CHAPTER 8: NONLINEAR OPTIMIZATION MODELS

INTRODUCTION

Many business processes behave in a nonlinear manner.

- The price of a bond is a nonlinear function of interest rates.
- The price of a stock option is a nonlinear function of the price of the underlying stock.
- The marginal cost of production often decreases with the quantity produced.
- The quantity demanded for a product is often a nonlinear function of the price.

A **nonlinear optimization problem** is any optimization problem in which at least one term in the objective function or a constraint is nonlinear.

A PRODUCTION APPLICATION – PAR, INC., REVISITED

Recall that Par, Inc., decided to manufacture standard and deluxe golf bags. In formulating the linear programming model for the Par Inc.'s problem, we assumed that it could sell all of the standard and deluxe bags it could produce. However, depending on the price of the golf bags, this assumption may not hold.

An inverse relationship usually exists between price and demand. As price goes up, the quantity demanded goes down. Let P_S denote the price Par, Inc., charges for each standard bag and P_D denote the price for each deluxe bag.

A PRODUCTION APPLICATION – PAR, INC., REVISITED (2 OF 4)

We can solve the equation, $S = 2250 - 15P_s$ for P_s

to show how the price of a standard bag is related to the number of standard bags sold:

$$P_s = 150 - (1/15)S.$$

Remember that the profit contribution for producing and selling *S* standard bags (revenue – cost) is

$$P_s S - 70S.$$

Substituting, gives the profit contribution for standard bags:

$$\begin{aligned} &P_s S - 70S \\ &= (150 - (1/15)S)S - 70S \end{aligned}$$

A PRODUCTION APPLICATION – PAR, INC., REVISITED

Suppose that the cost to produce each deluxe golf bag is \$150. Using the same logic we used to develop the profit contribution for standard bags, the profit contribution for deluxe bags is

$$P_D D - 150D = (300 - 1/15D)D - 150D - 1/5D^2$$

Total profit contribution is the sum of the profit contribution for standard bags and the profit contribution for deluxe bags. Thus, total profit contribution is written as

$$\text{Total profit contribution} = 80S - 1/15S^2 + 150D - 1/5D^2$$

A PRODUCTION APPLICATION – PAR, INC., REVISITED (4 OF 4)

Using a computer solution method such as LINGO, we find that the values of *S* and *D* that maximize the profit contribution function are $S = 600$ and $D = 375$.

The corresponding prices are $110 for standard bags and $225 for deluxe bags, and the profit contribution is $52,125.

These values provide the optimal solution for Par, Inc., if all production constraints are also satisfied.

A CONSTRAINED PROBLEM (1 OF 5)

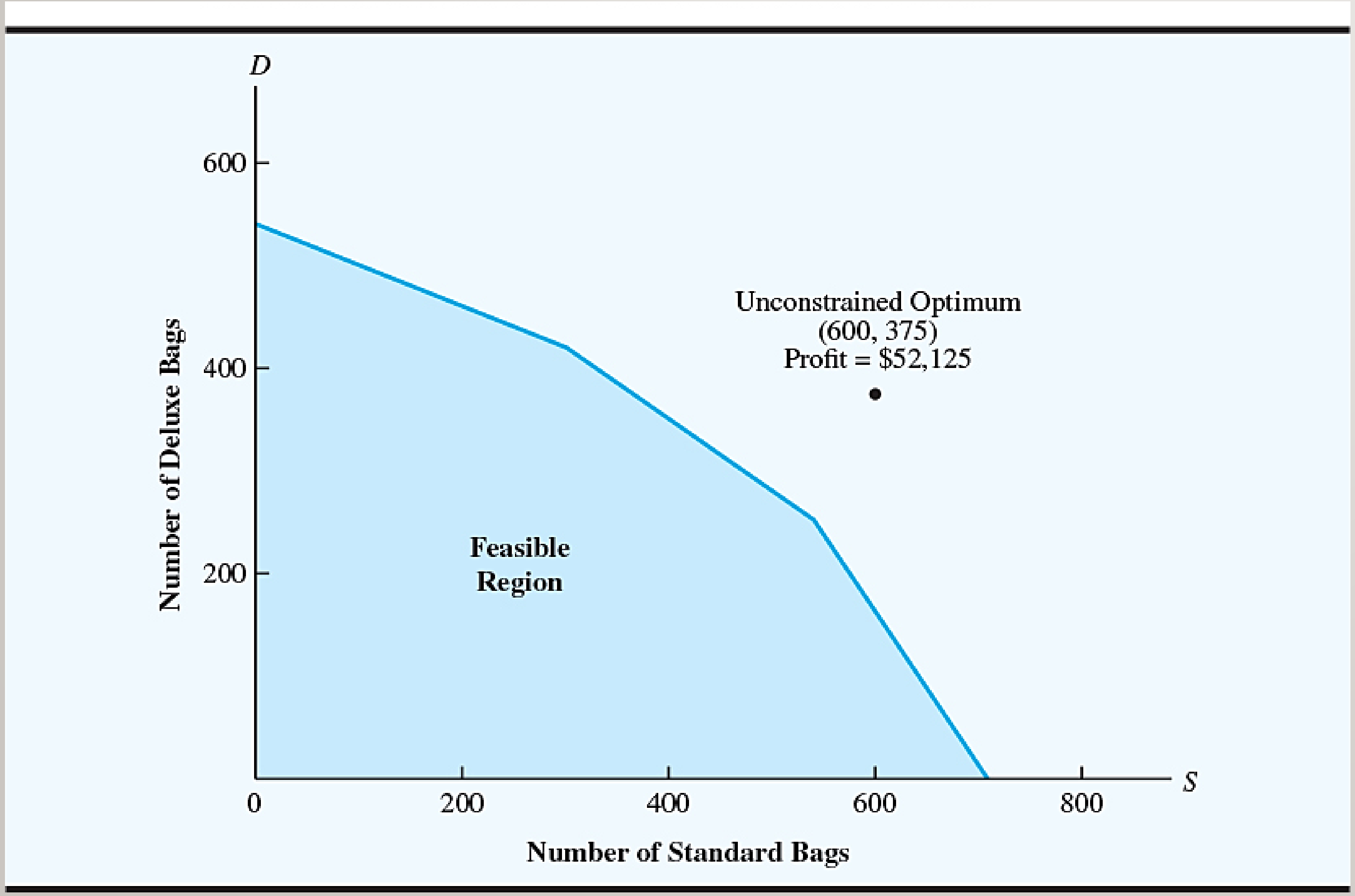
Unfortunately, Par, Inc., cannot make the profit contribution associated with the optimal solution to the unconstrained problem because the constraints defining the feasible region are violated. For instance, the cutting and dyeing constraint is

$$7/10S + D \leq 630$$

A production quantity of 600 standard bags and 375 deluxe bags will require 7/10(600) + 1(375) = 795 hours, which exceeds the limit of 630 hours by 165 hours.

A CONSTRAINED PROBLEM (2 OF 5)

The feasible region for the original Par, Inc., problem along with the unconstrained optimal solution point (600, 375) is shown here:



The unconstrained optimum of (600, 375) is outside the feasible region.

A CONSTRAINED PROBLEM (3 OF 5)

The complete mathematical model for the Par, Inc., constrained nonlinear maximization problem is:

$$\text{Max } 80S - \tfrac{1}{15}S^2 + 150D - \tfrac{1}{5}D^2$$

s.t.

$$\begin{aligned}
\tfrac{7}{10}S + D &\leq 630 && \text{Cutting and Dyeing} \\
\tfrac{1}{2}S + \tfrac{5}{6}D &\leq 600 && \text{Sewing} \\
S + \tfrac{2}{3}D &\leq 708 && \text{Finishing} \\
\tfrac{1}{10}S + \tfrac{1}{4}D &\leq 135 && \text{Inspection and Packaging} \\
S, D &\geq 0
\end{aligned}$$

A CONSTRAINED PROBLEM (4 OF 5)

The optimal value of the objective function is $49,920.55. The optimal solution is to produce 459.7166 standard bags and 308.1984 deluxe bags. In the Slack/Surplus column, the value of 0 in Constraint 1 means that the optimal solution uses all the labor hours in the cutting and dyeing department; but the nonzero values in rows 2–4 indicate that slack hours are available in the other departments.

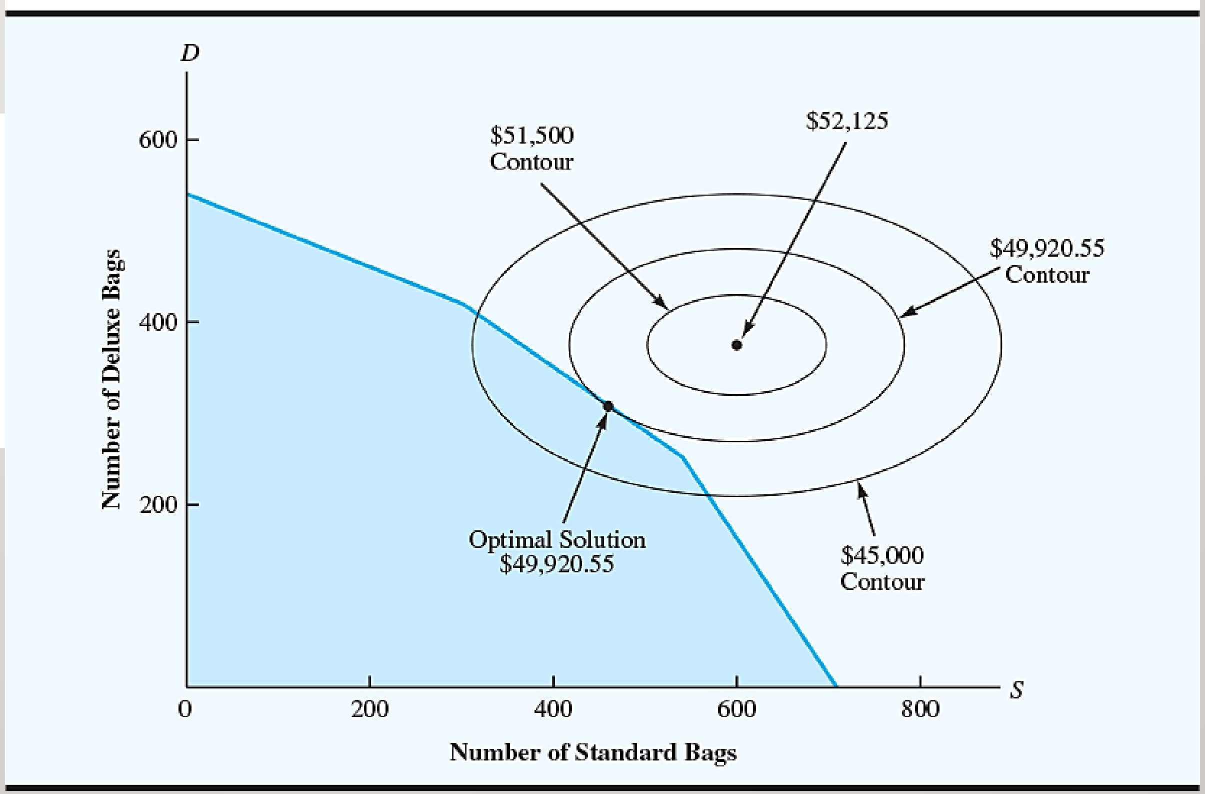
Optimal Objective Value = 49920.54655

Variable	Value	Reduced Cost
S	459.71660	0.00000
D	308.19838	0.00000
Constraint	**Slack/Surplus**	**Dual Value**
1	0.00000	26.7205
2	113.31074	0.00000
3	42.81679	0.00000
4	11.97875	0.00000

A CONSTRAINED PROBLEM (5 OF 5)

Here we see three profit contribution *contour lines*. Each point on the same contour line is a point of equal profit. Here, the contour lines show profit contributions of \$45,000, \$49,920.55, and \$51,500.

For the Par, Inc., problem with a quadratic objective function, the profit contours are ellipses.



LOCAL AND GLOBAL OPTIMA

A feasible solution is a <u>local optimum</u> if there are no other feasible solutions with a better objective function value in the immediate neighborhood.

- For a maximization problem the local optimum corresponds to a local maximum.
- For a minimization problem the local optimum corresponds to a local minimum.

- A feasible solution is a <u>global optimum</u> if there are no other feasible points with a better objective function value in the feasible region.
- A global optimum is also a local optimum.

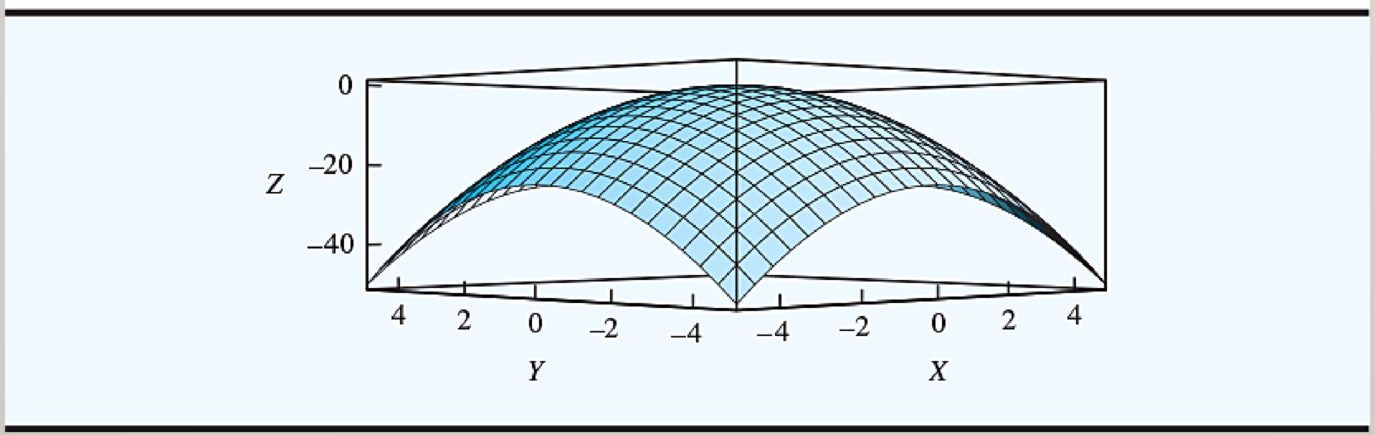
MULTIPLE LOCAL OPTIMA

- Nonlinear optimization problems can have multiple local optimal solutions, in which case we want to find the best local optimum.
- Nonlinear problems with multiple local optima are difficult to solve and pose a serious challenge for optimization software.
- In these cases, the software can get "stuck" and terminate at a local optimum.
- There can be a severe penalty for finding a local optimum that is not a global optimum.
- Developing algorithms capable of finding the global optimum is currently a very active research area.

SINGLE LOCAL MAXIMUM

Consider the function $f(X, Y) = -X^2 - Y^2$.

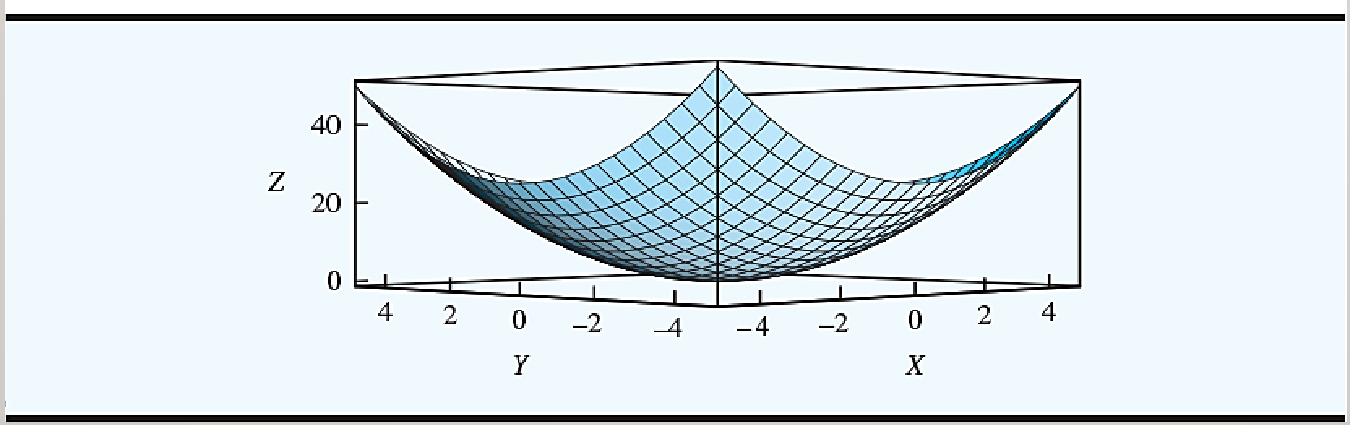
- A function that is bowl-shaped down is a concave function.
- The maximum value for this particular function is 0 and the point (0, 0) gives the optimal value of 0.
- Functions such as this one have a single local maximum that is also a global maximum.
- This type of nonlinear problem is relatively easy to maximize.



SINGLE LOCAL MINIMUM

Consider the function $f(X, Y) = X^2 + Y^2$.

- A function that is bowl-shaped up is a <u>convex function</u>.
- The minimum value for this particular function is 0 and the point (0, 0) gives the optimal value of 0.
- Functions such as this one have a single local minimum that is also a global minimum.
- This type of nonlinear problem is relatively easy to minimize.

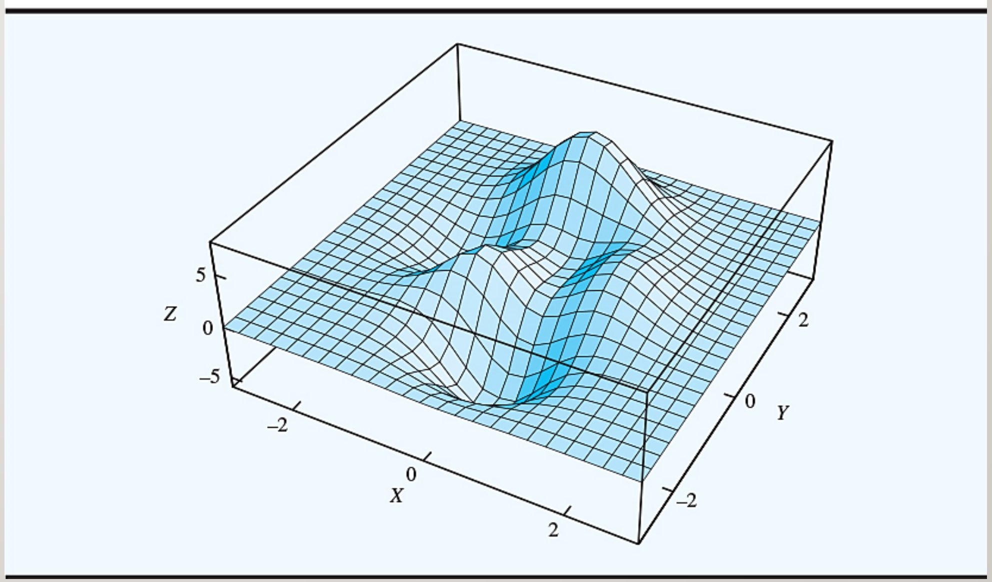


MULTIPLE LOCAL OPTIMA

Consider the function

$$f\left(X,Y\right)=3\left(1-X\right)^{2}e^{\left(-X^{2}-\left(Y+1\right)^{2}\right)}-10\left(X/5-X^{3}-Y^{5}\right)e^{\left(-X^{2}-Y^{2}\right)}-e^{\left(-\left(X+1\right)^{2}-Y^{2}\right)}/3$$

- The hills and valleys in the graph show that this function has several local maximums and local minimums.
- There are two local minimums, one of which is the the global minimum.
- There are three local maximums, one of which is the global maximum.



DUAL VALUES

- Recall that the dual value is the change in the value of the optimal solution per unit increase in the right-hand side of the constraint.
- The interpretation of the dual value for nonlinear models is exactly the same as it is for LPs.
- However, for nonlinear problems the allowable increase and decrease are not usually reported.
- This is because for typical nonlinear problems the allowable increase and decrease are zero.
- That is, if you change the right-hand side by even a small amount, the dual value changes.

CONSTRUCTING AN INDEX FUND (1 OF 5)

Index funds are a very popular investment vehicle in the mutual fund industry.

- Vanguard 500 Index Fund is the largest mutual fund in the U.S. with over $70 billion in net assets in 2005.
- An **index fund** is an example of passive asset management.
- The key idea behind an index fund is to construct a portfolio of stocks, mutual funds, or other securities that closely matches the performance of a broad market index such as the S&P 500.
- Behind the popularity of index funds is research that basically says "you can't beat the market."

CONSTRUCTING AN INDEX FUND (2 OF 5)

Let's revisit the Hauck Financial Services example from Chapter 5. Assume that Hauck has a substantial number of clients who wish to own a mutual fund portfolio with the characteristic that the portfolio, as a whole, closely matches the performance of the S&P 500 stock index.

What percentage of the portfolio should be invested in each mutual fund in order to most closely mimic the performance of the entire S&P 500 index?

CONSTRUCTING AN INDEX FUND (3 OF 5)

Here are the decision variables:

	Planning Scenarios				
Mutual Fund	**Year 1**	**Year 2**	**Year 3**	**Year 4**	**Year 5**
Foreign Stock	10.06	13.12	13.47	45.42	-21.93
Intermediate-Term Bond	17.64	3.25	7.51	-1.33	7.36
Large-Cap Growth	32.41	18.71	33.28	41.46	-23.26
Small-Cap Growth	33.44	19.40	3.85	58.68	-9.02
Small-Cap Value	24.56	25.32	-6.70	5.43	17.31
S&P 500 Return	**25.00**	**20.00**	**8.00**	**30.00**	**-10.00**

FS = proportion of portfolio invested in a foreign stock mutual fund

IB = proportion of portfolio invested in an intermediate-term bond fund

LG = proportion of portfolio invested in a large-cap growth fund

LV = proportion of portfolio invested in a large-cap value fund

SG = proportion of portfolio invested in a small-cap growth fund

SV = proportion of portfolio invested in a small-cap value fund

CONSTRUCTING AN INDEX FUND (4 OF 5)

The constraints are:

$$
\begin{aligned}
&\text{Min} \quad (R1 - 25)^2 + (R2 - 20)^2 + (R3 - 8)^2 + (R4 - 30)^2 + (R5 - [-10])^2 \\
&\text{s.t.} \\
&\quad R1 = 10.06FS + 17.64IB + 32.41LG + 32.36LV + 33.44SG + 24.56SV \\
&\quad R2 = 13.12FS + 3.25IB + 18.71LG + 20.61LV + 19.40SG + 25.32SV \\
&\quad R3 = 13.47FS + 7.51IB + 33.28LG + 12.93LV + 3.85SG - 6.70SV \\
&\quad R4 = 45.42FS - 1.33IB + 41.46LG + 7.06LV + 58.68SG + 5.43SV \\
&\quad R5 = -21.93FS + 7.36IB - 23.26LG - 5.37LV - 9.02SG + 17.31SV \\
&\qquad\qquad FS + IB + LG + LV + SG + SV = 1 \\
&\qquad\qquad FS, IB, LG, LV, SG, SV \geq 0
\end{aligned}
$$

CONSTRUCTING AN INDEX FUND (5 OF 5)

The optimal value of the objective function is 4.42689, the sum of the squares of the return deviations.

The portfolio calls for approximately 30% of the funds to be invested in the foreign stock fund (*FS* = 0.30334), 36% of the funds to be invested in the large-cap value fund (*LV* = 0.36498), 23% of the funds to be invested in the small-cap growth fund (*SG* = 0.22655), and 11% of the funds to be invested in the small-cap value fund (*SV* = 0.10513).

MARKOWITZ PORTFOLIO MODEL

- There is a key tradeoff in most portfolio optimization models between risk and return.
- The index fund model presented earlier managed the tradeoff passively.
- The Markowitz mean-variance portfolio model provides a very convenient way for an investor to actively trade-off risk versus return.
- We will now demonstrate the Markowitz portfolio model by extending the previous example.

EXAMPLE: MARKOWITZ PORTFOLIO MODEL (1 OF 2)

- The portfolio variance is the average of the sum of the squares of the deviations from the mean value under each scenario.
- The larger the variance value, the more widely dispersed the scenario returns are about the average return value.
- If the portfolio variance were equal to zero, then every scenario return R_i would be equal.

EXAMPLE: MARKOWITZ PORTFOLIO MODEL (2 OF 2)

- There are two basic ways to formulate the Markowitz model:
- (1) Minimize the variance of the portfolio subject to constraints on the expected return, and
- (2) Maximize the expected return of the portfolio subject to a constraint on risk.
- We will now demonstrate the first (1) formulation, assuming the client requires the expected portfolio return to be at least 10 percent.

BLENDING: THE POOLING PROBLEM (1 OF 2)

- Blending problems arise when a manager must decide how to blend two or more components (resources) to produce one or more products.
- It is often the case that while transporting or storing the blending components, the components must share a pipeline or storage tank.
- In this case, the components are called <u>pooled components</u>.

BLENDING: THE POOLING PROBLEM (2 OF 2)

Two types of decisions arise:

- What should the proportions be for the components that are to be pooled?
- How much of the pooled and non-pooled components will be used to make each of the final products?

EXAMPLE: BLENDING - THE POOLING PROBLEM

Grand Strand refinery wants to refine three petroleum components into regular and premium gasoline in order to maximize total profit contribution. Components 1 and 2 are pooled in a single storage tank. Component 3 has its own storage tank.

The maximum number of gallons available for the three components are 5000, 10,000, and 10,000, respectively. The three components cost \$2.50, \$2.60, and \$2.84, respectively. Regular gasoline sells for \$2.90 and premium sells for \$3.00. At least 10,000 gallons of regular gasoline must be produced.

The product specifications for regular and premium gasoline are shown on the next slide.

EXAMPLE: BLENDING - THE POOLING PROBLEM (2 OF 9)

Product Specifications

- Regular gasoline At most 30% component 1
  At least 40% component 2
  At most 20% component 3

- Premium gasoline At least 25% component 1
  At most 45% component 2
  At least 30% component 3

EXAMPLE: BLENDING - THE POOLING PROBLEM (3 OF 9)

Define the 6 Decision Variables

y_1 = gallons of component 2 in the pooling tank

y_2 = gallons of component 1 in the pooling tank

x_{pr} = gallons of pooled components 1 and 2 in regular gas

x_{pp} = gallons of pooled components 1 and 2 in premium gas

x_{3r} = gallons of component 3 in regular gasoline

x_{3p} = gallons of component 3 in premium gasoline

EXAMPLE: BLENDING - THE POOLING PROBLEM

Define the Objective Function

Maximize the total contribution to profit (which is revenue from selling regular and premium gasolines minus cost of buying components 1, 2, and 3):

$$\text{Max} \quad 2.90(x_{pr} + x_{3r}) + 3.00\left(x_{pp} + x_{3p}\right)$$
$$- 2.50 y_1 - 2.60 y_2 - 2.84\left(x_{3r} + x_{3p}\right)$$

(Note: $x_{pr} + x_{3r}$ = gallons of regular gasoline sold.

$x_{pp} + x_{3p}$ = gallons of premium gasoline sold.

$x_{3r} + x_{3p}$ = gallons of component 3 consumed.)

EXAMPLE: BLENDING - THE POOLING PROBLEM

Define the 11 Constraints

Conservation equation:

1) $y_1 + y_2 = x_{pr} + x_{pp}$

Component availability:

2) $y_1 \leq 5,000$

3) $y_2 \leq 10,000$

4) $x_{3r} + x_{3p} \leq 10,000$

Minimum production of regular gasoline:

5) $x_{pr} + x_{3r} \geq 10,000$

EXAMPLE: BLENDING - THE POOLING PROBLEM (6 OF 9)

Define the 11 Constraints (continued)

Premium gasoline specifications:

9) $\left(y_1 / \left(y_1 + y_2\right)\right) x_{pp} \le 0.25\left(x_{pp} + x_{3p}\right)$

10) $\left(y_2 / \left(y_1 + y_2\right)\right) x_{pp} \le 0.45\left(x_{pp} + x_{3p}\right)$

11) $x_{3p} \ge 0.3\left(x_{pp} + x_{3p}\right)$

Non-negativity: $x_{pr}, x_{pp}, x_{3r}, x_{3p}, y_1, y_2 \ge 0$

EXAMPLE: BLENDING - THE POOLING PROBLEM

Computer Output

Objective Function Value = 1045000.000

Variable	Value	Reduced Cost
x_{pr}	8000.000	0.000
x_{3r}	2000.000	0.000
x_{pp}	6000.000	0.000
x_{3p}	2571.429	0.000
y_{pr}	5000.000	0.000
y_1	9000.000	0.000

EXAMPLE: BLENDING - THE POOLING PROBLEM (8 OF 9)

Computer Output (continued)

Constraint	Slack/Surplus	Dual Value
1	0.000	1.412
2	1000.000	0.000
3	5428.571	0.000
4	0.000	-3.061
5	142.857	0.000
6	1142.857	0.000
7	0.000	0.000
8	0.000	-2.197
9	0.000	0.865
10	0.000	0.000
11	0.000	-0.123

EXAMPLE: BLENDING - THE POOLING PROBLEM (9 OF 9)

Solution

	Regular Gasoline	Premium Gasoline
Component 1	2,857.143 (28.57%)	2,142.857 (25.00%)
Component 2	5,142.857 (51.43%)	3,857.143 (45.00%)
Component 3	2,000.000 (20.00%)	2,571.429 (30.00%)
Total	10,000.000 (100.00%)	8,571.429 (100.00%)

FORECASTING ADOPTION OF A NEW PRODUCT (1 OF 6)

- Forecasting new adoptions (purchases) after a product introduction is an important marketing problem.
- We introduce here a forecasting model developed by Frank Bass.
- Nonlinear programming is used to estimate the parameters of the Bass forecasting model.

FORECASTING ADOPTION OF A NEW PRODUCT (2 OF 6)

The Bass model has three parameters that must be estimated.

- *m* is the number of people estimated to eventually adopt a new product
- *q* is the coefficient of imitation which measures the likelihood of adoption due to a potential adopter influenced by someone who has already adopted the product
- *p* is the coefficient of imitation which measures the likelihood of adoption assuming no influence from someone who has already adopted the product.

FORECASTING ADOPTION OF A NEW PRODUCT (3 OF 6)

Developing the Forecasting Model

F_t , the forecast of the number of new adopters during time period t , is

F_t = (likelihood of a new adoption in time period t)

x (number of potential adopters remaining at the end of time period t – 1)

FORECASTING ADOPTION OF A NEW PRODUCT (4 OF 6)

Developing the Forecasting Model

- Essentially, the likelihood of a new adoption is the likelihood of adoption due to innovation plus the likelihood of adoption due to imitation.
- Let C_{t-1} denote the number of people who have adopted the product up to time $t - 1$.
- Hence, C_{t-1}/m is the fraction of the number of people estimated to adopt the product by time $t - 1$.
- The likelihood of adoption due to imitation is $q(C_{t-1}/m)$.
- The likelihood of adoption due to innovation <u>and</u> imitation is $p + q(C_{t-1}/m)$.

FORECASTING ADOPTION OF A NEW PRODUCT

Developing the Forecasting Model

- The number of potential adopters remaining at the end of time period

$$t-1 \text{ is } m-C_{t-1}.$$

- Hence, the complete forecast model is given by

$$F_t = \left(p + q\left(C_{t-1}/m\right)\right)\left(m - C_{t-1}\right)$$

FORECASTING ADOPTION OF A NEW PRODUCT (6 OF 6)

Nonlinear Optimization Problem Formulation

$$\text{Min} \sum_{t=1}^{N} E_t^2$$

$$F_t = \left(p + q\left(C_{t-1} / m\right)\right)\left(m - C_{t-1}\right),$$

$$t = 1, \ldots., N$$

$$E_t = F_t - S_t, \quad t = 1, \ldots.,$$

$$N$$

where *N* = number of time periods of data available

E_t = forecast error for time period *t*

S_t = actual number of adopters (or a multiple of that number such as sales) in time period *t*