# Implied Binomial Tree and Copula Estimation

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# Outline

1. Implied Binomial Tree Derivation
2. Copula Estimation

# Implied Binomial Tree

- Construction adapted to the volatility smile
- Possibility to price derivative securities
- Calculation of the state price density (SPD)
- Calculation of the implied local volatility surface

# IBT algorithm

- Arrow-Debreu prices $\lambda_n^i$ (discounted risk-neutral probability) the price of an option that pays 1 in one and only one state i at $n$th level, and otherwise pays 0

$$\lambda_{n+1}^{1} = e^{-r\Delta t}\{(1-p_n^1)\lambda_n^1\}$$
$$\lambda_{n+1}^{i+1} = e^{-r\Delta t}\{\lambda_n^i p_n^i + \lambda_n^{i+1}(1-p_n^{i+1})\}$$
$$\lambda_{n+1}^{n+1} = e^{-r\Delta t}\{\lambda_n^n p_n^n\}$$

- Put option price $P(K, n\Delta t)$

$$P(K, n\Delta t) = \sum_{i=0}^{n} \lambda_{n+1}^{i+1} \max(K - S_{n+1}^{i+1}, 0)$$

- Define $S^i_{n+1} = S^1_1 = S, i = \frac{n}{2} + 1$, for n even
- K = S = $S^i_n$
- Risk-neutral condition:

$$F^i_n = p^n_i S^{i+1}_{n+1} + (1 - p^n_i) S^i_{n+1}$$

- Define

$$\rho_u = \sum_{j=i+1}^{n} \lambda^j_n (F^j_n - S^i_n), \rho_l = \sum_{j=1}^{i-1} \lambda^j_n (S^i_n - F^j_n)$$

$$
\begin{aligned}
P(S, n\Delta t) &= e^{-r\Delta t}[\lambda_n^1(1-p_n^1)\max(S - S_{n+1}^1, 0) \\
&+ \sum_{j=1}^{n-1}\{\lambda_n^j p_n^j + \lambda_n^{j+1}(1-p_n^{j+1})\}\max(S - S_{n+1}^{j+1}, 0) \\
&+ \lambda_n^n p_n^n \max(S - S_{n+1}^{n+1}, 0)] \\
&= e^{-r\Delta t}[\lambda_n^1(1-p_n^1)(S - S_{n+1}^1) \\
&+ \sum_{j=1}^{i-1}\{\lambda_n^j p_n^j + \lambda_n^{j+1}(1-p_n^{j+1})\}(S - S_{n+1}^{j+1})]
\end{aligned}
$$

$$
\begin{aligned}
P(S, n\Delta t) &= e^{-r\Delta t}\lambda_n^i(1 - p_n^i)(S - S_{n+1}^i) \\
&\quad + \sum_{j=1}^{i-1} \lambda_n^j \{(1 - p_n^j)(S - S_{n+1}^j) + p_n^j(S - S_{n+1}^{j+1})]\} \\
&= e^{-r\Delta t}\{\lambda_n^i(1 - p_n^i)(S - S_{n+1}^i) + \sum_{j=1}^{i-1} \lambda_n^j(S_n^i - F_n^j)\}
\end{aligned}
$$

Upward

$$S_{n+1}^{i+1} = \frac{S_{n+1}^{i}\{C(S,n\Delta t)e^{r\Delta t} - \rho_u\} - \lambda_n^i S(F_n^i - S_{n+1}^i)}{C(S,n\Delta t)e^{r\Delta t} - \rho_u - \lambda_n^i(F_n^i - S_{n+1}^i)}$$

Downward

$$S_{n+1}^{i} = \frac{S_{n+1}^{i+1}\{P(S,n\Delta t)e^{r\Delta t} - \rho_l\} - \lambda_n^i S(F_n^i - S_{n+1}^{i+1})}{P(S,n\Delta t)e^{r\Delta t} - \rho_l - \lambda_n^i(F_n^i - S_{n+1}^{i+1})}$$

# Copula

- Model dependencies between assets in a portfolio
- Measurement of risk of a portfolio
- Extrem outcomes are more likely if the assets in a portfolio are highly correlated
- Reduction of risk by reducing the correlation of the returns
- Possibility to model assets with different distributions

# Empirical Study

- Empirical study using Gaussian, t, Gumbel and Clayton copula
- *S&P* 500 stocks between 2006 and 2015
- Data from three sectors (aeronautic and defense, media and oil exploration) with four companies each

# Used Copulae

Gaussian Copula

$$C_p^{Gauss}(u_1, \dots, u_d) = \Phi_p(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d))$$

Student's t-copula

$$C^t_{\nu,\Psi}(u_1, \dots u_d) = t_{\nu,\Psi}(t_\nu^{-1}(u_1), \dots, t_\nu^{-1}(u_d))$$

# Used Copulae

Gumbel copula, $1 \leq \theta \leq \infty$

$$C_\theta(u_1, ..., u_d) = exp[-\{\sum_{i=1}^{d} (log\ \ u_i)^\theta\}^{\theta^{-1}}$$

Clayton copula, $0 < \theta$

$$C_\theta(u_1, ..., u_d) = \{(\sum_{i=1}^{d} u_i^{-\theta}) - d + 1\}^{-\frac{1}{\theta}}$$

# Procedure

- GARCH for stock return
- Using residuals to estimate dependence structure from copula functions
- Simulate d-dimensional dependence bades on copula
- Portfolio Value-at-Risk estimate

# Companies Included

Aeronautic and defense

- Boing, Lockheed Martin, Northrop Grumman, General Dynamics

Media

- CBS, Comcast, Time Warners, Viacom

Oil exploration and equipment

- Devon Energy, Occidental Petroleum, Marathon Oil, ConocoPhillips

# Tau(Clayton Copula)

| | Boing | LM | NG | GD |
|---|---|---|---|---|
| Boing | 1 | 0.06 | 0.52 | 0.43 |
| LM | 0.06 | 1 | 0.05 | 0.01 |
| NG | 0.52 | 0.05 | 1 | 0.58 |
| GD | 0.43 | 0.01 | 0.58 | 1 |
| CBS | 0.31 | 0 | 0.31 | 0.33 |
| Comcast | 0.32 | 0.05 | 0.38 | 0.35 |
| TW | 0.33 | 0.04 | 0.34 | 0.33 |
| Viacom | 0.22 | 0 | 0.21 | 0.22 |
| DE | 0.17 | 0 | 0.20 | 0.25 |
| OP | 0.19 | 0 | 0.27 | 0.29 |
| MO | 0.13 | 0 | 0.17 | 0.20 |
| CP | 0.22 | 0 | 0.25 | 0.25 |

# Tau(Clayton Copula)

| | CBS | Comcast | TW | Viacom |
|---|---|---|---|---|
| Boing | 0.31 | 0.32 | 0.33 | 0.22 |
| LM | 0 | 0.05 | 0.04 | 0 |
| NG | 0.31 | 0.38 | 0.34 | 0.21 |
| GD | 0.33 | 0.35 | 0.33 | 0.22 |
| CBS | 1 | 0.43 | 0.43 | 0.36 |
| Comcast | 0.43 | 1 | 0.45 | 0.31 |
| TW | 0.43 | 0.45 | 1 | 0.37 |
| Viacom | 0.36 | 0.31 | 0.37 | 1 |
| DE | 0.30 | 0.25 | 0.21 | 0.19 |
| OP | 0.28 | 0.27 | 0.23 | 0.19 |
| MO | 0.27 | 0.21 | 0.20 | 0.20 |
| CP | 0.29 | 0.24 | 0.23 | 0.19 |

# Tau(Clayton Copula)

| | DE | OP | MO | CP |
|---|---|---|---|---|
| Boing | 0.17 | 0.19 | 0.13 | 0.22 |
| LM | 0 | 0 | 0 | 0 |
| NG | 0.20 | 0.27 | 0.17 | 0.25 |
| GD | 0.25 | 0.29 | 0.20 | 0.25 |
| CBS | 0.30 | 0.28 | 0.27 | 0.29 |
| Comcast | 0.25 | 0.27 | 0.21 | 0.24 |
| TW | 0.21 | 0.23 | 0.20 | 0.23 |
| Viacom | 0.19 | 0.19 | 0.20 | 0.19 |
| DE | 1 | 0.56 | 0.58 | 0.56 |
| OP | 0.56 | 1 | 0.54 | 0.55 |
| MO | 0.58 | 0.54 | 1 | 0.57 |
| CP | 0.56 | 0.55 | 0.57 | 1 |

# Results

- Higher dependence of companies in the same sector
- Similar structure of the other Copulae
- Gumbel finds in general a higher dependence as the Clayton
- Gaussian and t-Copula very similar and with higher dependence as the other two