# CHAPTER 1: INTRODUCTION

1.1 - Problem Solving and Decision Making

1.2 - Quantitative Analysis and Decision Making

1.3 - Quantitative Analysis

1.4 - Models of Cost, Revenue, and Profit

1.5 - Management Science Techniques

# INTRODUCTION

The body of knowledge involving quantitative approaches to decision making is referred to as

- Management Science
- Operations Research
- Decision Science

It had its early roots in World War II and is flourishing in business and industry due, in part, to:

- numerous methodological developments (e.g. simplex method for solving linear programming problems)
- a virtual explosion in computing power

# PROBLEM SOLVING AND DECISION MAKING

7 Steps of **Problem Solving**

(First 5 steps are the process of **decision making**)

1. Define the problem.
2. Determine the set of alternative solutions.
3. Determine the criteria for evaluating alternatives.
4. Evaluate the alternatives.
5. Choose an alternative (make a decision).
6. Implement the selected alternative.
7. Evaluate the results.

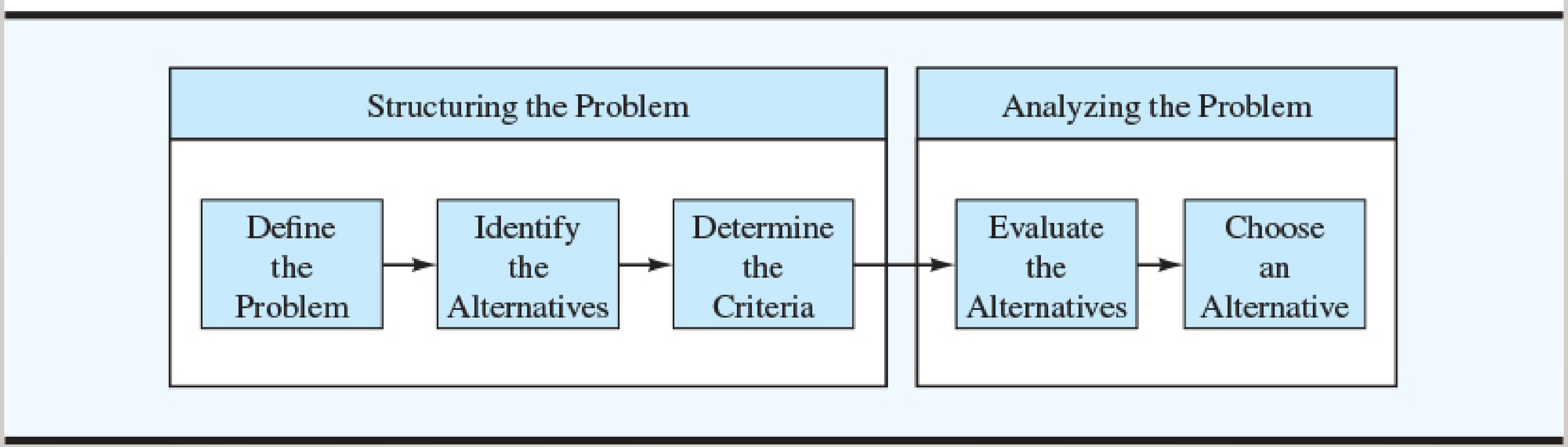
# QUANTITATIVE ANALYSIS AND DECISION MAKING (1 OF 2)

Decision-Making Process

- Problems in which the objective is to find the best solution with respect to one criterion are referred to as **single-criterion decision problems**.
- Problems that involve more than one criterion are referred to as **multicriteria decision problems**.

# QUANTITATIVE ANALYSIS AND DECISION MAKING (2 OF 2)

Decision-Making Process



# QUANTITATIVE ANALYSIS (1 OF 5)

## Analysis Phase of Decision-Making Process

Qualitative Analysis

- based largely on the manager's judgment and experience
- includes the manager's intuitive "feel" for the problem
- is more of an art than a science

# QUANTITATIVE ANALYSIS (2 OF 5)

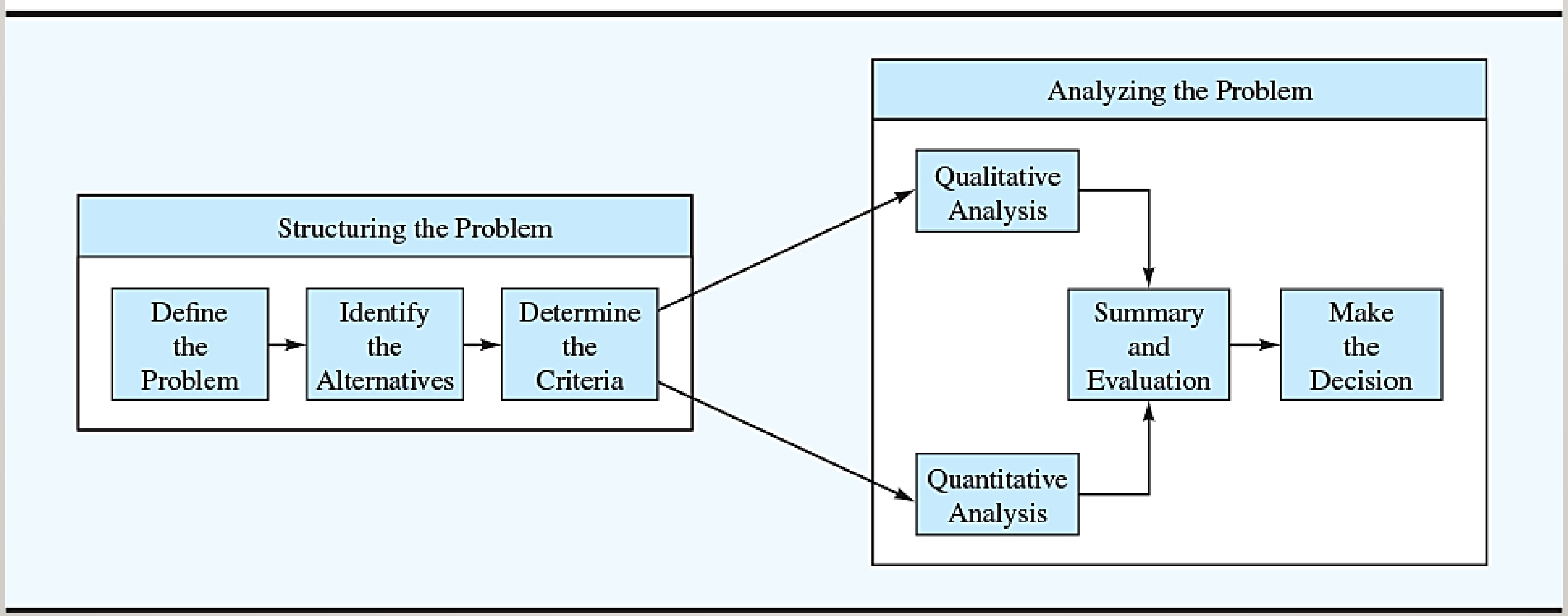
## Analysis Phase of Decision-Making Process

### Quantitative Analysis

- analyst will concentrate on the quantitative facts or data associated with the problem
- analyst will develop mathematical expressions that describe the objectives, constraints, and other relationships that exist in the problem
- analyst will use one or more quantitative methods to make a recommendation

# QUANTITATIVE ANALYSIS (3 OF 5)

Analysis Phase of Decision-Making Process



# QUANTITATIVE ANALYSIS (4 OF 5)

Potential Reasons for a Quantitative Analysis Approach to Decision Making

- The problem is complex.
- The problem is very important.
- The problem is new.
- The problem is repetitive.

# QUANTITATIVE ANALYSIS (5 OF 5)

Quantitative Analysis Process

- Model Development
- Data Preparation
- Model Solution
- Report Generation

# MODEL DEVELOPMENT (1 OF 2)

**Models** are representations of real objects or situations

Three forms of models are:

- **Iconic models** - physical replicas (scalar representations) of real objects
- **Analog models** - physical in form, but do not physically resemble the object being modeled
- **Mathematical models** - represent real world problems through a system of mathematical formulas and expressions based on key assumptions, estimates, or statistical analyses

# MODEL DEVELOPMENT (2 OF 2)

Generally, experimenting with models (compared to experimenting with the real situation):

- requires less time
- is less expensive
- involves less risk

The more closely the model represents the real situation, the accurate the conclusions and predictions will be.

# MATHEMATICAL MODELS 

**Objective Function** – a mathematical expression that describes the problem's objective, such as maximizing profit or minimizing cost

- Consider a simple production problem. Suppose $x$ denotes the number of units produced and sold each week, and the firm's objective is to maximize total weekly profit. With a profit of $10 per unit, the objective function is $10x$.

# MATHEMATICAL MODELS (2 OF 9)

**Constraints** – a set of restrictions or limitations, such as production capacities

- To continue our example, a production capacity constraint would be necessary if, for instance, 5 hours are required to produce each unit and only 40 hours are available per week. The production capacity constraint is given by $5x \leq 40$.
- The value of $5x$ is the total time required to produce $x$ units; the symbol  indicates that the production time required must be less than or equal to the 40 hours available.

# MATHEMATICAL MODELS (3 OF 9)

**Uncontrollable Inputs** – environmental factors that are not under the control of the decision maker

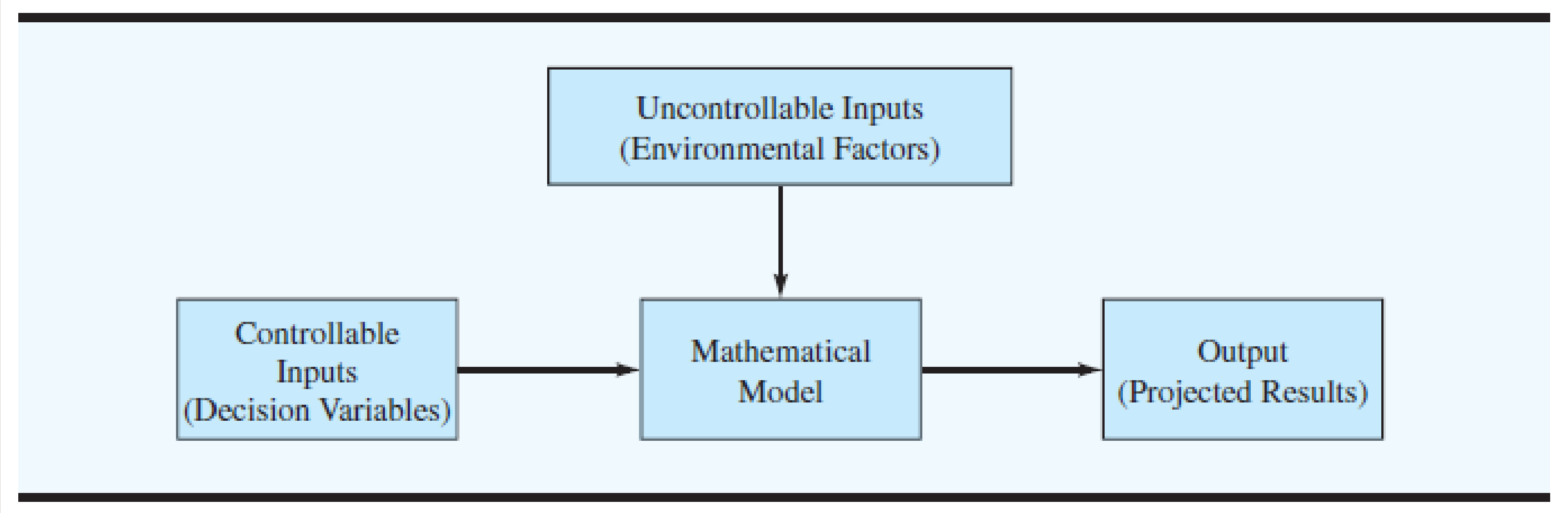
- In the preceding mathematical model, the profit per unit ($10), the production time per unit (5 hours), and the production capacity (40 hours) are environmental factors not under the control of the manager or decision maker.

# MATHEMATICAL MODELS (4 OF 9)

**Decision Variables** – controllable inputs; decision alternatives specified by the decision maker, such as the number of units of a product to produce.

- In the preceding mathematical model, the production quantity *x* is the controllable input to the model.

# MATHEMATICAL MODELS (5 OF 9)



# MATHEMATICAL MODELS (6 OF 9)

A complete mathematical model for our simple production problem is:

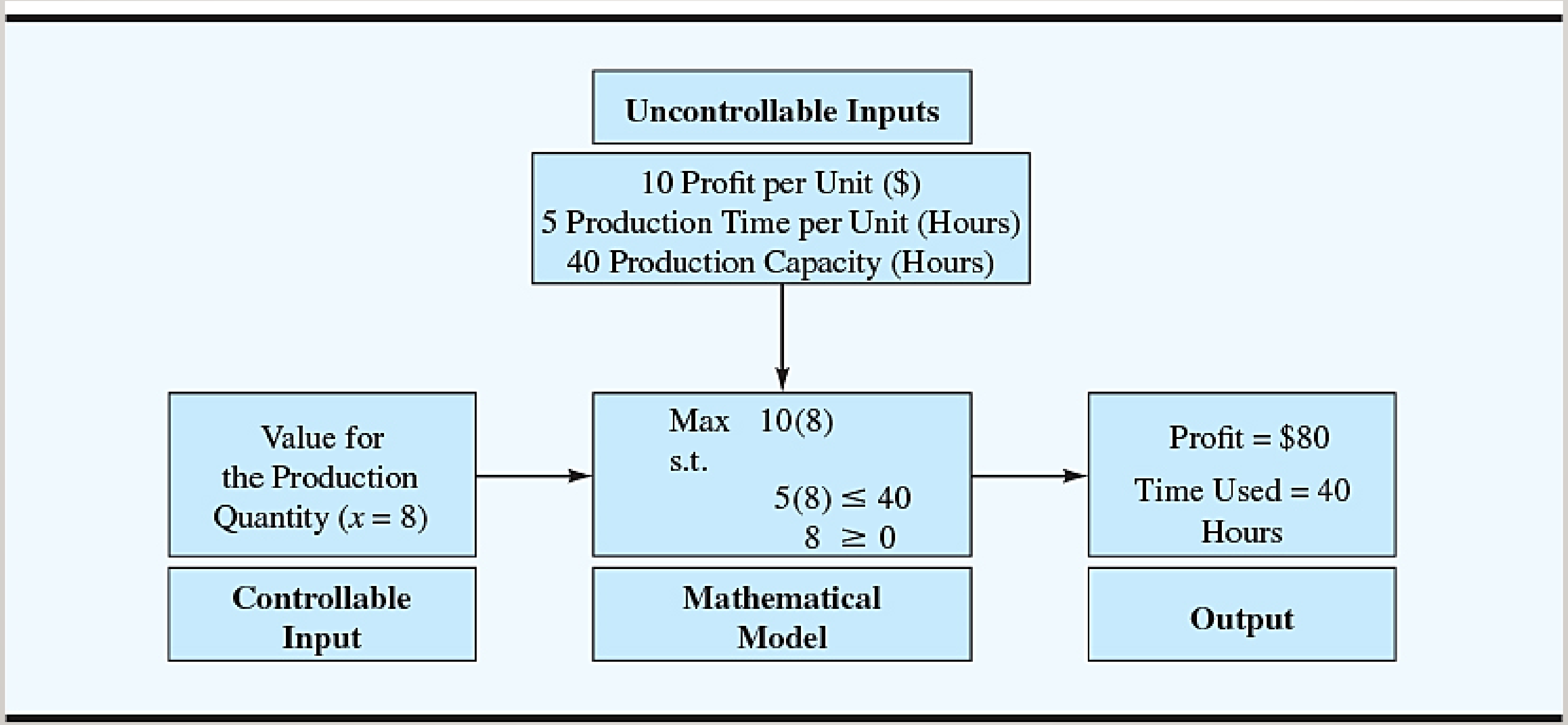
Maximize: $10x$ (objective function)

subject to: $5x \leq 40$ (constraint)

$x \geq 0$ (constraint)

The second constraint reflects the fact that it is not possible to manufacture a negative number of units.

# MATHEMATICAL MODELS (7 OF 9)



# MATHEMATICAL MODELS (8 OF 9)

**Deterministic Model** – if all uncontrollable inputs to the model are known and cannot vary

**Stochastic (or Probabilistic) Model** – if any uncontrollable are uncertain and subject to variation

- Stochastic models are often more difficult to analyze.
- In our simple production example, if the number of hours of production time per unit could vary from 3 to 6 hours depending on the quality of the raw material, the model would be stochastic.

# MATHEMATICAL MODELS (9 OF 9)

Data preparation is not a trivial step, due to the time required and the possibility of data collection errors.

- A model with 50 decision variables and 25 constraints could have over 1300 data elements!
- Often, a fairly large data base is needed.
- Information systems specialists might be needed.

# MODEL SOLUTION (1 OF 2)

The analyst attempts to identify the alternative (the set of decision variable values) that provides the "best" output for the model.

- The "best" output is the **optimal solution**.
- If the alternative does not satisfy all of the model constraints, it is rejected as being **infeasible**, regardless of the objective function value.
- If the alternative satisfies all of the model constraints, it is **feasible** and a candidate for the "best" solution.

# MODEL SOLUTION (2 OF 2)

Trial-and-Error Solution for Production Problem

| Production Quantity | Projected Profit | Total Hours of Production | Feasible Solution |
|---|---|---|---|
| 0 | 0 | 0 | Yes |
| 2 | 20 | 0 | Yes |
| 4 | 40 | 20 | Yes |
| 6 | 60 | 30 | Yes |
| 8 | 80 | 40 | Yes |
| 10 | 100 | 50 | No |
| 12 | 120 | 60 | No |

# MODEL TESTING AND VALIDATION

Often, goodness/accuracy of a model cannot be assessed until solutions are generated.

- Small test problems having known, or at least expected, solutions can be used for model testing and validation.
- If the model generates expected solutions, use the model on the full-scale problem.
- If inaccuracies or potential shortcomings inherent in the model are identified, take corrective action such as:
  - Collection of more-accurate input data
  - Modification of the model

# REPORT GENERATION

A managerial report, based on the results of the model, should be prepared.

- The report should be easily understood by the decision maker.
- The report should include:
  - the recommended decision
  - other pertinent information about the results (for example, how sensitive the model solution is to the assumptions and data used in the model)

# MODELS OF COST, REVENUE, AND PROFIT 

Some of the most basic quantitative models arising in business and economic applications are those involving the relationship between a volume variable—such as production volume or sales volume—and cost, revenue, and profit.

Through the use of these models, a manager can determine the projected cost, revenue, and/or profit associated with an established production quantity or a forecasted sales volume.

# MODELS OF COST, REVENUE, AND PROFIT (2 OF 9)

The cost of manufacturing or producing a product is a function of the volume produced. This cost can usually be defined as a sum of two costs: fixed cost and variable cost.

- **Fixed cost** is the portion of the total cost that does not depend on the production volume; this cost remains the same no matter how much is produced.
- **Variable cost**, on the other hand, is the portion of the total cost that is dependent on and varies with the production volume.

# MODELS OF COST, REVENUE, AND PROFIT (3 OF 9)

Nowlin Plastics produces a line of cell phone covers. Nowlin's best-selling cover is its Viper model, a slim but very durable black and gray plastic cover.

Several products are produced on the same manufacturing line, and a setup cost is incurred each time a changeover is made for a new product.

- Suppose that the setup cost for the Viper is \$3000. This setup cost is a fixed cost that is incurred regardless of the number of units eventually produced.
- In addition, suppose that variable labor and material costs are \$2 for each unit produced.

# MODELS OF COST, REVENUE, AND PROFIT (4 OF 9)

The cost–volume model for producing $x$ units of the Viper can be written as

$$C(x) = 3000 + 2x$$

where

$$x = \text{production volume in units}$$
$$C(x) = \text{total cost of producing } x \text{ units}$$

# MODELS OF COST, REVENUE, AND PROFIT 

**Marginal cost** is defined as the rate of change of the total cost with respect to production volume. That is, it is the cost increase associated with a one-unit increase in the production volume.

$$C(x) = 3000 + 2x$$

where

$$x = \text{production volume in units}$$
$$C(x) = \text{total cost of producing } x \text{ units}$$

In the cost model of equation (1.3), we see that the total cost $C(x)$ will increase by \$2 for each unit increase in the production volume. Thus, the marginal cost is \$2.

# MODELS OF COST, REVENUE, AND PROFIT (6 OF 9)

Suppose that each Viper cover sells for $5. The model for total revenue can be written as

$$R(x) = 5x$$

where

$$x = \text{sales volume in units}$$
$$R(x) = \text{total revenue associated with selling } x \text{ units}$$

<u>**Marginal revenue**</u> is defined as the rate of change of total revenue with respect to sales volume. That is, it is the increase in total revenue resulting from a one-unit increasein sales volume. In the model of equation above, we see that the marginal revenue is $5.

# MODELS OF COST, REVENUE, AND PROFIT (7 OF 9)

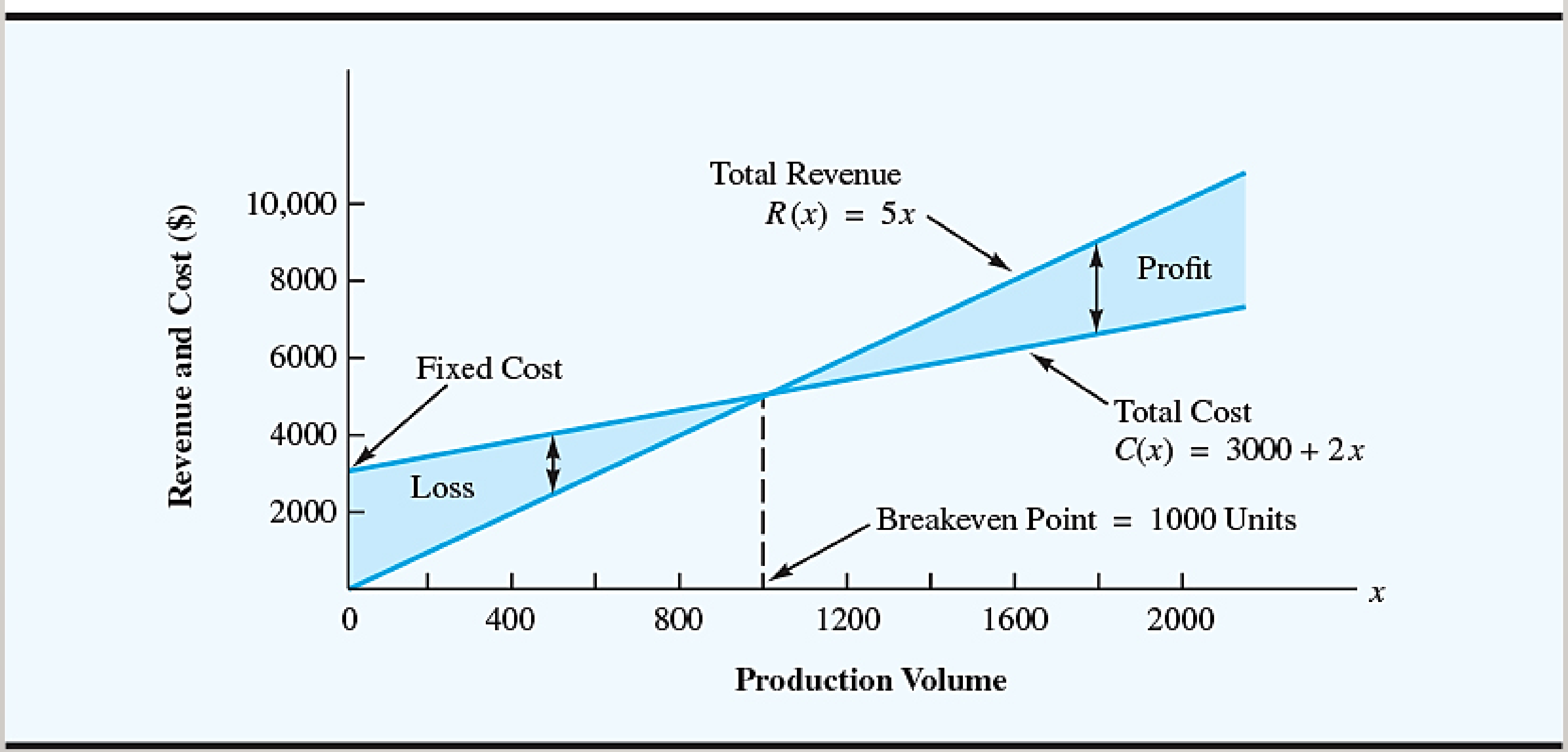
Total profit, denoted *P*(*x*), is total revenue minus total cost; therefore, the following model provides the total profit associated with producing and selling *x* units:

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 5x - (3000 + 2x) = -3000 + 3x \end{aligned}$$

Using this equation, we can now determine the total profit associated with any production volume *x*.

# MODELS OF COST, REVENUE, AND PROFIT 

The volume that results in total revenue equaling total cost (providing $0 profit) is called the **breakeven point**.



# MODELS OF COST, REVENUE, AND PROFIT (9 OF 9)

The breakeven point can be found by setting the total profit expression equal to zero and solving for the production volume. Using equation (1.5), we have

$$P(x) = -3000 + 3x$$
$$0 = -3000 + 3x$$
$$3x = 3000$$
$$x = 1000$$

With this information, we know that production and sales of the product must be greater than 1000 units before a profit can be expected.

# MANAGEMENT SCIENCE TECHNIQUES 

- Linear Programming
- Integer Linear Programming
- Nonlinear Programming
- PERT/CPM
- Inventory Models
- Walking Line Models
- Simulation
- Decision Analysis
- Goal Programming
- Analytic Hierarchy Process
- Forecasting
- Markov-Process Models
- Distribution/Network Models

# MANAGEMENT SCIENCE TECHNIQUES (2 OF 6)

- **Linear programming** is a problem-solving approach developed for situations involving maximizing or minimizing a linear function subject to linear constraints that limit the degree to which the objective can be pursued.
- **Integer linear programming** is an approach used for problems that can be set up as linear programs with the additional requirement that some or all of the decision recommendations be integer values.
- **Network models** are specialized solution procedures for problems in transportation system design, information system design, project scheduling, …..

# MANAGEMENT SCIENCE TECHNIQUES (3 OF 6)

- **Project scheduling: PERT** (Program Evaluation and Review Technique) **and CPM** (Critical Path Method) help managers responsible for planning, scheduling, and controlling projects that consist of numerous separate jobs or tasks performed by a variety of departments, individuals, and so forth.
- **Inventory models** are used by managers faced with the dual problems of maintaining sufficient inventories to meet demand for goods and, at the same time, incurring the lowest possible inventory holding costs.

# MANAGEMENT SCIENCE TECHNIQUES (4 OF 6)

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# MANAGEMENT SCIENCE TECHNIQUES (5 OF 6)

- **Waiting line (or queuing) models** help managers understand and make better decisions concerning the operation of systems involving waiting lines.
- **Simulation** is a technique used to model the operation of a system. This technique employs a computer program to model the operation and perform simulation computations.
- **Decision analysis** can be used to determine optimal strategies in situations involving several decision alternatives and an uncertain pattern of future events.
- **Forecasting methods** are techniques that can be used to predict future aspects of a business operation.

# MANAGEMENT SCIENCE TECHNIQUES (6 OF 6)

- **Goal programming** is a technique for solving multi-criteria decision problems, usually within the framework of linear programming.
- **Analytic hierarchy process** is a multi-criteria decision-making technique that permits the inclusion of subjective factors in arriving at a recommended decision.
- **Markov-process models** are useful in studying the evolution of certain systems over repeated trials (such as describing the probability that a machine, functioning in one period, will function or break down in another period).

# METHODS USED MOST FREQUENTLY

- Linear programming
- Integer programming
- Network models (such as transportation and transshipment models)
- Simulation

# CHAPTER 2: AN INTRODUCTION TO LINEAR PROGRAMMING

# LINEAR PROGRAMMING (1 OF 2)

- Linear programming has nothing to do with computer programming.
- The use of the word "programming" here means "choosing a course of action."
- Linear programming involves choosing a course of action when the mathematical model of the problem contains only linear functions.

# LINEAR PROGRAMMING (2 OF 2)

- The **maximization** or **minimization** of some quantity is the **objective** in all linear programming problems.
- All LP problems have **constraints** that limit the degree to which the objective can be pursued.
- A **feasible solution** satisfies all the problem's constraints.
- An **optimal solution** is a feasible solution that results in the largest possible objective function value when maximizing (or smallest when minimizing).
- A **graphical solution method** can be used to solve a linear program with two variables.

# GUIDELINES FOR MODEL FORMULATION

**Problem formulation or modeling** is the process of translating a verbal statement of a problem into a mathematical statement.

- Understand the problem thoroughly.
- Describe the objective.
- Describe each constraint.
- Define the decision variables.
- Write the objective in terms of the decision variables.
- Write the constraints in terms of the decision variables.

# A SIMPLE MAXIMIZATION PROBLEM (1 OF 4)

Par, Inc., is a small manufacturer of golf equipment and supplies whose management has decided to move into the market for medium- and high-priced golf bags. Par, Inc.'s distributor has agreed to buy all the golf bags Par, Inc., produces over the next three months.

Each golf bag produced will require the following operations:

1. Cutting and dyeing the material
2. Sewing
3. Finishing (inserting umbrella holder, club separators, etc.)
4. Inspection and packaging

# A SIMPLE MAXIMIZATION PROBLEM 

This production information is summarized in this table:

| | Production Time (hours) | |
|---|---|---|
| **Department** | **Standard Bag** | **Deluxe Bag** |
| Cutting and Dyeing | 7/10 | 1 |
| Sewing | 1/2 | 5/6 |
| Finishing | 1 | 2/3 |
| Inspection and Packaging | 1/10 | 1/4 |

# A SIMPLE MAXIMIZATION PROBLEM (3 OF 4)

- Par, Inc.'s production is constrained by a limited number of hours available in each department. The director of manufacturing estimates that 630 hours for cutting and dyeing, 600 hours for sewing, 708 hours for finishing, and 135 hours for inspection and packaging will be available for the production of golf bags during the next three months.
- The accounting department analyzed the production data and arrived at prices for both bags that will result in a profit contribution1 of $10 for every standard bag and $9 for every deluxe bag produced.

# A SIMPLE MAXIMIZATION PROBLEM (4 OF 4)

The complete model for the Par, Inc., problem is as follows:

| | Production Time (hours) | |
|---|---|---|
| **Department** | **Standard Bag** | **Deluxe Bag** |
| Cutting and Dyeing | 7/10 | 1 |
| Sewing | 1/2 | 5/6 |
| Finishing | 1 | 2/3 |
| Inspection and Packaging | 1/10 | 1/4 |

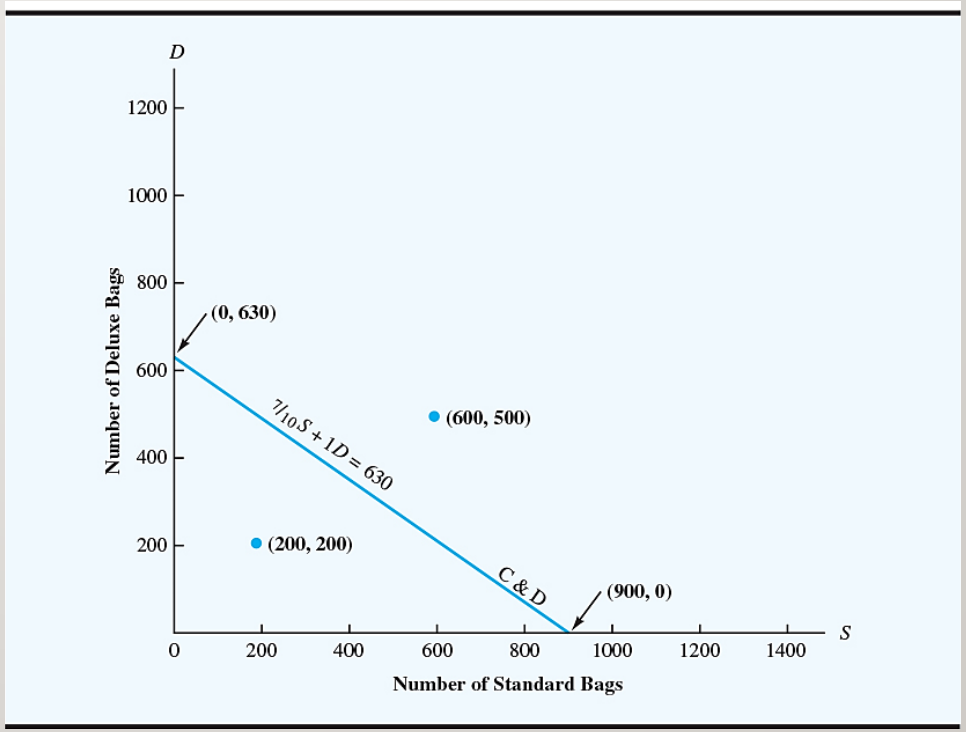
$$
\begin{aligned}
&\text{Max} \quad 10S + 9D \\
&\text{subject to (s.t.)} \\
&\qquad \tfrac{7}{10}S + 1D \le 630 \quad \text{Cutting and Dyeing} \\
&\qquad \tfrac{1}{2}S + \tfrac{5}{6}D \le 600 \quad \text{Sewing} \\
&\qquad 1S + \tfrac{2}{3}D \le 708 \quad \text{Finishing} \\
&\qquad \tfrac{1}{10}S + \tfrac{1}{4}D \le 135 \quad \text{Inspection and Packaging} \\
&\qquad S, D \ge 0
\end{aligned}
$$

# GRAPHICAL SOLUTION PROCEDURE (1 OF 5)

Earlier, we saw that the inequality representing the cutting and dyeing constraint is:

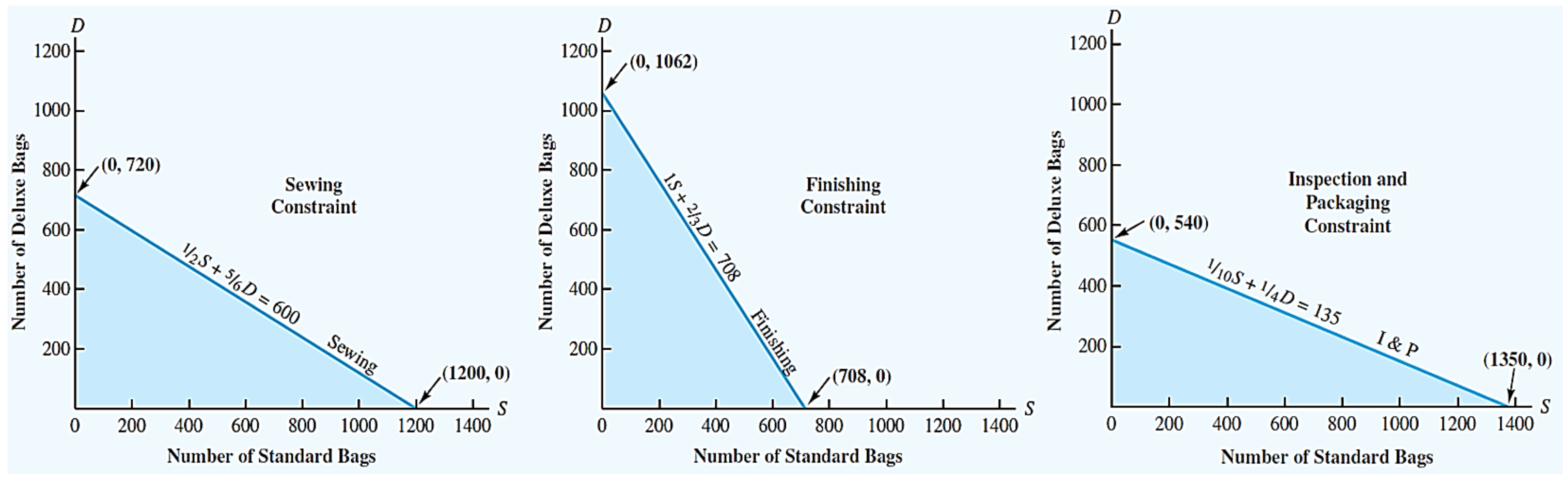
$$\frac{7}{10}S + 1D \leq 630$$

To show all solution points that satisfy this relationship, we start by graphing the solution points satisfying the constraint as an equality.



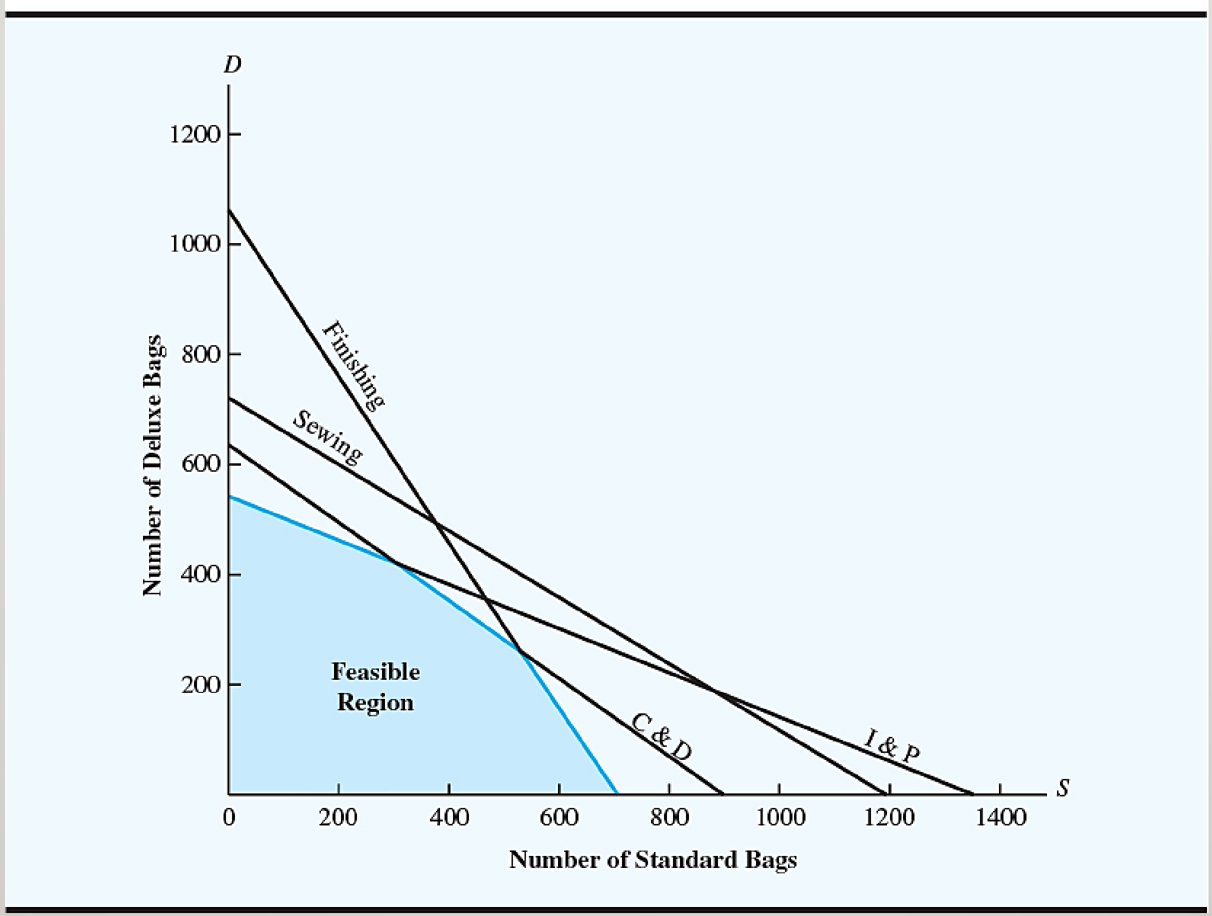
# GRAPHICAL SOLUTION PROCEDURE (2 OF 5)

We continue by identifying the solution points satisfying each of the other three constraints.



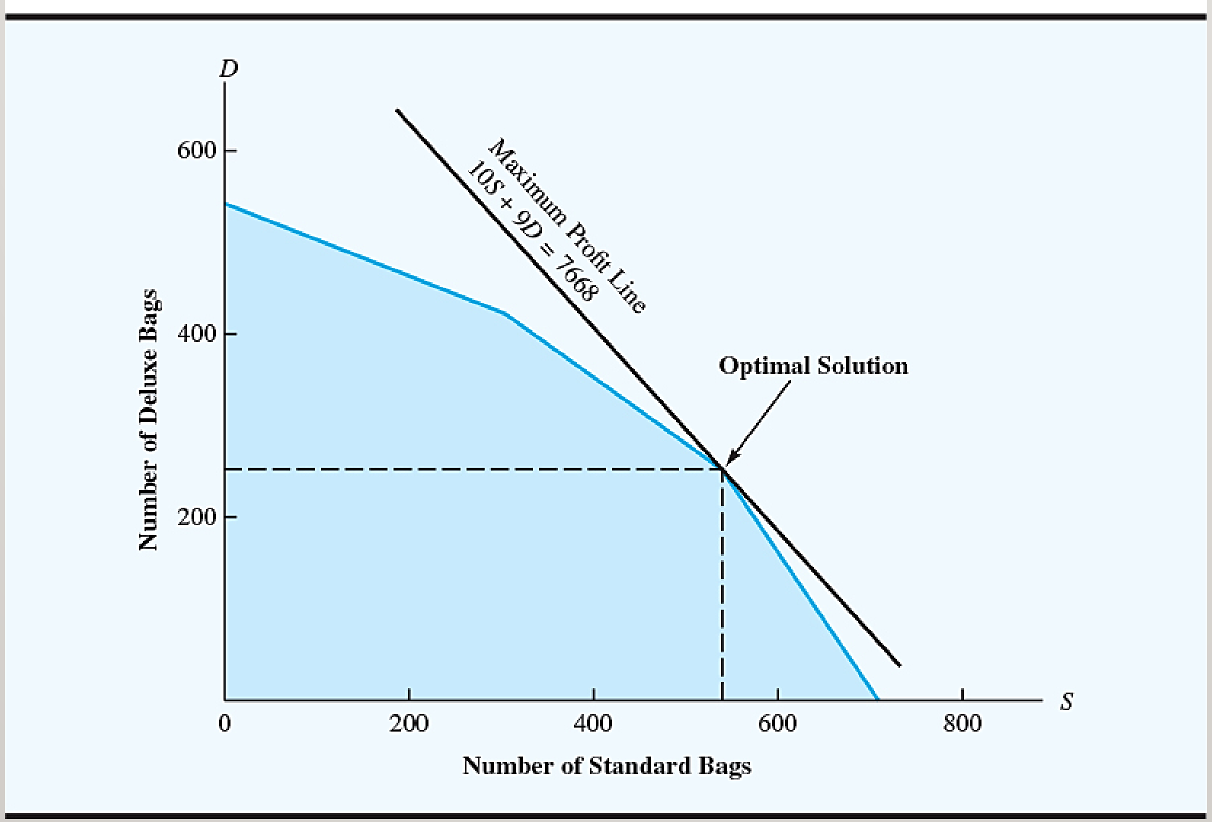
# GRAPHICAL SOLUTION PROCEDURE (3 OF 5)

The graph shown identifies the feasible region:



# GRAPHICAL SOLUTION PROCEDURE (4 OF 5)

The optimal solution point is at the intersection of the cutting and dyeing and the finishing constraint lines.



# GRAPHICAL SOLUTION PROCEDURE (5 OF 5)

The optimal values of the decision variables *S* and *D* must satisfy dyeing and the finishing constraints simultaneously.

$$\frac{7}{10}S + 1D = 630 \qquad \text{Dyeing Constraint}$$

$$1S + \frac{2}{3}D = 708 \qquad \text{Finishing Constraint}$$

This system of equations can be solved using substitution.

The exact location of the optimal solution point is $S = 540$ and $D = 252$. The optimal production quantities for Par, Inc., are 540 standard bags and 252 deluxe bags, with a resulting profit contribution of 10(540) + 9(252) = $7,668.

# SUMMARY OF THE GRAPHICAL SOLUTION PROCEDURE FOR MAXIMIZATION PROBLEMS

1. Prepare a graph of the feasible solutions for each of the constraints.
2. Determine the feasible region that satisfies all the constraints simultaneously.
3. Draw an objective function line.
4. Move parallel objective function lines toward larger objective function values without entirely leaving the feasible region.
5. Any feasible solution on the objective function line with the largest value is an optimal solution.

# SLACK AND SURPLUS VARIABLES (1 OF 2)

- A linear program in which all the variables are non-negative and all the constraints are equalities is said to be in **standard form**.
- Standard form is attained by adding **slack variables** to "less than or equal to" constraints, and by subtracting **surplus variables** from "greater than or equal to" constraints.
- Slack and surplus variables represent the difference between the left and right sides of the constraints.
- Slack and surplus variables have objective function coefficients equal to 0.

# SLACK AND SURPLUS VARIABLES (2 OF 2)

The complete solution tells management that the production of 540 standard bags and 252 deluxe bags will require all available cutting and dyeing time (630 hours) and all available finishing time (708 hours), while 600 - 480 = 120 hours of sewing time and 135 - 117 = 18 hours of inspection and packaging time will remain unused. The 120 hours of unused sewing time and 18 hours of unused inspection and packaging time are referred to as slack for the two departments.

| Constraint | Hours Required for *S* = 540 and *D* = 252 | Hours Available | Unused Hours |
|---|---|---|---|
| Cutting and Dyeing | 7/10(540) + 1(252) = 630 | 630 | 0 |
| Sewing | 1/2(540) + 5/6(252) = 480 | 600 | 120 |
| Finishing | 1(540) + 2/3(252) = 708 | 708 | 0 |
| Inspection and Packaging | 1/10(540) + 1/4(252) = 117 | 1/4 | 18 |

# SLACK VARIABLES (1 OF 2)

Often slack variables, are added to the formulation of a linear programming problem to represent the slack, or idle capacity. Unused capacity makes no contribution to profit; thus, slack variables have coefficients of zero in the objective function. After the addition of four slack variables, denoted as $S_1, S_2, S_3,$ and $S_4$, the mathematical model of the Par, Inc., problem becomes

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Max | $10S$ | + | $9D$ | + | $0S_1$ | + | $0S_2$ | + | $0S_3$ | + | $0S_4$ | |
| s.t. | | | | | | | | | | | | |
| | ${}^{7}/_{10}S$ | + | $1D$ | + | $1S_1$ | + | | + | | + | | = 630 |
| | ${}^{1}/_{2}S$ | + | ${}^{5}/_{6}D$ | + | | + | $1S_2$ | + | | + | | = 600 |
| | $1S$ | + | ${}^{2}/_{3}D$ | + | | + | | + | $1S_3$ | + | | = 708 |
| | ${}^{1}/_{10}S$ | + | ${}^{1}/_{4}D$ | + | | + | | + | | + | $1S_4$ | = 135 |
| | $S, D, S_1, S_2, S_3, S_4 \geq 0$ | | | | | | | | | | | |

# SLACK VARIABLES (2 OF 2)

Referring to the standard form of the Par, Inc., problem, we see that at the optimal solution (*S* = 540 and *D* = 252), the values for the slack variables are

| Constraint | Value of Slack Variable |
|---|---|
| Cutting and Dyeing | $S_1 = 0$ |
| Sewing | $S_2 = 120$ |
| Finishing | $S_3 = 0$ |
| Inspection and Packaging | $S_4 = 18$ |

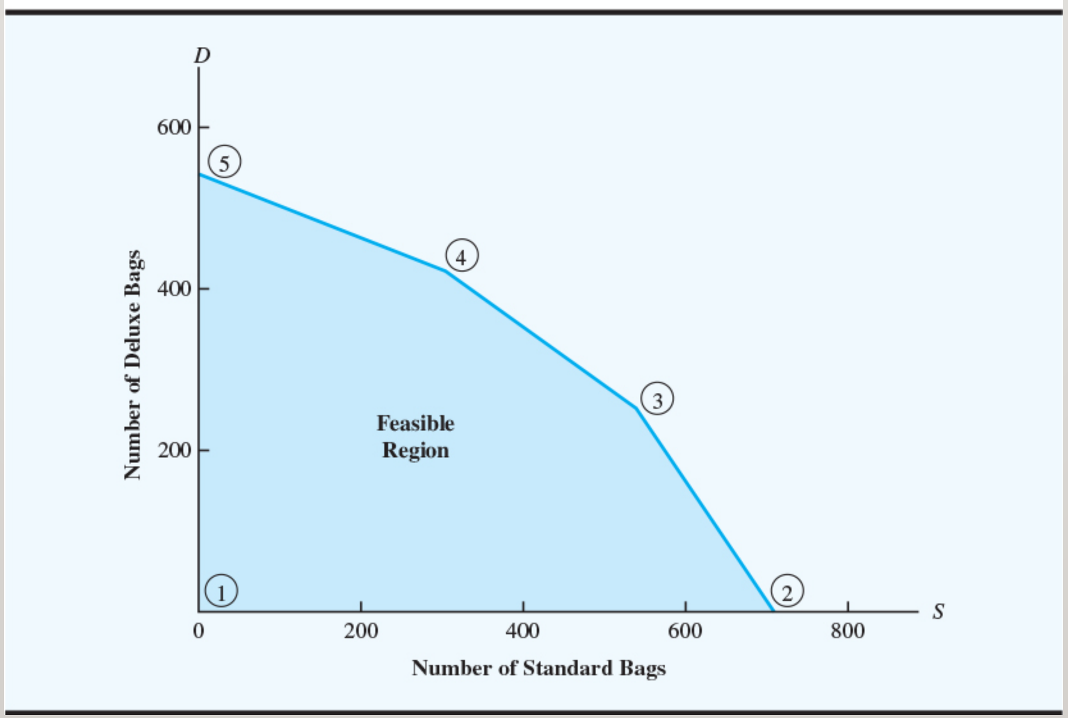
On the other hand, the sewing and the inspection and packaging constraints are not binding the feasible region at the optimal solution, which means we can expect some unused time or slack for these two operations.

# EXTREME POINTS AND THE OPTIMAL SOLUTION (1 OF 2)

- The corners or vertices of the feasible region are referred to as the **extreme points**.
- An optimal solution to an LP problem can be found at an extreme point of the feasible region.
- When looking for the optimal solution, you do not have to evaluate all feasible solution points.
- You have to consider only the extreme points of the feasible region.

# EXTREME POINTS AND THE OPTIMAL SOLUTION (2 OF 2)

Here are the 5 extreme points of the feasible region for the Par, Inc., Problem:



# COMPUTER SOLUTIONS (1 OF 3)

- LP problems involving 1000s of variables and 1000s of constraints are now routinely solved with computer packages.
- Linear programming solvers are now part of many spreadsheet packages, such as Microsoft Excel.
- Leading commercial packages include CPLEX, LINGO, MOSEK, Xpress-MP, and Premium Solver for Excel.

# COMPUTER SOLUTIONS (2 OF 3)

Here is a computer solution to the Par, Inc., Problem.

Optimal Objective Value = 7668.00000

| Variable | Value | Reduced Cost |
|---|---|---|
| S | 540.00000 | 0.00000 |
| D | 252.00000 | 0.00000 |

| Constraint | Slack/Surplus | Dual Value |
|---|---|---|
| 1 | 0.00000 | 4.37500 |
| 2 | 120.00000 | 0.00000 |
| 3 | 0.00000 | 6.93750 |
| 4 | 18.00000 | 0.00000 |

| Variable | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|
| S | 10.00000 | 3.50000 | 3.70000 |
| D | 9.00000 | 5.28571 | 2.33333 |

| Constraint | RHS Value | Allowable Increase | Allowable Decrease |
|---|---|---|---|
| 1 | 630.00000 | 52.36364 | 134.40000 |
| 2 | 600.00000 | Infinite | 120.00000 |
| 3 | 708.00000 | 192.00000 | 128.00000 |
| 4 | 135.00000 | Infinite | 18.00000 |

# COMPUTER SOLUTIONS (3 OF 3)

1. Prepare a graph of the feasible solutions for each of the constraints.
2. Determine the feasible region that satisfies all the constraints simultaneously.
3. Draw an objective function line.
4. Move parallel objective function lines toward <u>smaller</u> objective function values without entirely leaving the feasible region.
5. Any feasible solution on the objective function line with the <u>smallest</u> value is an optimal solution.

# A SIMPLE MINIMIZATION PROBLEM (1 OF 6)

M&D Chemicals produces two products that are sold as raw materials to companies manufacturing bath soaps and laundry detergents.

- M&D's management specified that the combined production for products A and B must total at least 350 gallons.
- A customer ordered 125 gallons of product A.
- Product A requires 2 hours of processing time per gallon.
- Product B requires 1 hour of processing time per gallon.
- 600 hours of processing time are available.
- M&D's objective is to satisfy these requirements at a minimum total production cost.
- Production costs are \$2 per gallon for product A and \$3 per gallon for product B.

# A SIMPLE MINIMIZATION PROBLEM (2 OF 6)

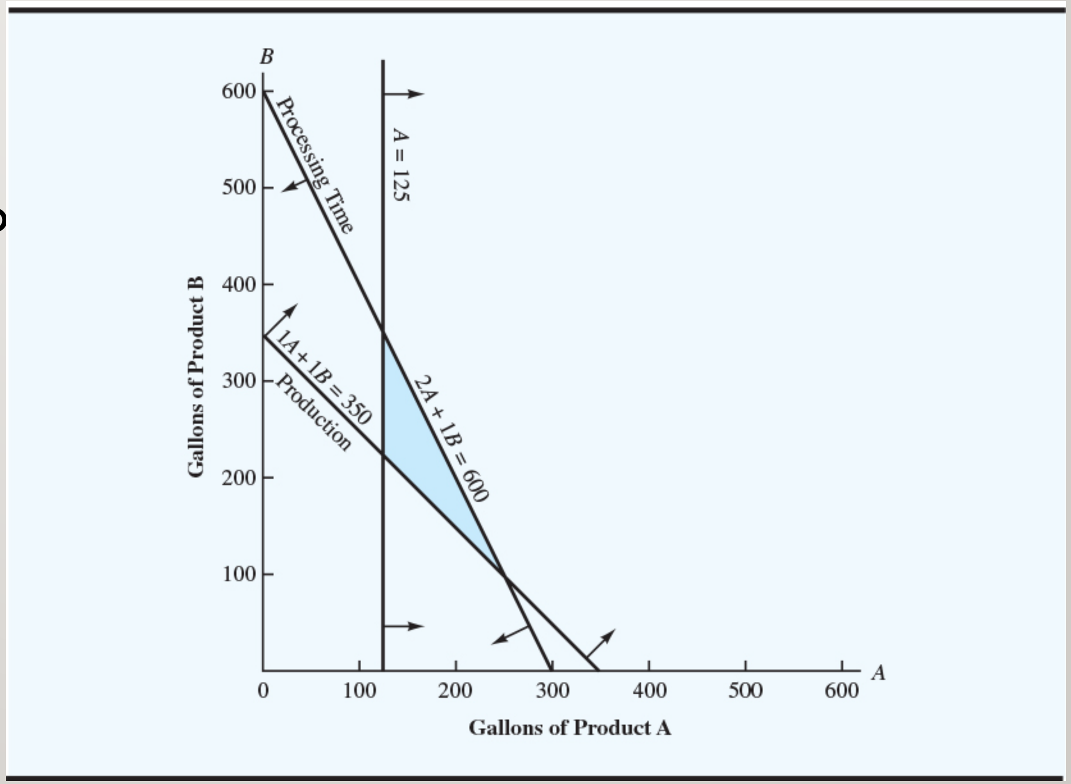
After adding the nonnegativity constraints ($A, B \geq 0$), we arrive at the following linear program for the M&D Chemicals problem:

$$
\begin{array}{llll}
\text{Min} & 2A + 3B & & \\
\text{s.t.} & & & \\
 & 1A \phantom{{}+1B} \geq 125 & & \text{Demand for product A} \\
 & 1A + 1B \geq 350 & & \text{Total production} \\
 & 2A + 1B \leq 600 & & \text{Processing time} \\
 & \quad A, B \geq 0 & &
\end{array}
$$

# A SIMPLE MINIMIZATION PROBLEM (3 OF 6)

Here is the feasible region for the M&D Chemicals problem:

Note that the objective functio
2A + 3B = 800 intersects the
feasible region at the extreme
point A = 250, B = 100.



# A SIMPLE MINIMIZATION PROBLEM (4 OF 6)

The optimal solution to the M&D Chemicals problem shows that the desired total production of A + B = 350 gallons is achieved by using all processing time: 2A + 1B = 2(250) + 1(100) = 600 hours.

Note that the constraint requiring that product A demand be met has been satisfied with A = 250 gallons. In fact, the production of product A exceeds its minimum level by 250 – 125 = 125 gallons.

This excess production for product A is referred to as *surplus*.

# A SIMPLE MINIMIZATION PROBLEM (5 OF 6)

Including two surplus variables, $S_1$ and $S_2$, for the ≥ constraints and one slack variable, $S_3$, for the ≤ constraint, the linear programming model of the M&D Chemicals problem becomes

$$
\begin{array}{ll}
\text{Min} & 2A + 3B + 0S_1 + 0S_2 + 0S_3 \\
\text{s.t.} & \\
& 1A - 1S_1 = 125 \\
& 1A + 1B - 1S_2 = 350 \\
& 2A + 1B + 1S_3 = 600 \\
& \quad A, B, S_1, S_2, S_3 \geq 0
\end{array}
$$

All the constraints are now equalities.

# A SIMPLE MINIMIZATION PROBLEM (6 OF 6)

At the optimal solution of A = 250 and B = 100, the values of the surplus and slack variables are as follows:

| Constraint | Value of Surplus or Slack Variables |
|---|---|
| Demand for product A | $S_1 = 125$ |
| Total production | $S_2 = 0$ |
| Processing time | $S_3 = 0$ |

# COMPUTER SOLUTION

Optimal Objective Value = 800.00000

| Variable | Value | Reduced Cost |
|---|---|---|
| A | 250.00000 | 0.00000 |
| B | 100.00000 | 0.00000 |

| Constraint | Slack/Surplus | Dual Value |
|---|---|---|
| 1 | 125.00000 | 0.00000 |
| 2 | 0.00000 | 4.00000 |
| 3 | 0.00000 | –1.00000 |

| Variable | Objective Coefficient | Allowable Increase | Allowable Decrease |
|---|---|---|---|
| A | 2.00000 | 1.00000 | Infinite |
| B | 3.00000 | Infinite | 1.00000 |

| Constraint | RHS Value | Allowable Increase | Allowable Decrease |
|---|---|---|---|
| 1 | 125.00000 | 125.00000 | Infinite |
| 2 | 350.00000 | 125.00000 | 50.00000 |
| 3 | 600.00000 | 100.00000 | 125.00000 |

# FEASIBLE REGION

- The feasible region for a two-variable LP problem can be nonexistent, a single point, a line, a polygon, or an unbounded area.
- Any linear program falls in one of four categories:
  - is infeasible
  - has a unique optimal solution
  - has alternative optimal solutions
  - has an objective function that can be increased without bound
- A feasible region may be unbounded and yet there may be optimal solutions. This is common in minimization problems and is possible in maximization problems.

# SPECIAL CASES (1 OF 5)

## Alternative Optimal Solutions

In the graphical method, if the objective function line is parallel to a boundary constraint in the direction of optimization, there are **alternate optimal solutions**, with all points on this line segment being optimal.

# SPECIAL CASES (2 OF 5)

Let's return to the Par, Inc., problem. However, now assume that the profit for the standard golf bag (*S*) has decreased to \$6.30. The revised objective function becomes 6.3*S* + 9*D*.

The objective function values at these two extreme points are identical:

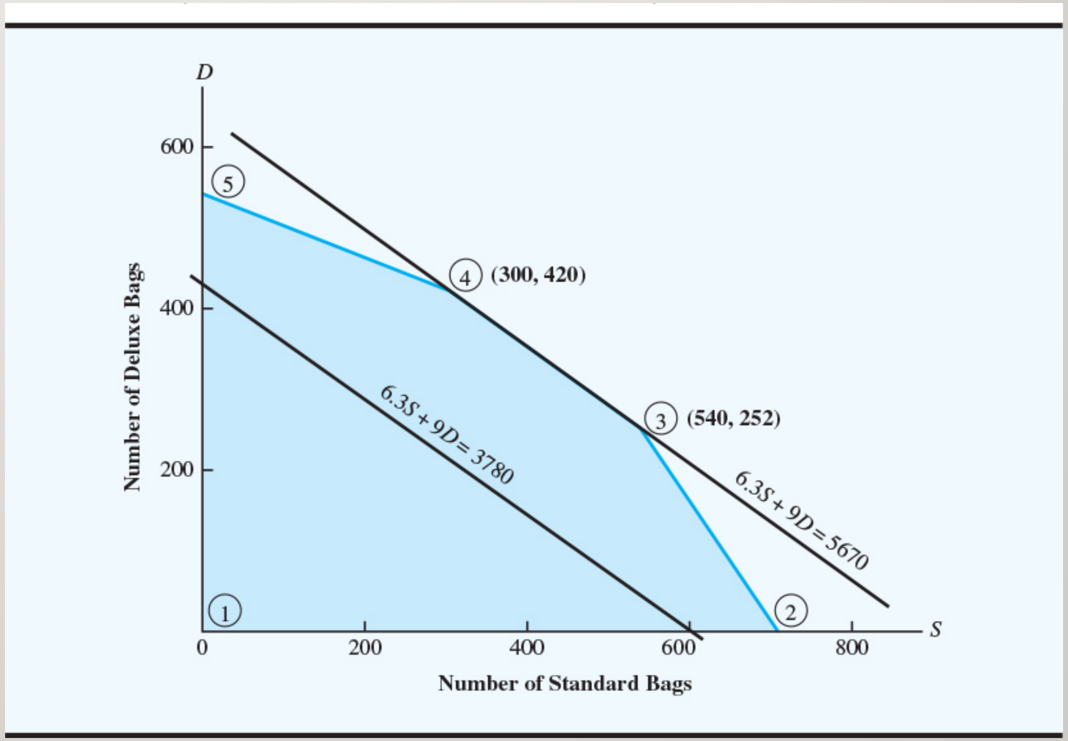
6.3S + 9D =

6.3(300)+9(420) = 5670

and

6.3S + 9D =

6.3(540)+9(252) = 5670



# SPECIAL CASES (3 OF 5)

Furthermore, any point on the line connecting the two optimal extreme points also provides an optimal solution.

For example, the solution point ($S$ = 420, $D$ = 336), which is halfway between the two extreme points, also provides the optimal objective function value of $6.3S + 9D = 6.3(420) + 9(336) = 5670$.

A linear programming problem with alternative optimal solutions is generally a good situation for the manager or decision maker. It means that several combinations of the decision variables are optimal and that the manager can select the most desirable optimal solution.

# SPECIAL CASES (4 OF 5)

Infeasibility

- No solution to the LP problem satisfies all the constraints, including the non-negativity conditions.
- Graphically, this means a feasible region does not exist.
- Causes include:
  - A formulation error has been made.
  - Management's expectations are too high.
  - Too many restrictions have been placed on the problem (i.e. the problem is over-constrained).

# SPECIAL CASES (5 OF 5)

Unboundeded

- The solution to a maximization LP problem is unbounded if the value of the solution may be made indefinitely large without violating any of the constraints.
- For real problems, this is the result of improper formulation.  (Quite likely, a constraint has been inadvertently omitted.)

# GENERAL LINEAR PROGRAMMING NOTATION (1 OF 3)

We selected decision-variable names of S and D in the Par, Inc., problem and A and B in the M&D Chemicals problem to make it easier to recall what these decision variables represented in the problem.

Although this approach works well for linear programs involving a small number of decision variables, it can become difficult when dealing with problems involving a large number of decision variables.

# GENERAL LINEAR PROGRAMMING NOTATION (2 OF 3)

A more general notation that is often used for linear programs uses the letter *x* with a subscript.

In the Par, Inc., problem, we could have defined the decision variables:

$x_1$ = number of standard bags

$x_2$ = number of deluxe bags

In the M&D Chemicals problem, the same variable names would be used, but their definitions would change:

$x_1$ = number of gallons of product A

$x_2$ = number of gallons of product B

# GENERAL LINEAR PROGRAMMING NOTATION (3 OF 3)

A disadvantage of using general notation for decision variables is that we are no longer able to easily identify what the decision variables actually represent in the mathematical model.

The advantage of general notation is that formulating a mathematical model for a problem that involves a large number of decision variables is much easier.