CHAPTER 7: INTEGER LINEAR PROGRAMMING

7.1 - Types of Integer Linear Programming Models

7.2 - Graphical and Computer Solutions for an All-Integer Linear Program

7.3 - Applications Involving 0-1 Variables

7.4 - Modeling Flexibility Provided by 0-1 Variables

TYPES OF INTEGER PROGRAMMING MODELS

- An LP in which all the variables are restricted to be integers is called an **all-integer linear program** (ILP).
- The LP that results from dropping the integer requirements is called the **LP Relaxation** of the ILP.
- If only a subset of the variables are restricted to be integers, the problem is called a **mixed-integer linear program** (MILP).
- Binary variables are variables whose values are restricted to be 0 or 1. If all variables are restricted to be 0 or 1, the problem is called a **0-1 or binary integer linear program**.

GRAPHICAL AND COMPUTER SOLUTIONS FOR AN ALL-INTEGER LINEAR PROGRAM (1 OF 3)

- Eastborne Realty has $2 million for new rental property.
- Eastborne wants to invest in townhouses and apartment buildings.
- Each townhouse can be purchased for $282,000. 5 are available.
- Each apartment building can be purchased for $400,000, and the developer will construct as many as Eastborne wants to purchase.
- Eastborne's property manager can devote up to 140 hours per month to these new properties.
- Each townhouse is expected to require 4 hours per month.
- Each apartment building is expected to require 40 hours per month.
- The annual cash flow, after deducting mortgage payments and operating expenses, is estimated to be $10,000 per townhouse and $15,000 per apartment building.
- Eastborne's owner would like to determine the number of townhouses and the number of apartment buildings to purchase to maximize annual cash flow.

GRAPHICAL AND COMPUTER SOLUTIONS FOR AN ALL-INTEGER LINEAR PROGRAM (2 OF 3)

The Decision Variables

T = number of townhouses to purchase

A = number of apartment buildings to purchase

The objective function for cash flow (in thousands of dollars) is:

$$\text{Max } 10T + 15A$$

Three constraints must be satisfied:

$282T + 400A \leq 2000$	Funds available ($1000s)
$4T + 40A \leq 140$	Manager’s time (hours)
$T \leq 5$	Townhouses available

The variables T and A must be nonnegative integers.

GRAPHICAL AND COMPUTER SOLUTIONS FOR AN ALL-INTEGER LINEAR PROGRAM (3 OF 3)

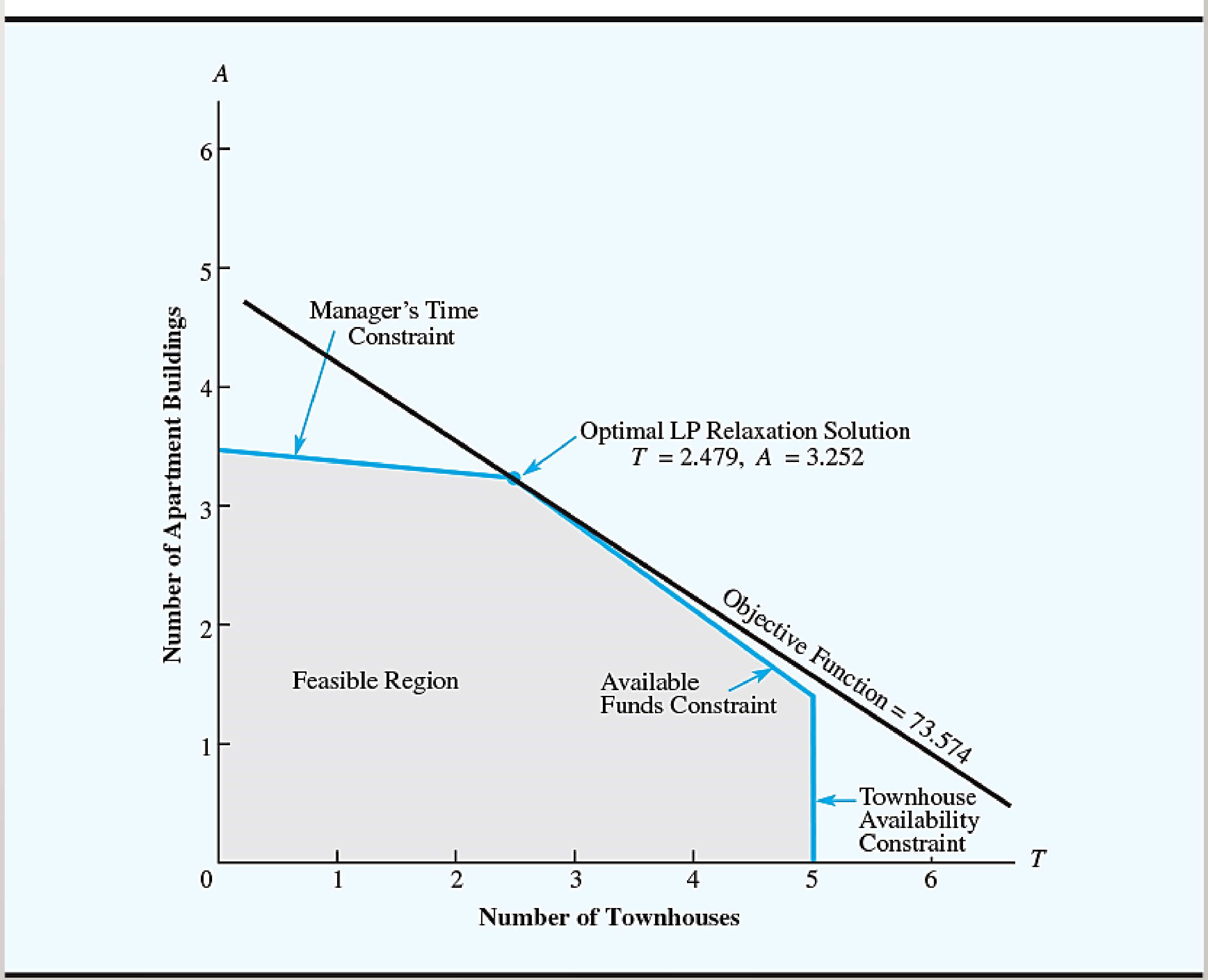
Suppose that we drop the integer requirements for *T* and *A* and solve the LP Relaxation of the Eastborne Realty problem.

Solution:

T = 2.479 townhouses and

A = 3.252 apartment buildings. The optimal value of the objective function is 73.574, which indicates an annual cash flow of $73,574.

But, Eastborne cannot purchase fractional numbers of townhouses and apartment buildings.

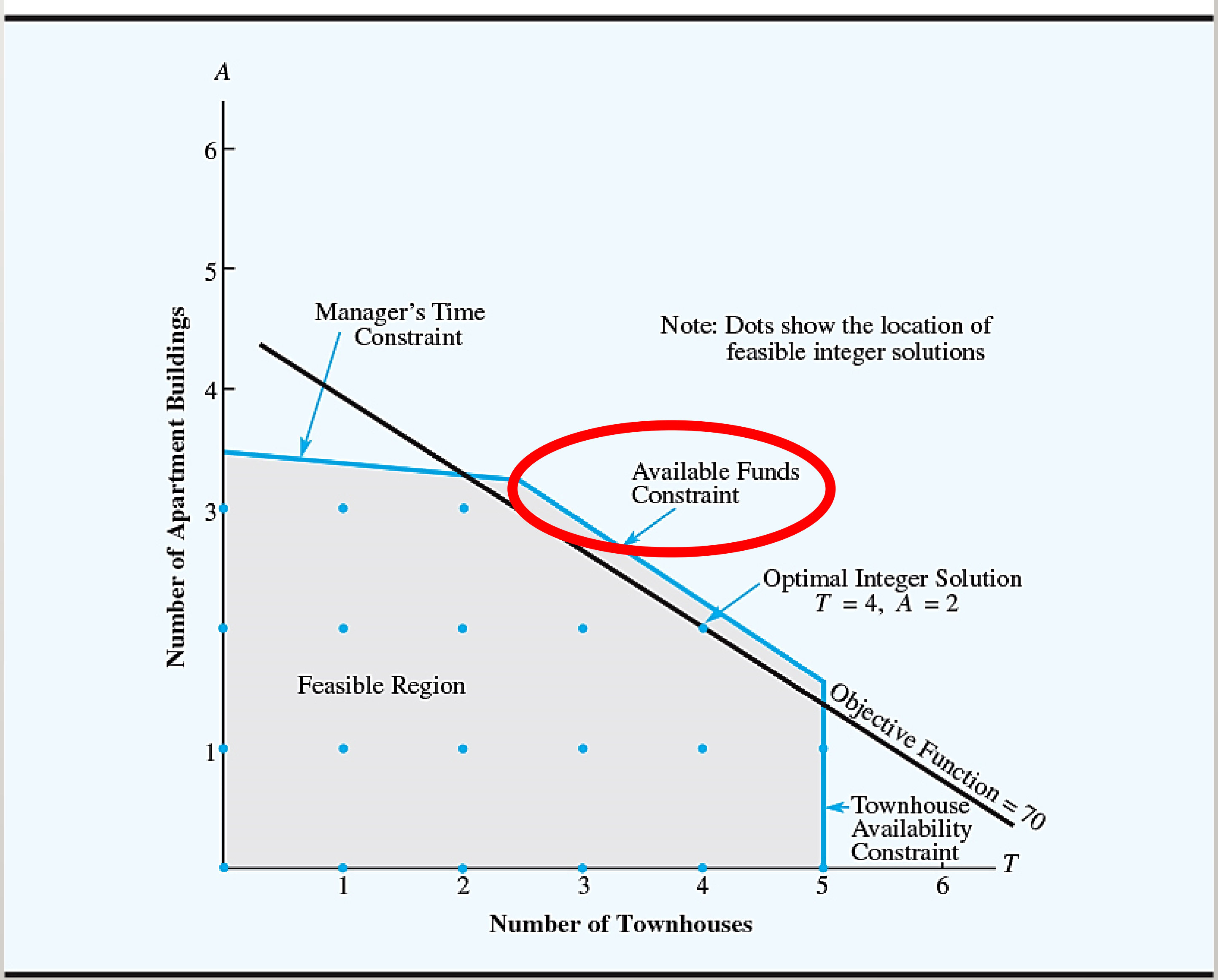


ROUNDING TO OBTAIN AN INTEGER SOLUTION

- In many cases, a noninteger solution can be rounded to obtain an acceptable integer solution. However, rounding may not always be a good strategy.
- Suppose that we round the solution to the LP Relaxation to obtain the integer solution $T = 2$ and $A = 3$, with an objective function value of $10(2) + 15(3) = 65$. The annual cash flow of $65,000 is substantially less than the annual cash flow of $73,574 provided by the solution to the LP Relaxation.
- $T = 3$ and $A = 3$ is infeasible because it requires more funds than the $2,000,000 Eastborne has available. The rounded solution of $T = 2$ and $A = 4$ is also infeasible for the same reason.

GRAPHICAL SOLUTION OF THE ALL-INTEGER PROBLEM (1 OF 2)

- The graph of the feasible region is drawn exactly as in the LP Relaxation of the problem.
- Show the integer solutions as dots.
- Move the objective function line to the best extreme point in the feasible region.



GRAPHICAL SOLUTION OF THE ALL-INTEGER PROBLEM (2 OF 2)

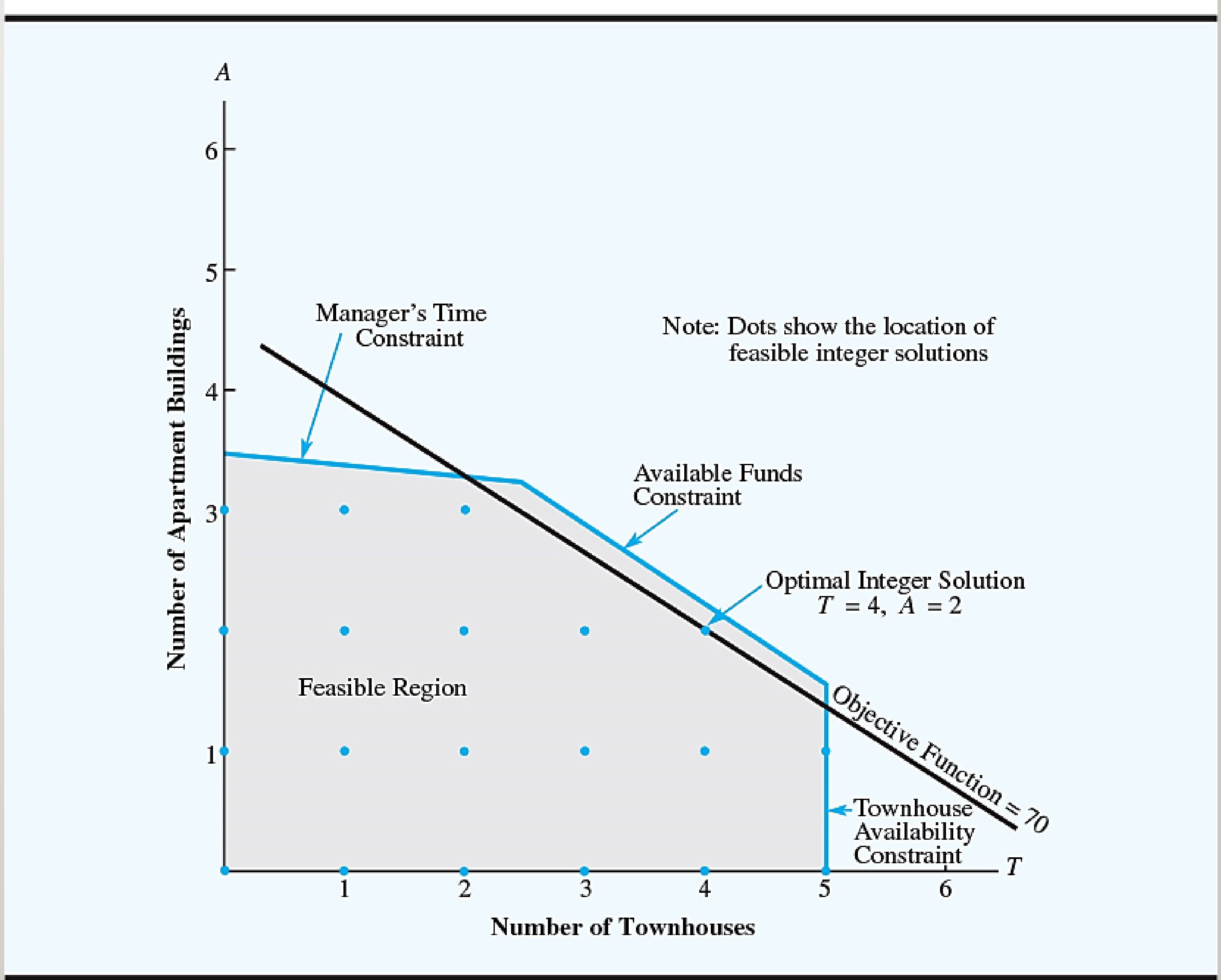
Optimal Integer Solution:

T = 4 and A = 2

Annual Cash Flow =

$70,000

This solution is significantly better than the solution found by rounding (Annual cash flow = $65,000)



USING THE LP RELAXATION TO ESTABLISH BOUNDS

For integer linear programs involving maximization, the value of the optimal solution to the LP Relaxation provides an upper bound on the value of the optimal integer solution. For integer linear programs involving minimization, the value of the optimal solution to the LP Relaxation provides a lower bound on the value of the optimal integer solution.

The value of the optimal integer solution is $70,000, and the value of the optimal solution to the LP Relaxation is $73,574. Thus, we know from the LP Relaxation solution that the upper bound for the value of the objective function is $73,574.

COMPUTER SOLUTION

The solution of *T* = 4 townhouses and *A* = 2 apartment buildings has a maximum annual cash flow of $70,000. The values of the slack variables tell us that the optimal solution has $72,000 of available funds unused, 44 hours of the manager's time still available, and 1 of the available townhouses not purchased.

Optimal Objective Value = 70.00000	
Variable	Value
T	4.00000
A	2.00000
Constraint	Slack/Surplus
1	72.00000
2	44.00000
3	1.00000

APPLICATIONS INVOLVING 0-1 VARIABLES

Much of the modeling flexibility provided by integer linear programming is due to the use of 0-1 variables.

In many applications, 0-1 variables provide selections or choices with the value of the variable equal to 1 if a corresponding activity is undertaken and equal to 0 if the corresponding activity is not undertaken.

The capital budgeting, fixed cost, distribution system design, bank location, and product design/market share applications presented in this section make use of 0-1 variables.

EXAMPLE: CAPITAL BUDGETING (1 OF 4)

The Ice-Cold Refrigerator Company is considering investing in several projects that have varying capital requirements over the next four years. Faced with limited capital, management would like to select the most profitable projects. The estimated net present value for each project, the capital requirements, and the available capital over the four-year period are:

	Project				
	Plant Expansion	**Warehouse Expansion**	**New Machinery**	**New Product Research**	
Present Value	$90,000	$40,000	$10,000	$37,000	Total Capital Available
Year 1 Cap Rqmt	$15,000	$10,000	$10,000	$15,000	$40,000
Year 1 Cap Rqmt	$20,000	$15,000		$10,000	$50,000
Year 1 Cap Rqmt	$20,000	$20,000		$10,000	$40,000
Year 1 Cap Rqmt	$15,000	$5,000	$4,000	$10,000	$35,000

EXAMPLE: CAPITAL BUDGETING (2 OF 4)

Decision Variables

The four 0-1 decision variables are as follows:

P = 1 if the plant expansion project is accepted;

0 if rejected

W = 1 if the warehouse expansion project is accepted;

0 if rejected

M = 1 if the new machinery project is accepted;

0 if rejected

R = 1 if the new product research project is accepted;

0 if rejected

EXAMPLE: CAPITAL BUDGETING (3 OF 4)

In a capital budgeting problem, the company's objective function is to maximize the net present value of the capital budgeting projects. This problem has four constraints: one for the funds available in each of the next four years.

$$\text{Max } 90P + 40W + 10M + 37R$$

$$\begin{aligned}
\text{s.t.} \quad & 15P + 10W + 10M + 15R \le 40 && \text{(Yr. 1 capital avail.)} \\
& 20P + 15W + 10R \le 50 && \text{(Yr. 2 capital avail.} \\
& 20P + 20W + 10R \le 40 && \text{(Yr. 3 capital avail.)} \\
& 15P + 5W + 4M + 10R \le 35 && \text{(Yr. 4 capital avail.)} \\
& P, W, M, R = 0, 1
\end{aligned}$$

EXAMPLE: CAPITAL BUDGETING (4 OF 4)

Optimal Solution:

$P = 1, W = 1, M = 1, R = 0.$
Total estimated net present value = $140,000

- The company should fund the plant expansion, the warehouse expansion, and the new machinery projects.
- The new product research project should be put on hold unless additional capital funds become available.
- The company will have $5,000 remaining in year 1, $15,000 remaining in year 2, and $11,000 remaining in year 4. Additional capital funds of $10,000 in year 1 and $10,000 in year 3 will be needed to fund the new product research project.

EXAMPLE: FIXED COST (1 OF 5)

Three raw materials are used to produce 3 products: a fuel additive, a solvent base, and a carpet cleaning fluid. The profit contributions are $40 per ton for the fuel additive, $30 per ton for the solvent base, and $50 per ton for the carpet cleaning fluid.

Each ton of fuel additive is a blend of 0.4 tons of material 1 and 0.6 tons of material 3. Each ton of solvent base requires 0.5 tons of material 1, 0.2 tons of material 2, and 0.3 tons of material 3. Each ton of carpet cleaning fluid is a blend of 0.6 tons of material 1, 0.1 tons of material 2, and 0.3 tons of material 3.

EXAMPLE: FIXED COST (2 OF 5)

RMC has 20 tons of material 1, 5 tons of material 2, and 21 tons of material 3, and is interested in determining the optimal production quantities for the upcoming planning period.

There is a fixed cost for production setup of the products, as well as a maximum production quantity for each of the three products.

Product	Setup Cost	Maximum Production
Fuel additive	\$200	50 tons
Solvent base	\$50	25 tons
Cleaning fluid	\$400	40 tons

EXAMPLE: FIXED COST (3 OF 5)

Decision Variables

F = tons of fuel additive produced

S = tons of solvent base produced

C = tons of carpet cleaning fluid produced

SF = 1 if the fuel additive is produced; 0 if not

SS = 1 if the solvent base is produced; 0 if not

SC = 1 if the cleaning fluid is produced; 0 if not

EXAMPLE: FIXED COST (4 OF 5)

Problem Formulation

$$
\begin{array}{llr}
\text{Max} & 40F + 30S + 50C - 200SF - 50SS - 400SC & \\
\text{s.t.} & & \\
& 0.4F + 0.5S + 0.6C \leq 20 & \text{Material 1} \\
& 0.2S + 0.1C \leq 5 & \text{Material 2} \\
& 0.6F + 0.3S + 0.3C \leq 21 & \text{Material 3} \\
& F \leq 50SF & \text{Maximum } F \\
& S \leq 25SS & \text{Maximum } S \\
& C \leq 40SC & \text{Maximum } C \\
& F, S, C \geq 0;\ SF, SS, SC = 0, 1 &
\end{array}
$$

EXAMPLE: FIXED COST (5 OF 5)

Optimal Solution

Produce 25 tons of fuel additive.
Produce 20 tons of solvent base.
Produce 0 tons of cleaning fluid.

The value of the objective function after deducting the setup cost is \$1350. The setup cost for the fuel additive and the solvent base is \$200 + \$50 = \$250.

The optimal solution shows *SC* = 0, which indicates that the more expensive \$400 setup cost for the carpet cleaning fluid should be avoided.

EXAMPLE: DISTRIBUTION SYSTEM DESIGN

The Martin-Beck Company operates a plant in St. Louis with an annual capacity of 30,000 units. Product is shipped to regional distribution centers located in Boston, Atlanta, and Houston. Because of an anticipated increase in demand, Martin-Beck plans to increase capacity by constructing a new plant in one or more of the following cities: Detroit, Toledo, Denver, or Kansas City.

EXAMPLE: SUPPLY CHAIN DESIGN (1 OF 5)

The estimated annual fixed cost and the annual capacity for the four proposed plants are as follows:

Proposed Plant	Annual Fixed Cost	Annual Capacity
Detroit	$175,000	10,000
Toledo	$300,000	20,000
Denver	$375,000	30,000
Kansas City	$500,000	40,000

EXAMPLE: SUPPLY CHAIN DESIGN (2 OF 5)

The company's long-range planning group developed forecasts of the anticipated annual demand at the distribution centers as follows:

Distribution Center	Annual Demand
Boston	30,000
Atlanta	20,000
Houston	20,000

EXAMPLE: SUPPLY CHAIN DESIGN (3 OF 5)

The shipping cost per unit from each plant to each distribution center is shown below.

	Distribution Centers		
Plant Site	**Boston**	**Atlanta**	**Houston**
Detroit	5	2	3
Toledo	4	3	4
Denver	9	7	5
Kansas City	10	4	2
St. Louis	8	4	3

EXAMPLE: SUPPLY CHAIN DESIGN (4 OF 5)

Decision Variables

y_1 = 1 if a plant is constructed in Detroit; 0 if not

y_2 = 1 if a plant is constructed in Toledo; 0 if not

y_3 = 1 if a plant is constructed in Denver; 0 if not

y_4 = 1 if a plant is constructed in Kansas City; 0 if not

x_{ij} = the units shipped (in 1000s) from plant i to distribution center j, with i = 1, 2, 3, 4, 5 and j = 1, 2, 3

EXAMPLE: SUPPLY CHAIN DESIGN (5 OF 5)

Problem Formulation

$$\text{Min } 5x_{11} + 2x_{12} + 3x_{13} + 4x_{21} + 3x_{22} + 4x_{23} + 9x_{31} + 7x_{32} + 5x_{33} + 10x_{41} + 4x_{42}$$
$$+ 2x_{43} + 8x_{51} + 4x_{52} + 3x_{53} + 175y_1 + 300y_2 + 375y_3 + 500y_4$$

s.t.

$x_{11} + x_{12} + x_{13} \le 10\mathrm{y}_1$ Detroit capacity

$x_{21} + x_{22} + x_{23} \le 20\mathrm{y}_2$ Toledo capacity

$x_{31} + x_{32} + x_{33} \le 30\mathrm{y}_3$ Denver capacity

$x_{41} + x_{42} + x_{43} \le 40\mathrm{y}_4$ Kansas City capacity

$x_{51} + x_{52} + x_{53} \le 30$ St. Louis capacity

$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} = 30$ Boston demand

$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} = 20$ Atlanta demand

$x_{13} + x_{23} + x_{33} + x_{43} + x_{53} = 20$ Houston demand

EXAMPLE: DISTRIBUTION SYSTEM DESIGN

Optimal Solution

Construct a plant in Kansas City ($y_4 = 1$).

Ship 20,000 units: Kansas City to Atlanta ($x_{42} = 20$),

Ship 20,000 units: Kansas City to Houston ($x_{43} = 20$),

Ship 30,000 units: St. Louis to Boston ($x_{51} = 30$).

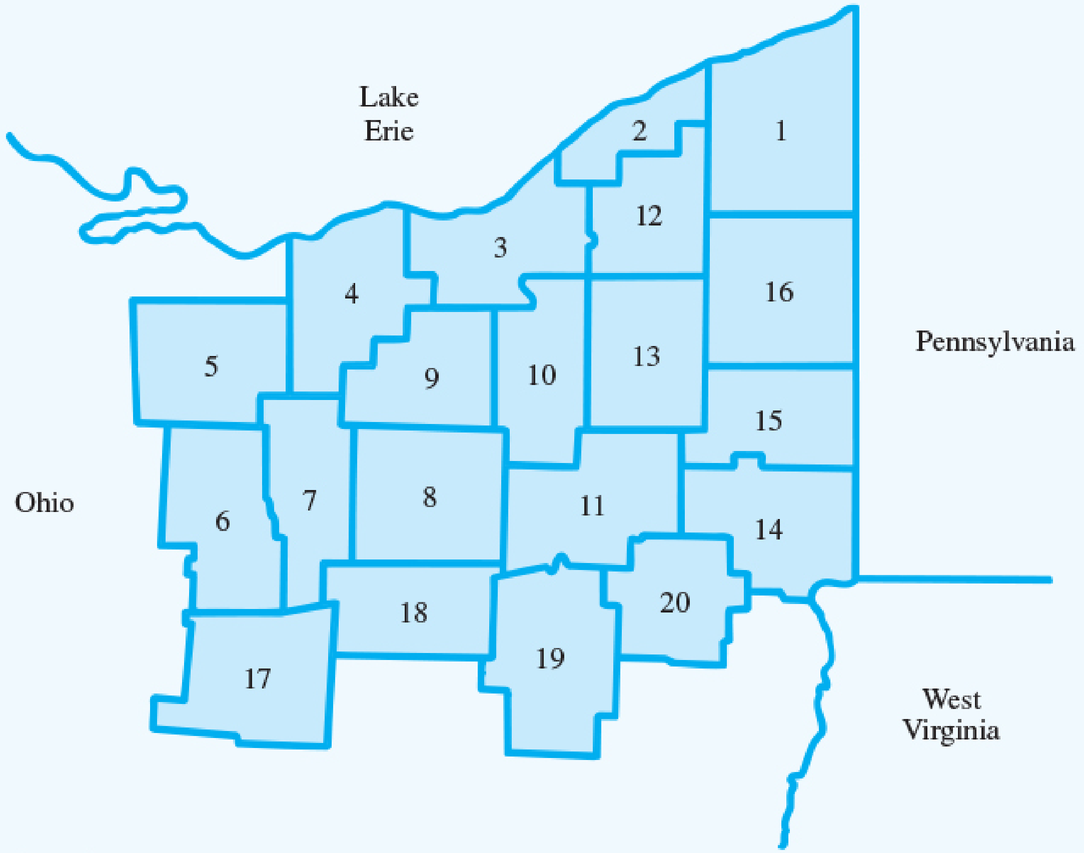
Total cost: $860,000 including fixed cost of $500,000.

EXAMPLE: BANK LOCATION (1 OF 7)

The long-range planning department for the Ohio Trust Company is considering expanding its operation into a 20-county region in northeastern Ohio. Ohio Trust does not have, at this time, a principal place of business in any of the 20 counties.

According to the banking laws in Ohio, if a bank establishes a principal place of business (PPB) in any county, branch banks can be established in that county and in any adjacent county. To establish a new PPB, Ohio Trust must either obtain approval for a new bank from the state's superintendent of banks or purchase an existing bank.

EXAMPLE: BANK LOCATION (2 OF 7)



Counties

1. Ashtabula	6. Richland	11. Stark	16. Trumbull
2. Lake	7. Ashland	12. Geauga	17. Knox
3. Cuyahoga	8. Wayne	13. Portage	18. Holmes
4. Lorain	9. Medina	14. Columbiana	19. Tuscarawas
5. Huron	10. Summit	15. Mahoning	20. Carroll

EXAMPLE: BANK LOCATION (3 OF 7)

The 20 counties in the region and adjacent counties are listed on the next slide. For example, Ashtabula County is adjacent to Lake, Geauga, and Trumbull counties; Lake County is adjacent to Ashtabula, Cuyahoga, and Geauga counties; and so on.

As an initial step in its planning, Ohio Trust would like to determine the minimum number of PPBs necessary to do business throughout the 20-county region. A 0-1 integer programming model can be used to solve this **location problem** for Ohio Trust.

EXAMPLE: BANK LOCATION (4 OF 7)

Counties Under Consideration	Adjacent Counties (by Number)
1. Ashtabula	2, 12, 16
2. Lake	1, 3, 12
3. Cuyahoga	2, 4, 9, 10, 12, 13
4. Lorain	3, 5, 7, 9
5. Huron	4, 6, 7
6. Richland	5, 7, 17
7. Ashland	4, 5, 6, 8, 9, 17, 18
8. Wayne	7, 9, 10, 11, 18
9. Medina	3, 4, 7, 8, 10
10. Summit	3, 8, 9, 11, 12, 13
11. Stark	8, 10, 13, 14, 15, 18, 19, 20
12. Geauga	1, 2, 3, 10, 13, 16
13. Portage	3, 10, 11, 12, 15, 16
14. Columbiana	11, 15, 20
15. Mahoning	11.13. 14, 16
16. Trumbull	1, 12, 13, 15
17. Knox	6, 7, 18
18. Holmes	7, 8, 11, 17, 19
19. Tuscarawas	11, 18, 20
20. Carroll	11, 14, 19

EXAMPLE: BANK LOCATION (5 OF 7)

Decision Variables:

x_i = 1 if a PBB is established in county *i*; 0 otherwise

Problem Formulation:

$$
\begin{array}{lllllll}
\text{Min} & x_1 + x_2 + & \cdots & & + x_{20} & & \\
\text{s.t.} & & & & & & \\
& x_1 + x_2 & + x_{12} + x_{16} & & & \geq 1 & \text{Ashtabula} \\
& x_1 + x_2 + x_3 & + x_{12} & & & \geq 1 & \text{Lake} \\
& \cdot & & & & \cdot & \\
& \cdot & & & & \cdot & \\
& \cdot & & & & \cdot & \\
& x_{11} & + x_{14} + x_{19} & & + x_{20} & \geq 1 & \text{Carroll} \\
& x_i = 0, 1 \quad i = 1, 2, \ldots, 20 & & & & &
\end{array}
$$

EXAMPLE: BANK LOCATION (6 OF 7)

Optimal Solution:

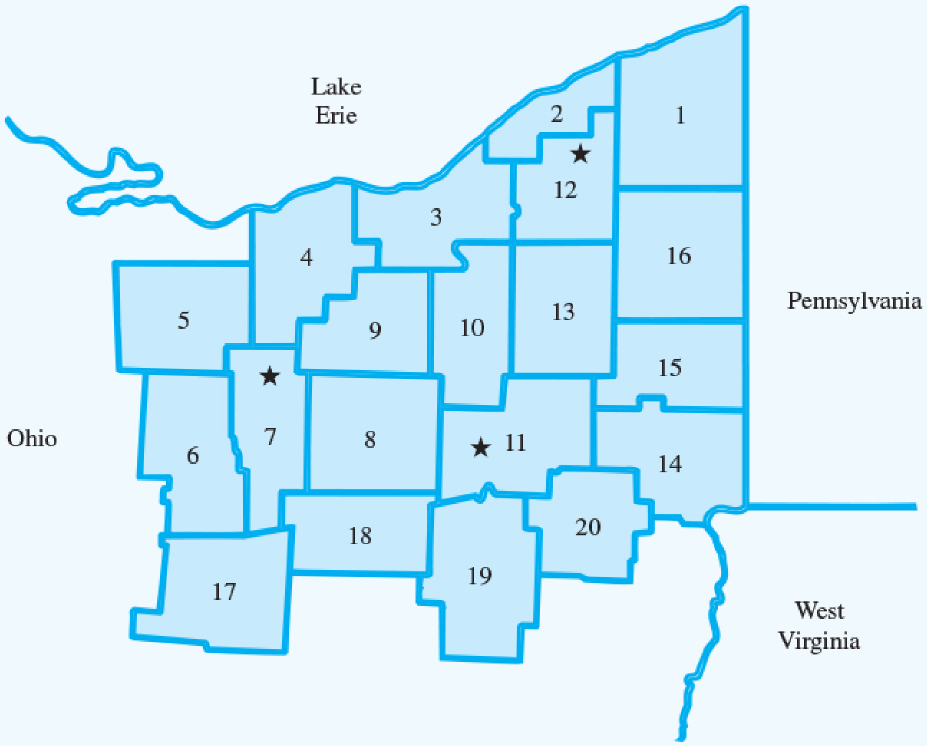
For this 20-variable, 20-constraint problem:

Establish PPBs in Ashland, Stark, Geauga counties.

(With PPBs in these three counties, Ohio Trust can place branch banks in all 20 counties.)

All other decision variables have an optimal value of zero, indicating that a PPB should not be placed in these counties.

EXAMPLE: BANK LOCATION (7 OF 7)



Counties

1. Ashtabula
2. Lake
3. Cuyahoga
4. Lorain
5. Huron
6. Richland
7. Ashland
8. Wayne
9. Medina
10. Summit
11. Stark
12. Geauga
13. Portage
14. Columbiana
15. Mahoning
16. Trumbull
17. Knox
18. Holmes
19. Tuscarawas
20. Carroll

★ A principal place of business should be located in these counties.

PRODUCT DESIGN AND MARKET SHARE OPTIMIZATION (1 OF 8)

Conjoint analysis is a market research technique that can be used to learn how prospective buyers of a product value the product's attributes. In this section we will show how the results of conjoint analysis can be used in an integer programming model of a **product design and market share optimization problem**.

PRODUCT DESIGN AND MARKET SHARE OPTIMIZATION

Antonio's and King's, have the major share of the market. In trying to develop a sausage pizza that will capture a significant share of the market, Salem determined that the four most important attributes when consumers purchase a frozen sausage pizza are crust, cheese, sauce, and sausage flavor.

- The crust attribute has two levels (thin and thick);
- The cheese attribute has two levels (mozzarella and blend);
- The sauce attribute has two levels (smooth and chunky); and
- The sausage flavor attribute has three levels (mild, medium, and hot).

PRODUCT DESIGN AND MARKET SHARE OPTIMIZATION (3 OF 8)

In a typical conjoint analysis, a sample of consumers are asked to express their preference for specially prepared pizzas with chosen levels for the attributes.

Then regression analysis is used to determine the part-worth for each of the attribute levels. In essence, the part-worth is the utility value that a consumer attaches to each level of each attribute.

	Crust		Cheese		Sauce		Sausage Flavor		
Consumer	**Thin**	**Thick**	**Mozarella**	**Blend**	**Smooth**	**Chunky**	**Mild**	**Medium**	**Hot**
1	11	2	6	7	3	17	26	27	8
2	11	7	15	17	16	26	14	1	10
3	7	5	8	14	16	7	29	16	19
4	13	20	20	17	17	14	25	29	10
5	2	8	6	11	30	20	15	5	12
6	12	17	11	9	2	30	22	12	20
7	9	19	12	16	16	25	30	23	19
8	5	9	4	14	23	16	16	30	3

PRODUCT DESIGN AND MARKET SHARE OPTIMIZATION (4 OF 8)

For consumer 1, the part-worths for the crust attribute are 11 for thin crust and 2 for thick crust, indicating a preference for thin crust. For the cheese attribute, the part-worths are 6 for the mozzarella cheese and 7 for the cheese blend; thus, consumer 1 has a slight preference for the cheese blend. From the other part-worths, we see that consumer 1 shows a strong preference for the chunky sauce over the smooth sauce (17 to 3) and has a slight preference for the medium-flavored sausage.

	Crust		Cheese		Sauce		Sausage Flavor		
Consumer	**Thin**	**Thick**	**Mozarella**	**Blend**	**Smooth**	**Chunky**	**Mild**	**Medium**	**Hot**
1	11	2	6	7	3	17	26	27	8
2	11	7	15	17	16	26	14	1	10
3	7	5	8	14	16	7	29	16	19
4	13	20	20	17	17	14	25	29	10
5	2	8	6	11	30	20	15	5	12
6	12	17	11	9	2	30	22	12	20
7	9	19	12	16	16	25	30	23	19
8	5	9	4	14	23	16	16	30	3

PRODUCT DESIGN AND MARKET SHARE OPTIMIZATION (5 OF 8)

The decision variables are defined as follows:

l_{ij} = 1 if Salem chooses level l for attribute j; 0 otherwise

y_k = 1 if consumer k chooses the Salem brand; 0 otherwise

The objective is to choose the levels of each attribute that will maximize the number of consumers preferring the Salem brand pizza. Because the number of customers preferring the Salem brand pizza is just the sum of the y_k variables, the objective function is

$$\text{Max} \quad y_1 + y_2 + \ldots + y_8$$

PRODUCT DESIGN AND MARKET SHARE OPTIMIZATION (6 OF 8)

One constraint is needed for each consumer in the sample. For consumer 1, the utility of a particular type of pizza can be expressed as the sum of the part-worths:

Utility for Consumer 1 =

$$11l_{11} + 2l_{21} + 6l_{12} + 7l_{22} + 3l_{13} + 17l_{23} + 26l_{14} + 27l_{24} + 8l_{34}$$

In order for consumer 1 to prefer the Salem pizza, the utility for the Salem pizza must be greater than the utility for consumer 1's current favorite. Recall that consumer 1's current favorite brand of pizza is Antonio's, with a utility of 52. Thus, consumer 1 will only purchase the Salem brand if the levels of the attributes for the Salem brand are chosen such that

$$11l_{11} + 2l_{21} + 6l_{12} + 7l_{22} + 3l_{13} + 17l_{23} + 26l_{14} + 27l_{24} + 8l_{34} > 52$$

PRODUCT DESIGN AND MARKET SHARE OPTIMIZATION (7 OF 8)

The Model:

$$11l_{11} + 2l_{21} + 6l_{12} + 7l_{22} + 3l_{13} + 17l_{23} + 26l_{14} + 27l_{24} + 8l_{34} \geq 1 + 52y_1$$
$$11l_{11} + 7l_{21} + 15l_{12} + 17l_{22} + 16l_{13} + 26l_{23} + 14l_{14} + 1l_{24} + 10l_{34} \geq 1 + 58y_2$$
$$7l_{11} + 5l_{21} + 8l_{12} + 14l_{22} + 16l_{13} + 7l_{23} + 29l_{14} + 16l_{24} + 19l_{34} \geq 1 + 66y_3$$
$$13l_{11} + 20l_{21} + 20l_{12} + 17l_{22} + 17l_{13} + 14l_{23} + 25l_{14} + 29l_{24} + 10l_{34} \geq 1 + 83y_4$$
$$2l_{11} + 8l_{21} + 6l_{12} + 11l_{22} + 30l_{13} + 20l_{23} + 15l_{14} + 5l_{24} + 12l_{34} \geq 1 + 58y_5$$
$$12l_{11} + 17l_{21} + 11l_{12} + 9l_{22} + 2l_{13} + 30l_{23} + 22l_{14} + 12l_{24} + 20l_{34} \geq 1 + 70y_6$$
$$9l_{11} + 19l_{21} + 12l_{12} + 16l_{22} + 16l_{13} + 25l_{23} + 30l_{14} + 23l_{24} + 19l_{34} \geq 1 + 79y_7$$
$$5l_{11} + 9l_{21} + 4l_{12} + 14l_{22} + 23l_{13} + 16l_{23} + 16l_{14} + 30l_{24} + 3l_{34} \geq 1 + 59y_8$$

$$l_{11} + l_{21} = 1$$
$$l_{12} + l_{22} = 1$$
$$l_{13} + l_{23} = 1$$
$$l_{14} + l_{24} + l_{34} = 1$$

PRODUCT DESIGN AND MARKET SHARE OPTIMIZATION (8 OF 8)

The optimal solution to this integer linear program is

$$l_{11} = l_{22} = l_{23} = l_{14} = 1 \text{ and } y_1 = y_2 = y_6 = y_7 = 1$$

The value of the optimal solution is 4, indicating that if Salem makes this type of pizza, it will be preferable to the current favorite for four of the eight consumers.

With $l_{11} = l_{22} = l_{23} = l_{14} = 1$, the pizza design that obtains the largest market share for Salem has a thin crust, a cheese blend, a chunky sauce, and mild-flavored sausage.

Note also that with $y_1 = y_2 = y_6 = y_7 = 1$, consumers 1, 2, 6, and 7 will prefer the Salem pizza. With this information Salem may choose to market this type of pizza.

MODELING FLEXIBILITY PROVIDED BY 0-1 VARIABLES

When x_i and x_j represent binary variables designating whether projects *i* and *j* have been completed, the following special constraints may be formulated:

- At most *k out of n* projects will be completed:

$$\sum_j x_j \le k$$

- Project *j* is conditional on project *i*:

$$x_j - x_i \le 0$$

- Project *i* is a corequisite for project *j*:

$$x_j - x_i = 0$$

- Projects *i* and *j* are mutually exclusive:

$$x_i - x_j \le 1$$

CAUTIONARY NOTE ABOUT SENSITIVITY ANALYSIS

- Sensitivity analysis often is more crucial for ILP problems than for LP problems.
- A small change in a constraint coefficient can cause a relatively large change in the optimal solution.
- Recommendation: Resolve the ILP problem several times with slight variations in the coefficients before choosing the "best" solution for implementation.