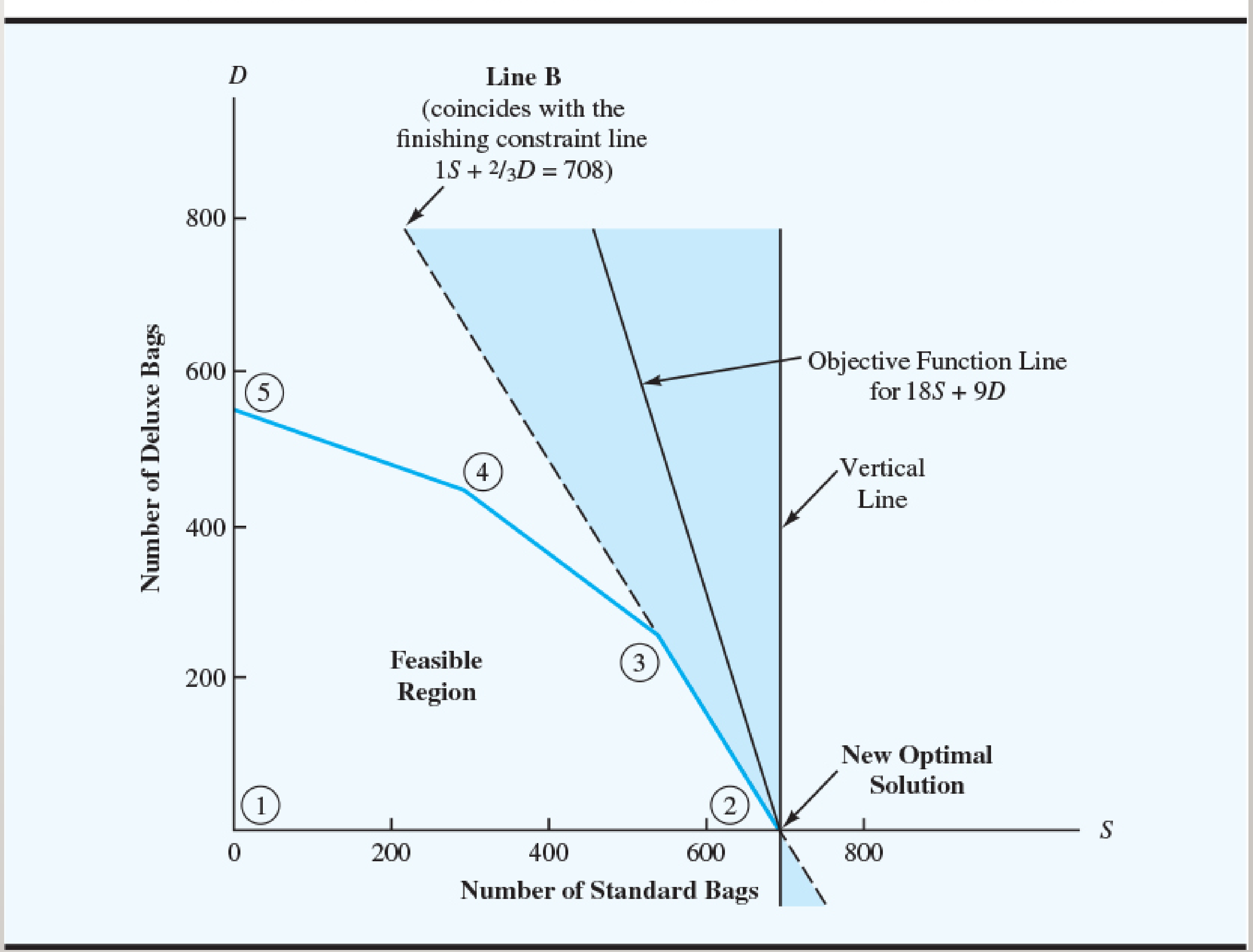
Therefore $$-\frac{3}{2} \le -\frac{C_S}{9} \le -\frac{7}{10}$$

and $$6.3 \le C_S \le 13.5$$

The range of optimality for CS tells Par, Inc.'s management that, with other coefficients unchanged, the profit contribution for the standard bag can be anywhere between \$6.30 and \$13.50 and the production quantities of 540 standard bags and 252 deluxe bags will remain optimal.

RANGE OF OPTIMALITY (6 OF 7)

In cases where the rotation of the objective function line about an optimal extreme point causes the objective function line to become vertical, there will be either no upper limit or no lower limit for the slope.



RANGE OF OPTIMALITY (7 OF 7)

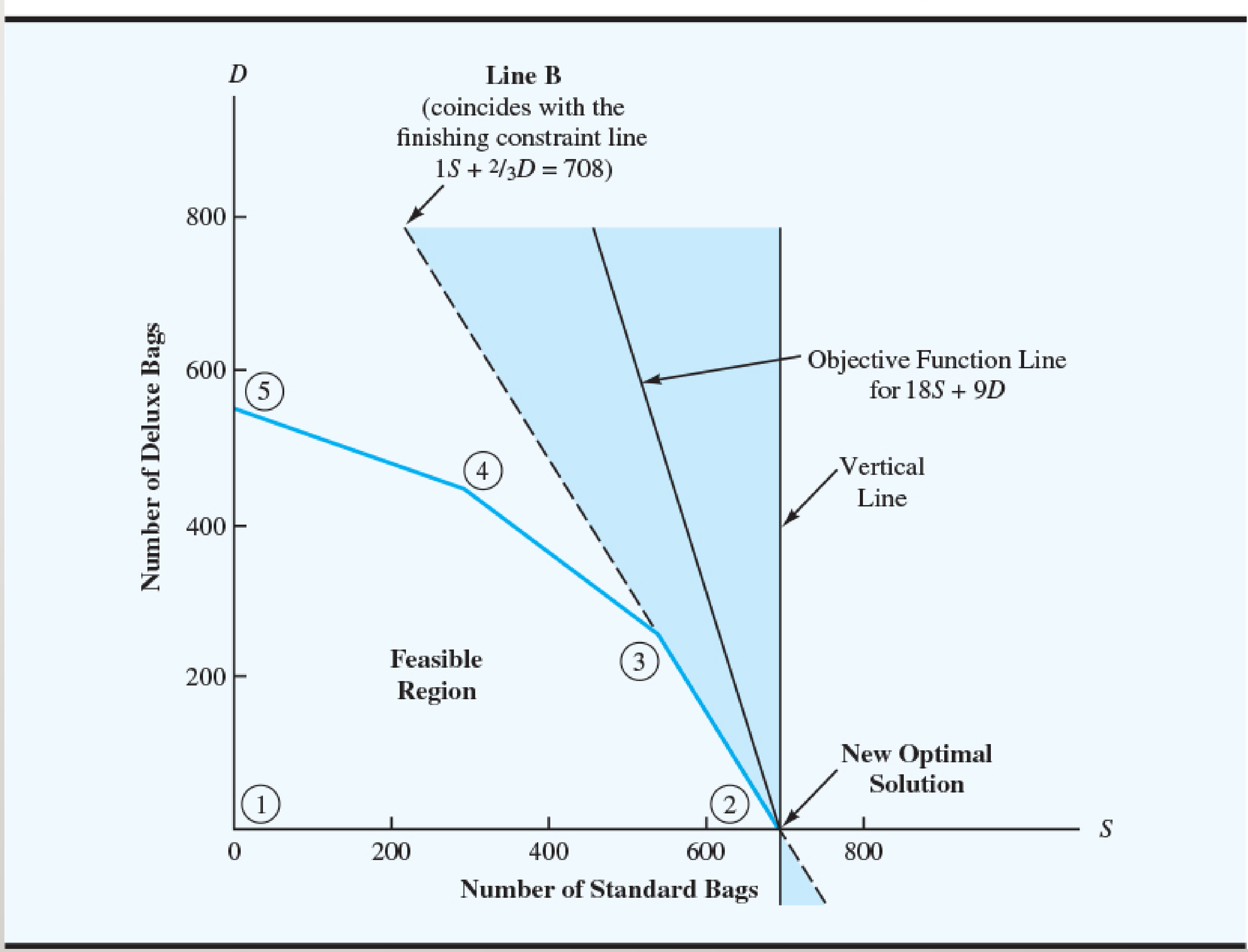
In this case $$-\frac{C_S}{C_D} \leq -\frac{3}{2}$$

Following the previous procedure of holding *C* constant at its original value,

$C_D = 9$ we have

$$-\frac{C_S}{9} \leq -\frac{3}{2}$$

$$13.5 \leq C_S < \infty$$



SIMULTANEOUS CHANGES (1 OF 3)

If two or more objective function coefficients are changed simultaneously, further analysis is necessary to determine whether the optimal solution will change.

When solving two-variable problems graphically, simply compute the slope of the objective function

$$(-C_S / C_D)$$

for the new coefficient values. If this ratio is greater than or equal to the lower limit on the slope of the objective function and less than or equal to the upper limit, then the changes made will not cause a change in the optimal solution.

SIMULTANEOUS CHANGES (2 OF 3)

Consider changes in both of the objective function coefficients for the Par, Inc., problem. Suppose the profit contribution per standard bag is increased to $13 and the profit contribution per deluxe bag is simultaneously reduced to $8. Recall that the ranges of optimality for C_S and C_D (both computed in a one-at-a-time manner) are

$$6.3 \le C_S \le 13.5$$

$$6.67 \le C_D \le 14.29$$

For these ranges of optimality, we can conclude that changing either C_S to $13 or C_D to $8 (but not both) would not cause a change in the optimal solution of S = 540 and D = 252.

SIMULTANEOUS CHANGES (3 OF 3)

However in recomputing the slope of the objective function with simultaneous changes for both C_S and C_D, we saw that the optimal solution did change.

A range of optimality, by itself, can only be used to draw a conclusion about changes made to one objective function coefficient at a time.

RIGHT-HAND SIDES (1 OF 4)

Let us consider how a change in the right-hand side for a constraint might affect the feasible region and perhaps cause a change in the optimal solution.

- The change in the value of the optimal solution per unit increase in the right hand side of the constraint is called the **dual value**.
- As the RHS increases, other constraints will become binding and limit the change in the value of the objective function.

RIGHT-HAND SIDES (2 OF 4)

Let us consider what happens if an additional 10 hours of production time become available in the cutting and dyeing department of Par, Inc. The right-hand side of the cutting and dyeing constraint is changed from 630 to 640, and the constraint is rewritten as

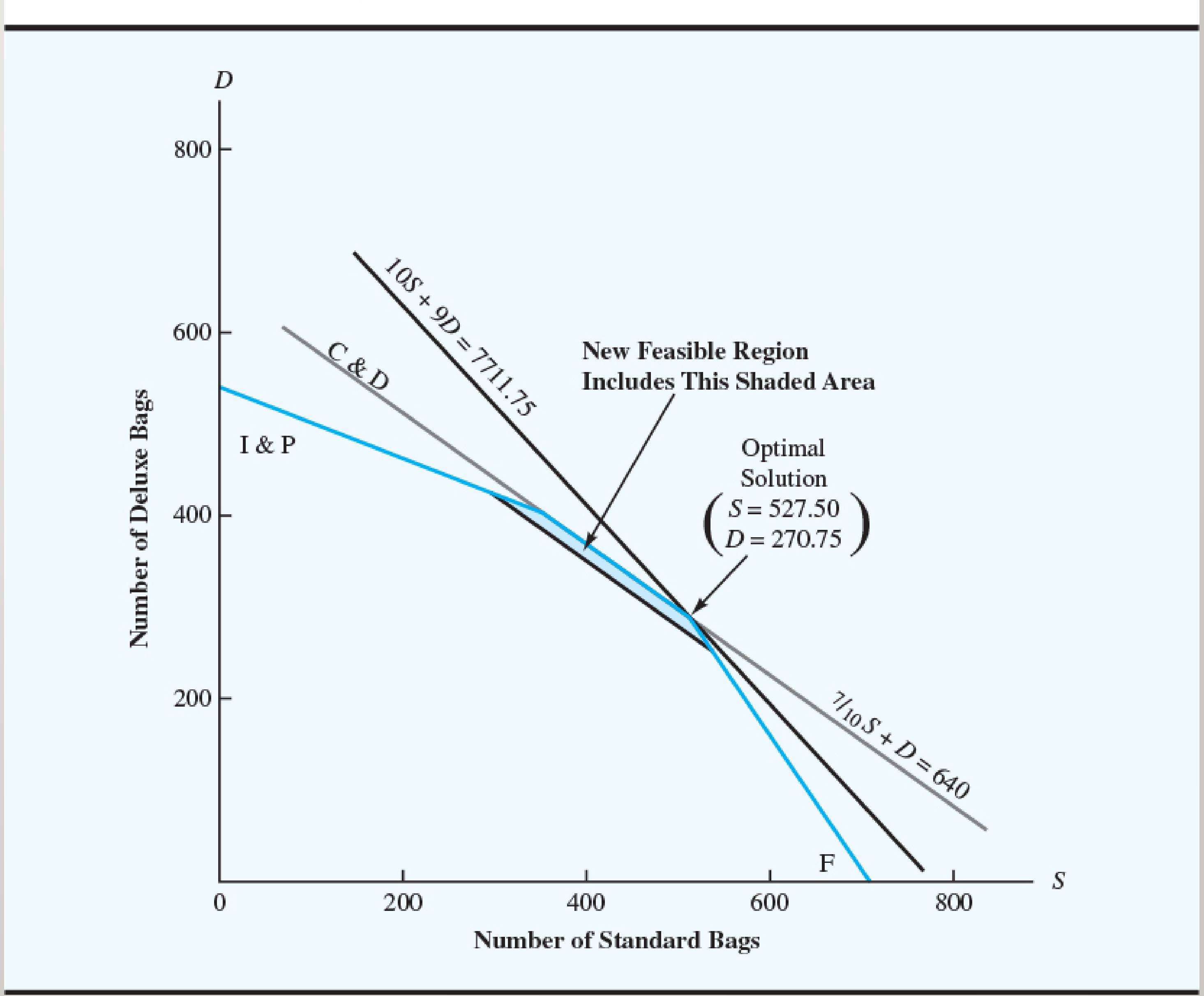
$$\frac{7}{10}S + 1D \leq 640$$

By obtaining an additional 10 hours of cutting and dyeing time, we expand the feasible region for the problem.

RIGHT-HAND SIDES (3 OF 4)

Using the graphical solution procedure with the enlarged feasible region shows that the extreme point S = 527.5 and D = 270.75 now provides the optimal solution.

The increase in profit is \$7711.75 - \$7668.00 = \$43.75.



RIGHT-HAND SIDES (4 OF 4)

The *change* in the value of the optimal solution per unit increase in the right-hand side of the constraint is called the **dual value.**

- Here, the dual value for the cutting and dyeing constraint is $4.375
- If we increase the right-hand side of the cutting and dyeing constraint by 1 hour, the value of the objective function will increase by $4.375.
- Conversely, if the right-hand side of the cutting and dyeing constraint were to decrease by 1 hour, the objective function would go down by $4.375.

SENSITIVITY ANALYSIS: COMPUTER SOLUTION (1 OF 2)

Software packages such as *LINGO* and *Microsoft Excel* provide the following LP information:

- Information about the objective function:
  - its optimal value
  - coefficient ranges (ranges of optimality)
- Information about the decision variables:
  - their optimal values
  - their reduced costs
- Information about the constraints:
  - the amount of slack or surplus
  - the dual prices
  - right-hand side ranges (ranges of feasibility)

SENSITIVITY ANALYSIS: COMPUTER SOLUTION

Let's look at the Par, Inc., problem restated below:

$$
\begin{array}{lrcll}
\text{Max} & 10S + 9D & & & \\
\text{s.t.} & & & & \\
& \tfrac{7}{10}S + 1D & \leq & 630 & \text{Cutting and dyeing} \\
& \tfrac{1}{2}S + \tfrac{5}{6}D & \leq & 600 & \text{Sewing} \\
& 1S + \tfrac{2}{3}D & \leq & 708 & \text{Finishing} \\
& \tfrac{1}{10}S + \tfrac{1}{4}D & \leq & 135 & \text{Inspection and packaging} \\
& S, D & \geq & 0 &
\end{array}
$$

THE SOLUTION FOR THE PAR, INC., PROBLEM (1 OF 5)

Optimal Objective Value = 7668.00000

Variable	Value	Reduced Cost
S	540.00000	0.00000
D	252.00000	0.00000

Constraint	Slack/Surplus	Dual Value
1	0.00000	4.37500
2	120.00000	0.00000
3	0.00000	6.93750
4	18.00000	0.00000

Variable	Objective Coefficient	Allowable Increase	Allowable Decrease
S	10.00000	3.50000	3.70000
D	9.00000	5.28571	2.33333

Constraint	RHS Value	Allowable Increase	Allowable Decrease
1	630.00000	52.36364	134.40000
2	600.00000	Infinite	120.00000
3	708.00000	192.00000	128.00000
4	135.00000	Infinite	18.00000

THE SOLUTION FOR THE PAR, INC., PROBLEM (2 OF 5)

The Dual Value column contains information about the marginal value of each of the four resources at the optimal solution.

Optimal Objective Value = 7668.00000

Variable	Value	Reduced Cost
S	540.00000	0.00000
D	252.00000	0.00000
Constraint	**Slack/Surplus**	**Dual Value**
1	0.00000	4.37500
2	120.00000	0.00000
3	0.00000	6.93750
4	18.00000	0.00000

The dual values of 4.375 for constraint 1 (cutting and dyeing) and 6.9375 for constraint 3 (finishing) tell us that an additional hour of cutting and dyeing time increases the value of the optimal solution by $4.37, and an additional hour of finishing time increases the value of the optimal solution by $6.94.

THE SOLUTION FOR THE PAR, INC., PROBLEM (3 OF 5)

The **reduced cost** is equal to the dual value for the nonnegativity constraint associated with the variable.

Optimal Objective Value = 7668.00000		
Variable	Value	Reduced Cost
S	540.00000	0.00000
D	252.00000	0.00000

The reduced cost on variable S and D is zero. The nonnegativity constraint is $S \geq 0$. The current value of S is 540, so changing the nonnegativity constraint to $S \geq 1$ has no effect on the optimal solution value. Because increasing the right-hand side by one unit has no effect on the optimal objective function value, the dual value (i.e., reduced cost) of this nonnegativity constraint is zero.

THE SOLUTION FOR THE PAR, INC., PROBLEM (4 OF 5)

Variable S, which has a current profit coefficient of 10, has an allowable increase of 3.5 and an allowable decrease of 3.7.

Variable	Objective Coefficient	Allowable Increase	Allowable Decrease
S	10.00000	3.50000	3.70000
D	9.00000	5.28571	2.33333

Therefore, as long as the profit contribution associated with the standard bag is between \$10 - \$3.7 = \$6.30 and \$10 - \$3.5 = \$13.50, the production of S = 540 standard bags and D = 252 deluxe bags will remain the optimal solution.

THE SOLUTION FOR THE PAR, INC., PROBLEM (5 OF 5)

The final section of the computer output provides the allowable increase and allowable decrease in the right-hand sides of the constraints relative to the dual values holding.

Constraint	RHS Value	Allowable Increase	Allowable Decrease
1	630.00000	52.36364	134.40000
2	600.00000	Infinite	120.00000
3	708.00000	192.00000	128.00000
4	135.00000	Infinite	18.00000

As long as the constraint right-hand side is not increased (decreased) by more than the allowable increase (decrease), the associated dual value gives the exact change in the value of the optimal solution per unit increase in the right-hand side.

RELEVANT COST AND SUNK COST

- A resource cost is a **relevant cost** if the amount paid for it is dependent upon the amount of the resource used by the decision variables.
- Relevant costs **are** reflected in the objective function coefficients.
- A resource cost is a **sunk cost** if it must be paid regardless of the amount of the resource actually used by the decision variables.
- Sunk resource costs are **not** reflected in the objective function coefficients.

CAUTIONARY NOTE ON THE INTERPRETATION OF DUAL VALUES

When the cost of a resource is *sunk,* the dual value can be interpreted as the maximum amount the company should be willing to pay for one additional unit of the resource.

When the cost of a resource used is relevant, the dual value can be interpreted as the amount by which the value of the resource exceeds its cost.

THE MODIFIED PAR, INC., PROBLEM (1 OF 4)

The graphical solution procedure is useful only for linear programs involving two decision variables. In practice, the problems solved using linear programming usually involve a large number of variables and constraints.

For instance, the Management Science in Action, Determining Optimal Production Quantities at GE Plastics, describes how a linear programming model with 3100 variables and 1100 constraints was solved in less than 10 seconds to determine the optimal production quantities at GE Plastics.

THE MODIFIED PAR, INC., PROBLEM (2 OF 4)

In this section we discuss the formulation and computer solution for two linear programs with three decision variables. In doing so, we will show how to interpret the reduced-cost portion of the computer output.

$$\begin{array}{lrll}
\text{Max} & 10S + 9D & & \\
\text{s.t.} & & & \\
 & \tfrac{7}{10}S + 1D & \leq 630 & \text{Cutting and dyeing} \\
 & \tfrac{1}{2}S + \tfrac{5}{6}D & \leq 600 & \text{Sewing} \\
 & 1S + \tfrac{2}{3}D & \leq 708 & \text{Finishing} \\
 & \tfrac{1}{10}S + \tfrac{1}{4}D & \leq 135 & \text{Inspection and packaging} \\
 & S, D & \geq 0 &
\end{array}$$

THE MODIFIED PAR, INC., PROBLEM (3 OF 4)

Suppose that management is also considering producing a lightweight model designed specifically for golfers who prefer to carry their bags.

- The design department estimates that each new lightweight model will require 0.8 hours for cutting and dyeing, 1 hour for sewing, 1 hour for finishing, and 0.25 hours for inspection and packaging.
- Because of the unique capabilities designed into the new model, Par, Inc.'s management feels they will realize a profit contribution of $12.85 for each lightweight model produced during the current production period.

THE MODIFIED PAR, INC., PROBLEM (4 OF 4)

After adding *L* to the objective function and to each of the four constraints, we obtain the following linear program for the modified problem:

$$
\begin{array}{lrcl}
\text{Max} & 10S + 9D + 12.85L & & \\
\text{s.t.} & & & \\
& \tfrac{7}{10}S + 1D + 0.8L \leq 630 & & \text{Cutting and dyeing} \\
& \tfrac{1}{2}S + \tfrac{5}{6}D + 1L \leq 600 & & \text{Sewing} \\
& 1S + \tfrac{2}{3}D + 1L \leq 708 & & \text{Finishing} \\
& \tfrac{1}{10}S + \tfrac{1}{4}D + \tfrac{1}{4}L \leq 135 & & \text{Inspection and packaging} \\
& S, D, L \geq 0 & &
\end{array}
$$

SOLUTION TO THE MODIFIED PAR, INC., PROBLEM (1 OF 5)

Optimal Objective Value = 8299.80000

Variable	Value	Reduced Cost
S	280.00000	0.00000
D	0.00000	–1.15000
L	428.00000	0.00000

Constraint	Slack/Surplus	Dual Value
1	91.60000	0.00000
2	32.00000	0.00000
3	0.00000	8.10000
4	0.00000	19.00000

Variable	Objective Coefficient	Allowable Increase	Allowable Decrease
S	10.00000	2.07000	4.86000
D	9.00000	1.15000	Infinite
L	12.85000	12.15000	0.94091

Constraint	RHS Value	Allowable Increase	Allowable Decrease
1	630.00000	Infinite	91.60000
2	600.00000	Infinite	32.00000
3	708.00000	144.63158	168.00000
4	135.00000	9.60000	64.20000

SOLUTION TO THE MODIFIED PAR, INC., PROBLEM (2 OF 5)

The optimal solution calls for the production of 280 standard bags, 0 deluxe bags, and 428 of the new lightweight bags; the value of the optimal solution is $8299.80.

Optimal Objective Value = 8299.80000

Variable	Value	Reduced Cost
S	280.00000	0.00000
D	0.00000	–1.15000
L	428.00000	0.00000

SOLUTION TO THE MODIFIED PAR, INC., PROBLEM (3 OF 5)

Let us now look at the Reduced Cost column. The reduced costs are the dual values of the corresponding nonnegativity constraints.

Optimal Objective Value = 8299.80000

Variable	Value	Reduced Cost
S	280.00000	0.00000
D	0.00000	–1.15000
L	428.00000	0.00000

The reduced costs for *S* and *L* are zero because these decision variables already have positive values in the optimal solution. However, the reduced cost for decision variable *D* is –1.15. The interpretation of this number is that if the production of deluxe bags is increased from the current level of 0 to 1, then the optimal objective function value will decrease by 1.15.

SOLUTION TO THE MODIFIED PAR, INC., PROBLEM (4 OF 5)

The dual values for constraints 3 and 4 are 8.1 and 19, respectively, indicating that these two constraints are binding in the optimal solution.

Constraint	Slack/Surplus	Dual Value
1	0.00000	0.00000
2	56.75676	0.00000
3	0.00000	8.10000
4	0.00000	19.00000

Each additional hour in the finishing department would increase the value of the optimal solution by $8.10, and each additional hour in the inspection and packaging department would increase the value of the optimal solution by $19.00.

SOLUTION TO THE MODIFIED PAR, INC., PROBLEM (5 OF 5)

Because of a slack of 91.6 hours in the cutting and dyeing department and 32 hours in the sewing department, management might want to consider the possibility of utilizing these unused labor-hours in the finishing or inspection and packaging departments.

Constraint	Slack/Surplus	Dual Value
1	0.00000	0.00000
2	56.75676	0.00000
3	0.00000	8.10000
4	0.00000	19.00000

For example, some of the employees in the cutting and dyeing department could be used to perform certain operations in either the finishing department or the inspection and packaging department.

LIMITATIONS OF CLASSICAL SENSITIVITY ANALYSIS (1 OF 4)

Classical sensitivity analysis obtained from computer output can provide useful information on the sensitivity of the solution to changes in the model input data.

Classical sensitivity analysis provided by most computer packages does have its limitations. Three such limitations are:

1. Simultaneous changes in input data
2. Changes in constraint coefficients
3. Nonintuitive dual values

LIMITATIONS OF CLASSICAL SENSITIVITY ANALYSIS (2 OF 4)

Limitation: Simultaneous changes in input data

- The sensitivity analysis information in computer output is based on the assumption that only one coefficient changes; it is assumed that all other coefficients will remain as stated in the original problem. Thus, the range analysis for the objective function coefficients and the constraint righthand sides is only applicable for changes in a single coefficient.

LIMITATIONS OF CLASSICAL SENSITIVITY ANALYSIS (3 OF 4)

Limitation: Changes in constraint coefficients

- Classical sensitivity analysis provides no information about changes resulting from a change in the coefficient of a variable in a constraint. Instead, we must simply change the coefficient and rerun the model.

LIMITATIONS OF CLASSICAL SENSITIVITY ANALYSIS (4 OF 4)

Limitation: Nonintuitive Dual Values

- Constraints with variables naturally on both the left-hand and right-hand sides often lead to dual values that have a nonintuitive explanation.