Mathematical Foundations for Finance and Insurance

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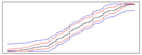
Quiz 19

Prove Theorem 52 as an application of Theorem 38, instead of using Lemma 51.

Outline

1. Motivation ✓
2. Theorems
3. Proof



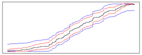
Theorem 38

Theorem (Dvoretzky, Kiefer, and Wolfowitz)

Let F be defined on $\mathbb{R}$, $\exists C > 0$ (not depending on F) such that

$$P(D_n > d) < C exp(-2nd^2), \quad d > 0$$

$\forall n = 1, 2, ...$



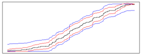
Theorem 52

Theorem (52)

Let $0 < p < 1$. Suppose that ξ_p is the unique solution x of $F(x-) \leq p \leq F(x)$. Then for every $\epsilon > 0$

$$P(\, | \, \widehat{\xi_{pn}} - \xi_p | > \epsilon) \leq 2exp(-2n\delta_\epsilon^2), \forall n$$

where $\delta_\epsilon = min(F(\xi_p + \epsilon) - p, p - F(\xi_p + \epsilon)$



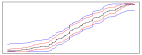
Proof

Theorem (52)

$$P(\,|\,\widehat{\xi_{pn}} - \xi_p| > \epsilon) \leq 2exp(-2n\delta_\epsilon^2)$$

Theorem (38)

$$P(D_n > d) < Cexp(-2nd^2)$$



Proof

Theorem (52)

$$P(\,|\,\widehat{\xi_{pn}} - \xi_p| > \epsilon) \leq 2exp(-2n\delta_\epsilon^2)$$

Theorem (38)

$$P(D_n > d) < Cexp(-2nd^2)$$

$D_n = KS - Distance = sup|F_n(x) - F(x)|$

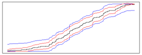
Proof

Theorem (52)

$$P(\,|\,\widehat{\xi_{pn}} - \xi_p| > \epsilon) \leq 2exp(-2n\delta_\epsilon^2)$$

Theorem (38)

$$P(sup|F_n(x) - F(x)| > d) < Cexp(-2nd^2)$$



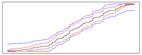
Proof

Theorem (52)

$$P(\,|\,\widehat{\xi_{pn}} - \xi_p| > \epsilon) \leq 2exp(-2n\delta_\epsilon^2)$$

Theorem (38)

$$P(|F_n(x) - F(x)| > d) \leq \; Cexp(-2nd^2)$$



Proof

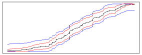
Theorem (52)

$$P(\,|\,\widehat{\xi_{pn}} - \xi_p| > \epsilon) \leq 2exp(-2n\delta_\epsilon^2)$$

Theorem (38)

$$P(|F_n(x) - F(x)| > d) \leq \; Cexp(-2nd^2)$$

Using the Strong Consistency property and the following definition: Given a sample $X_1, ..., X_n$ of observations on F, the sample pth quantile $\widehat{\xi_{pn}}$ or $\widehat{\xi_p}$, is defined as the pth quantile of the sample distribution function F_n



Proof

Theorem (52)

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Theorem (38)

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Proof

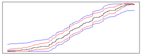
Theorem (52)

$$P(\,|\,\widehat{\xi_{pn}} - \xi_p| > \epsilon) \leq 2exp(-2n\delta_\epsilon^2)$$

Theorem (38)

$$P(\,|\,\widehat{\xi_{pn}} - \xi_p| > \epsilon) \leq \; Cexp(-2nd^2)$$

Using the following notation: $\delta_\epsilon = F(\zeta_p + \epsilon) - p \rightarrow \delta_\epsilon \sim \epsilon$



Proof

Theorem (52)

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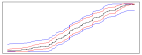
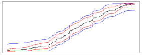


Proof

For any integer n and any λ not less than $\sqrt{log(2)/2}$ and $\gamma n^{-1/6}$where $\gamma = 1.0841$, we have:

$$P(D_n > n) \leq exp(-2\lambda^2)$$

Proof.
First Step:
where $n \geq 39$ and $\lambda \leq \frac{\sqrt{n}}{2}$: $C = 3.61$
Second Step:
where $n \leq 38$ and $\lambda > \frac{\sqrt{n}}{2}$: $C = 2$ $\square$



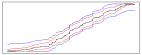
Proof

Theorem (52)

$$P(\,|\,\widehat{\xi_{pn}} - \xi_p| > \epsilon) \leq \; 2exp(-2n\delta_\epsilon^2)$$

Theorem (38)

$$P(\,|\,\widehat{\xi_{pn}} - \xi_p| > \epsilon) \leq \; 2exp(-2n\delta_\epsilon^2)$$



Further Information

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The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality
The Annals of Probability, Vol 18, No.3, 1269-1283 (1990)

Dudley
Uniform Central Limit Theorems
Cambridge University Press, 2. Ed. (2014)