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MARC HENRARD

Interest Rate Modelling in the Multi-curve Framework

Foundations, Evolution and Implementation

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Preface

The first lines of this book were written in 2006. At the time the term *multi-curve framework*, which is used for the book's title, had not been coined and my idea was only to write a couple of pages for a note. In the meantime, August 2007 changed the course of writing on interest rate curve modelling for ever.

> Festina lente.
>
> *Latin saying*

> Personal translation: Haste slowly.

Why did it take me so long?

> Chacun sa méthode... Moi, je travaille en dormant et la solution de tous les problèmes, je la trouve en rêvant. (Each his own method ... Myself, I work sleeping and the solution to all problems, I find it dreaming).
>
> *Drôle de drame (1937) – Marcel Carné*

This means a lot of nights spent working to dream up all these pages.

The starting point of the reflection was my quest to answer the question 'What is the present value of an FRA(Forward Rate Agreement)?' in a convincing way. I could not find a satisfactory answer in the literature. The answers I could find were either 'it is trivial', or a description of a replication argument for which it was not acceptable to discuss the numerous hidden hypotheses. Discussing the hypothesis was not politically correct as a scientist nor as a business executive. For the former, it questioned a foundation of quantitative finance, and thus the developments built on those foundations. For the latter, it was not seen positively by board members and supervisors as it casted doubts on accounting figures – maybe rightly so – and had legal implications.

The reason for my interest in discounting is that

> Gentlemen prefer bonds.
>
> *Andrew Mellon – 1855–1937*

Ironically, the first article to come out of those reflections, titled *Irony in derivatives discounting*, was published just one month before the now famous August 2007. The publication was not a prediction of what would happen in the derivative market just after and should not be seen as a premonition. Neither should it be seen as a cause of the crisis. It was nevertheless an indication of inconsistencies in

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1 Introduction

1.1 9 August 2007

9 August 2007 is probably the day that the *multi-curve framework* became paramount, even if the term had not yet been coined at that time. What happened on that date can be found in history books. From an interest rate curve modelling perspective, it can be summarised by

> Houston, we have a problem!
>
> *Attributed to the crew of the US's Apollo 13 – 14 April 1970*

From the pre-crisis period to the time of finishing this book, the pricing of Ibor[1] and overnight-related derivatives has changed significantly. Before 2007, the standard textbook formulas,[2] using a unique multipurpose curve in each currency, were generally used in banks and software packages.

Nevertheless not everybody believed this was a perfect approach. Several banks, at least in their front-office, were using different pragmatic approaches incorporating cost of funding, risk-free rates different from Libor and similar issues. I have personally developed and used related techniques for pricing on a trading desk from 2003 onward. I would not, however, call the techniques used at that time a theoretically sound framework. It was more a set of ad hoc adjustments done for specific products with a narrow range of applicability. A theoretical foundation was missing and the coherency between instruments was not guaranteed.

For non-par swaps,[3] noticeable quote differences could be observed between banks with different credit qualities and quantitative development levels. It was a clear sign that adjustments to the textbook formulas were done in some places (and

[1] We use the term Ibor as a generic term for all rates fixing with similar rules. In particular the term covers Libor (originally ten currencies and since May 2013 reduced to six currencies), Euribor (EUR), Tibor (JPY), Cibor (DKK), and so on. See Appendix B for the conventions in the main currencies and Quantitative-Research (2012c) for more currencies and details.

[2] Some of them are described in the first part of Chapter 2.

[3] The interbank swaps are mainly traded at par, with the noticeable exception of the asset swaps.

probably not in others). Since then, several stories have appeared in newspapers and journals describing how some more knowledgeable banks may have used that knowledge to make profits at the detriment of their counterparts and clients. It is difficult to know how much was systematic misinformation and how much was simply taking the risk of using techniques other than the consensus. It should serve the readers as a warning that the current multi-curve framework, as described in this book, is today's equivalent of the 2007 textbook formulas. The framework is still evolving. and the approach is not future proof. As this book is published, there are a lot of people researching and implementing similar subjects in academia and in the industry. In particular, Chapter 8 is a work in progress, both for its theoretical developments and its implementation in practice.

Since the development of the framework became public some people have claimed that the framework was used as early as the mid-1990s in some places. I have not seen any evidence of those claims and all the texts from that period that are now publicly surfacing fall in the *precursor* category, which is discussed in the next section. They described some incoherencies in the single-curve textbook formulas, and proposed some local adjustments, but failed to propose a global, theoretically sound and practically implementable alternative.

Certainly the framework was not developed by one person or in one place. It is the collective work of many people, coming to the problem from a theoretical or practical point of view. It is a child with a lot of fathers and mothers, a child that is still growing fast.

What is certain is that the framework became paramount even before most people knew it existed, and that happened in August 2007.

1.2 Foundations, evolution, and implementation

This book's subtitle is *Foundations, Evolution, and Implementation.* The subtitle acts as a quick summary of the book content.

The *foundations* are paramount. As the framework is different from the previous one-curve approach in fundamental aspects for index-linked financial instruments, one can not rely on a 'copy and paste' style approach. Any claim, even the most basic one, should be backed by clear definitions and clear proofs. Before building the skyscraper that is the pricing of an exotic derivative, one has not only to be sure that the foundations are sound but also indicate explicitly where they are. In some cases, mathematically sophisticated developments are made, only to notice later that their domain of applicability is empty. Their starting hypotheses, which seem acceptable in a one-curve world, lead to contradictions for the multi-curve framework. This is why the book systematically uses an 'axiomatic approach'. The fundamental hypotheses are displayed in the text as quotes with a bold letter in front reminding us of the content of the hypothesis – such as **D** for the discounting hypothesis at the beginning of Section 2.2. These *axioms* or fundamental hypotheses are the foundations of the framework. No development in interest rate modelling can be done without referring to them or to their equivalent in a different framework.

The *evolution* is important to understand how this new framework started and where it is coming from. Several choices made in the new framework are recycling previous approaches with appropriate and justified twists. Why and where those twists were made is an important piece of information. Looking at the final product, it may seem to have been produced by black magic. It is not any more when seen through the eyes of Darwin, as the result of evolution.

In chronological order, the evolution came before the foundations. The theory is more an explanation of the practice obtained by tinkering with different approaches than a new, ground-breaking theory that started to be used from scratch. We refer the reader to the book *Antifragile* by Taleb (2012) for more on how practice often precedes theory. In this book, we reverse the chronological order and start with the foundations.

> Si, à l'égard de plusieurs questions traitées dans cette étude, j'ai comparé les résultats de l'observation à ceux de la théorie, ce n'est pas pour vérifier des formules établies par des méthodes mathématiques, mais pour montrer seulement que le marché, à son insu, obéit à une loi qui le domine : la loi de la probabilité.
>
> *Louis Bachelier, Théorie de la spéculation, 1900*
>
> Personal translation: If, regarding several questions analysed in this study, I compared the observed results to those of the theory, it is not to verify the formulas obtained by mathematical methods, but only to show that the market, unwittingly, complies to a law that dominates it: the law of probability.

This quote, which is the final sentence of Bachelier's 1900 thesis, can be applied in some ways to this book. We do not claim that the theory described appeared in 2007 and started to be used by the market. It first started to be used, and then its use was codified. It was first used in a particular context, on interest rate derivatives desks, and then coherency requirements, analysed by the theory, lead to its general use in finance. The market, unwittingly, complies to a law and the framework tries to axiomatise it.

The *implementation* is the 'raison d'être' of the framework and of this book. The foundations make sense only if they are used in practice. Every piece of theory, every example in this book, was created because it is useful in practice and was implemented for a practical reason to serve a practitioner request. It is possible to implement all of it in a unique and coherent framework. Moreover, the framework is not a goal by itself but is the foundation and an initial building block of interest rate modelling, including forex – as interest rates in several currencies – and inflation.

When you have read and fully implemented all the details of this book you will have completed only the first stage of your journey! Thankfully, you need not set out upon the project on your own. You can join an open community that has worked on it, through its foundations, evolution and implementation.

1.3 Standard textbook framework

The book by Hull (2006), one of the most popular introductory textbooks to derivative pricing, is used as an example of the way earlier literature treats the curves question. In the section on the *type of rates* in the *Interest rates* chapter (p. 76) the existence of both Libor and Libid is acknowledged. The latter is described in the above book as the rate at which a cash rich investor can invest in the interbank market. In 2006, those rates were relatively close to the OIS (Overnight Indexed Swap) rates.

When it comes to valuing the first derivatives (Section 4.7: Forward Rate Agreements) the explanation is 'the assumption underlying the contract is that the borrowing or lending would normally be done at Libor'. This is first a misleading statement on the instrument itself. The reality is that the contract *settlement amount* is computed by discounting with a *Libor fixing* rate between the end of the accrual period and its start. There is *no actual borrowing or lending* and there is *no assumption* in the FRA[4] contract, only a clear (contingent) settlement formula. For the valuation of the instrument before its fixing date, the approach described in the above book is to use the same Libor rate to discount the resulting quantity to the valuation date. But there is no justification in the text for choosing that particular rate from the different ones described in the previous sections. The choice of Libor for that purpose is a modelling choice and not a contractual obligation. This vagueness is certainly a witness of the consensus at that time: if the instrument is related to an Ibor rate, use similar rates for everything, even if there is no modelling or legal reason to do so. The hidden explanation is that it is easier and everybody is doing it.

Another standard textbook on swaps and curve construction is Sadr (2009). In the description of the pricing of a swap, the terms used to describe the rates linked to the Ibor leg are 'hypothetical loan', 'deposits', 'funding' and 'risk-free rates'. The text indicates that to obtain the standard one-curve formula, those different rates have to be the same. Using the equivalence between those rates the book proceeds through standard arbitrage-free arguments to obtain the standard swap pricing formula.

[4] The FRA contract details are described in Appendix B.3.

The approach is described in the chapter called 'Swaps: It's still About Discounting', which for a 2009 book is maybe an unfortunate title. There is no analysis of what happens if the rates are different. The book also describes a method that separates the forward and discount curves, in a way similar to Fruchard et al. (1995) described below. This can be viewed as an example of pre-crisis consensus where Libor, risk-free and funding were somehow considered equivalent.

In his book on model risk, Morini (2011) has a good description of the multiple hidden hypotheses that were used to price simple instruments in that framework.

It is a valuable exercise to read the past approaches with the hindsight of the recent developments and to ask why it was done in that way. Every reader interested in model risk or model validation should do the exercise as a warning to any model, including the ones presented here.

More details on the standard textbook framework are provided at the start of Chapter 2. We use this introduction as a starting point for our *evolution* analysis and as a justification on the choice of fundamental hypotheses and definitions for the rest of the book.

1.4 The precursors

We now review some of the *precursors* literature. We describe why the questions asked and the features described in those articles were important, but also why we do not incorporate them in the *multi-curve framework* literature stricto sensu.

These precursors articles include Fruchard et al. (1995), Tuckman and Porfirio (2003) and Boenkost and Schmidt (2004).

Fruchard et al. (1995) describes the existence of a cross-currency basis and proposes an approach to the pricing of cross-currency swaps. One of the hypotheses (Hypothesis 4 of the paper) is that floating references are independent of their payment frequency, and implicitly that they can be obtained from the same pseudo-discount factor curve. The justification is: 'This is a generally accepted consequence of the risk neutral probability approach.' This approach is based on a market reference for liquidity, the USD Libor (Hypothesis 5). In that main currency, there is a unique curve for discounting and forward estimation. In other currencies there is one discounting and one forward curve. The discounting curves in the other currencies are deduced from the (interest rate swaps) IRS fixed rates and cross-currency basis spread. The paper explicitly excludes the possibility of basis between Ibor rates of different tenors. This basis is one of the features of the post August 2007 market. The approach is able to take into account one cross-currency basis between the base currency and each other currency. The explanation of the difference between currencies is based on an hypothetical difference of liquidity. It potentially explains a difference between currencies but not within a given currency. The formulas to obtain the pricing are quite involved and use several layers of derivatives and integrals.

Tuckman and Porfirio (2003) also focuses on cross-currency swaps. The authors clearly point to incoherences in the then approach for the pricing of those instruments. The solution proposed uses two swaps of the same tenor, one of them not available in the market – a zero-coupon swap – but does not propose a theoretically satisfactory solution. It is an important hint, as it uses two different types of asset to build the curves, like the current multi-curve framework uses two types of instrument. Unfortunately, one of the two assets proposed by the article does not exist in the market and the solution can not be implemented in practice.

Boenkost and Schmidt (2004) also analyses cross-currency swaps starting from the visible incoherencies. They propose two different approaches. One of them is based on two curves in each currency. The two curves are not used for different financial purposes but one is used for each leg of a swap. The result is a different set of incoherencies, like a swap in its last period with zero netted cash-flows having a non-zero present value.

All the above articles asked very good questions, and...

> In mathematics, the art of posing a question is more important than the art of solving one.
>
> *Georg Cantor, 1867*

1.5 Early multi-curve framework literature

To my knowledge, the first article to propose a coherent valuation framework where the discounting is explicitly differentiated from the index forward estimation was *The irony in the derivatives discounting* (Henrard 2007). It was published in July 2007, just one month before the crisis started. To my knowledge there is no link between the article publication and the crisis.

The article focuses on discounting in interest rate derivatives. The starting observation is that different curves are used to value different instruments (OIS, IRS, cross-currency swaps), which is clearly incoherent and creates portfolio level arbitrages.

The approach proposed described only one Ibor curve; it is not an important feature of the approach but a description of the then reality. The approach can be used with several Ibor related curves without changes. The spread between the curves is described in a multiplicative way and is equivalent to the β we define in Chapter 2; our notation is adapted from that used in the paper. Only the case of a constant spread is discussed in the original paper; again, it is not an important feature of the approach but more a description of the then reality.

The first instrument for which the price is discussed is the FRA and the paper provides a pricing formula, including all the hypotheses required to obtain the result. The valuation of swaps in the framework and the impact on the value and risk for not-at-par instruments is discussed. The approach is extended in a coherent way to (Short Term Interest Rate) STIR futures, caps/floors and swaptions. The Ibor fixings are clearly dissociated from the discounting rate through the introduction of an exogenous spread. The spread can be obtained from the market value of different instruments; this is equivalent to the curve building process we propose in Chapter 5.

For practical examples, the article uses an OIS-based discounting curve (called Libid in the paper). The article was probably overly simplistic, as it used only a constant spread between the curves and did not use explicitly different curves for the different Ibor tenors. Nevertheless it was an early proposal leading to today's most commonly used approach.

The importance of the multi-curve framework, with each curve having a specific purpose, is attested by the numerous related literature that appeared in the following years. Throughout the book we have tried to cite as many sources as possible in the relevant sections. We want to give readers the opportunity to go back to the original sources. In this section we comment on a couple of the early papers which describe some particular aspects of the framework. In the literature the framework has received many names. Some of those names are: *two curves*, *multi-curve framework*, *derivative tenor curves*, *funding-Ibor*, *discounting-estimation*, *discounting-forecast*, *discounting-forward*, and *multi-curve market*. Through the book, we will use the name used in the title: *multi-curve framework*.

Ametrano and Bianchetti (2009) is the earliest paper describing the impact of the multi-curve framework on curve construction. Kijima et al. (2009) is important as the first paper describing the impact of collateral. Our approach to collateral, described in Chapter 8, is different but leads to the same fundamental result. Morini (2009) links the framework to the credit risk of the Libor basket obligors. Chibane et al. (2009) describes a multi-curve framework similar to the one described here and a bootstrapping approach to curve calibration. Bianchetti (2010) proposes a description of a multi-curve approach. Piterbarg (2010) is an early description of collateral when collateral rates and risk-free rates are equal. Moreni and Pallavicini (2010) propose a parsimonious simultaneous modelling of both discounting and forward curves. Mercurio (2009), Mercurio (2010a) and Mercurio (2010c) propose a comprehensive Libor Market Model approach for discounting and forward curves. Traven (2010) presents a simplified analysis of the framework based on a restricted set of instruments and on spread around a dominant curve. Bianchetti and Carlicchi (2011) analyses the impact of the framework on optional instruments like swaptions and cap/floor.

In Figure 1.1 the spreads between EUR swaps with different floating leg indexes are displayed in relation to the above literature for the period 2004–13. Note the

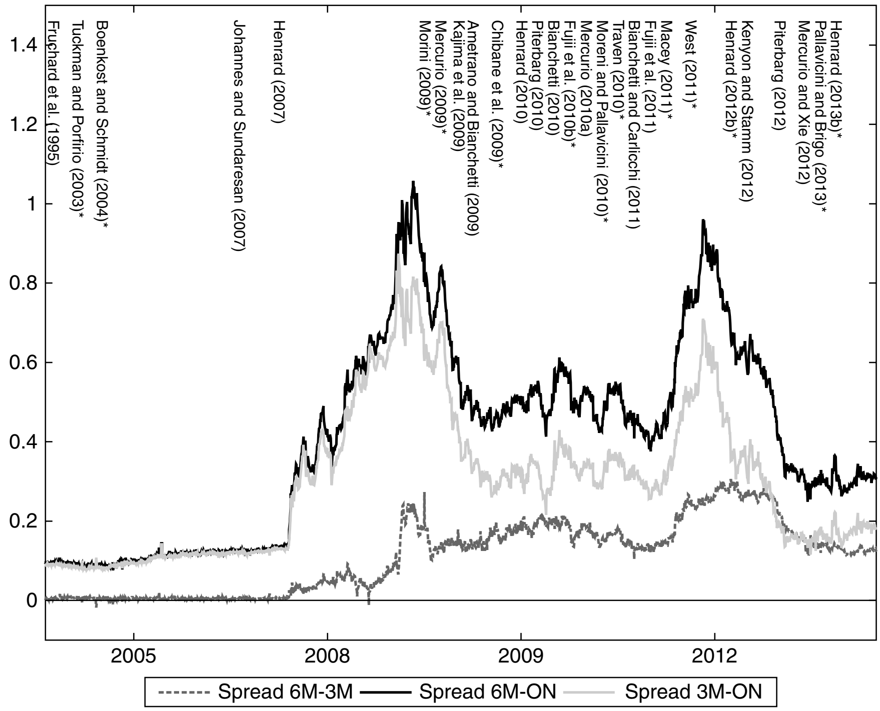


Figure 1.1 Relationship between swap spreads and multi-curves related literature. The spreads are between six months Euribor and three months Euribor and between six months Euribor and OIS, both for swaps with two years tenor.

burst of activity a couple of years after the start of the period with larger spreads. The dates on the graph are approximative in order to avoid text overlap. Literature with a * indicates working papers that have not been published in peer reviewed journals. Obviously the dates proposed are biased, as the publication in peer reviewed journals is strongly impacted by the referee process and the publication back-log. The chronology nevertheless clearly indicates a burst of activity soon after the actual widening of the different spreads, a period where the impact of the new approach was larger.

In the years following the spread widening, a large effort has been made by most banks and software developers to adjust their systems to the new reality. Several financial institutions had started these developments in one form or another before the actual crisis. Nevertheless, robust and general implementations as described here take time and they were not available immediately.

The foundations of the multi-curve framework presented in this book are mainly based on the approach described in Henrard (2010) and Henrard (2013a), the first version of which dates back to mid-2008. It is to my knowledge the first theoretically sound publicly available description of what is now called the *multi-curve framework*. We changed some of the original definition to introduce a more general framework as described in Henrard (2012c).

In that new multi-curve framework context, all of the previous textbook approaches to derivative pricing, and in particular the valuation of vanilla products such as IRS, FRA and futures need to be reviewed carefully. Even for some of the vanilla instruments, strong modelling assumptions are required to obtain simple formulas.

The notions of *yield curves* and *discount factors* need to be handled carefully. The framework presented here is based on a unique curve used to discount all the cash-flows related to the derivatives, whatever the tenors they relate to, and other curves for index estimation. Even if those estimation curves are different from the discounting and old curves, we will in some circumstances use the same names of discount factor, yield curve and forward rate.

The choice of the discounting curve is by itself an open question. Different people will choose different curves. One possibility is to relate it to the rate that can be achieved for the residual cash; the name used then is the *funding* approach. Another possible starting point is that most of the interbank transactions are done on a collateral basis, and in many contracts the collateral renumeration is set to the same rate as the one used in OIS. In particular, Kijima et al. (2009) showed that in the presence of collateral, the derivative pricing is related to discounting at collateral rate. The exact meaning of the claim needs to be made precise; this will be done in Chapter 8.

In Chapter 2, we suppose that the curves are given. The actual curve calibration process, that is how the market information can be used to obtain those theoretical objects, is proposed in Chapter 5. The curves related to indexes are often called *estimation*, *projection*, *forecast* or *forward* curves. Here the plural is used as there is one curve for each index. In practice, the most popular tenors are one, three, six and twelve months and each currency has its favourite family of indexes – Libor, Euribor, CDOR and so on. In theory one should have one curve for each tenor (two, four, and so on months) and for each index family for each currency. For those curves we will also use the terms yield, discount factor and forward rate at the risk of slight abuse of terminology. The forward curves will never be used for discounting.[5] In the next section, we set the main hypothesis used in the framework. Then we price the simple derivatives (IRS, FRA, STIR futures). The term 'simple' has to be taken with a pinch of salt. In this framework, OISs and FRAs are really contingent claims and a curve hypothesis is needed to price them. Of course STIR futures require a so-called convexity adjustment and a model must be specified. In the subsequent chapter, we describe curve construction in detail. We not only

[5] Except for FRAs where the discounting using the Ibor fixing rate is part of the contract specification.

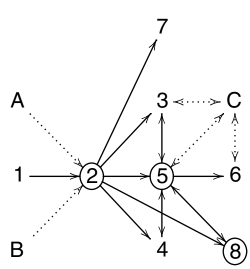
describe the basic requirements for such a process but also describe some nice-to-haves that are useful tools to have in practice. To our knowledge this is the most generic curve calibration process described in the literature, from both a theoretical and an implementation point of view.

1.6 Collateral and funding

The description in the first chapters of the multi-curve framework is in a credit risk-free world without the market reality of the *collateral* and *funding* issues. Even if the central theme of the book is the multi-curve framework and not the collateral and funding issues, a book on the curve framework resulting from the crisis would be missing an important part of the reality if the crisis impact on collateral and funding was not discussed at all. Chapter 8 describes some of the issues related to those features. It provides results that were initially presented in Kijima et al. (2009), Macey (2011), Pallavicini et al. (2012) and Piterbarg (2012). It also presents extensions of the above results. In particular we generalise the definition of collateral and propose results about collateral with assets themselves collateralised – called collateral square. Those results are original and were first published in Henrard (2013b). Our approach to proofs borrows heavily on Macey (2011) and Piterbarg (2012) even if we add the missing self-financing property to the original sketches.

1.7 How to read this book

The easiest way to read this book is to start at Chapter 1 and go to Chapter 8, with a stop in the Appendices when required – you are allowed to eat and sleep between chapters. This is certainly not the only way to read it. In the graph below I have represented the *dependency graph* between the different chapters of the book. The dotted lines for the Appendices indicate where they fit best, but some readers may want to skip them entirely. The links with bidirectional arrows indicate that in some way the two chapters depend on each other and are best read in parallel if the reader has that capacity.



From the graph, the reader can see that, in the mind of the author, the most important chapters are Chapter 2 (Foundations), Chapter 5 (Curve calibration), and Chapter 8 (Collateral).

I tried to write this book in the way I would like to read it. The different *axioms* are clearly evidenced and for each result I have tried to be clear on which hypotheses are required. I have added as many references as possible. The reader should have the opportunity to go to the original literature and compare different approaches. I have also added a lot of cross references between sections, to allow the reader to move between sections with related subjects.

1.8 What is not in this book

A large part of the collateral and funding issues are not discussed in this book. For more in-depth analysis of these issues, we refer to the recent book Brigo et al. (2013). In particular we do not analyse non-perfect collateral. In general the credit risk issues, including Credit Value Adjustment (CVA), are not part of this book. We again refer to the above recent book for more on that subject.

The relation between the Ibor index level and the credit risk embedded in the basket is not discussed. A early account of it can be found in Morini (2009) and a more recent explanation in Crépey (2013) and Filipovic and Trolle (2013). In this book we do not try to explain from first principles the level of the spreads; we take them as an input to the framework.

The inflation derivatives can be viewed as a particular case of Ibor related derivatives in the sense that an index, different from the value of risk-free bonds, is used to reset floating payments. But the inflation market is sufficiently different, with its own idiosyncrasies, and would require an extra chapter or two, not a simple paragraph. We have not tried to cover the subject here.

This book discuss only derivatives, and not assets such as bonds or equities. The bond market would be a mixture of credit, discounting and maybe Ibor (for Floating Rate Notes). This is not covered in this book.

2 The Multi-curve Framework Foundations

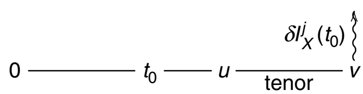
2.1 One-curve world

We start this chapter with a description of some of the one-curve framework pricing formulas. They are important from an historical point of view but also because they explain the evolution of the multi-curve framework, and where its formulas and definitions come from. Chapter 3 proposes different implementations of the framework described here.

Below, we start by describing briefly what an Ibor coupon is. This is the basic financial instrument from which the multi-curve framework is built. More details regarding the conventions related to different coupons and swaps are available in Appendix B. Here we only recall the details important for the understanding of the theory. Note also that we will refrain from using the term *Forward Rate Agreement* (FRA) for one period swaps or single payment Ibor floating coupons. The term FRA has clear market terms and conditions, which are described in Section B.3, including a special settlement procedure which makes them unsuitable as a starting instrument. Some documents in the literature use the term FRA in a loose way as a starting instrument to simplify the description. In my opinion this creates more confusion than simplification.

The general idea of the one-curve world is that all interest rate derivatives depend on only one curve, which is supposed to be at the same time the risk-free curve and the curve relevant for Ibor. The fixed future cash-flows are discounted with that curve. The discount factor at time t for a maturity u is denoted $P(t, u)$. The theoretical deposits underlying the Ibor indexes are priced using the same curve. At any fixing date, the deposit that pays the notional at the settlement date and receives the notional plus the Ibor interest at maturity is supposed to be a fair deposit; this is a deposit for which the total present value is zero. The total value includes the initial settlement of notional and the final payment of notional plus interest. For the purpose of computing the present value, the unique curve is used.

Let j designate an index, such as 'USD LIBOR three months'. We define a j-Ibor floating coupon as a financial instrument which pays at the end of the underlying period the Ibor rate set at the start of the period. The details of the instrument are as follows. The rate is set or fixed at a date t_0 for the period $[u,v]$ $(t_0 \leq u < v)$ of length j; at the end date v the amount paid in currency X is the Ibor fixing $I_X^j(t_0)$ multiplied by the conventional accrual factor δ for the period. The lag between t_0 and u is called the *spot lag* or *settlement lag*. The difference between u and v is the tenor of the index j. All periods and accrual factors should be calculated according to the day count, business day convention, calendar and end-of-month rule appropriate to the relevant Ibor or overnight rate. We describe these conventions for the main currencies in Appendix B. The graphical representation of the coupon, with 0 being today, is given below. The vertical undulating curves represent floating payments and the vertical plain lines represent fixed payments.

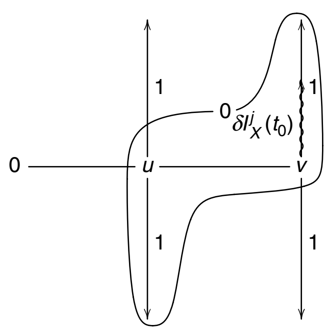


We would like to price this instrument using simple no arbitrage arguments. At this stage we suppose that we are in a credit risk-free world and that the Ibor fixing is equal to the risk-free rate to be paid over the period $[u,v]$ if a deposit is agreed in t_0.

There's no such thing as a free lunch.

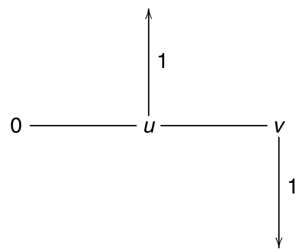
Milton Friedman, 1975

To obtain the price we add synthetic payments and reception of the notional both at the start date and at the end date. Ignoring the settlement risk, paying and receiving the same amount at the same date has a zero value. Adding these cash-flows has not changed the total present value of our instrument. This is represented in the next graph.



We link three of the five payments together: the payment of the initial notional, and the receipt of the final notional and of the interest $\delta I_X^j(t_0)$. The value of this bundle of three cash-flows is very easy to compute. The interest received is the fair interest for a deposit over the period $[u, v]$ if decided in t_0, the notional flows are the standard settlement of the deposit at the start and repayment of the notional at maturity. Viewed from t_0, this instrument has, whatever the fair deposit rate is in t_0, a value of 0; there is no advantage or disadvantage to entering into such a transaction. What is the value today of a transaction that has in all states of the world a value of 0 in t_0? Obviously by no arbitrage argument the value is 0. The three cash-flows circled in the above graph have a total value of 0. We simply remove them and are left with the value described in the next graph.

Note that the fixing date t_0 no longer appears in the graph. This does not mean that any fixing date can be chosen. We were able to simplify the representation only because the coupon period $[u, v]$ is the standard period for the Ibor index fixed in t_0. Even if the date does not appear explicitly at this stage, it appears implicitly through the conventions of the underlying Ibor index and in the simplification done previously. This remark will be important when we deal with in-arrear or delayed payments.



The total value of the remaining cash-flows, which is also the value of the coupon, can be written in 0 as

$$\text{cpn} = P(0, u) - P(0, v).$$

This description of the value of a Ibor coupon is often called the *cash-flow equivalent* valuation of the coupon. The floating coupon is replaced by two fixed cash-flows for valuation purposes. This description is valid only under the hypothesis that the Ibor rate is a risk-free rate corresponding to the curve P. The description will be modified later when we introduce actual Ibor indexes, which are not equivalent to risk-free deposits.

The above formula can be modified as

$$\text{cpn} = P(0, v)\left(\frac{P(0, u)}{P(0, v)} - 1\right) = P(0, v)\delta\frac{1}{\delta}\left(\frac{P(0, u)}{P(0, v)} - 1\right).$$

If we use the definition

$$F = \frac{1}{\delta}\left(\frac{P(0, u)}{P(0, v)} - 1\right) \tag{2.1}$$

and call the number F the *estimated forward rate*, we have the formula

$$cpn = P(0,v)\delta F.$$

The value of the coupon is the discounted (from payment date to today) value of the estimated forward rate. This description of the value is called *discounting forward rates*. This is a second approach to Ibor coupon valuation that will also appear when we introduce the multi-curve framework in the next sections.

We also price a market FRA in this framework. An FRA is described by dates similar to a coupon (see Appendix B for details): a fixing date t_0 and a period $[u,v]$ with accrual factor δ. A FRA is also described by a reference rate K. The settlement is done in u with the receipt of the amount $\delta(I_X^j(t_0)-K)/(1+\delta I_X^j(t_0))$ by the FRA buyer. If the rate $I_X^j(t_0)$ is the risk-free rate for the period $[u,v]$, the above payment is equal in present value to the (risk-free) amount $\delta(I_X^j(t_0)-K)$ in v. The value of the FRA is equivalent to the value of receiving the floating rate $I_X^j(t_0)$ and paying the fixed rate K (multiplied by the relevant accrual factor) in v. This last value is the same as that of a one period swap with fixed rate K.

Under all the simplifying assumptions used above, the value of an FRA is equivalent to the value of a one period swap. But this is not a financial description, it only shows that the two products have an equal present value under a specific strong modelling assumption; this is an equal theoretical result under that specific model.

In this sense, it can be claimed that an approach using the modified, simplified and non-market version of an FRA makes sense only if one does not work in a true multi-curve framework. The use of the simplified version is an indication that one works in a single-curve framework.

2.2 Discounting curve

The multi-curve framework starts with the description of the fundamental curve, which is the first of the multiple curves. The starting point is the discounting of known cash-flows. This is the first hypothesis on which the multi-curve framework is based. The cash-flows considered are credit risk-free cash-flows. The impact of collateral on the framework is discussed in Chapter 8.

D: The instrument paying one unit of currency X in u is an asset for each u and each currency X. Its value in t is denoted $P_X^D(t,u)$. The value is continuous in t.

Remember that time is money.

Benjamin Franklin – 1748

With these curves we are able to value fixed cash-flows. The discounting curves are functions $P_X^D : [0, T] \to (0, +\infty)$. To be able to model curves with the usual tools of quantitative finance one needs to fix an upper bound on the time frame on which the modelling is done. This is described in, for example, (Hunt and Kennedy, 2004, Section 7.4.4). This is why we impose an upper bound T. If the bound T is taken sufficiently large, it will have no impact in practice.

We do not restrict discount factors to be below 1 as we want to allow negative rates. Similarly we do not impose that P_X^D is monotonically decreasing in the second variable. In this sense the treatment presented here is more general than the one presented in (Andersen and Piterbarg, 2010, Section 6.1.1). We restrict only the discount factors to be positive. We want to work in an arbitrage-free world. As the payment at maturity u is $1 > 0$, its present value today t, represented by the discount factor $P_X^D(t, u)$, should be strictly positive.

2.3 Forward curves

Our goal is to price Ibor- and overnight-related derivatives, in particular IRSs and OISs. We need an hypothesis to say that these instruments exist in the framework we are describing.

In this section, we adopt the misuse of language of calling instruments *Ibor coupons* even if the underlying index is not strictly speaking an Ibor index, but an index playing the same role or an overnight index.

As the period addition, t + period j, is used often we adopt the notation $t + j$ for that date, without clarifying in which unit the j is; it is usually clear from the context.

Our existence hypothesis for the Ibor coupons reads as

$\mathbf{I}^{\mathbf{CPN}}$: The value of a j-Ibor floating coupon is an asset for each tenor j, each fixing date and each currency. Its value is a continuous function of time.

The subscript CPN stands for CouPoN. We will use the same notation convention for the different quantities related to the Ibor coupons.

This hypothesis is implicit in most of the literature. It is important to state it explicitly as this is not a consequence of the existence of the discounting curve. This hypothesis is one of the *foundations* described in the introduction.

Once we have assumed that the instrument is an asset, we can give its value a name. We do this indirectly through the curves F_X^j below. These curves are called *forward curves*, *projections curves* or *estimation curves* in the literature. We will use the term *forward curves* most of the time. At this stage we insist that the curves F_X^j are pure definition and the use of the word *curve* should be understood in the mathematical sense, not in the financial sense.

Definition 2.1 (Forward coupon rate). *The* forward curve $F_X^{\mathrm{CPN},j}$ *is the continuous function such that,*

$$P_X^D(t,v)\delta F_X^{\mathrm{CPN},j}(t,u,v) \tag{2.2}$$

is the price in t of the j-Ibor coupon with fixing date t_0, start date u and maturity date v ($t \leq t_0 \leq u = \mathrm{Spot}(t_0) < v$).

The reason for this definition is to keep the usual formulas involving forward rate computation[1]. The standard terms of *discounting curve* and *forward rate* will still be used. But at this stage we insist that they are only definitions and names. One should not attach too strong a financial intuition to those mathematical objects even if we use standard financial terms to designate them.

La mathématique est l'art de donner le même nom à des choses différentes.

Henri Poincaré.

Personal translation: Mathematics is the art of giving the same name to different items.

If the floating coupons have a price, the functions $F_X^{\mathrm{CPN},j}$ exist. At this stage the only link between the curves and market rates is that the Ibor rate fixing in t_0 for the period j, denoted $I_X^j(t_0)$, is

$$I_X^j(t_0) = F_X^{\mathrm{CPN},j}(t_0,u,v). \tag{2.3}$$

To obtain this equality the value time-continuity was used. Take a coupon with fixing in t_0. Its value at $t_0 - \epsilon$ is

$$P_X^D(t_0-\epsilon,v)\delta F_X^{\mathrm{CPN},j}(t_0-\epsilon,u,v).$$

After the fixing, the floating coupon becomes a fixed coupon and its value in $t_0+\epsilon$ is

$$P_X^D(t_0+\epsilon,v)\delta I_X^j(t_0).$$

The value is obtained simply by discounting the known amount from v. By taking the limit on both values when $\epsilon \xrightarrow{>} 0$, one gets the result for $I_X^j(t_0)$.

The text is written with the intuition that the coupon is related to I_X, which is an interest rate index over the period $[u,v]$. Fundamentally there are two types of assets: the risk-free bonds and the index paying coupons; nothing would be different in this chapter if I_X^j was not an interest rate but any other type of index, such

[1] We use the simple interest notation as this is the most used convention in the market. We could also have used something like $P_X^D(t,v)((1+F_X^{\mathrm{CPN},j}(t,u,v))^\delta - 1)$ if we were mainly working in the Brazilian market, where rates are quoted on a daily compounded basis.

as the amount of snowfall in a city or the golf ball production by a manufacturer. Obviously the products would be called *snow swap* or *golf ball swap* but the theory and mathematics would remain the same.

Note also that as the coupons are assets in our economy from hypothesis $\mathbf{I}^{\mathbf{CPN}}$, using Definition 2.1, the forward rates $F_X^{\text{CPN},j}(.,u,v)$ are martingales in the $P_X^D(.,v)$-numeraire. This is simply coming from the fact that F_X^j is the value of an asset divided by the numeraire. This remark will be very important when we model the forward rates. Even if it seems trivial at this stage, we write that result as a theorem.

Theorem 2.1 (Martingale property of forward rates). *Under the hypotheses* $\mathbf{D}$ *and* $\boldsymbol{I^{CPN}}$, *the forward rates* $F_X^{\text{CPN},j}(.,u,v)$ *described in Definition 2.1 are martingales in the* $P_X^D(.,v)$*-numeraire.*

This result could have been used as the definition of forward rate. It would be possible to define the rate $F_X^{\text{CPN},j}(t,u,v)$ through the expectation

$$F_X^{\text{CPN},j}(t,u,v) = \mathrm{E}^v\left[I_X^j(t_0) \middle| \mathcal{F}_t \right].$$

We prefer the definition through the present value as it defines the quantities used through financial realities and not to mathematical abstraction. We will use a different approach when the mathematical abstraction definition gives a more general approach to a problem, like in the case of collateral definition in Chapter 8.

The above theorem justifies the development of an Libor Market Model (LMM) on those quantities. The forward rates are martingales and can be modelled by driftless stochastic equations. This will be discussed in Chapter 7.

Several implementations of these definitions are possible. They will be described in Chapter 3.

We should also add a couple of remarks on the dates. We use the notation $t+j$ as if the time displacement was a real addition. This is not the case. There is no inverse due to non-business dates, several dates u can lead to the same $u+j$. This is not an exceptional case; three days a week, in the standard following rule, have the date $u+j$ on the same Monday.

We also define the forward risk-free rate as follows.

Definition 2.2 (Forward risk-free rate). *The risk-free forward rate over the period* $[u,v]$ *is given at time* t *by*

$$F_X^D(t,u,v) = \frac{1}{\delta}\left(\frac{P_X^D(t,u)}{P_X^D(t,v)} - 1 \right). \tag{2.4}$$

This definition will be useful when we introduce spreads. Moreover some definitions and results are written in a more symmetric way with this notation.

2.4 Interest rate swap

With hypothesis $\mathbf{I}^{\mathbf{CPN}}$ and the related definition, the computation of the present value of vanilla IRSs is straightforward. The definition was selected for that reason. An IRS is described by a set of fixed coupons or cash-flows c_i at dates $\tilde{t}_i$ $(1 \leq i \leq \tilde{n})$. For those flows, the discounting curve is used. It also contains a set of floating coupons over the periods $[t_{i-1}, t_i]$ with $t_i = t_{i-1} + j$ $(1 \leq i \leq n)$. The accrual factors for the periods $[t_{i-1}, t_i]$ are denoted δ_i. The value of a (fixed rate) receiver IRS in $t < t_0$ is

$$\sum_{i=1}^{\tilde{n}} c_i P_X^D(t, \tilde{t}_i) - \sum_{i=1}^{n} P_X^D(t, t_i)\delta_i F_X^{\mathrm{CPN},j}(t, t_{i-1}, t_i). \tag{2.5}$$

In the textbook one-curve pricing approach, the IRSs are usually priced through either the *discounting forward rate* approach or the *cash-flow equivalent* approach. The *discounting forward rate* approach is similar to the above formula.

The *cash-flow equivalent* approach consists in replacing, for valuation purposes, the (receiving) floating leg by receiving the notional at the period start and paying the notional at the period end. We would like to have a similar result in our new framework. To this end we have the following definition.

Definition 2.3 (Multiplicative coupon spread). *The* multiplicative spread *between a forward curve and the discounting curve is*

$$\beta_X^{\mathrm{CPN},j}(t, u, v) = \left(1 + \delta F_X^{\mathrm{CPN},j}(t, u, v)\right) \frac{P_X^D(t, v)}{P_X^D(t, u)} = \frac{1 + \delta F_X^{\mathrm{CPN},j}(t, u, v)}{1 + \delta F_X^D(t, u, v)}. \tag{2.6}$$

This type of multiplicative spread is quite natural in the interest rate context. The factor β is the investment factor to be composed with the risk-free investment factor to obtain the Ibor investment factor. It could also be written as

$$\left(1 + \delta F_X^D(t, u, v)\right)\left(1 + \delta F_X^{\beta}\right) = \left(1 + \delta F_X^{\mathrm{CPN},j}(t, u, v)\right)$$

for a clear definition of F_X^{β}.

Obviously the value of this variable is constant at 1 if $F_X^D = F_X^j$, in particular in the one-curve world. With the above definition, a j-Ibor coupon present value is

$$\begin{aligned}
P_X^D(t, v)\delta F_X^{\mathrm{CPN},j}(t, u, v) &= P_X^D(t, v)\left(\beta_X^{\mathrm{CPN},j}(t, v, u)\frac{P_X^D(t, u)}{P_X^D(t, v)} - 1\right) \\
&= \beta_X^{\mathrm{CPN},j}(t, u, v)P_X^D(t, u) - P_X^D(t, v).
\end{aligned}$$

The value is equal to the value of receiving β_X^j notional at the period start and paying the notional at the period end.

For the forward risk-free rate F^D_X defined above, the floating coupon price can also be written as

$$P^D_X(t,v)\delta\left(\beta_X^{\text{CPN},j}(t,u,v)F^D_X(t,u,v)+\frac{1}{\delta}\left(\beta_X^{\text{CPN},j}(t,u,v)-1\right)\right)$$

which is the formula originally proposed in Henrard (2007).

The cash-flow equivalent description can be extended to the valuation of swaps. The present value of the swap from Equation (8.18) is

$$\sum_{i=1}^{\tilde{n}} c_i P^D_X(t,\tilde{t}_i)-\sum_{i=1}^{n}\left(\beta_X^{\text{CPN},j}(t,t_{i-1},t_i)P^D_X(t,t_{i-1})-P^D(t,t_i)\right).$$

Grouping the terms with the same discount factor together and defining $\bar{n}$, $\bar{t}_i$ and d_i appropriately, one has

$$\sum_{i=0}^{\bar{n}} d_i(t)P^D_X(t,\bar{t}_i).$$

The quantities d_i depend on the curves and are not really constant. With some hypotheses that we will use later, the quantities will be constant and the situation will be in some sense equivalent to the one-curve world.

A consequence of hypothesis $\mathbf{I}^{\mathbf{CPN}}$ and the definition of $\beta_X^{\text{CPN},j}$ is that $\beta_X^{\text{CPN},j}(.,t_{i-1},t_i)$ is a martingale in the $P^D_X(.,t_{i-1})$ numeraire. The Ibor coupon value is $\beta_X^{\text{CPN},j}(t,t_{i-1},t_i)P^D_X(t,t_{i-1})-P^D_X(t,t_i)$. The coupon is an asset due to hypothesis $\mathbf{I}^{\mathbf{CPN}}$ and so its value divided by the numeraire $P^D_X(t,t_{i-1})$ is a martingale. The second term, the zero-coupon $P^D_X(t,t_i)$, is also an asset, hence its rebased value is also a martingale. The rebased first term is thus also a martingale and its value is $\beta_X^{\text{CPN},j}P^D_X(t,t_{i-1})/P^D_X(t,t_{i-1})=\beta_X^{\text{CPN},j}(t)$. This proves that $\beta_X^{\text{CPN},j}(.,t_{i-1},t_i)$ is a martingale under the $P^D(.,t_{i-1})$-measure.

On the other side, P^j_X cannot be an asset nor its rebased value a martingale. If it were, we would have for any numeraire N,

$$\begin{aligned}P^j_X(0,t)&=N_0\,\mathrm{E}^N\left[N_t^{-1}P^j_X(t,t)\right]\\&=N_0\,\mathrm{E}^N\left[N_t^{-1}1\right]\\&=N_0\,\mathrm{E}^N\left[N_t^{-1}P^D_X(t,t)\right]=P^D_X(0,t)\end{aligned}$$

that is, we would have two assets paying the same amount in t in all states of the universe and they would be equal at any previous date. The two curves P^D_X and P^j_X would be identical, which is a contradiction with the starting hypothesis that we have two different assets.

As with the one-curve framework, we can define a forward swap rate. This is the fixed rate for which the vanilla IRS price is 0:

$$S_t^j = \frac{\sum_{i=1}^{n} P_X^D(t, t_i)\delta_i F_X^{\mathrm{CPN},j}(t, t_{i-1}, t_i)}{\sum_{i=1}^{\tilde{n}} P_X^D(t, \tilde{t}_i)\tilde{\delta}_i}. \tag{2.7}$$

Note that the swap rates of multi-period swaps depend on both the forward curve and the discounting curve. In that aspect, the forward rates of Ibor coupons and the forward rates of multi-period swaps are fundamentally different.

The OIS swaps, paying a composition of overnight rates, have a different term sheet than the IRS described above. They are discussed in Section 2.7.

In practice: swap fixing

The description of swaps above is valid for swaps with one fixed leg and the other leg made of floating coupons only. In practice, this is not the case for most of the swaps on the trade date. The standard swaps are traded with an effective date equal to the spot date. The definition of spot is different for each currency and is described in Section B.8.

When the Ibor index fixing for the day has already taken place (after 11:00 am London time for Libor), the first coupon of the floating leg is not a floating coupon anymore but a fixed coupon. In that case, the value of the swap is

$$\sum_{i=1}^{\tilde{n}} c_i P_X^D(t, \tilde{t}_i) - P_X^D(t, t_1)\delta_1 I_X^j(t^*) - \sum_{i=2}^{n} P_X^D(t, t_i)\delta_i F_X^{\mathrm{CPN},j}(t, t_{i-1}, t_i)$$

where t^* is the fixing time. In practice, it is important in the valuation, including when calibrating the curves, to clearly distinguish between the coupons that have already been fixed and the ones that have not.

More generally the swaps that have *aged* or *seasoned*, that is, swap after their trading date, will also have one or more of the floating coupons already fixed. Most of the time, one of the coupons is fixed. If the valuation time is between the fixing of one coupon, except the first one, and the payment date of the previous one, there will be two fixed coupons.

In practice: dates mismatch in swaps

In the description of j-Ibor floating coupons we indicated that the payment is done at the end of the fixing period, that is, we set our notation with the v used in P_X^D and the one used in $F_X^{\mathrm{CPN},j}$ the same. As indicated in Appendix B.6, due to business day conventions, there can be a mismatch between those dates. The *payment date* (v in P_X^D) can be several days before the *fixing period end date* (v in $F_X^{\mathrm{CPN},j}$). The difference between these two dates is often one or two days but can be up to six.

To avoid introducing cumbersome notation we will not make the distinction between these dates. In theory the mismatch should lead to convexity adjustment for those payments.

In Section 6.2 we analyse the impact of this convexity adjustment on pricing in the Gaussian HJM model. As the impact is negligible, we generally ignore it. In all the other computations we use the actual dates, computed using all the appropriate conventions, but ignore the related convexity adjustments.

2.5 Forward rate agreement

An FRA with *FRA discounting* settlement rule is an instrument linked to a period of length j, a fixing date t_0, an accrual factor δ and a fixed rate K. At the fixing date t_0, the Ibor rate $I^j_X(t_0)$ is recorded. The contractual payment in $u = \text{Spot}(t_0)$ (the start date) received by the contract buyer is

$$\frac{\delta(I^j_X(t_0) - K)}{1 + \delta I^j_X(t_0)}.$$

More details about the FRA conventions are given in Appendix B.3.

The origin of the formula can be traced back to the computation of the value as the difference between the Ibor fixing and the fixed rate discounted at the Ibor rate from the end date to the settlement date. The rate is not paid at the end of its natural period but at the start, and is discounted with the fixing rate itself; we insist that the Ibor discounting used in the above formula is not a modelling choice but part of the contract term sheet. It is not directly a floating coupon or a one period swap as defined in the previous section. In that sense, the above definition of an FRA, in line with a market FRA term sheet, is different from those of Ametrano and Bianchetti (2009), Bianchetti (2010), Chibane et al. (2009) and Mercurio (2009).

We would like to value this instrument in a simple way in our economy. In a general contingent claim formula with a generic numeraire N and associated expected value $\mathrm{E}^N\,[]$, its value in 0 is

$$N_0\,\mathrm{E}^N\left[N_u^{-1}\frac{\delta(I^j_X(t_0) - K)}{1 + \delta I^j_X(t_0)}\right].$$

Here we cannot use the usual trick of postponing the payment to $v = u + j$ by deciding to invest in t_0 from u to v at the rate $I^j_X(t_0)$ (multiplying by $1 + \delta I^j_X(t_0)$) and selecting $P^j_X(.,v)$ as the numeraire to simplify the formula. The reason is that the investment cannot be done at the Ibor rate but only at the risk-free rate.

Our goal at this stage is to obtain a relatively simple, coherent and practical approach to Ibor derivatives pricing. To achieve the simplicity, our next hypothesis is related to the spreads between the curves, as defined through the quantities $\beta_X^j(t)$. Other spread hypotheses will be analysed in later chapters.

S0$^{\mathbf{CPN}}$: The multiplicative spreads $\beta_X^j(t,u,u+j)$, as defined in Equation (2.6), are constant through time: $\beta_X^j(t,u,u+j) = \beta_X^j(0,u,u+j)$ for all t and u.

The hypothesis **S0$^{\mathbf{CPN}}$** can be viewed as the equivalent of the constant continuously compounded spread used in Henrard (2007). Here, the spread is not constant across maturities but deterministic and given by its initial values. The continuously compounded spread is $\ln(\beta_X^j(u,v))/(u-v)$. In that sense, the framework described here is a direct extension of the one developed in Henrard (2007) but adapted to the situation where the spreads are not equal for all maturities.

The hypothesis **S0$^{\mathbf{CPN}}$** is equivalent to the hypothesis that β_X^j is deterministic. The equivalence can be obtained easily through a martingale argument. We know that $\beta_X^j(t,u,v)$ is a $P_X^D(.,u)$ martingale. If $\beta_X^j(.,u,v)$ is deterministic, $\beta_X^j(0,u,v) = \mathrm{E}^u\left[\beta_X^j(t,u,v)\right] = \beta_X^j(t,u,v)$. This proves that when β is deterministic, it is constant in t.

Theorem 2.2 *Under hypotheses **D**, **I^{CPN}** and **S0^{CPN}** the price in t of the FRA of length j with fixing date t_0, accrual factor δ and reference rate K is*

$$P_X^D(t,u)\frac{\delta(F_X^{\mathrm{CPN},j}(t,u,v)-K)}{1+\delta F_X^{\mathrm{CPN},j}} = P_X^D(t,v)\frac{\delta(F_X^{\mathrm{CPN},j}(t,u,v)-K)}{\beta_X^j(t,u,v)}$$

where $u = \mathrm{Spot}(t_0)$ *and* $v = \mathrm{Spot}(t_0)+j$.

Proof: We use the strategy to invest in t_0 the value from u to v. With the $P_X^D(.,v)$ numeraire, the value is

$$P_X^D(t,v)\,\mathrm{E}^v\left[(1+\delta F_X^D(t_0,u,v))\frac{\delta(I_X^j(t_0)-K)}{1+\delta I_X^j(t_0)}\middle|\mathcal{F}_t\right].$$

We replace $I_X^j(t_0)$ by $F_X^{\mathrm{CPN},j}(t_0,u,v)$ and use the definition of $\beta_X^j(t)$ to obtain

$$P_X^D(t,v)\,\mathrm{E}^v\left[(1+\delta F_X^D(t_0,u,v)) - (1+\delta K)(\beta_X^j(t_0,u,v))^{-1}\middle|\mathcal{F}_t\right].$$

We use the constancy of β_X^j and the martingale property of F_X^D to obtain the results.

□

The above result relies heavily on hypothesis **S0$^{\mathbf{CPN}}$**. Even if the floating coupon value exists by **I$^{\mathbf{CPN}}$** and the related definition, this is not enough to price the

FRA; an extra hypothesis is necessary to obtain a simple formula. Discounting the FRA with the Ibor rate between the start date and the end date creates an adjustment represented by the coefficient β_X^j. The above formula is used by most systems decoupling discounting and forward curves. The hypothesis $\mathbf{S0^{CPN}}$ justifying such a formula is seldom provided.

(Mercurio 2010a, Appendix A) analyses the impact of non-constant spread on the valuation of market FRAs. His analysis shows that the adjustment required when the spread is not constant depends on the spread size and its volatility. With realistic market data, he obtains adjustments of up to 0.5 bps for a ten year FRA. For FRAs below one year, as typically used in curve constructions, the adjustment he obtains is below 0.1 bps.

An FRA is said to be at-the-money when the rate K is such that the instrument value is 0. Using the above result, this is the case when

$$K = F_0^j = \frac{1}{\delta}\left(\beta_X^j(t_1,t_2)\frac{P^D(0,t_1)}{P^D(0,t_2)} - 1\right).$$

This is essentially the FRA fair rate obtained in Mercurio (2009) with, in his notation,

$$\beta_X^j = \frac{1}{R + (1-R)\,\mathrm{E}^{\cdot}\left[Q(t_1,t_2)\right]}.$$

The adjustment factor above is linked in Mercurio (2009) to credit-related parameters such as recovery rate and default probability. It is also linked to the quantity $K_L(t,t_1,t_2)$ defined in (Kijima et al., 2009, Equation (4.7)). In their article the spread is linked to some model parameters; in the approach described here the spread is fitted to the market curves.

Note also that to compute the FRA fair rate, it is enough to know the forward curve F_X^j while to compute its present value, one needs to know both the forward curve and the discounting curve.

2.6 STIR Futures

A general pricing formula for interest rate futures in the one-factor Gaussian HJM model in the one-curve framework was proposed in Henrard (2005). The formula extended a previous result proposed in Kirikos and Novak (1997). The pricing in a displaced diffusion LMM with skew is analysed in Jäckel and Kawai (2005). Piterbarg and Renedo (2004) analyse the pricing of futures in some general stochastic volatility model to study the impact of the smile; they also analyse the pricing of options on futures.

The Gaussian HJM formula was extended to the multi-curve framework in Henrard (2007). In this section, we describe the pricing of futures under the hypotheses

$\mathbf{I^{CPN}}$ and $\mathbf{S0^{CPN}}$ in a multi-factor Gaussian HJM multi-curve framework. The exact notation for the multi-factor Gaussian HJM framework used here is given in Appendix 9. The pricing of the futures in the LMM with stochastic basis is proposed in Mercurio (2010b). In Mercurio and Xie (2012), the pricing of futures for an additive stochastic basis spread is analysed. The pricing of futures in a framework with multiplicative stochastic basis is proposed in Chapter 7.

The future *fixing* or *last trading date* is denoted t_0. The fixing is on the Ibor rate between $u = \text{Spot}(t_0)$ and $v = u + j$. The fixing accrual factor for the period $[u, v]$ is δ. More details on the STIR futures conventions can be found in Appendix B.4.

The futures price is denoted Φ_t^j. On the fixing date, the relation between the futures price and the fixing rate is

$$\Phi_{t_0}^j = 1 - I_X^j(t_0).$$

The futures margining is done on the futures price (multiplied by the notional and the futures accrual factor).

When analysing futures prices, we will use the generic futures price process theorem (Hunt and Kennedy 2004, Theorem 12.6), which states that

$$\Phi_t^j = \mathrm{E}^{\mathbb{N}}\left[\Phi_{t_0}^j \middle| \mathcal{F}_t\right]$$

where $\mathrm{E}^{\mathbb{N}}[]$ is the cash account numeraire expectation. A generalisation of this theorem is proved in Chapter 8. The formula above can be obtained as a special case of formula (8.3) for a collateral rate $c_t = 0$. The link between pricing under collateral and futures is explained in that chapter.

Theorem 2.3 *Let $0 \le t \le t_0 \le u < v$. In the multi-factor Gaussian HJM multi-curve framework under the hypotheses **D**, $\mathbf{I^{CPN}}$ and $\mathbf{S0^{CPN}}$, the price in t of the futures fixing in t_0 for the period $[u, v]$ with fixing accrual factor δ is given by*

$$\Phi_t^j = 1 + \frac{1}{\delta} - \frac{1}{\delta}(1 + \delta F_X^{\text{CPN},j}(t, u, v))\gamma(t, t_0, u, v) \tag{2.8}$$

where the convexity adjustment factor $\gamma(t, t_0, u, v)$ is given in Appendix A.

Proof: The above number can be written as

$$\begin{aligned} 1 - I_X^j(t_0) &= 1 + \frac{1}{\delta} - \frac{1}{\delta}\left(1 + \delta I_X^j(t_0)\right) \\ &= 1 + \frac{1}{\delta} - \frac{1}{\delta}\left(1 + \delta F_X^{\text{CPN},j}(t_0, u, v)\right). \end{aligned}$$

The investment factor $1 + \delta F_X^j(t_0, u, v)$ can be written as the product of $\beta_X^j(t_0)$ and a ratio of discount factors. Using Lemma A.1, the ratio of discount factors is

$$\frac{P^D(t_0, v)}{P^D(t_0, u)} = \frac{P^D(t, v)}{P^D(t, u)} \exp\left(-\alpha(t, t_0, u, v)X_{t,t_0} - \frac{1}{2}\alpha^2(t, t_0, u, v)\right)\gamma(t, t_0, u, v)$$

By hypothesis $\mathbf{S0^{CPN}}$, the coefficient $\beta_X^j(t_0)$ is constant. The expected value of the above exponential is 1 and the announced result follows. □

The pricing formula mixes the rate in the forward curve $F_X^{\text{CPN},j}$ and the constant γ. The constant depends on the model parameters of the discounting curve P^D. The result incorporates the forward rate, the discounting curve model and the interaction between them. The result strongly depends on the interaction between the forward curve and the discounting curve given by $\mathbf{S0^{CPN}}$.

The convexity adjustment is the difference between the futures rate and the forward rate, that is, the convexity adjustment is given by

$$1 - \Phi_t^j - F_X^{\text{CPN},j}(t,u,v) = \frac{1}{\delta}(1 + \delta F_X^{\text{CPN},j}(t,u,v))(\gamma(t,t_0,u,v) - 1).$$

The adjustment is positive when $\gamma > 1$.

The quantity γ, given by Equation (A.3), is in most practical cases above 1. Recently the convexity adjustment has been negative in EUR as described in Carver (2013). Pozdnyakov and Steele (2009) describe technical conditions in which the adjustment in the one-curve world is positive. From the formula of γ, it will be the case if $\nu(s,u) \cdot \nu(s,v) > |\nu(s,u)|^2$ for all s $(t \leq s \leq t_0)$. This can be interpreted geometrically as the projection of $\nu(s,v)$ on $\nu(s,u)$ is larger than $\nu(s,u)$, or, for a certain definition of 'volatility', the volatility of the longer term bond of maturity v is larger than the shorter bond of maturity u.

In Chapter 7 we will propose several other approaches to futures pricing when the spread between the curves, represented by β, is not constant but stochastic.

2.7 Overnight indexed swaps

The coupons of OISs differ from those of Ibor swaps. The coupons on an OIS are computed by compounding the overnight rates and are paid at the end of a given period (often three months or one year). Moreover, the payment is done with a lag (usually two days after the last fixing publication in EUR and USD). More details on the OIS conventions can be found in Appendix B.9.

In this section we analyse the impact of the compounding on the pricing. The impact of compounding in the multi-curve framework was originally analysed in Quantitative Research (2012d).

The impact of the delayed payment is ignored here. It was proved in Henrard (2004) that the adjustment is minimal and can be neglected in practice. We will come back to the feature and propose its analysis in Chapter 6.

The description of an overnight indexed coupon is as follows. The times associated are denoted t_i $(i = 0, \ldots, n)$. They correspond to successive business days in the relevant calendar. The fixing for the period $[t_{i-1}, t_i]$ is denoted $I_X^O(t_{i-1})$ $(i = 1, \ldots, n)$ and the accrual factor in the index convention is δ_i.

The overnight indexed coupon pays the amount

$$\left(\prod_{i=1}^{n}(1+\delta_i I_X^O(t_{i-1}))\right) - 1$$

in t_p. In this section we suppose that $t_p = t_n$.

Let $P_X^O(s,t)$ denote the OIS forward curve in the multi-curve framework. The meaning of OIS forward curve is the one described in $\mathbf{I^{CPN}}$ and its related definition: the pricing of a one period (from t_s to t_e, paying $\delta_e I_e^O$) overnight index coupon is given in s by

$$P_X^D(s,t_e)\delta F_X^O(s,t_s,t_e).$$

As before we define

$$\beta_X^O(s,t,t+1d) = (1+\delta F_X^O(s,t,t+1d))\frac{P_X^D(s,t+1d)}{P_X^D(s,t)} \tag{2.9}$$

where $t+1d$ has to be understood as t plus one business day.

We work under the constant spread hypothesis $\mathbf{S0^{CPN}}$ with $j = O$. The present value of the coupon (in the $P_X^D(.,t_n)$ numeraire) is given by

$$P_X^D(0,t_n)\,\mathrm{E}^n\left[\prod_{i=1}^{n}\Big(1+\delta_i F_X^O(t_{i-1},t_{i-1},t_i)\Big) - 1\right] \tag{2.10}$$

Obviously the part of the value that requires analysis is the product. Using the definition of β^O and the constance of $\beta_X^O(s,t,t+1d)$ over time, we have

$$\begin{aligned}
&\mathrm{E}^n\left[\left(\prod_{i=1}^{n}(1+\delta_i F_X^O(t_{i-1},t_{i-1},t_i))\right)\right]\\
&\quad = \mathrm{E}^n\left[\left(\prod_{i=1}^{n}\beta_X^O(t_{i-1},t_i)\frac{P_X^D(t_{i-1},t_{i-1})}{P_X^D(t_{i-1},t_i)}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}^n\left[\left(\prod_{i=1}^{n-1} \beta_X^O(t_{i-1},t_i)\frac{P_X^D(t_{i-1},t_{i-1})}{P_X^D(t_{i-1},t_i)}\beta_X^O(t_{n-1},t_n)\right.\right.\\
&\qquad \left.\left.\times \mathrm{E}^n\left[\left.\frac{P_X^D(t_{n-1},t_{n-1})}{P_X^D(t_{n-1},t_n)}\right|\mathcal{F}_{t_{n-2}}\right]\right)\right]\\
&= \mathrm{E}^n\left[\left(\prod_{i=1}^{n-1} \beta_X^O(t_{i-1},t_i)\frac{P_X^D(t_{i-1},t_{i-1})}{P_X^D(t_{i-1},t_i)}\beta_X^O(t_{n-1},t_n)\frac{P_X^D(t_{n-2},t_{n-1})}{P_X^D(t_{n-2},t_n)}\right)\right]\\
&= \mathrm{E}^n\left[\left(\prod_{i=1}^{n-2} \beta_X^O(t_{i-1},t_i)\frac{P_X^D(t_{i-1},t_{i-1})}{P_X^D(t_{i-1},t_i)}\beta_X^O(t_{n-2},t_{n-1})\right.\right.\\
&\qquad \left.\left.\times \beta_X^O(t_{n-1},t_n)\frac{P_X^D(t_{n-2},t_{n-2})}{P_X^D(t_{n-2},t_n)}\right)\right]\\
&= \mathrm{E}^n\left[\left(\prod_{i=1}^{n-2} \beta_X^O(t_{i-1},t_i)\frac{P_X^D(t_{i-1},t_{i-1})}{P_X^D(t_{i-1},t_i)}\beta_X^O(t_{n-2},t_{n-1})\right.\right.\\
&\qquad \left.\left.\times \beta_X^O(t_{n-1},t_n)\frac{P_X^D(t_{n-3},t_{n-2})}{P_X^D(t_{n-3},t_n)}\right)\right].
\end{aligned}
$$

In the second equality we use the fact that $P_X^D(s,t)$ is $\mathcal{F}_{t_{n-2}}$-measurable for $s \leq t_{n-2}$ and that $\beta_X^O(t_{n-1},t_n)$ is constant. In the third equality we use the martingale property of the asset $P_X^D(.,x)$ in the $P_X^D(.,t_n)$ numeraire. The fourth equality is simply the simplification of fractions. In the fifth equality we use the same technique as in the second and third.

The same procedure as the one in the second and third equalities can be repeated down for all factors in the product and we obtain

$$
\mathrm{E}^n\left[\frac{P_X^D(t_0,t_0)}{P_X^D(t_0,t_n)}\left(\prod_{i=1}^{n}\beta_X^O(t_{i-1},t_i)\right)\right].
$$

After once more applying the martingale property, we obtain the following result.

Theorem 2.4 *Under the hypotheses **D**, **I^{CPN}** and **S0^{CPN}**, the price of the overnight indexed coupon described above is given by*

$$
P_X^D(0,t_n)\left(\prod_{i=1}^{n}(1+\delta_i F_X^0(0,t_{i-1},t_i)) - 1\right).
$$

This means that with the hypothesis **S0$^{\mathbf{CPN}}$**, the overnight indexed coupon can be priced as if the forward rate were composed and then discounted.

When the pseudo-discount factors description of the overnight forward curves is used, as described in Chapter 3, the value of the coupon can be simplified further.

The value can be computed directly as a ratio of discount factors, without requiring to compute the actual composition of the daily rates. In practice, when dealing with overnight forward curves, we use the pseudo-discount factor approach. This is faster due to the above simplification. Moreover as the index is very short term, there is virtually no arbitrary part in the pseudo-discount factor curve.

2.8 Forex and cross-currency swaps

Up to now, everything discussed is valid for any currency and we have reviewed only single-currency instruments. We have also supposed that the risk-free curves are given.

For cross-currency instruments like forex swaps and cross-currency swaps, cash-flows in different currencies are involved. Our convention is that the present value of each cash-flow is evaluated in its own currency; at this stage no conversion into another currency is done.

Forex spot, forward or swap are simply sets of fixed cash-flows in each currency. The cash-flows in one currency are valued using the relevant discounting curve and produce a result in that currency. The total present value is two amounts in two different currencies.

As described in Appendix B.14, the forex swaps are mainly interest rate products. The main market information they convey is the difference in interest rate for a given period between two currencies. As such, they create links between the curves in different currencies.

Through multi-currency instruments a link between the discounting curves in different currencies is imposed by the market. To keep the coherence of the framework, the curve in one currency follows from the choice in any other currency. The valuation of cross-currency swaps is done as for single-currency swaps. The present value of each coupon is computed (in its own currency) using the previous formulas, and they are added. The total present value is two amounts, one in each of the swap currencies. To convert the multi-currency amount to one currency, today's exchange rate is used.

The existence of par forex and cross-currency swaps, that is, with zero total present value, will impose a relationship between discounting curves in different currencies and exchange rates. Note that it is the total converted present value which is null, not each of the present values in each currency. Each leg of a forex swap transaction does not have a zero present value. Even if the forex swaps are mainly interest rate products, and this is why they are used in curve calibration, they create small currency exposures. The currency exposure exists from the trading moment and does not only appear due to market movements. The only case where there is no currency exposure is when the interest rate in the first currency of the swap is zero over the swap period. Paying one amount and receiving the same amount at the end is a fair deal by itself, without introducing the second currency. This is a very rare occurrence.

The introduction of multi-currency instruments and the multi-currency coherence they impose also means that interest rates are becoming forex rate dependent; changing the exchange rate and keeping the market quotes of forex and cross-currency swaps will have an impact on the interest rate obtained. Some remarks on the forex sensitivity of interest rate curves is proposed in the curve calibration chapter.

3 Variation on a Theme

Up to now, we have worked with generic forward curves $F_X^{\mathrm{CPN},j}(t,u,v)$ and we have not discussed their implementation. If there were infinitely many market instruments, one for each starting date u – or even better, one for each starting time – the general curve would be a good enough description of the full economy. The market would provide the forward rate for each possible date and the curve description would merely be a data storage and not a data modelling tool.

In practice, there are a lot less market instruments on which we can build the forward curves; there are a lot less than one instrument for each time u. Often one has monthly or quarterly information on the short part of the curve, provided by FRA, futures or short term swaps, and annual information at best on the long part of the curve, provided by swaps or swap futures. One needs to resort to a modelling mechanism of some sort to find the intermediary value, often combined with interpolation. Chapter 4 discusses the subject of interpolation.

In the next two sections, we outline two ways to describe the curves; the two methods lead to different implementations. Each of them can be combined with different interpolation schemes. This dual approach was initially described in Henrard (2012c). A third approach is proposed in Fries (2013) based on coupons present values.

The first implementation uses pseudo-discount factors. This is certainly the most commonly used approach in practice; in a lot of literature and software packages it is even the definition of multi-curve framework. Starting from the description of the multi-curve framework we proposed in the previous chapter, it may seems strange to use such a convoluted approach to describe the curves. This is mainly due to the *evolution* of the framework from the one-curve framework. It allows us to have the same description of the curves for the discounting curve and the forward curves. As we will see in the next chapter, a lot of the curve interpolation literature focuses on the impact of interpolation in that specific framework.

The second implementation, which in many ways should really be the first, is to describe the forward curve directly by modelling the forward rates themselves. As we will see later in this chapter and in the next one, many of the drawbacks of simple interpolation, such as linear interpolation, can be reduced by using the direct curve description in terms of forward rates. Some similarities between discounting and forward curves are lost. As the theoretical descriptions of the two types of curve are very different, having different implementations is not a problem. On the positive side, the intuition of the forward curves is more direct and not obtained through a ratio of discount factors.

In the last section of this chapter we describe an approach to the multi-curve framework different from the one described in the rest of the book. The section can be viewed as a side story, where the story of the main character's cousin is told. It can be skipped without losing essential information for the rest of this book. Nevertheless it is an interesting simplified version of the 'real world' where there is not only one set of instruments but several sets distinguished by their collateral rules. This explanation will really make sense only after Chapter 8. In the alternative framework, the forward curves are based on STIR futures instead of being based on Ibor coupons. This is certainly a meaningful approach if one has a portfolio composed mainly of futures with only a limited number of swaps.

3.1 Forward curves through pseudo-discount factors

The pseudo-discount factor forward curves are defined as follows.

Definition 3.1 (Coupon pseudo-discount factor curves). *The forward curve* $P_X^{\text{CDF},j}(t,s)$ *is the continuous function defined for* $t \leq s$ *such that* $P_X^{\text{CDF},j}(t,t) = 1$, $P_X^{\text{CDF},j}(t,s)$ *is an arbitrary strictly positive function for* $t \leq s < \text{Spot}(t) + j$*, and for* $t_0 \geq t$, $u = \text{Spot}(t_0)$ *and* $v = u + j$ *one has*

$$F_X^{\text{CPN},j}(t,u,v) = \frac{1}{\delta}\left(\frac{P_X^{\text{CDF},j}(t,u)}{P_X^{\text{CDF},j}(t,v)} - 1\right). \tag{3.1}$$

The origin of the above definition can be traced back to the one-curve world, where the forward rate was written with a similar formula described by Equation (2.1). Nevertheless the substance of the two formulas is very different. The one-curve formula is a result obtained from different hypotheses and from no arbitrage condition. The above formula is merely a definition. The pseudo-discount factor should be viewed as the 'wrong number used in the wrong formula to obtain the correct result' type of approach. The formula is better understood through the *evolution* than through the *foundations*.

Definition 3.1, which refers to an arbitrary function, is itself arbitrary in more than one way. The definition fixes the first j-period as the arbitrary part; one could instead fix any other j period and deduce the rest of the curve from there.

In Baviera and Cassaro (2012) the authors propose to use the arbitrariness on the second period by imposing a specific interpolation in that period and building the first period in a way similar to our definition. One could even take an arbitrary decomposition of the j-period time interval into subintervals and distribute those subintervals arbitrarily on the real axis in such a way that, modulo the j-periods, they recompose the initial j-period. As another arbitrary choice, one could also change the value of $P_X^{\text{CDF},j}(t,t)$ to any value different from 1. As only the ratios between two values are used and never a value on its own, the choice of initial value has no impact on the end results. One could also impose an arbitrary value for $P_X^{\text{CDF},j}$ in $(t, \text{Spot}(t))$ instead of in (t,t). As the curve is used only for forward computation, it is used only with time $s > \text{Spot}(t)$; the value for shorter times is irrelevant.

This arbitrariness of the pseudo-discount factors makes it very difficult to design a model for those values. All the relevant models I'm aware of model the value $F_X^{\text{CPN},j}(t,u,v)$ and not the pseudo-discount factors $P_X^{\text{CDF},j}(t,u)$. This modelling problem with pseudo-discount factors is discussed further in Section 7.1.

Curve calibration

The curve calibration is the subject of Chapter 5. In this section we describe a very simple curve calibration to emphasise the difference between pseudo-discount factor curve description and the direct rate curve description of the next section.

A relatively standard way to calibrate the curves P_X^D and $P_X^{\text{CDF},j}$ is to select a set of market instruments for which the market quotes are known and an equal number of node points. An interpolation scheme is selected and the discount factors (or the associated rates) on the node points are calibrated to reproduce the market quotes. The market forward rates $F_X^{\text{CPN},j}(0,u,u+j)$ can be computed from the forward pseudo-discount factors curve using Formula (3.1). The pseudo-discount factors are calibrated using the formula and the instruments are priced from the pseudo-discount factors using the same formula. Everything is coherent and we could stop here. The big difference with a direct approach is the interpolation. Do you interpolate the pseudo-discount factors or the forward rates?

A typical forward rate curve using the pseudo-discount factors approach is displayed in black in Figure 3.1. The swap data used to calibrate the curve are those proposed in (Andersen and Piterbarg, 2010, Section 6.2) in a one-curve setting. The interpolation scheme is linear on (continuously compounded) rates. For our calibration, the swaps are fixed versus three months Ibor. We use a discounting curve with flat OIS market rates at 4%. The discounting curve is mostly irrelevant for the discussion in this section.

The familiar sawtooth pattern can be seen. There are two angles in the curve for each node point. One when the fixing period end date is on a node and one when the fixing period start date is on the node. One of the reasons for this unpleasant

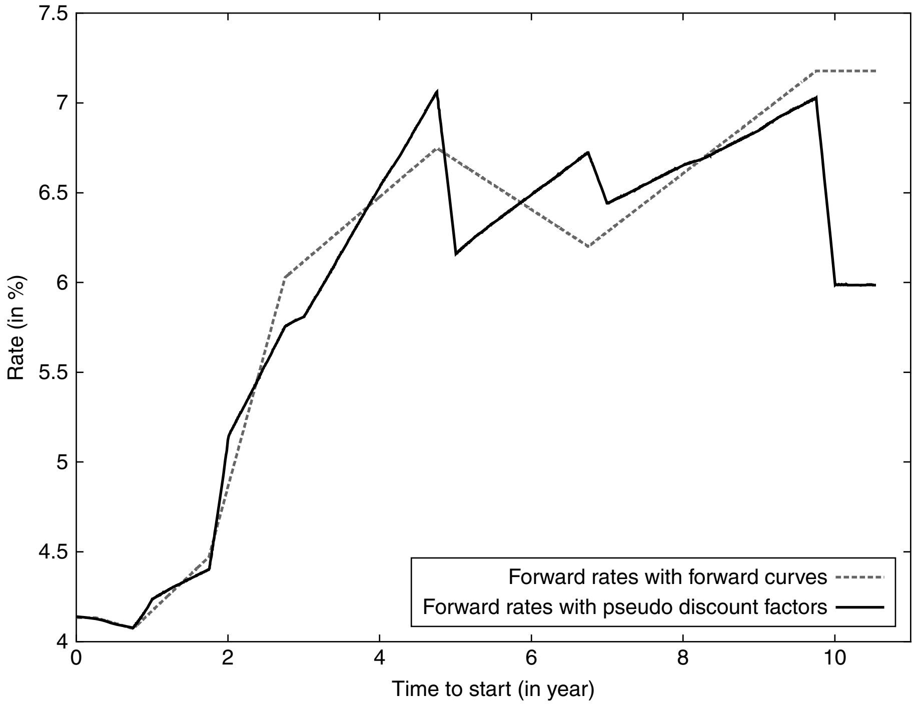


Figure 3.1 Forward Ibor three months rates computed using pseudo-discount factors in black and direct forward rate curve in dotted line. Both curves are calibrated using the same market data (see text).

shape is probably that we have an intuition on a market quantity – the forward rate – but model it indirectly through a ratio of discount factors where our intuition is diluted. The graph represents the forward rate, which is of interest to the trader and risk manager, but the data is stored and calibrated using a different mechanism.

3.2 Direct forward curves

In this approach we do not need an intermediary function. We work directly with the curve $F_X^{\text{CPN},j}(0,u,v)$, with $u > \text{Spot}(0)$ the variable of this one-dimensional function. This approach, which seems the natural approach in the multi-curve framework, has not been documented until recently and software implementations seems to mostly ignore it.

Note that in this framework, the *Ibor discounting* is impossible as there is no discount factor associated to the Ibor curves. There is no longer an arbitrary part to the curve. The curve is defined unambiguously (as long as the corresponding market instruments exist) for all $u \geq \mathrm{Spot}(0)$.

> Modeling the forward rate by constructing the discount or spot rates is similar to fixing a pair of eyeglasses while wearing gloves. [...] Instead, if one models the forward curve directly and embeds all of the desired properties into it, then by construction the spot and discount rates will have the characteristics that the modeler requires.
>
> *Jerassy-Etzion, Stripping the Yield Curve With Maximally Smooth Forward Curves, 2010*

Curve calibration

The advantage of this approach is that the market rates on which we have some intuition are modelled directly. In some senses, and borrowing a well known name, it could be called the *Libor Market Model* of forward curve description – not of curve dynamic as its namesake.

There is no longer a requirement for an arbitrary part like in Definition 3.1 of the pseudo-discount factor approach. The interpolation and constraints can be imposed directly on the market quantities. In Figure 3.1, the forward rate using the same data as the previous approach and the same linear interpolation scheme, but with the interpolation directly on the forward rate, is presented in dotted line.

It is to each market maker or risk manager to decide which one he prefers. With the reported data, the forward rates display less sawtooth effect with the direct forward rate approach. With some other market rates, the picture may be different.

In Table 3.1 we give the figures of the maximum variation of the three months rates three months apart. It is expected that the view of three months rates at some distant time in the future will not change too widely for relatively small differences in dates. Larger variations may indicate a problem with the way we estimate forward curves.

The three month period was chosen as this is the problematic period artificially introduced by the specific form of Definition 3.1. As can be seen from the numbers,

Table 3.1 The maximum changes of three months rates over a three month period – maximum increase and maximum decrease

	Pseudo-discount factors		Direct forward	
Interpolation	**Increase**	**Decrease**	**Increase**	**Decrease**
Linear	69.1	104.1	37.4	27.5
Natural cubic spline	36.7	15.4	38.5	17.5

Figures in basis points.

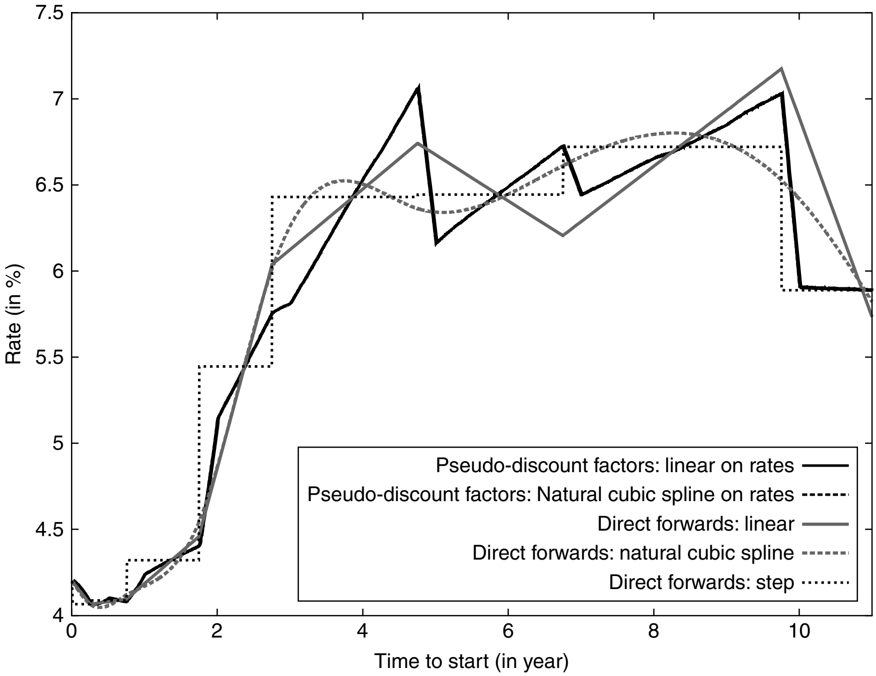


Figure 3.2 Forward Ibor three months rates computed using pseudo-discount factors (in black) and direct forward rate curve (in grey) with natural cubic spline interpolation. The dashed lines are using linear interpolation and the dotted line a step function.

the interpolation on pseudo-discount factors introduces a significantly larger variation. The figures using the linear interpolation on the direct forward rate are not very different from the cubic spline interpolation on the pseudo-discount factors.

In Figure 3.2, we did the same comparison as in the previous figure, this time using a smoother interpolation mechanism. The interpolation scheme is natural cubic spline.

For the direct rates, we have also added an interpolation through step function (dotted line). As the curve represents the forward rate and not the rate to maturity, the step function is not necessarily the worst choice. In all cases, it avoids unnecessary swings in the forward rate function.

The choice of the curve implementation and interpolation mechanism not only has an impact at the level of the forward rates but also on the implied hedging. One quality often requested from interpolation mechanisms is locality; this is for example the third desirable feature quoted in Hagan and West (2006). In Table 3.2 we give the sensitivity to the instrument quotes used to build the forward curve using the two implementations proposed above and two interpolation schemes. The instrument we want to hedge is a 1Yx5Y forward swap.

To value the above expected value we use Lemma A.2 twice and we have to compute

$$\frac{P_X^D(0,t_p)}{P_X^D(0,v)}\beta_X^{\text{CPN},j}(u,v)\frac{P_X^D(0,u)}{P_X^D(0,v)}$$
$$\mathrm{E}^v\left[\exp\left(-\int_0^{t_0}(\nu(\tau,t_p)-\nu(\tau,v))\cdot dW_\tau^v-\frac{1}{2}\int_0^{t_0}|\nu(\tau,t_p)-\nu(\tau,v)|^2ds\right)\right.$$
$$\left.\exp\left(-\int_0^{t_0}(\nu(\tau,u)-\nu(\tau,v))\cdot dW_\tau^v-\frac{1}{2}\int_0^{t_0}|\nu(\tau,u)-\nu(\tau,v)|^2ds\right)\right].$$

The variance of the stochastic integral in the exponentials is equal to the sum of squared plus the term

$$\int_0^{t_0}(\nu(\tau,t_p)-\nu(\tau,v))(\nu(\tau,u)-\nu(\tau,v))ds.$$

The expected value is thus reduced to

$$\zeta(t_0,u,v,t_p)=\exp\left(\int_0^{t_0}(\nu(\tau,t_p)-\nu(\tau,v))(\nu(\tau,u)-\nu(\tau,v))d\tau\right).$$

Theorem 6.2 *Under the hypotheses* ***D***, $\boldsymbol{I}^{\boldsymbol{CPN}}$ *and* $\boldsymbol{S0}^{\boldsymbol{CPN}}$, *the present value in* 0 *of the coupon with early payment date* t_p *is, in the HJM model on the discounting curve,*

$$P_X^D(0,t_p)F_X^{\text{CPN},j}(0,u,v)\zeta(t_0,u,v,t_p)+\frac{1}{\delta}P_X^D(0,t_p)(\zeta(t_0,u,v,t_p)-1).$$

When the coupon is a standard coupon with $t_p=v$, the adjustment ζ is equal to 1 and the standard formula is recovered. In all cases, $\zeta>1$ for Hull–White one-factor models.

The rate adjustment, that is the amount by which the rate should be changed in the standard valuation formula, is given by

$$\left(\frac{1}{\delta}-F_X^{\text{CPN},j}(0,u,v)\right)(\zeta-1).$$

A graphical representation of the adjustment is proposed in Figure 6.1. The model used was a constant volatility Hull–White model with a mean reversion of 0.01 and a volatility of 0.02. The volatility used is probably larger that the market volatility, which is closer to 0.01. The adjustment produced is more on the high side than a realistic estimate. The delays $v-t_p$ used are one, two and seven days. The one and two day delays are encountered once a week each. If holidays are included the differential can be larger. Some examples of larger date mismatches are provided in Section B.3. For standard coupons, the seven day example is certainly an upper limit. The payment periods used in the examples are three and six months. The adjustment is largely independent of the rate level and of the payment period.

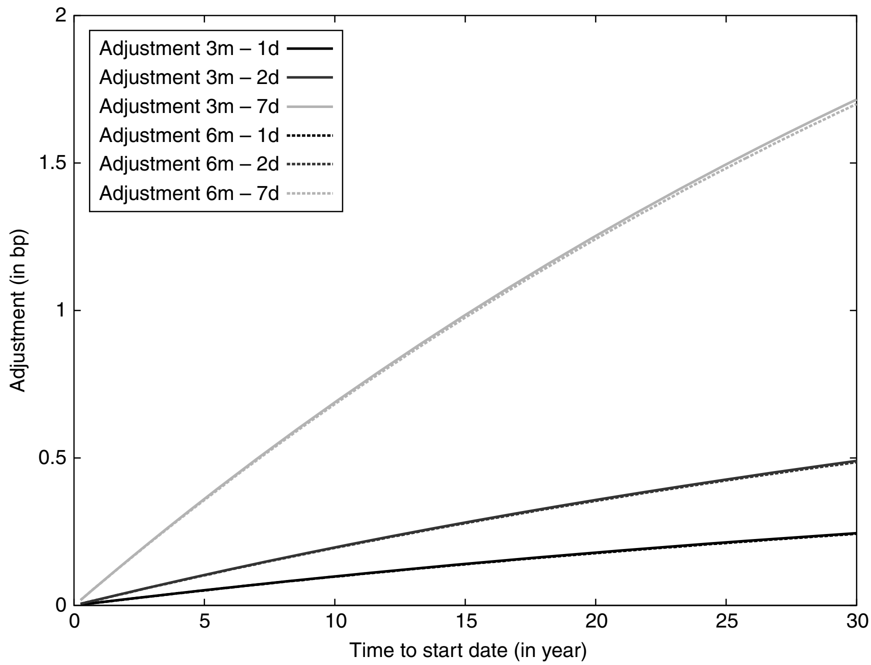


Figure 6.1 Timing adjustment for Ibor rate payments in case of dates mismatches. Impacts for payments one, two and seven days before the underlying index date. Payment periods are three and six months.

If we approximate the impact to be a one day lag in one day out of seven and a two day lag in one day out of seven and take the values from the graph, we obtain an approximate a total impact on a 30 year swap rate of 0.05 basis points. This is probably too small to significantly impact a market quote but large enough that risk managers want to check those numbers for large books. As the impacts are date dependent and a one day change in the starting date can have an important impact, it is not inconceivable for a large book with small total interest rate sensitivity to have a large timing adjustment impact.

6.3 Compounded coupons

The description of a compounded coupon is the following. The times associated are denoted t_i $(i = 0,\ldots,n)$. The fixing take place at dates $t^f_{i-1} \leq t_{i-1}$. The difference between t^f_{i-1} and t_{i-1} is the spot lag. The rates for the periods $[t_{i-1}, t_i]$ are denoted $I^j_X(t^f_{i-1})$ $(i = 1,\ldots,n)$ and the accrual factors in the index convention are δ_i.

The coupon pays in t_n the amount (to be multiplied by the notional)

$$\left(\prod_{i=1}^{n}(1+\delta_i I_X^j(t_{i-1}^f))\right)-1.$$

From Equation 2.3, each fixing is linked to the curve on the fixing date by

$$I_X^j(t_{i-1}^f)=F_X^{\mathrm{CPN},j}(t_{i-1}^f,t_{i-1},t_i).$$

In this section, we work under the standard simplifying assumption of constant spread $\mathbf{S0^{CPN}}$.

Theorem 6.3 *Under the hypotheses* $\boldsymbol{D}$, $\boldsymbol{I^{CPN}}$ *and* $\boldsymbol{S0^{CPN}}$, *the present value of the compounded Ibor coupon described above is in t*

$$P_X^D(t,t_n)\left(\prod_{i=1}^{n}(1+\delta_i F_X^{\mathrm{CPN},j}(t,t_{i-1},t_i))\right)-1.$$

Proof: The present value (in the $P_X^D(.,t_n)$ numeraire) is given by

$$P^D(t,t_n)\,\mathrm{E}^n\left[\left(\prod_{i=1}^{n}\left(1+\delta_i F_X^j(t_{i-1}^f,t_{i-1},t_i)\right)\right)-1\,\middle|\,\mathcal{F}_t\right] \tag{6.1}$$

Obviously the part that requires analysis is the product. Using the shortened notation β_{i-1} for $\beta_X^{\mathrm{CPN},j}(0,t_{i-1},t_i)$, we have

$$\begin{aligned}
&\mathrm{E}^n\left[\prod_{i=1}^{n}\left(1+\delta_i F_X^{\mathrm{CPN},j}(t_{i-1}^f,t_{i-1},t_i)\right)\middle|\,\mathcal{F}_t\right]\\
&=\mathrm{E}^n\left[\prod_{i=1}^{n}\beta_{i-1}\frac{P^D(t_{i-1}^f,t_{i-1})}{P^D(t_{i-1}^f,t_i)}\middle|\,\mathcal{F}_t\right]\\
&=\mathrm{E}^n\left[\left(\prod_{i=1}^{n-1}\beta_{i-1}\frac{P^D(t_{i-1}^f,t_{i-1})}{P^D(t_{i-1}^f,t_i)}\right)\beta_{n-1}\,\mathrm{E}^n\left[\frac{P^D(t_{n-1}^f,t_{n-1})}{P^D(t_{n-1}^f,t_n)}\middle|\,\mathcal{F}_{t_{n-2}^f}\right]\middle|\,\mathcal{F}_t\right]\\
&=\mathrm{E}^n\left[\left(\prod_{i=1}^{n-1}\beta_{t_{i-1}^f}^j\frac{P^D(t_{i-1}^f,t_{i-1})}{P^D(t_{i-1}^f,t_i)}\right)\beta_{t_{n-1}}^j\frac{P^D(t_{n-2}^f,t_{n-1})}{P^D(t_{n-2}^f,t_n)}\middle|\,\mathcal{F}_t\right]\\
&=\mathrm{E}^n\left[\left(\prod_{i=1}^{n-2}\beta_{t_{i-1}^f}^j\frac{P^D(t_{i-1}^f,t_{i-1})}{P^D(t_{i-1}^f,t_i)}\right)\beta_{t_{n-2}^f}^j\beta_{t_{n-1}^f}^j\frac{P^D(t_{n-2}^f,t_{n-2})}{P^D(t_{n-2}^f,t_n)}\middle|\,\mathcal{F}_t\right]\\
&=\mathrm{E}^n\left[\left(\prod_{i=1}^{n-2}\beta_{t_{i-1}^f}^j\frac{P^D(t_{i-1}^f,t_{i-1})}{P^D(t_{i-1}^f,t_i)}\right)\beta_{t_{n-2}^f}^j\beta_{t_{n-1}^f}^j\frac{P^D(t_{n-3}^f,t_{n-2})}{P^D(t_{n-3}^f,t_n)}\middle|\,\mathcal{F}_t\right].
\end{aligned}$$

In the second equality we use the fact that $P_X^D(s,t)$ is $\mathcal{F}_{t_{n-2}}$-measurable for $s \leq t_{n-2}$ and that β_{n-1} is constant. In the third equality we use the martingale property of the asset $P_X^D(.,x)$ in the $P_X^D(.,t_n)$ numeraire. The fourth equality is simply the simplification of fractions. In the fifth equality we use the same technique as in the second and third but with t_{n-3}^f.

The same procedure can be repeated down for all factors in the product and we obtain

$$\mathrm{E}^n\left[\left(\prod_{i=1}^{n}\beta_{i-1}\right)\frac{P_X^D(t_0^f,t_0)}{P_X^D(t_0^f,t_n)}\middle|\mathcal{F}_t\right].$$

Applying the martingale property once more we obtain for the coupon value

$$P_X^D(t,t_n)\left(\prod_{i=1}^{n}\beta_{i-1}\frac{P_X^D(t,t_0)}{P_X^D(t,t_n)}-1\right)$$

The announced result is obtained by re-decomposing the product and using the definition of β_{i-1}. □

A compounded Ibor coupon can be priced as the discounted value of compounded forward. This is a strong result that requires the strong hypothesis $\mathbf{S0}^{\mathbf{CPN}}$.

In the case of a forward curve described by pseudo-discount factors, the result can be simplified further and reads as

$$P_X^D(t,t_n)\left(\frac{P_X^j(t,t_0)}{P_X^j(t,t_n)}-1\right).$$

This means that the compounded coupon can be priced as if there were no composition but simply one payment on one rate.

6.4 Federal Funds swaps

The description of the Fed Fund swaps and their conventions can be found in Appendix B.12.

6.4.1 Approximation of the arithmetic average

When computing the price of compounded rate with pseudo-discount factors implementation, if we use the simplifying assumption of deterministic spread (which is trivially satisfied when pricing OISs with OIS discounting), we do not need to compute every overnight rate and actually compound them. It is possible to use the direct formula of the ratio of start and end discount factors

$$P^D(0,t_n)\,\mathrm{E}^n\left[\left(\prod_{i=1}^{n}\frac{P^O(t_{i-1},t_{i-1})}{P^O(t_{i-1},t_i)}\right)-1\right]=P^D(0,t_n)\left(\frac{P^O(0,t_0)}{P^O(0,t_n)}-1\right).$$

This result is described in Section 2.7.

This type of direct computation result, which speeds-up the computation dramatically, is not available for arithmetic average. Nevertheless some approximations are available and we describe one here. It was initially introduced in Takada (2011).

Let A_c be the amount paid for the compounded rates, that is,

$$A_c=\left(\prod_{i=1}^{n}(1+\delta_i I^O(t_{i-1}))\right)-1.$$

To the first order,

$$1+\delta_i I^O(t_{i-1})\simeq\exp(\delta_i I^O(t_{i-1})).$$

As δ_i is small, the approximation is relatively good. Let A_a denote the amount paid for the arithmetic average. We have

$$\begin{aligned}A_a&=\sum_{i=1}^{n}\delta_i I^O(t_{i-1})\\&\simeq\ln\left(\prod_{i=1}^{n}(1+\delta_i I^O(t_{i-1}))\right)=\ln(1+A_c).\end{aligned}$$

As A_c can be computed efficiently, if the approximation is good enough and the adjustment due to the non-linearity of the function is not too large, we have an efficient approximate way to compute the arithmetic average coupon.

In Table 6.2 we analyse the quality of the approximation for several levels of rates. The curves we use are constant (on zero-coupon continuously compounded rates) with level 1%, 5% and 10%. We have computed the difference between the exact arithmetic average and the one provided by the above approximation. The coupons are three months long with the dates computed using market conventions. We run the test with a sample of 36 different starting dates to cover different month lengths and weekend effects. Depending on the level of rates, the error is from below 0.01 basis points to 0.25 basis points. The first error is certainly small enough; it is that of a 1% curve level, which is roughly the rate level in the main currencies at the time of writing. The error for a rate level of 10% is probably beyond the level of precision one would like for market making.

For low rates ($< 5\%$), the approximation is precise enough with the error below the quotation precision of the relevant market instrument, which is at best of 0.1 basis points.

Table 6.2 Error between exact arithmetic average and proposed approximation. Results for three month coupons

Error – curve 1%		Error – curve 5%		Error – curve 10%	
Min.	**Max.**	**Min.**	**Max.**	**Min.**	**Max.**
0.002	0.003	0.062	0.064	0.248	0.256

Figures in basis points. Figures computed with exact market dates for a sample of 36 starting dates.

Table 6.3 Performance for exact and approximated formulas of the arithmetic average coupons

Exact formula			Approximated formula			Ratio		
PV	**PVCS**	**C. + PV**	**PV**	**PVCS**	**C. + PV**	**PV**	**PVCS**	**C. + PV**
200	215	470	5	12	250	40.0	17.9	1.9

Figures for 10,000 three month coupons. Time in milliseconds. 'PV' corresponds to the present value, 'PVCS' to the present value curve sensitivity and 'C. + PV' to the coupon construction (all daily dates) and the present value.

Now that we have established that the approximation is good enough in the current rate environment, we have to analyse if it is useful. Is the computation time of the approximation lower enough to justify the use of the approximation? The answer to that question is summarised in Table 6.3. We have analysed the performance in three ways: the calculation itself for present value and curve sensitivity (bucketed delta) and the performance of the instrument construction and the present value. The last measure requires probably a little bit of explanation. For simple instruments, the time consuming part is often to construct the dates associated to the instrument following the different conventions, not to compute the relatively simple present value. This is certainly the case for overnight-related products, where a lot of dates are calculated. If the actual value computation time was dwarfed by the instrument construction time, the approximation exercise would be relatively meaningless. Note also that the curve sensitivity is computed by algorithmic differentiation and the results are (almost) independent of the number of points used to calibrate the curve and the number of node points impacting the instrument.

The conclusion of the table is that for the computation of present value and curve sensitivity, the approximation formula is significantly faster (between 20 and 40 times). This is not surprising as we use only two dates (start and end) to compute the result instead of all the daily points (around 60 dates for three months). If we add the instrument construction time, the impact is lesser but still relatively important. The time for construction and present value is divided by two. Which of those two numbers is relevant in practice: 2 or 20? A little bit in between! If you want only to compute the present value of a new instrument, the ratio 2 is the realistic one. But if you build curves with those instruments, the instrument description is computed once and the present value and curve sensitivities are computed numerous times. Suppose that to calibrate the curves you need for each instrument one instrument construction, ten present values, and two sensitivities, the ratio in that case would be around eight. The small approximation error is probably a price you would accept to divide the curve construction time by eight.

At this stage, the analysis covers only the *static approximation*, that is the computation of the arithmetic average when the rates are known. A second level of approximation is required as the rates in the arithmetic average are not paid at the right time; they are paid at the end of the accrual period and not at the end of their reference period.

6.4.2 Convexity adjustment

In this section we compute the Fed Fund swaps value including the convexity adjustment. The amount

$$\sum_{i=1}^{n} \delta_i I_X^O(t_{i-1})$$

is paid in t_n.

We work in the multi-curve framework under the assumption $\mathbf{S0^{CPN}}$ of constant spread $\beta^O(t_{i-1}, t_i)$ between the discounting curve and the overnight forward curve. This assumption is trivially satisfied if the discounting curve is equal to the overnight forward curve. The model used is the multi-factor HJM model on the discounting curve as described in Appendix A. The forward overnight rate seen from t is denoted $F_X^O(t, t_{i-1}, t_i)$. We will abbreviate it as $F_i^O(t)$.

The value of the coupon is, using the cash account numeraire,

$$\begin{aligned} N_0 \,\mathrm{E}^N &\left[(N_{t_n})^{-1} \sum_{i=1}^{n} \delta_i I_X^O(t_{i-1}) \right] \\ &= \sum_{i=1}^{n} \mathrm{E}^N \left[(N_{t_n})^{-1} \delta_i F_i^O(t_{i-1}) \right] \\ &= \sum_{i=1}^{n} \mathrm{E}^N \left[(N_{t_n})^{-1} \left((1 + \delta_i F_X^D(t_{i-1}, t_{i-1}, t_i))\beta^O - 1 \right) \right]. \end{aligned}$$

We compute the value of one of these expectations. For that we use standard HJM results given by Lemma A.1 and A.3:

$$N_{t_n}^{-1} = P_X^D(0, t_n) \exp\left(-\int_0^{t_n} \nu(s, t_n) \cdot dW_s - \frac{1}{2} \int_0^{t_n} |\nu(s, t_n)|^2 ds \right)$$

and

$$
\begin{aligned}
1+\delta_i F^D(t_{i-1},t_{i-1},t_i) = (1+\delta_i F_i^D(0,t_{i-1},t_i)) \\
\times \exp\left(-\int_0^{t_{i-1}} (\nu(s,t_i)-\nu(s,t_{i-1}))\cdot dW_s \right.\\
\left. -\frac{1}{2}\int_0^{t_{i-1}} \left(|\nu(s,t_i)|^2-|\nu(s,t_{i-1})|^2\right) ds\right).
\end{aligned}
$$

In what follows, we use the usual extension of $\nu(s,t)$ for values $s>t$ with 0.

Combining the two HJM results above, the expected value to be computed for each fixing date is an exponential martingale plus some residual factors coming from the completion of (variance) squares. The expected value of the exponential martingales are 1. The residual terms are (after a slightly lengthy but easy computation)

$$
\zeta_i = \int_0^{t_n} (\nu(s,t_i)-\nu(s,t_{i-1}))\,(\nu(s,t_n)-\nu(s,t_{i-1}))\,ds.
$$

The final valuation result is given by the following theorem.

Theorem 6.4 *In the multi-curve framework under hypothesis* $\boldsymbol{S0}^{\boldsymbol{CPN}}$ *on the basis between discounting and overnight forwards, the present value of a Fed Fund swap coupon in the multi-factor HJM model on the discounting curve is*

$$
P^D(0,t_n)\sum_{i=1}^{n}\left(\delta_i F^O(0,t_{i-1},t_i)\zeta_i + (\zeta_i - 1)\right).
$$

To compute the value exactly, we need to compute all the forwards and an adjustment for each of them.

6.4.3 Convexity adjustment and approximation

We can combine the approximations discussed above with a convexity adjustment. The amount paid can be approximated by

$$
A_a \simeq \ln\left(\prod_{i=1}^{n}(1+\delta_i I^O(t_i))\right) = \ln(1+A_c).
$$

When the discounting curve is the overnight forward rate curve, the discretely compounded rate itself can be approximated by a continuous composition

$$
A_c \simeq \exp\left(\int_{t_0}^{t_n} r_\tau\,d\tau\right) - 1.
$$

The present value of such a coupon is thus approximated by

$$
M_0\,\mathrm{E}^M\left[(M_{t_n})^{-1}\int_{t_0}^{t_n} r_\tau\,d\tau\right]
$$

for any numeraire M. Here we chose the t_n-forward numeraire, that is $P^D_X(.,t_n)$. The change of numeraire in the HJM model is given by

$$dW^{t_n}_t = dW_t + \nu(t,t_n)\,dt.$$

We use the result on the dynamic of the cash account twice (once to t_0 and once to t_n) and the above change of numeraire to obtain

$$\int_{t_0}^{t_n} r_\tau\,d\tau = \ln\left(\frac{P^D(0,t_0)}{P^D(0,t_n)}\right)\left(\int_0^{t_0}\nu(s,t_0)\cdot dW^{t_n}_s + \int_0^{t_n}\nu(s,t_n)\cdot dW^{t_n}_s \right.$$
$$\left. - \frac{1}{2}\int_0^{t_n}|(\nu(s,t_n)-\nu(s,t_0)|^2 ds\right).$$

The expectation of the two stochastic integrals is 0 and the final result is as follows.

Theorem 6.5 *In the multi-curve framework under hypothesis* $\boldsymbol{S0^{CPN}}$ *on the basis between discounting and overnight forwards, the approximated present value of a Fed Fund swap coupon in the multi-factor HJM model on the discounting curve is*

$$P^D(0,t_n)\left(\ln\left(\frac{P^D(0,t_0)}{P^D(0,t_n)}\right)+\zeta\right)$$

with

$$\zeta = \frac{1}{2}\int_0^{t_n}|(\nu(s,t_n)-\nu(s,t_0)|^2 ds.$$

6.5 Federal Funds futures

The *30-Day Federal Funds futures* (simply called Fed Funds futures) are based on the monthly average of overnight Fed Funds rate for the contract month.

Let $0 < t_0 < t_1 < \cdots < t_n$ be the relevant dates for the Fed Funds futures, with t_0 the first business day of the reference month, t_{i+1} the business day following t_i and t_n the first business day of the following month. Let δ_i be the accrual factor between t_i and t_{i+1} ($0 \le i \le n-1$) and δ the accrual factor for the total period $[t_0,t_n]$.

The overnight rates fixing between t_i and t_{i+1} are denoted $I^O_X(t_i)$ with $F^O_X(t_i,t_i,t_{i+1}) = I^O_X(t_i)$. The future price on the final settlement date t_n is

$$\Phi_{t_n} = 1 - \frac{1}{\delta}\left(\sum_{i=0}^{n-1}\delta_i I^O_X(t_i)\right).$$

The margining is done on the price multiplied by the notional and divided by the one month accrual fraction (1/12).

The model used is the Hull–White model on the discounting curve as described in Appendix A. The result uses the deterministic spread hypothesis $\mathbf{S0^{CPN}}$ between the curves.

Theorem 6.6 *In the HJM model on the discount curve in the multi-curve framework and with the deterministic hypothesis between spread and discount factor ratio* $\mathbf{S0^{CPN}}$*, the price of the average overnight future for the period* $[t_0, t_n]$ *is given for* $t_j \leq t < t_{j+1}$ $(j \in \{0, 1, \ldots, n\})$ *by*

$$\Phi_t = 1 - \frac{1}{\delta}\left(\sum_{i=1}^{j} \delta_i I_X^O(t_i) + \sum_{i=j+1}^{n-1} 1 - (1 + \delta_i F_X^O(t, t_i, t_{i+1}))\gamma(t, t_i, t_i, t_{i+1})\right)$$

where γ *is given in Appendix A.*

The formula is divided into two parts to cope with the case where the averaging period has started already. In those cases the convexity adjustments γ are very small.

Proof: The generic futures pricing formula applied to the instrument is

$$\begin{aligned}\Phi_t &= \mathrm{E}^N\left[1 - \frac{1}{\delta}\left(\sum_{i=0}^{n-1} \delta_i F_X^O(t_i, t_i, t_{i+1})\right)\middle|\, \mathcal{F}_t\right] \\ &= 1 - \frac{1}{\delta}\left(\sum_{i=1}^{j} \delta_i I_X^O(t_i) + \sum_{i=j+1}^{n-1} 1 - \mathrm{E}^N\left[1 + \delta_i F_X^O(t_i, t_i, t_{i+1})\middle|\, \mathcal{F}_t\right]\right).\end{aligned}$$

We are left with the same technical computation as in Section 2.6, once for each period. The result follows immediately. □

6.6 Deliverable swaps futures

The results presented in this section were first proposed in Henrard (2012b). The description of the instrument itself is provided in Appendix B.13.

6.6.1 Pricing

The arbitrage-free price of the future Φ_θ in θ satisfies, for PV_θ the present value of the underlying swap without up-front payment in θ and with notional 1 to the long party,

$$(1 - \Phi_\theta) N P^D(\theta, t_0) + \mathrm{PV}_\theta\, N = 0$$

or

$$\Phi_\theta = 1 + \frac{\mathrm{PV}_\theta}{P^D(\theta, t_0)}.$$

After delivery, the swap is cleared on CME clearing. The swaps are collateralised according to CME variation margin methodology. Here we suppose that the methodology is equivalent to a discounting with a given discounting curve as described in the previous chapters of the book.

The results presented here are similar to the one presented in Kennedy (2010) for SwapNote futures and in Henrard (2006b) for bond futures. They are more general as they are for multi-factor models and in the multi-curve framework.

Using the generic pricing future price process theorem (Hunt and Kennedy, 2004, Theorem 12.6),

$$\Phi_0 = \mathrm{E}^{\mathbb{N}}\left[\Phi_\theta\right].$$

Using the swap pricing formula,

$$\Phi_\theta = 1 + \sum_{i=0}^{n} d_\theta^i \frac{P_X^D(\theta, t_i)}{P_X^D(\theta, t_0)}.$$

The pricing formula uses the description of the swap in terms of equivalent cash-flows as provided in Section 2.4.

Using Lemma A.1, if d_θ^i is independent of the ratio of discount factors, the futures can be written as a function of a random variable X_i:

$$1 + \sum_{i=0}^{n} d_t^i \frac{P_X^D(0, t_i)}{P_X^D(0, t_0)} \exp\left(-\alpha(0,\theta,t_0,t_i)X_i - \frac{1}{2}\alpha^2(0,\theta,t_0,t_i)\right)\gamma(0,\theta,t_0,t_i).$$

The exponential terms have a expectation of 1. We thus obtain the following result.

Theorem 6.7 *In the multi-curve framework with independent spread hypothesis in the Gaussian HJM model, the price of a swap futures with expiry date θ is given by*

$$F_0 = 1 + \frac{1}{P_X^D(0, t_0)} \sum_{i=0}^{n} \mathrm{E}^{\mathbb{N}}\left[d_\theta^i\right] P_X^D(0, t_i)\gamma(0,\theta,t_0,t_i).$$

With hypothesis $\mathbf{S0^{CPN}}$, the price is

$$F_0 = 1 + \frac{1}{P_X^D(0, t_0)} \sum_{i=0}^{n} d_0^i P_X^D(0, t_i)\gamma(0,\theta,t_0,t_i). \tag{6.2}$$

6.6.2 Numerical examples

In this section we give some numerical examples of the impact of the futures features on the price and risks in the case of constant spread hypothesis $\mathbf{S0^{CPN}}$. The analysis is done in a Hull–White one-factor model. We compare the futures to the

Table 6.4 Difference in future price and swap present value for different expiries, tenor and moneyness

		Tenor			
Expiry	Moneyness	2Y	5Y	10Y	30Y
3M	−100 bps	−1.56	−3.73	−6.94	−15.85
	0 bps	−0.26	−0.62	−1.14	−2.47
	+100 bps	1.04	2.49	4.66	10.91
9M	−100 bps	−6.11	−14.51	−26.72	−59.16
	0 bps	−2.22	−5.25	−9.59	−20.73
	+100 bps	1.67	4.02	7.54	17.69

Price differences in basis points.

underlying swap. The comparison is on the price (for market making) and the curve risk (for hedging). The comparisons are done with a flat curve at 3% (continuously compounded) rate and a 2% Hull–White constant volatility.

From Equation (6.2), it is clear that the impact on the value is twofold. First is the multiplication by the common factor $1/P^D(0,t_0)$. This factor is always larger than 1 when interest is positive. This represents the fact that the profit is paid immediately (through the margin) and not at final settlement. As the futures usually have an expiry up to three or six months in the future, the impact is up to 0.5 times the short term rate. With the very low rates prevalent at the time of writing this impact is very small: below 0.10%. For higher rates, the impact can be non-negligible. When interest is at 3%, the impact on a six month future is around 1.5%.

The differences between the futures price (with the 1 subtracted) and the underlying swap present value are displayed in Table 6.4. The difference clearly depends on three items: expiry, tenor and moneyness. The moneyness is the difference between the current forward swap rate and the fixed rate embedded in the futures. If markets have moved significantly between the issuance time and the current level, the swaps can be far off-the-money.

On the curve risk side, the derivative with respect to the rate will be roughly $t_n - t_0$ for the future and t_n for the swap. The ratio swap notional/future notional will be impacted by this. Using this very crude approximation, for a three month futures, the ratio is around 1.125 for a 2 year tenor and 1.01 for a 30 year tenor. Actual ratios for different scenarios are given in Table 6.5. The ratios strongly depend on the future expiry and underlying swap tenor but very weakly on the moneyness.

The second impact is the so-called convexity adjustment $\gamma(0,\theta,t_0,t_i)$. As mentioned earlier, these adjustments are always less than 1 and this will decrease the price. The factors depend on the level of volatility (the adjustment is larger for higher volatilities) and on the time to maturity (the adjustment is larger for longer

Table 6.5 Swap/futures nominal total sensitivity hedging

Expiry	Moneyness	2Y	5Y	10Y	30Y
3M	−100 bps	1.13	1.05	1.02	1.01
	+100 bps	1.14	1.05	1.02	1.01
6M	−100 bps	1.32	1.10	1.04	1.01
	+100 bps	1.32	1.10	1.05	1.01
9M	−100 bps	1.58	1.16	1.07	1.02
	+100 bps	1.58	1.17	1.07	1.02

Hedging ratios: the notional of futures required to hedge the same amount of swap.

maturities). As the underlying tenors are between 2 and 30 years, the impact will be considerably different between the different futures. Also the adjustment factor impact will depend on the *moneyness*. As the coupon is fixed on the first trading date and the futures starts to trade several quarters before the delivery date, the difference can be important.

We also analyse the bucketed sensitivities. The difference appears mainly around the delivery date. The futures has an extra sensitivity on the discounting curve, as expected from the $1/P^D(0, t_0)$ term in Equation (6.2). In Table 6.6 we display the sensitivities of a futures on the two years swap with a coupon close to the money and an expiry in three months time (futures with delivery on 19 December 2012 valued on 20 September 2012). The sensitivities are the derivatives with respect to the rates (continuously compounded) at the node point indicated and for a notional of 1. We work in a framework where the discounting curve is calibrated to OISs and use OIS to hedge part of the difference between swaps and futures. The OIS we use is a swap with maturity on the futures delivery date. We hedge the underlying swap with a notional of 100 million using the futures and the OIS. The quantity of the futures is obtained by hedging the total sensitivity to the forward curve. The quantity of the OIS is obtained by hedging the total sensitivity of the swap and its futures hedge to the discounting curve. The residual is the result of hedging a 100 million receiver swap with the above described portfolio. The portfolio consists of selling 993 contracts[1] and receiving OIS for 102.4 million; its risk is displayed in the last columns.

Table 6.7 displays the same information for a 30 year tenor futures with expiry in six months. Note that the hedging quality on the forward curve is not uniform through the tenors. This is expected, as in Equation 6.2 the cash-flows are multiplied by convexity adjustments that are different for each cash-flow date.

[1] The figure provided are for a not rounded hedge. In practice it is not possible to trade fractions of contracts. The figures achievable in practice will be slightly different.

Table 6.6 Swap/futures/OIS sensitivities and hedging efficiency for 2Y futures

	Futures		Swap		OIS		Residual	
Tenor	**Dsc**	**Fwd**	**Dsc**	**Fwd**	**Dsc**	**Fwd**	**Dsc**	**Fwd**
3m	0.24	0.25	0.00	0.24	−0.23	0.00	0.37	−0.02
6m	0.00	−0.01	0.00	−0.01	0.00	0.00	−0.08	0.03
1y	0.01	−0.02	0.01	−0.02	0.00	0.00	−0.10	0.10
2y	0.01	−1.99	0.01	−1.98	0.00	0.00	−0.18	−0.08
5y	0.00	−0.18	0.00	−0.17	0.00	0.00	−0.01	−0.02
10y	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
30y	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

The sensitivity is the derivative with respect to the rates for a notional of 1 for the three instruments. The residual is the sensitivity of a portfolio of a 100 million receiver swap, selling 993 futures contracts and paying 102.4 million OIS up to expiry. The sensitivities of the portfolio are in USD for a one basis point move.

Table 6.7 Swap/futures/OIS sensitivities and hedging efficiency for 30Y futures

	Futures		Swap		OIS		Residual	
Tenor	**Dsc**	**Fwd**	**Dsc**	**Fwd**	**Dsc**	**Fwd**	**Dsc**	**Fwd**
3m	0.01	0.01	0.00	0.01	0.00	0.00	−107.98	−0.05
6m	0.49	0.48	0.00	0.48	−0.48	0.00	219.91	−2.68
1y	0.00	−0.02	0.00	−0.02	0.00	0.00	−0.47	0.54
2y	0.00	−0.14	0.00	−0.13	0.00	0.00	−2.36	2.98
5y	0.00	−0.54	0.00	−0.53	0.00	0.00	−8.91	10.56
10y	0.02	−3.33	0.01	−3.29	0.00	0.00	−50.02	45.97
30y	−0.03	−16.96	−0.04	−16.75	0.00	0.00	−50.17	−57.31

The sensitivity is the derivative with respect to the rates for a notional of 1 for the three instruments. The residual is the sensitivity of a portfolio of a 100 million receiver swap, selling 987 futures contracts and paying 103.7 million OIS up to expiry. The sensitivities of the portfolio are in USD scaled to one basis point.

6.6.3 Curve calibration and sensitivity

Like any instrument presented in this chapter, the swap futures can be used for curve calibration. The calibration requires the instrument quotes, a curve parameterisation and the model – HJM in our case – parameters.

The curve calibration approach described in Chapter 5 is very generic and can be applied to any instrument, even newly created ones.

What is different in the swap futures case with respect to all other interest rate products we described up to now is the quote 'dimension'. Most of the interest rate products are quoted through rate or yield related quotes, the exceptions being the forex swaps and the swap futures. The swap futures described in this section are quoted on a present-value-like basis.

When the sensitivity of a portfolio with respect to the inputs is computed, the figures obtained for swap futures will be dramatically different from the sensitivities for swaps.

6.7 Portfolio hedging

This section is not really an extra instrument but describes an approach to computing hedges with different instruments. The material could have been in another chapter but is a little bit short to be a chapter by itself. For that reason I have appended it to the end of the instruments chapter.

6.7.1 Introduction

When computing the sensitivities of a portfolio to the market quotes used in the curve construction, one implicitly computes the quantity of the instruments used in the curve construction required to hedge the portfolio. In some cases one would like to compute the (optimal) hedge with a different set of reference securities. One possibility in that case is to compute the synthetic market quotes of the reference securities, reconstruct the curves from those instruments and recompute the sensitivities.

Another possibility is to use the original sensitivities and the sensitivities of the hedging instruments and find a *good* hedge, optimal in a sense to be described. A method to do that in the case where the original sensitivities are considered independent is described in (Andersen and Piterbarg, 2010, Section 6.4). Similar techniques are used in a different context to minimise delta/normal VaR using the covariance matrix to indicate the importance of and interaction between the sensitivities. In the book by Andersen and Piterbarg (2010), the method is referred to as the *Jacobian method*. As we suggest systematically computing the Jacobian matrices of curve construction and other processes, we will refrain from using that terminology, which can be confusing, and refer to it instead as the *hedging with reference securities* method.

6.7.2 Notation

Suppose that the curves are constructed with n_m securities for which the market quotes are $m = (m_i)_{1 \leq i \leq n_m}$. The market quotes are the parameters with respect to which the sensitivities are available. In general this is a par rate, a spread or a price. Any other parameter/shock sensitivity would work from a theoretical point of view.

Suppose we have a portfolio P made of many securities. We consider it as a unique entity for hedging purposes. We want to hedge that portfolio with other reference securities. The risk of the portfolio with respect to the curve market quotes is known. It is denoted $\partial P = (\partial_i P)_{i=1,\dots n_m}$ and is the short notation for $\partial_i P = \partial_{m_i} P(m)$; ∂P is represented by a column vector.

The values of the reference instruments to be used in the hedging are denoted R_j with $j = 1, \dots, n_r$. Their sensitivity with respect to the market quotes are also supposed to be known.

The quantity of each reference security used in the hedging portfolio is denoted by $q = (q_j)_{j=1,\dots,n_r}$. The goal of this method is to find the optimal quantities q. The value of the total hedging portfolio is

$$H = H(q) = \sum_{j=1}^{n_r} q_j R_j = q^T R.$$

The sensitivity of the hedging portfolio with respect to the market quotes is, using as notation that $\partial_i R$ is the column vector with the derivatives of the different components of R with respect to m_i,

$$\partial_i H = q^T \partial_i R.$$

We denote by ∂R the matrix with the $\partial_i R$ as columns. The total sensitivity of the original portfolio P plus the hedging portfolio H is (the column vector)

$$\partial P + (q^T \partial R)^T = \partial P + \partial R^T q.$$

The quantities q are the number we are trying to find to obtain the optimal hedging portfolio. The portfolio is optimal in the sense discussed below.

6.7.3 Optimal hedging

The notion of *optimal hedging portfolio* can cover a lot of financial realities. Here we restrict ourself to the following sense.

We want, when possible, to have the individual residual sensitivities to be small (zero would be perfect). The sensitivities to some market quotes may be more important than others. We usually want to associate a weight to each of the quotes to indicate how much we want to have them small. Also there may be quotes that we consider as closely related and if each of them cannot be small then we want their total to be small. This last point is particularly important in the multi-curve framework.

To represent the related rates, we use a weight matrix. The matrix is an n_w lines and n_m columns matrix denoted W. The hedging mismatch penalty is

$$W(\partial P + \partial R^T p)$$

and we try to optimise it in the least-square sense. Some examples of possible weights and their intuition are discussed in Section 6.7.4.

The summation in the penalty computation makes sense only if all the numbers are converted to the same currency. We suppose that this has been done.

The weight matrix is related to the curves used (n_m is the number of instruments in the curve construction) but not a priori to the reference instruments used for hedging. The number of lines n_w is the number of penalties we want to minimise. In general it is larger than n_m.

With this possibility of adding weights to related securities, our analysis is more general than that of Andersen and Piterbarg (2010). Also we do not associate cost to the hedging instruments; in that respect our approach covers less applications.

The (least-square) problem we have to solve is to find the quantity vector q which minimises the quantity

$$(\partial P + \partial R^T q)^T W^T W (\partial P + \partial R^T q). \tag{6.3}$$

The solution of the above problem is given by the solution of the linear system

$$(\partial R W^T W \partial R^T) q = -\partial R W^T W \partial P. \tag{6.4}$$

6.7.4 Examples

First suppose you have a curve set made of a discounting curve and a forward curve, each of them with two points: a 1Y and a 5Y point. We order them as first the two discounting points and then the two forward points. You want, if possible to have all sensitivities small, but mainly you want the total 1Y and the total 5Y risks to be small. You can chose the following weight matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & 0 & 5 & 0 \\ 0 & 5 & 0 & 5 \end{bmatrix}$$

The first four lines give a standard weight to each individual sensitivity. The next two lines give a larger weight to the total 1Y and 5Y sensitivities.

For the second example, suppose you have a discounting curve, a forward curve and a government curve, each of them with two points (a total of six quotes). We have a swap desk and want to hedge the swaps residual risks with bond futures, and we chose bond futures as our reference instruments. We know that a direct point by point comparison will be useless: the portfolio and the hedging instruments do not depend on the same curve. Our only interest is in the total risk for each maturity.

A potential matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

At the opposite end of the spectrum, suppose you have a diversified portfolio of swaps and government bonds. You are concerned to have the total sensitivity of the discounting and forward curves together and the government curve on its own to be small. You can choose

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{bmatrix}.$$

Suppose you have a unique curve with four (ordered) points and want to indicate that adjacent points compensate partially. You can use a tri-diagonal matrix like

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

A residual sensitivity of $(1,-1,0,0)$ is in this case better than $(1,0,-1,0)$.

In the VaR-like approach, the matrix $W^T W$ would be the variance-covariance matrix. As only the matrix $W^T W$ is used in the computation, any decomposition of the covariance matrix will provide the same result. In the delta/normal approach, the α-quantile VaR is $N^{-1}(\alpha)\sqrt{p^T Vp}$. The VaR is minimal when $p^T Vp$ is minimal. Our optimal hedging portfolio has minimal VaR when W is such that $V = W^T W$.

7 Options and Spread Modelling

In this chapter we describe several modelling approaches to curve and spread dynamic. The dynamic refers to the term structure of the discounting curve but also, and this is the difference with the one-curve framework, the term structure of the different forward rates or spreads. The first and simplest approach to spread modelling is the one considered previously, that is the constant spread approach described by hypothesis $\mathbf{S0^{CPN}}$.

From Section 7.4 onward, each section of this chapter proposes a different modelling approach. Each section can be read separately, even if we propose links and comparisons between the approaches. The order of presentation should not be interpreted as an order of sophistication or quality.

For historical and financial reasons, the liquid optional instruments are in separated niches. For example, Ibor and overnight-linked swaps are both very liquid even if on different parts of the curve. The swaptions on Ibor swaps are also very liquid but the swaptions on overnight-linked swaps are virtually nonexistent. Similarly, basis swaps are relatively liquid, especially since the crisis, but the swaptions on basis swaps are almost not traded at all. If one looks at the exchange traded world, the STIR futures and their options, which are the equivalent of OTC (over-the-counter) FRA and cap/floors, are liquid. But there is very little equivalent to swaps and swaptions. New instruments, like the deliverable swap futures described in Appendix B.13, have recently been added to the offering and have gained a decent liquidity. But there is virtually no equivalent to the swaptions. For that reason we propose a piece of 'financial fiction' in Section 7.8.6 by proposing a new potential exchange traded instrument that could complete the offering.

From a model calibration perspective, the absence of a certain number of natural volatility instruments means that it is often difficult to find the appropriate data. Classical term structure models apply to the discounting curve. In the collateral discounting approach, described in Chapter 8, the discounting curve is often the curve calibrated to OISs. But there are no direct OIS volatility instruments and so no way to directly calibrate those classical models. We often need additional hypotheses on the behaviour of the spread to be able to price even the simplest instruments. It will thus be important to be able to move information from one type of instrument to another. As usual it will be important to have a coherent approach where the same model can price different instruments and be able to compare risk and hedging strategies.

7.1 Short rate models

Historically, short rate models have been heavily used to model the interest rate term structure. This approach is certainly still important to model the risk-free curve P_X^D.

The same approach cannot be extended easily to the forward curves. In the generic approach we present, the notion of short rate for the forward curves is ill-defined. To define the short rate one needs an equivalent of the instantaneous forward, which itself requires an equivalent of the discount factors to obtain the equivalent of Definition 4.1. This type of pseudo-discount factor curve is defined only in the special case described in Section 3.1, where those objects are defined to use methods similar to the one used for the discounting curve. This is useful for interpolation schemes and to define curves through spreads.

Even when the pseudo-discount factors are used, they are, by Definition 3.1, arbitrary up to the first fixing end date and even never used up to the settlement date. A short rate model on those quantities would be based on arbitrary quantities which have no impact whatsoever in the framework. Apart from the theoretical weakness of such an approach, it is difficult to imagine numerical schemes that would be stable enough to accept arbitrary number as inputs and still produce stable and meaningful outputs.

Technically, one important point on the forward curves is that the pseudo-discount factors $P_X^{\text{CDF},j}(.,u)$, when they are defined, are not assets for any u in our economy. If any factor was an asset, its value today could be written using any numeraire $q_.$ and its associated measure $\mathbb{Q}$ as

$$
\begin{aligned}
P_X^j(t,u) &= q_t \,\mathrm{E}^{\mathbb{Q}}\left[q_u^{-1} P_X^{\text{CDF},j}(u,u) \middle| \mathcal{F}_t \right] \\
&= P_X^D(t,u)\,\mathrm{E}^u\left[P_X^D(u,u)^{-1} P_X^{\text{CPN},j}(u,u) \middle| \mathcal{F}_t \right] = P_X^D(t,u)
\end{aligned}
$$

where $\mathrm{E}^u\,[]$ is the expectation associated to the $P_X^D(.,u)$ numeraire. This proves that if $P_X^{\text{CPN},j}(.,u)$ was an asset, it would always be equal to $P_X^D(.,u)$. This is in contradiction to the core subject of this book, which is the differentiation between the discounting curve and the forward index estimation curve.

The same can be said in the case when the pseudo-discount factor is defined by a formula such as

$$P_X^j(t,u) = \mathrm{E}^{\mathbb{Q}}\left[f(r_\cdot^j) \middle| \mathcal{F}_t \right]$$

for some function f of a pseudo-short rate $r_\cdot^j$.

Mal nommer les choses, c'est ajouter au malheur du monde.

Albert Camus

Personal translation: Badly naming items, it is adding to the world's tragedy.

7.2 Spreads description

Our hypothesis $\mathbf{S0}^{\mathbf{CPN}}$ is one way to link the curves together to be able to price contingent claims. Other possibilities are presented below. Most of the analysis presented below has been initially presented in Henrard (2010).

7.2.1 Swap market spread

Under this hypothesis, the spreads between the market standard tenor swap rate and other forwards are known for every fixing. Let M be the *standard market index*, the one for which we have market data, and S be the swap forward rate.

The spreads or basis

$$B^j(t_0,t_1) = S_{t_0}^j(t_0,t_1) - S_{t_0}^M(t_0,t_1) \tag{7.1}$$

are known for every fixing date t_0 and every tenor j.

Using the martingale property of the swap rate it is easy to verify that a deterministic hypothesis is equivalent to a constant forward spread.

If one has a model for S^M, the swap rates related to other Ibor indexes can be obtained by a deterministic shift. This is particularly adapted to a Bachelier type model, where the volatility would be the same for swap rates with different indexes. This can be used in a Black-like model under the understanding that if the market rate is log-normal, the others are shifted log-normal.

This hypothesis is in practice very close to $\mathbf{S0}^{\mathbf{CPN}}$, which can be viewed as an approach with known continuously compounded spreads.

7.2.2 Black spread

The simplest model used to model interest rates derivatives is the Black model on forward rates. The base equation is

$$dS_t^M = \sigma S_t^M dW_t. \tag{7.2}$$

The volatility σ is that given by the market for the standard tenor and specific expiry, tenor and strike. In this new hypothesis, the similar rate for the forward (non-market convention) curve is supposed to follow a similar equation

$$dS_t^j = \sigma S_t^j dW_t.$$

The forward swap rate follows a Black equation (between 0 and expiry) with the same volatility and Brownian motion as the rate in the market convention.

The other tenor swap rates are proportional to the market swap rate:

$$S_t^j = \frac{S_0^j}{S_0^M} S_t^M.$$

Under that approach the spread between the rates $S^M - S^j$ is not constant nor deterministic. It is a constant proportion of the rate. The spread grows (and reduces) with the rate:

$$S_t^j - S_t^M = \left(\frac{S_0^j}{S_0^M} - 1 \right)$$

7.2.3 SABR/Constant elasticity of variance (CEV) spread

In the SABR (Stochastic Alpha Beta Rho model) framework, the base equations are

$$dS_t^M = \alpha_t (S_t^M)^\beta dW_t^1. \tag{7.3}$$

$$d\alpha_t = \nu \alpha_t dW_t^2 \tag{7.4}$$

A possible spread hypothesis based on these equations is as follows.

SA: The forward swap rate follows an SABR equation (between 0 and expiry) with the same parameters and Brownian motion as the rate in the market convention.

The impact on the spreads will depend on the β parameter, like the computation of the delta in the SABR framework. A β close to 1 will give a result close to Black spread and a β close to 0 will give a result close to market spread.

In the Black and SABR approaches, the spreads would increase when the rates increase. In the recent crisis, the spreads have increased while the rates were decreasing. This approach could be counterintuitive.

The three approaches mentioned above are only convenient if the objective is only to model one (swap) rate and not the whole term structure. These approaches would not be the most convenient for exotics (Bermuda, callable, and so on). By opposition, the hypothesis **S0**$^{\mathbf{CPN}}$, even if too simplistic, is a term structure hypothesis.

7.2.4 Continuously compounded constant spread

The same financial reality can be expressed in different ways, with different conventions. Traditionally each currency and instrument type has its own convention, as described in Appendix B. A spread in one convention is not equal to the spread in another convention. The translation is often not static and depends on the rate level itself. Consequently it is not possible to use one number as a fixed spread and use it for all conventions without adjustment.

Let's look at the standard money market simple rate and see if it is possible to use a constant spread in a coherent way. Suppose one rate is denoted L and the other is F with a spread S. The basic spread equation is

$$L = F + S. \tag{7.5}$$

Is it possible to use a constant spread S in a consistent way for all tenors in a simple interest market convention?

To see that this is not the case, take two consecutive periods of length δ. The same rate F_1 is used for the two periods. The rate over the total period is F_2 and the relation between them is $1 + 2\delta F_2 = (1 + \delta F_1)^2$. Then the relation on the starting rate L is

$$\begin{aligned} 1 + 2\delta F_2 + 2\delta S_2 = 1 + 2\delta L_2 = (1 + \delta L_1)^2 &= (1 + \delta F_1 + \delta S_1)^2 \\ &= (1 + \delta F_1)^2 + 2\delta S_1 \left(1 + \delta F_1 - \tfrac{1}{2}\delta S_1\right) \end{aligned}$$

This means that

$$S_2 = S_1 \left(1 + \delta F_1 - \tfrac{1}{2}\delta S_1\right)$$

To the first order in δ the spreads are the same for both tenors. Nevertheless they are not constant and the spread has a dependency on the rate level. A constant spread cannot be used in the simple interest in a coherent way. Not all current rates and forwards in the market conventions can be modelled with the same spread.

The situation is different in the continuously compounded zero-coupon framework. Let's denote by $\bar{F}$ and $\bar{L}$ the continuously compounded equivalent of F and L:

$$(1 + \delta F) = \exp(\delta \bar{F}) \quad \text{and} \quad (1 + \delta L) = \exp(\delta \bar{L})$$

The spread is expressed in that convention with $\bar{L} = \bar{F} + \bar{S}$. In this context it can be shown that a constant spread in spot and forward rates is coherent. Take three dates $t_0 < t_1 < t_2$ and the corresponding rates $F_{i,j}$ and $L_{i,j}$ between the dates t_i and t_j. The relation between the F rates is $\exp(\bar{F}_{0,2}(t_2 - t_0)) = \exp(\bar{F}_{0,1}(t_1 - t_0)) \exp(\bar{F}_{1,2}(t_2 - t_1))$, that is, $\bar{F}_{0,2}(t_2 - t_0) = \bar{F}_{0,1}(t_1 - t_0) + \bar{F}_{1,2}(t_2 - t_1)$. Then one has

$$\begin{aligned} \exp(\bar{L}_{0,2}(t_2 - t_0)) &= \exp((\bar{F}_{0,2} + \bar{S})(t_2 - t_0)) \\ &= \exp((\bar{F}_{0,1} + \bar{S})(t_1 - t_0)) \exp((\bar{F}_{1,2} + \bar{S})(t_2 - t_1)) \\ &= \exp(\bar{L}_{0,1}(t_1 - t_0)) \exp(\bar{L}_{1,2}(t_2 - t_1)) \end{aligned}$$

The composition relation on $\bar{F}$ implies a similar relation on $\bar{L}$ for a constant spread.

The simplest way to represent spreads according to the above analysis is through continuously compounded rates. The investment factors can be written as

$$\exp\left((r+s)(v-u)\right) = \exp\left(r(v-u)\right)\exp\left(s(v-u)\right).$$

This is the type of description that motivated the spread definition (Definition 2.3) with $\beta(u,v) = \exp\left(s(v-u)\right)$.

7.3 Constant multiplicative spread

The simplest hypothesis to model the spread is to use a constant spread hypothesis. This simplifying assumption, denoted $\mathbf{S0^{CPN}}$, is the one we used to provide formulas for FRAs and STIR futures in Chapter 2.

The same hypothesis can be combined with different term structure models on the discounting curve to obtain models in the multi-curve framework starting from standard one-curve models. In this section we give some examples of pricing with HJM and LMM models.

7.3.1 Heath-Jarrow-Morton model

The foundations and main results for the HJM model are described in Appendix A. The model describes the evolution of the risk-free curve, that is the evolution of $P_X^D(.,u)$. Using the hypothesis $\mathbf{S0^{CPN}}$, the values of any forward rate $F_X^{\mathrm{CPN},j}(t,u,v)$ is provided through the formula

$$1+\delta F_X^{\mathrm{CPN},j}(t,u,v) = \beta_X^{\mathrm{CPN},j}(0,u,v)\frac{P_X^D(t,v)}{P_X^D(t,u)}.$$

Whatever the numerical method used to obtain the values of P_X^D, being analytical formulas, numerical integration, PDE or Monte Carlo, the constant β can be multiplied and the values of $F_X^{\mathrm{CPN},j}$ obtained. In this way it is possible to modify all single-curve methods to also work in the multi-curve framework. The hypothesis $\mathbf{S0^{CPN}}$ is restrictive in the dynamic of the spread but has the advantage of being able to model any instrument that was modelled in the old framework. A particular case of such a modelling, for European physical delivery swaptions, is provided in Section 7.3.3.

7.3.2 Libor Market Model

The case of the Libor Market Model (LMM) is different from the HJM one, even thought the two modelling approaches are *twin brothers* (see Gątarek (2005)) in the single-curve framework. There are two ways to translate the Libor rates dynamic in the multi-curve framework: it can be translated as the dynamic of the forward risk-free rate (F^D of Definition 2.2) or it can be translated as the dynamic of the forward coupon rate ($F^{\text{CPN},j}$ of Definition 2.1). In this section we discuss only the constant spread hypothesis and ignore the more involved models on the dynamic of both sets of rates that are presented in Section 7.9.

In this section we sketch the former, modelling the forward risk-free rates. In terms of name, this is probably not what was intended by the original model, as the starting point is not a Libor-like rate as quoted by the market, but a synthetic forward risk-free rate. Nevertheless, in terms of financial mathematics, it may be closer to the original techniques. A rate, which can be written as a ratio of discount factors, is modelled; the model is initially done on a discrete set of forward rates and is linked to an HJM model of the risk-free curve.

The starting point is any version of the single-curve LMM applied to the discounting curve. Using the modelling of those rates, one can price any risk-free dependent contingent claims. Using hypothesis $\mathbf{S0^{CPN}}$, once the values of the forward risk-free rates are obtained, the actual Ibor forward rates are only one multiplication away. The Ibor forward rates are given by

$$1 + \delta F_X^{\text{CPN},j}(t,u,v) = \beta_X^{\text{CPN},j}(0,u,v)(1 + \delta F_X^D(t,u,v)).$$

In terms of implementation, the computation time of a Monte Carlo should be very similar to the one in the one-curve framework. Simply compute the relevant multiplicative spreads at the start, do the difficult Monte Carlo work on the risk-free forwards using any one-curve LMM technique and multiply by the spread at the end. The couple of extra divisions/multiplications for each cash-flow add a negligible time to the operation.

7.3.3 Swaptions in HJM

We use the cash-flow equivalent of the swap described in Section 2.4:

$$\sum_{i=0}^{\bar{n}} d_i P_X^D(t,\bar{t}_i).$$

Under hypothesis $\mathbf{S0^{CPN}}$, the quantities d_i are constant.

Using this approach, the pricing of swaption in the one-factor Gaussian HJM model, originally proposed in Henrard (2003) in the one-curve framework, can be written explicitly. The Gaussian HJM framework is described in Appendix A.

Theorem 7.1. (Exact swaption price in one-factor HJM) *Suppose we work in the Gaussian HJM model with a separable volatility as described in Definition A.1 and in the multi-curve framework with hypotheses* $\mathbf{D}$, $\mathbf{I^{CPN}}$ *and* $\mathbf{S0^{CPN}}$. *Let* $\theta \leq \bar{t}_0 < \cdots < \bar{t}_n$, $d_0 < 0$ *and* $d_i \geq 0$ $(1 \leq i \leq n)$. *The present value of a European receiver swaption (with physical delivery), with expiry* θ *on a swap with cash-flows equivalent* $(\bar{t}_i, d_i)$ *is given* *at time t by the $\mathcal{F}_t$-measurable random variable*

$$\sum_{i=0}^{n} d_i P^D(t,\bar{t}_i) N(\kappa + \alpha(0,\theta,\theta,\bar{t}_i))$$

where κ is the $\mathcal{F}_t$-measurable random variable defined as the (unique) solution of

$$\sum_{i=0}^{n} d_i P^D(t,\bar{t}_i) \exp\left(-\frac{1}{2}\alpha^2(0,\theta,\theta,\bar{t}_i) - \alpha(0,\theta,\theta,\bar{t}_i)\kappa\right) = 0. \tag{7.6}$$

The price of the payer swaption is

$$-\sum_{i=0}^{n} d_i P^D(t,\bar{t}_i) N(-\kappa - \alpha(0,\theta,\theta,\bar{t}_i))$$

Proof: The present value of the swaption is, in the $P_X^D(.,\theta)$ numeraire,

$$P^D(0,\theta)\,\mathrm{E}^{\theta}\left[\left(\sum_{i=0}^{n} d_i P_X^D(\theta,\bar{t}_i)\right)^+\right].$$

Using Lemma A.2, the value becomes

$$\mathrm{E}^{\theta}\left[\left(\sum_{i=0}^{n} d_i P_X^D(0,\bar{t}_i) \exp\left(-\alpha(0,\theta,\theta,\bar{t}_i) X_{0,\theta} - \frac{1}{2}\alpha^2(0,\theta,\theta,\bar{t}_i)\right)\right)^+\right].$$

The value inside the expected value is positive when $X < \kappa$. To see this, note that $\alpha(0,\theta,\theta,.)$ is increasing. The quantity is positive when

$$\exp(-\alpha(0,\theta,\theta,\bar{t}_0)x)\left(\sum_{i=1}^{n} d_i P_X^D(0,\bar{t}_i) \exp\left(-\frac{1}{2}(\alpha^2(0,\theta,\theta,\bar{t}_i)\right)\right.$$
$$\left.\exp\left(-(\alpha(0,\theta,\theta,\bar{t}_i) - \alpha(0,\theta,\theta,\bar{t}_0))x\right) + d_0\right) > 0$$

The large sum is a sum of negative exponentials with positive coefficients. It is a strictly decreasing function with limits $+\infty$ at $-\infty$ and limit 0 at $+\infty$. It crosses the value $-d_0 > 0$ at a unique value (called κ). It is above $-d_0$ when $x < \kappa$ and below it when $x > \kappa$.

The expected value can be written in terms of integrals by

$$\sum_{i=0}^{n} d_i P_X^D(0,t_i) \int_{-\infty}^{\kappa} \exp\left(-\alpha(0,\theta,\theta,t_i)x - \frac{1}{2}\alpha^2(0,\theta,\theta,t_i)\right) \exp\left(-\frac{1}{2}x^2\right) dx.$$

After changes of variable, the integrals can be written as normal cumulative densities and the result follows. □

7.4 Ibor forward rate modelling

The standard one-curve framework approach to modelling cap/floor is to notice that the forward rate is a martingale in the payment date forward numeraire. From there one can choose one's own preferred way to model the rate with a drift-less stochastic equation and obtain a coherent pricing. A very similar approach can be used in the multi-curve framework.

7.4.1 Black model

The forward rate $F_X^{\text{CPN},j}(.,u,v)$ is a martingale in the $P_X^D(.,v)$ numeraire under the hypotheses **D** and $\mathbf{I^{CPN}}$. This was described in Theorem 2.1. As F is a martingale, it can be modelled by a drift-less stochastic equation. The simplest approach is to suppose the rate follows a geometric Brownian motion:

$$dF_X^{\text{CPN},j}(t,u,v) = F_X^{\text{CPN},j}(t,u,v)\eta(t,u) \cdot dW_t^v. \tag{7.7}$$

The Brownian motion W_t^v is multidimensional with the dimension of η equal to the dimension of the Brownian motion. This is often referred to as the Black (1976) model for Ibor rates.

The solution of the above equation is an exponential martingale

$$F_X^{\text{CPN},j}(t,u,v) = F_X^{\text{CPN},j}(0,u,v) \exp\left(\int_0^t \eta(\tau,u) \cdot dW_\tau - \frac{1}{2}\int_0^t |\eta(\tau,u)|^2 d\tau \right).$$

Let

$$\sigma_\eta^2(t) = \int_0^t |\eta(\tau,u)|^2 d\tau$$

be the period volatility. With that notation, the solution can be written as

$$F_X^{\text{CPN},j}(t,u,v) = F_X^{\text{CPN},j}(0,u,v) \exp\left(\sigma_\eta X - \frac{1}{2}\sigma_\eta^2 \right)$$

for X a standard normally distributed random variable of dimension 1.

An Ibor caplet or floorlet with strike K and expiry θ on the period $[u,v]$ for an expiry equal to the fixing date of the rate $I_X^j(\theta)$ on the same period is a financial product that pays in v the amount

$$\left(\omega(I_X^j(\theta) - K) \right)^+$$

with $\omega = 1$ for a caplet and $\omega = -1$ for a floorlet. In practice, the amount is multiplied by a notional and an accrual factor. The factors act as multipliers of the total amount. To lighten the notation, we ignore those multiplicative factors in the analysis. They should appear as a multiplicative factor also in the final result.

Theorem 7.2 *In the multi-curve framework with hypotheses* **D** *and* $\boldsymbol{I}^{\boldsymbol{CPN}}$, *if the forward rate follows the dynamic given by Equation (7.7), the present value of a caplet/floorlet of strike K and expiry* θ *is given by*

$$P_X^D(0,v)\omega\left(F_X^{\mathrm{CPN},j}(0,u,v)N(\omega d_+) - KN(\omega d_-)\right)$$

where

$$d_\pm = \frac{\ln\left(\dfrac{F_X^{\mathrm{CPN},j}(0,u,v)}{K}\right) \pm \dfrac{1}{2}\sigma_\eta^2(\theta)}{\sigma_\eta(\theta)}.$$

This approach does not require any explicit hypothesis regarding the spread between the risk-free rate and the forward Ibor rate. The simple approach to modelling each rate is possible due to the definition of forward rate given by Definition 2.1. The forward rate was defined as an asset – the Ibor coupon – divided by an other asset – the discount factor – playing the role of a numeraire, facilitating the modelling of such quantities.

Note also that this approach is modelling only one specific forward rate and can be used to price instruments depending on that rate only. The approach does not describe the full forward curve but only one point on it. Also it does not describe the discounting curve dynamic, except very indirectly by saying that the forward rate dynamic is valid in a numeraire attached to the discounting curve. For this approach all the intuition and risk management techniques developed in the one-curve framework can be used with almost no change.

Even if the approach is not a full term structure and multi-curve description, it can be embedded in such models as the LMM. Some approaches to such models are described in Section 7.9.

7.4.2 SABR model

The same approach can be used for any model which is based on a stochastic equation of the forward rate in the $P_X^D(.,v)$ numeraire. In particular it can be applied to the SABR model from Hagan et al. (2002) which is very popular in interest rate modelling to describe the smile.

The term SABR is an acronym for Stochastic Alpha Beta Rho model. It is a stochastic volatility model on the forward rate:

$$\begin{aligned} dF_X^{\mathrm{CPN},j}(t,u,v) &= \alpha_t(F_X^{\mathrm{CPN},j}(t,u,v))^\beta dW_t^1 \\ d\alpha_t &= \nu\alpha_t dW_t^2 \end{aligned}$$

with W^1 and W^2 two correlated Brownian motions with $[W^1, W^2]_t = \rho t$. The state variables are F_t, the rate, and α_t, the volatility. In some presentations the volatility is denoted σ_t, like in the Black framework (see in particular (Rebonato et al., 2009, Section 3.2, Footnote 1)). The model parameters are β, the elasticity, ν, the volatility of volatility and ρ, the correlation.

Under that approach the price of cap/floor can be approximated by a Black formula similar to the one of Theorem 7.2 but with the period volatility replaced by $\sigma(K, F_X^{\mathrm{CPN},j})\sqrt{\theta}$ with

$$\sigma(K,f) = \frac{\alpha}{(fK)^{(1-\beta)/2}\left(1+\frac{(1-\beta)^2}{24}\ln^2\frac{f}{K}+\frac{(1-\beta)^4}{1920}\ln^4\frac{f}{K}\right)}\frac{z}{x(z)}$$
$$\times\left(1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right)\theta\right)$$

where

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\ln\frac{f}{K}$$

and

$$x(z) = \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right).$$

Like for the pure Black approach the framework results in conclusions very similar to those obtained in the one-curve framework and all the techniques developed there can be reused.

7.5 Swap rate modelling

The standard approach to swap rate modelling in the one-curve framework is very similar to the one described in the previous section. One notices that the swap rate is a martingale in the present value of a basis point numeraire and proceeds from there.

The notation regarding the vanilla interest rate swaps we use here are the same as in Section 2.4. Let

$$A_t = \sum_{i=1}^{\tilde{n}} P_X^D(t,\tilde{t}_i)\tilde{\delta}_i$$

be a positive random variable called *present value of a basis point*. The quantity is also called *delivery annuity* or *level*.

The swap rate $S_\cdot^j$ described in Equation (8.19) is a martingale in the A_t numeraire under the hypotheses **D** and $\mathbf{I}^{\mathbf{CPN}}$. To see that this is a martingale, it is enough to notice that the rate is a linear combination of assets – the floating coupons – divided by the numeraire.

As it is a martingale, it can be modelled by a drift-less stochastic equation. The simplest approach in this case is to suppose a geometric Brownian motion for the rate in the $A_.$ numeraire:

$$dS_t^j = S_t^j \eta(t,u) \cdot dW_t^A. \tag{7.8}$$

with W_t^A a Brownian motion in the measure associated to the A_t numeraire. The Brownian motion is a multidimensional random variable with the dimension of η equal to the dimension of the Brownian motion. This is often referred to as the Black (1976) model for swap rates.

The solution of the above equation is an exponential martingale

$$S_t^j = S_0^j \exp\left(\int_0^t \eta(\tau,u)\cdot dW_\tau - \frac{1}{2}\int_0^t |\eta(\tau,u)|^2 d\tau\right).$$

Let σ_η, defined by

$$\sigma_\eta^2(t) = \int_0^t |\eta(\tau,u)|^2 d\tau,$$

be the period volatility. With that notation, the solution can be written as

$$S_t^j = S_0^j \exp\left(\sigma_\eta(t)X - \frac{1}{2}\sigma_\eta^2(t)\right)$$

for X a standard normally distributed random variable of dimension 1.

A physical delivery swaption with strike K and expiry θ is a financial instrument that gives its owner the right in θ to enter into a vanilla fixed for Ibor interest rate swap with fixed rate K. The details of the swap, such as its convention and tenor, are agreed between the parties when the option is written.

On the expiry date, the value of the swaption on a payer swap can be written as

$$A_t(S_\theta^j - K)^+.$$

The value is simply the positive value of a receiver par swap plus the payer swap with fixed rate K. The total value of the par swap is 0 by definition and the values of the two floating legs cancel.

From that equation, one can proceed exactly like in the cap/floor case, just replacing the discount factor numeraire by the annuity numeraire.

Theorem 7.3 *In the multi-curve framework with hypotheses* **D** *and* $\mathbf{I^{CPN}}$, *if the swap rate follows the dynamic given by Equation (7.8), the present value of a swaption of strike K and expiry θ is given by*

$$A_0\omega\left(S_0^j N(\omega d_+) - KN(\omega d_-)\right)$$

where

$$d_{\pm} = \frac{\ln\left(\frac{S_0^j}{K}\right) \pm \frac{1}{2}\sigma_\eta^2(\theta)}{\sigma_\eta(\theta)}$$

and $\omega = +1$ *for a payer swaption and* $\omega = -1$ *for a receiver swaption.*

Like in the previous section, we note that the equation was obtained without explicit hypotheses on the spread between the risk-free rates and the forward Ibor rates. The only hypothesis is the dynamic of the swap rate, involving the risk-free and Ibor curves, in a specific numeraire, itself involving the risk-free curve.

7.5.1 Modified convention

The development for the SABR model in the previous section could be repeated for the swaption. We do not do it, but simply summarise the situation by the fact that swaptions are priced with a Black-like formula using an implied volatility which is strike, forward, time and model parameters dependent. This is summarised by the present value P given by

$$P = A_0 \text{Black}(S_0, K, \sigma(\theta, T, S_0, K, p)).$$

In the above approach we have supposed that only one convention is used for the swaption underlying swaps. In practice, the conventions are often mixed in one portfolio. The standard interbank market will use only one specific convention for each currency. Those conventions for swaps are described in Section B.8. But certain swaptions may have been traded with a corporate that requested a different convention, for example to hedge a bond.

As the modelling above is a modelling of a specific swap rate, it is implicitly a modelling in a specific convention. One way to deal with the change of convention problem is to have one volatility input for each convention. This approach is not feasible in practice as the required data is not available.

To ensure absence of arbitrage when the conventions are changed, modified versions of standard figures are introduced. Let C be the standard convention.

The convention C-modified delivery annuity is

$$A_t^C = \sum_{i=1}^{n} \delta_i^C P_X^D(t, t_i)$$

where δ_i^C is the accrual factor in the C convention for the same dates as the original present value of a basis point. The convention-modified swap rate is

$$S_t^C = \frac{\sum_{i=1}^n P_X^D(t,t_i)\delta_i F_X^{\text{CPN},j}(t,t_{i-1},t_i)}{A_t^C}.$$

To cover the case where the fixed rate is not the same for all coupons, a *strike equivalent* is introduced:

$$K = \frac{\sum_{i=1}^{n} \delta_i K_i P^D(0,t_i)}{A_0}.$$

Note that the strike equivalent is curve dependent.

The convention-modified strike equivalent is

$$K^C = \frac{\sum_{i=1}^{n} \delta_i K_i P^D(0,t_i)}{A_0^C}.$$

The price, taking into account the fact that the implied volatilities are provided for standard-convention instruments, is

$$P = A_0^C \text{Black}(S_0^C, K^C, \sigma(\theta, T, S_0^C, K^C, p)).$$

Note that the convention change in the swap rate and the strike equivalent is the inverse of the one in the annuity. The modification impact appears only through the volatility; if the smile is flat, the modification has no impact. If the swaption has a standard convention, there is no impact at all.

In this way, the potential arbitrage coming from a change of convention disappears. A swaption with strike 3.60% in the ACT/360 convention has exactly the same cash-flows as a swaption with strike 3.65% in the ACT/365 convention. We have to make sure that our pricing approach provides the same prices for both and does not introduce arbitrage. The strike and forward dependency in the SABR implied volatility would create this arbitrage if the strike and forward were not adjusted for the convention.

7.6 Parsimonious HJM multi-curve framework

7.6.1 The model

Let W_t^v be a multidimensional Brownian motion in the measure associated to the numeraire $P^D(.,v)$. The associated filtration is denoted $\mathcal{F}_.$.

In the parsimonious HJM approach proposed by Moreni and Pallavicini (2010), the equations for the continuously compounded instantaneous forward risk-free rates (see Section A.1 for the definition) and the market Ibor rates are

$$df(t,v) = \sigma(t,v) \cdot dW_t^v \tag{7.9}$$

$$dF_X^{\text{CPN},j}(t,u,v) = \left(F_X^{\text{CPN},j}(t,u,v) + a^j(v)\right) \eta^j(t,v) \cdot dW_t^v \tag{7.10}$$

The model for the risk-free rate is a Gaussian HJM model and we use the notation and results of Appendix A to analyse the approach.

In the cash account numeraire, the equations for the risk-free rates and the market Ibor forward rates are

$$
\begin{aligned}
df(t,v) &= \sigma(t,v)\cdot \nu(t,v)dt + \sigma(t,v)\cdot dW_t \\
dF_X^{\mathrm{CPN},j}(t,u,v) &= \left(F_X^{\mathrm{CPN},j}(t,u,v) + a^j(v)\right)\eta^j(t,v)\cdot \nu(t,v)dt \\
&\quad + \left(F_X^{\mathrm{CPN},j}(t,u,v) + a^j(v)\right)\eta^j(t,v)\cdot dW_t
\end{aligned}
$$

The solution of the Ibor rates equations in the cash account numeraire can be written as

$$
\begin{aligned}
F_X^{\mathrm{CPN},j}(t,u,v) + a^j(v) &= \left(F_X^{\mathrm{CPN},j}(s,u,v) + a^j(v)\right) \\
&\quad \times \exp\left(\int_s^t \eta^j(\tau,v)\cdot dW_\tau - \frac{1}{2}\int_s^t |\eta^j(\tau,v)|^2 d\tau + \int_s^t \eta^j(\tau,v)\cdot \nu(\tau,v)d\tau\right)
\end{aligned}
$$

We give a name to several integrated volatilities. The volatility of the forward rate is

$$
\sigma_\eta^2 = \sigma_\eta^2(\theta_0,\theta_1,v) = \int_{\theta_0}^{\theta_1} |\eta^j(s,v)|^2 ds.
$$

Similarly the interaction between the risk-free rates and Ibor forward rates is used for different instruments:

$$
\sigma_{\eta,N} = \sigma_{\eta,N}(\theta_0,\theta_1,v) = \int_{\theta_0}^{\theta_1} \eta^j(\tau,v)\cdot \nu(\tau,v)d\tau
$$

and

$$
\gamma_\eta = \gamma_\eta(\theta_0,\theta_1) = \gamma_\eta(\theta_0,\theta_1,v) = \exp\left(\sigma_{\eta,N}(\theta_0,\theta_1,v)\right).
$$

Using this notation, the forward rate solution can be written in the cash account numeraire as

$$
\begin{aligned}
F_X^{\mathrm{CPN},j}(t,u,v) + a^j(v) &= \left(F_X^{\mathrm{CPN},j}(s,u,v) + a^j(v)\right) \\
&\quad \times \exp\left(-X_\eta - \tfrac{1}{2}\sigma_\eta(s,t,v)^2\right)\gamma_\eta(s,t) \qquad (7.11)
\end{aligned}
$$

with X_η the normally distributed random variable $X_\eta = -\int_s^t \eta^j(\tau,v)\cdot dW_\tau$ independent of $\mathcal{F}_s$ with standard deviation $\sigma_\eta(s,t,v)$.

7.6.2 Parsimonious HJM analysis

With that result in mind, Equation (7.10) can be viewed as an extension of the constant spread hypothesis $\mathbf{S0}^{\mathbf{CPN}}$. For a multiplicative spread $\beta_0^j(v)$, the equation for the forward rate is Equation (7.10) with a displacement $a^j(v) = \beta_0^j(v)/\delta$ and a volatility $\eta^j(t,v) = \nu(t,v) - \nu(t,v)$.

Equation (7.10) can also be applied for the additive spread approach. Suppose that $F_t^j(v) = F_t^D(v) + s(v)$ with $s(v)$ a deterministic spread. Then the equation is satisfied with $a^j(v) = 1/\delta - s^j(v)$ and $\eta^j(t,v) = \nu(t,v) - \nu(t,v)$.

The choice of σ and η_i separately can lead to behaviour not in line with financial reality. Take $a^j(v) = 1/\delta$ and $\eta^j(t,v) = c(\nu(t,v) - \nu(t,v))$. The simplified intuition behind that choice is that the Ibor rates have a dynamic similar to the one coming from the short rate model but with a larger volatility (multiplied by c), one part coming from the rates and one from the credit. What is the market spread $F_X^{\text{CPN},j}(t,u,v) - F_X^D(t,u,v)$ implied by such a model? The solution can be written explicitly in that case.

For the first graphical representation, we take $t = 1$, $\sigma_2 = 0.01$, $F_0^D = 0.02$, $F_0^j = 0.0220$ and $c = 1.05$. The results for the rates and spreads are displayed in Figure 7.1. The X-axis represents the size of the normally distributed random variable underlying the dynamic. The Y-axis represents the rate or spread in percent. The horizontal dotted line is the initial level of the spread.

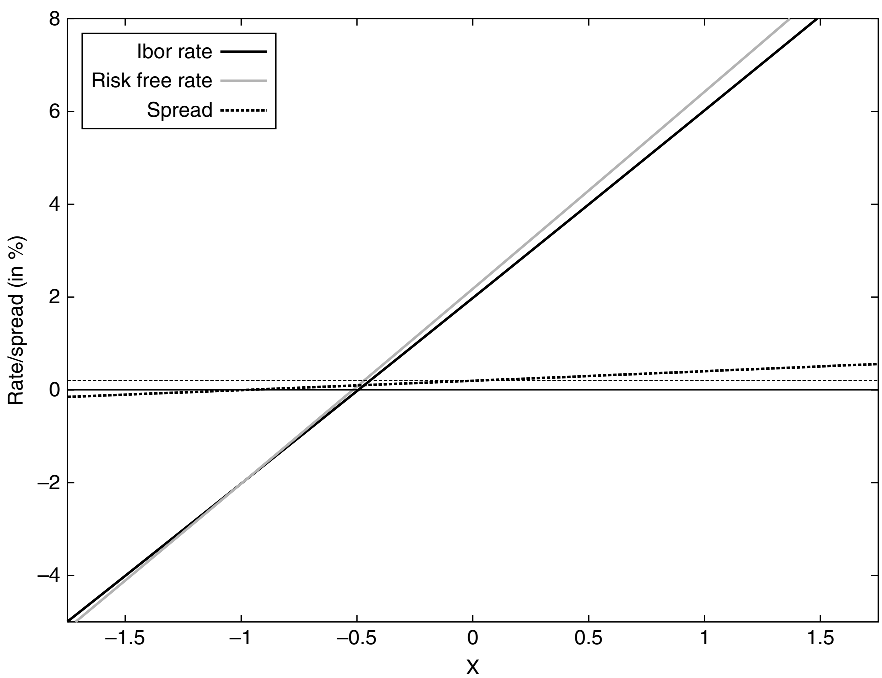


Figure 7.1 Representation of the rates relative dynamic and the implied spread dynamic in the parsimonious HJM approach. Parameter choice described in the text.

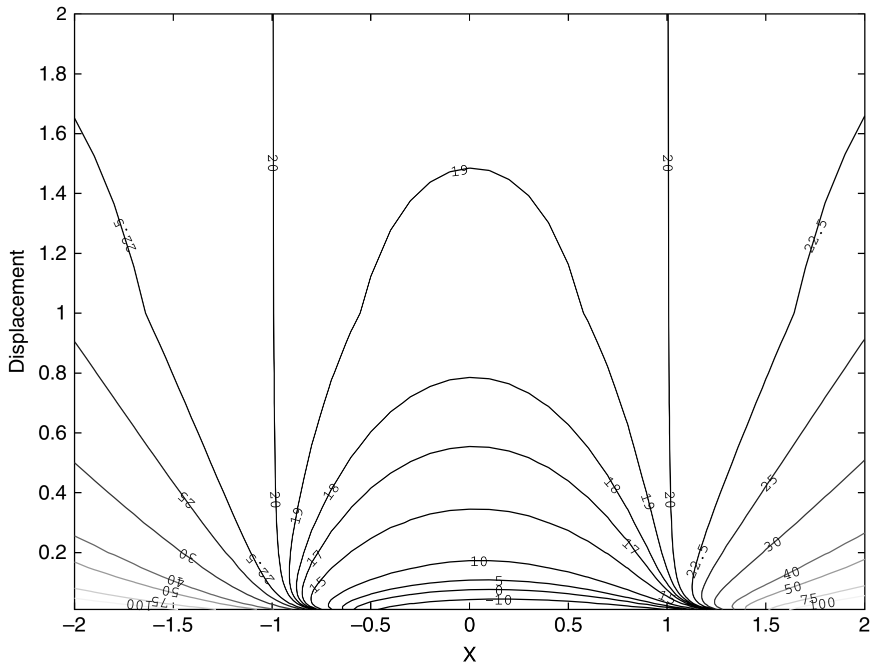


Figure 7.2 Spread levels for different values of the underlying standard normal random variable on the horizontal axis and different levels of displacement on the vertical axis. See the text for the choice of parameters.

As can be seen from the graph, the spread will depend on the level of rates. When the rate increases, the spread increases; when the rate decreases, the spread decreases. The dynamic is such that with the reasonable decrease of rate – normal distribution change of one standard deviation – the spread is becoming negative, in contradiction with financial intuition. Apart from the trivial constant spread case, it may be difficult to find actual implementation with non-trivial spread dynamic and at the same time intuitive spreads.

In the second graphical representation in Figure 7.2, we used the same starting points for the rate and the spread and the same time to expiry. This time we fix the volatility of the risk-free rate and vary the dynamic of the forward rate. The variation is obtained by changing the displacement coefficient. The coefficients used are between 0, corresponding to a log-normal model, and 2. A displacement of 4 would be equivalent to the implicit displacement of the risk-free rate of $1/\delta$ for a quarterly rate. The difference between a displacement of 2 and 4 is very minimal and would not add any information on the graph. The level of volatility η is chosen such that the approximate equivalent volatility $\nu(F^D(0) - 1/\delta)/(F^j(0) - a^j)$ is preserved. At the starting point, F^D and F^j have the same local volatility.

What is interesting from the graph is that even if the volatility is locally the same at the start point, the implied dynamics are very different. For small displacements, the spread can change from its initial value of 20 basis points to a range of -10 to $+100$ for not extreme movements of rate.

7.6.3 Cap/floor

The pricing of cap and floor in this approach is relatively easy. The forward rates follow a simple displaced diffusion formula in the numeraire on the discount factor to their maturity. A generalised Black formula is sufficient to price them.

Theorem 7.4 *In the parsimonious HJM framework, the present value of a cap with expiry θ and strike K on a rate over the period $[u, v]$ is given in 0 by*

$$C_0 = \left(F_X^j(0,u,v) + a^j(v)\right) N(\kappa) - \left(K + a^j(v)\right) N(\kappa - \sigma_\eta(0,\theta,v))$$

where

$$\kappa = \frac{1}{\sigma_\eta(0,\theta,v)} \ln\left(\frac{F_X^{\mathrm{CPN},j}(0,u,v) + a^j(v)}{\tilde{K} + a^j(v)}\right) + \frac{1}{2}\sigma_\eta(0,\theta,v).$$

This is a direct application of the formula for cap/floor in the displaced diffusion framework. For cap/floor the results are similar to those proposed in Section 7.4 but with a displaced log-normal diffusion instead of a pure log-normal diffusion.

7.6.4 STIR futures

STIR futures are described in Appendix B.4. The futures is characterised by a fixing date t_0 and a reference Ibor rate on the period $[u, v]$ with an accrual factor δ. The futures price is denoted Φ_t^j.

The pricing formula for such futures is given by the following theorem.

Theorem 7.5 (STIR futures in parsimonious HJM framework). *Let $0 \leq t \leq t_0 \leq u < v$. In the Gaussian parsimonious HJM model in the multi-curve framework with hypotheses **D** and $\boldsymbol{I}^{\boldsymbol{CPN}}$, the price of the futures fixing in t_0 for the period $[u, v]$ with accrual factor δ is given in t by*

$$\Phi_t^j = 1 - F^{\mathrm{CPN},j}(t,u,v)\gamma(t,t_0) + a^j(v)\left(1 - \gamma_\eta(t,t_0)\right). \tag{7.12}$$

Proof: Using the generic pricing future price process theorem,

$$\Phi_t^j = \mathrm{E}^N\left[1 - I_X^j(t_0)\middle| \mathcal{F}_t\right]$$

where $\mathrm{E}^N\,[]$ is the cash account numeraire expectation.

Using the formula for the forward rate described in the previous section, we have

$$
\begin{aligned}
&\mathrm{E}^{N}\left[F_X^{\mathrm{CPN},j}(t_0,u,v) \middle| \mathcal{F}_t \right] \\
&= \mathrm{E}^{\mathbb{N}}\left[\left(F_X^{\mathrm{CPN},j}(t,u,v)+a^j(v)\right)\exp\left(-X_\eta-\frac{1}{2}\sigma_\eta^2\right)\gamma_\eta(t,t_0,v)-a^j(v) \middle| \mathcal{F}_t \right] \\
&= \left(F_X^{\mathrm{CPN},j}(t,u,v)+a^j(v)\right)\gamma_\eta(t,t_0,v)-a^j(v)
\end{aligned}
$$

From the above result, the futures price is easily obtained:

$$
\Phi_t^j = 1-F_X^{\mathrm{CPN},j}(t,u,v)\gamma_\eta(t,t_0,v)-a^j(v)\left(\gamma_\eta(t,t_0.v)-1\right).
$$

□

The result is very close to the one proposed for Gaussian HJM with hypothesis $\mathbf{S0}^{\mathbf{CPN}}$ but with a different convexity adjustment as described by γ_η. Note also that in a Gaussian HJM model with deterministic spread, the shift a is equal to $1/\delta$. The formula proposed is a natural extension of the previous one in this more general framework.

Note that the futures price depends on the forward rate $F^{\mathrm{CPN},j}$ and also on the joined dynamic of the risk-free rate and the forward rate through γ_η.

7.6.5 STIR futures options – margin

In this section we analyse the options on STIR futures with daily margining. The options are subject to margining processes similar to the margining process on the underlying futures. Let θ be the option expiry date and K its strike price. For the futures itself, we use the same notation as in the previous section.

The futures options usually have an American feature. Due to the margining process, the American options have the same price as the European options. This general observation for options with continuous margining can be found in Chen and Scott (1993).

We denote by $R_t = 1-\Phi_t^j$ the futures rate and by $\tilde{K} = 1-K$ the strike rate.

Due to the margining process on the option, the price of the option with margining is

$$
\mathrm{E}^{\mathbb{N}}\left[(\Phi_\theta-K)^+\right] = \mathrm{E}^{\mathbb{N}}\left[(\tilde{K}-R_\theta)^+\right].
$$

Theorem 7.6 *Let $0 \le \theta \le t_0 \le u < v$. The value of a STIR futures call (European or American) option of expiry θ and strike K with continuous margining in the parsimonious Gaussian HJM multi-curve framework is given in 0 by*

$$
C_0 = (\tilde{K}+a^j(v))N(-\kappa_\gamma)-\left(F_X^{\mathrm{CPN},j}(0,u,v)+a^j(v)\right)\gamma_\eta(0,t_0,v)N(-\kappa_\gamma-\sigma_1(0,\theta,v))
$$

where κ is defined by

$$\kappa_\gamma = \frac{1}{\sigma_\eta(0,\theta,v)} \ln\left(\gamma_\eta(0,t_0)\frac{F_X^{\text{CPN},j}(0,u,v)+a^j(v)}{\tilde{K}+a^j(v)}\right) - \frac{1}{2}\sigma_\eta(0,\theta,v).$$

The price of a STIR futures put option is given by

$$P_0 = \left(F_X^{\text{CPN},j}(0,u,v)+a^j(v)\right)\gamma_\eta(0,t_0)N(\kappa+\sigma_\eta(0,\theta,v)) - (\tilde{K}+a^j(v))N(\kappa).$$

Proof: Using the generic pricing theorem we have

$$\begin{aligned} C_0 &= \mathrm{E}^N\left[((\Phi_\theta - K)^+)\right] \\ &= \mathrm{E}^N\left[\left(\tilde{K} - \left(F_X^{\text{CPN},j}(\theta,u,v)+a^j(v)\right)\gamma_\eta(\theta,t_0,v)+a^j(v)\right)^+\right]. \end{aligned}$$

Using Equation (7.11) once more, we have

$$\begin{aligned} C_0 = \mathrm{E}^{\mathbb{N}}\Bigg[\Bigg(\left(\tilde{K}+a^j(v)\right) - \left(F_X^{\text{CPN},j}(0,u,v)+a^j(v)\right) \\ \times \exp\left(-X_\eta - \frac{1}{2}\sigma_\eta(0,\theta,v)^2\right)\gamma_\eta(0,t_0,v)\Bigg)^+\Bigg]. \end{aligned}$$

where we have used the fact that $\gamma_\eta(0,\theta,v)\gamma_\eta(\theta,t_0,v) = \gamma_\eta(0,t_0,v)$.

In the expectation, the parenthesis is positive when

$$\tilde{K}+a^j(v) > \gamma(0,t_0)\left(F_X^{\text{CPN},j}(0,u,v)+a^j(v)\right)\exp\left(-X_\eta - \frac{1}{2}\sigma_\eta(0,\theta,v)^2\right)$$

or when $X_\eta > \sigma_\eta\kappa_\gamma$. With this notation, the expected value is equal to

$$\begin{aligned} \frac{1}{\sqrt{2\pi}}\int_{x>\kappa_\gamma} &\Bigg(\tilde{K}+a^j(v) \\ &- \left(F_X^{\text{CPN},j}(0,u,v)+a^j(v)\right)\exp\left(-\sigma_\eta x - \frac{1}{2}\sigma_\eta^2\right)\gamma(0,t_0)\Bigg)\exp\left(-\frac{1}{2}x^2\right)dx \end{aligned} \tag{7.13}$$

The results follow using the definition of cumulative normal density N. □

7.6.6 Forward rate agreement

In this section we analyse the pricing of a Forward Rate Agreement (FRA) as described in Appendix B.3.

Let t_0 be the fixing date of the FRA on a reference rate for the period $[u,v]$. The date u is the spot lag after t_0 and the end date v is a j-period after u using the appropriate business day convention. The FRA payment is done in u, this is at the start of the reference period and the amount is, for a fixing $I_X^j(t_0)$,

$$\delta \frac{(I_X^j(t_0) - K)}{1 + \delta I_X^j(t_0)}.$$

We incorporate this analysis in this chapter as it is important for building curves from market instruments to have present values for the most liquid market instruments. The techniques used in the pricing are very similar to those used for options on futures.

With the deterministic hypothesis $\mathbf{S0}^{\mathbf{CPN}}$, the fair rate for market FRAs is similar to the fair rate for simplified textbook FRAs. It is then possible to construct curves correctly even with the erroneous description of FRAs. The simplified version and the real version give the same market quote under hypothesis $\mathbf{S0}^{\mathbf{CPN}}$. This luck is only valid for the curve construction process, the present values of the actual portfolios are still erroneous.

This is only by luck, not by design. If a new approach to the spread is used, like in the parsimonious HJM model, one has to analyse the present value and the par rate.

Let

$$X_1 = \int_0^{t_0} \eta^j(\tau, v) \cdot dW_\tau^v$$

and

$$X_2 = \int_0^{t_0} (v(s,u) - v(s,v)) dW_\tau^v$$

The two dimensional random variable $X = (X_1, X_2)$ has a covariance

$$\begin{pmatrix} \alpha_1^2(0, t_0, v) & \alpha_{1,2}(0, t_0, u, v) \\ \alpha_{1,2}(0, t_0, u, v) & \alpha_2^2(0, t_0, u, v) \end{pmatrix}$$

with

$$\alpha_{1,2} = \alpha_{1,2}(\theta_0, \theta_1, u, v) = \int_{\theta_0}^{\theta_1} \eta^j(\tau, v) \cdot (v(s,u) - v(s,v)) d\tau$$

From Section 7.6.1, the Ibor rate satisfies

$$F_X^{\mathrm{CPN},j}(t_0, u, v) + a^j = \left(F_X^{\mathrm{CPN},j}(0, u, v) + a^j\right) \exp\left(-X_1 - \frac{1}{2}\sigma_1^2\right).$$

The expected value associated to the numeraire $P_X^D(., v)$ is denoted $\mathrm{E}^v[]$. The value of the FRA is given by

$$\mathrm{E}^v\left[\frac{P^D(t_0,u)}{P^D(t_0,v)} \delta \frac{(F_X^{\mathrm{CPN},j}(t_0,u,v) - K)}{1 + \delta F_X^{\mathrm{CPN},j}(t_0,u,v)}\right] = \mathrm{E}^v\left[\frac{P^D(t_0,u)}{P^D(t_0,v)}\left(1 + \frac{1+\delta K}{1 + \delta F_X^{\mathrm{CPN},j}(t_0,u,v)}\right)\right]$$

All the different parts can be computed explicitly:

$$\frac{P^D(0,u)}{P^D(0,v)} \exp\left(-\frac{1}{2}\alpha_2^2\right) \mathrm{E}^v \left[\exp(-X_2)\left(1+\frac{1}{c_1+c_2\exp(-X_1)}\right)\right]$$

with $c_1 = (1-\delta a_j)/(1+\delta K)$ and $c_2 = \delta(F_0^j + a^j)\exp(-\sigma_1^2/2)/(1+\delta K)$.

The above expectation can be computed easily by numerical integration on a two dimensional integral. What we want to demonstrate by the above result is that even in a relatively simple model and for a relatively simple quasi-linear instrument, the valuation formulas can be complex in a multi-curve framework with spread.

7.7 Additive stochastic spread

7.7.1 The model

The additive stochastic basis approach developed by Mercurio and Xie (2012) allows for any model for the risk-free curve.

The additive spread is defined by

Definition 7.1 (Additive spread). *The* additive spread *between the j-Ibor forward rate and the risk-free forward rate at time t for the period* $[u,v]$ *is given by*

$$S_t^j(v) = S_t^j(u,v) = F_X^{\mathrm{CPN},j}(t,u,v) - F_X^D(t,u,v).$$

The idea of the framework is to model the additive spread $S_t^j(v)$ as a function of the risk-free forward rate $F_X^D(t,u,v)$ and of an independent martingale $\mathcal{X}_t^j(v)$ through

$$S_t^j(v) = \phi_v^j(F_X^{\mathrm{CPN},j}(t,u,v), \mathcal{X}_t^j(v)). \tag{7.14}$$

The simplest function introduced in the above article and used here is

$$S_t^j(v) = S_0^j(0) + \alpha_v^j\left(F_X^{\mathrm{CPN},j}(t,u,v) - F_X^{\mathrm{CPN},j}(0,u,v)\right) + \beta_v^j\left(\mathcal{X}_t^j(v) - \mathcal{X}_0^j(v)\right)$$

with $\mathcal{X}_0^j(v) = 1$. The parameter β_v^j and the volatility of $\mathcal{X}$ have similar roles in adjusting the size of the independent part of the stochastic spread.

Note that in this approach, no choice of α, β and $\mathcal{X}$ can reproduce the deterministic multiplicative spread hypothesis $\mathbf{S0^{CPN}}$.

7.7.2 Model analysis

We first look at the relation between the level of rates and the implied spread. The spread is made of a function of the rate level, the one multiplied by α, and an independent part, the one multiplied by β. In Figure 7.3, we have represented some

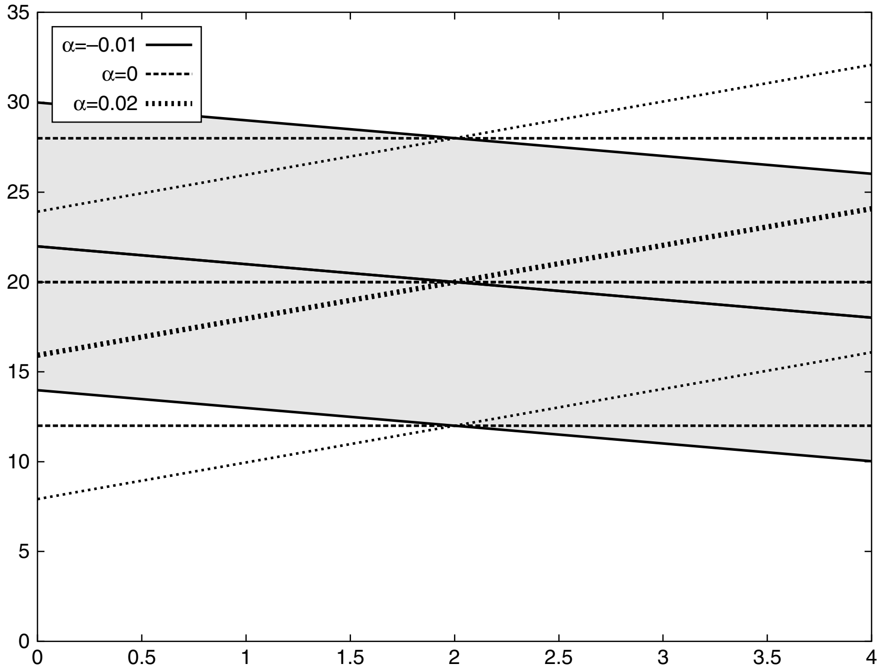


Figure 7.3 The additive spread $F_X^{\text{CPN},j}(t,u,v) - F_X^D(t,u,v)$ for different levels of risk-free rate $F_X^D(t,u,v)$. The rate is in percent and the spread in basis points.

examples of the framework with different coefficients. The starting forward risk-free rate $F^D(0)$ is 2.00%, the Ibor forward rate is $F^j(0) = 2.20\%$ and the horizon on which we look at the spread is one year. We use for $\mathcal{X}_t^j(v)$ a martingale of the form

$$\mathcal{X}_t^j(v) = \exp\left(-X_t^{\mathcal{X}}(v) - \frac{1}{2}\sigma_{\mathcal{X}}^2\right)$$

with $X^{\mathcal{X}}$ a martingale in the $P_X^D(.,v)$ numeraire normally distributed with mean 0 and variance $\sigma_{\mathcal{X}}^2$. In the example, we use $\sigma_{\mathcal{X}} = 0.0002$ and $\beta = 1$.

7.7.3 STIR futures

The pricing formula for STIR futures is given by the following theorem, which is proposed in Mercurio and Xie (2012). As for all the approaches, it is important to obtain relatively simple STIR futures formulas as the futures are one base ingredient for curve calibration.

Theorem 7.7 (STIR futures pricing in additive stochastic framework). *Let $0 \le t \le t_0 \le u < v$. In the additive stochastic basis multi-curve framework with hypotheses* **D**

and $\boldsymbol{I}^{\boldsymbol{CPN}}$, the price of the futures fixing in t_0 for the period $[u,v]$ with accrual factor δ is given in t by

$$\begin{aligned}\Phi_t^j =& 1 - F_X^{\mathrm{CPN},j}(t,u,v) \\ &+ \frac{(1+\alpha)}{\delta}\left(\mathrm{E}^{\mathbb{N}}\left[1+\delta F_X^D(t_0,u,v)\middle|\mathcal{F}_t\right] - (1+\delta F_X^D(t,u,v))\right)\end{aligned} \tag{7.15}$$

Proof: Using the generic pricing future price process theorem,

$$\Phi_t^j = \mathrm{E}^N\left[1 - F_X^{\mathrm{CPN},j}(t_0,u,v)\middle|\mathcal{F}_t\right]$$

where $\mathrm{E}^N[]$ is the cash account numeraire expectation. The forward rate can be written as

$$\begin{aligned}F_X^{\mathrm{CPN},j}(t_0,u,v) &= F_X^D(t_0,u,v) + S_{t_0}^j(v) \\ &= F_X^{\mathrm{CPN},j}(t,u,v) + (1+\alpha)\left(F_X^D(t_0,u,v) - F_X^D(t,u,v)\right) + \beta\left(\mathcal{X}_{t_0}^j(v) - \mathcal{X}_t^j(v)\right).\end{aligned}$$

As $\mathcal{X}_t^j$ is a cash account martingale, the expectation is reduced to

$$1 - F_X^{\mathrm{CPN},j}(t,u,v) + (1+\alpha)\left(\mathrm{E}^{\mathbb{N}}\left[F_X^D(t_0,u,v)\middle|\mathcal{F}_t\right] - F_X^D(t,u,v)\right).$$

□

The price of a futures on the Ibor rate is reduced to the price of a forward adjusted by the convexity adjustment on the risk-free rate and by the multiplicative factor describing the volatility of the spread dependency on rates.

Theorem 7.8 *Let $0 \le t \le t_0 \le u < v$. In the additive stochastic basis for the multi-curve framework, if the risk-free rate curve follows a Gaussian HJM model as describes in Appendix A, the price of the futures fixing in t_0 for the period $[u,v]$ with accrual factor δ is given in t by*

$$\Phi_t^j = 1 - F_X^{\mathrm{CPN},j}(t,u,v) + (1+\alpha)\left(F_X^D(t,u,v) + \frac{1}{\delta}\right)(\gamma(t,t_0,u,v) - 1). \tag{7.16}$$

The result is a direct consequence of the previous result and of the dynamic of the risk-free forward as described in Appendix A.

Note that the price depends on the forward Ibor rate $F_X^{\mathrm{CPN},j}$ but also on the risk-free forward rate F_X^D and the risk-free rate dynamic through γ. The fact that both $F_X^{\mathrm{CPN},j}$ and F_X^D appear in the pricing formula make it more difficult to price options on futures in the framework. The relatively simple results for futures options obtained in the parsimonious HJM model of Section 7.6 and in the multiplicative stochastic basis framework of Section 7.8 can not be achieved in this framework.

7.8 Multiplicative stochastic spread

In this section, we propose to analyse the multi-curve framework with a stochastic multiplicative spread approach and in the Gaussian HJM model for the risk-free curve described in Appendix A.

The idea of the stochastic spread framework is to model the multiplicative spread $\beta_t^j(v)$ as a function of the risk-free rate level and of an independent martingale $\mathcal{X}_t^j(v)$ with independent relative increments:

$$1+\delta F_X^{\text{CPN},j}(t,u,v) = \beta^j(F_X^D(t,u,v),\mathcal{X}_t^j(v))(1+\delta F_X^D(t,u,v)) \tag{7.17}$$

The dynamic of the risk-free curve will be based on a Brownian motion W_t which is $\mathcal{G}_t^1$-adapted. Let $\mathcal{G}_t^2$ be a filtration independent of $\mathcal{G}_t^1$. The random variable $\mathcal{X}_t^j(v)$ is a $\mathcal{G}_t^2$-adapted martingale in the $P^D(.,v)$ numeraire with independent relative increments and with $\mathcal{X}_0^j(v)=1$. Independent relative increments for $\mathcal{X}$ means that $\mathcal{X}_t/\mathcal{X}_s$ with $t>s$ is independent of $\mathcal{G}_s^2$. The filtration generated by $\mathcal{G}_t^1$ and $\mathcal{G}_t^2$ is denoted $\mathcal{F}_t$.

Note that due to the fact that $P^D(.,v)$ and $N_.$ are $\mathcal{G}_t^1$-adapted with $\mathcal{G}_t^1$ independent of $\mathcal{G}_t^2$, $\mathcal{X}_t$ is also a martingale in the $N_.$ numeraire.

Obviously one cannot chose any β^j and have a coherent framework. From its definition and hypothesis $\mathbf{I}^{\mathbf{CPN}}$, the quantity $1+\delta F_X^{\text{CPN},j}(t,u,v)$ is a martingale in the $P^D(.,v)$-numeraire as explained in Theorem 2.1. This requirement needs to be checked for each particular function β^j.

In this section we focus on a specific form of spread. The multiplicative spread is a function of the level of the risk-free rate, an independent random variable and a deterministic function x^j:

$$\beta^j(F_X^D(t,u,v),\mathcal{X}_t^j(v)) = \beta_0^j(v)\mathcal{X}_t^j(v)x^j(t,u,v)\left(\frac{1+\delta F_X^D(t,u,v)}{1+\delta F_X^D(0,u,v)}\right)^{a^j(v)}. \tag{7.18}$$

The multiplicative spread is the product of the independent part and of the dependent part written as an exponent function. The function $x^j(t,u,v)$ with $x^j(0,u,v)=1$ is deterministic and to be chosen in such a way that the martingale property is satisfied. A priori, the random variables $\mathcal{X}_t^j(v)$ and the coefficient $a^j(v)$ can be different for each index j and maturity date v. In the next sections we analyse futures and their options, which are linked to only one rate. Only one of those random variables and coefficients is used for each futures.

Note that when $a^j=0$ and $\mathcal{X}_t^j=\mathcal{X}_0^j$, we recover the constant multiplicative spread hypothesis $\mathbf{S0}^{\mathbf{CPN}}$.

In the applications, we will use for $\mathcal{X}_t^j(v)$ a martingale of the form

$$\mathcal{X}_t^j(v) = \exp\left(-X_t^{\mathcal{X},j}(v) - \frac{1}{2}\sigma_{\mathcal{X},j}^2(t,v)\right) \tag{7.19}$$

with $X^{\mathcal{X},j}$ normally distributed in the $P^D(.,v)$ numeraire with mean 0 and variance $\sigma^2_{\mathcal{X},j}(t,v)$. The volatility $\sigma_{\mathcal{X},j}$ is such that $\sigma_{\mathcal{X},j}(0,v)=0$. This variable could be, for example, the solution of a geometric Brownian motion equation.

The evolution of β^j_t can be written in terms of an earlier value β^j_s at s as

$$\beta^j_t(v)=\beta^j_s(v)\frac{\mathcal{X}^j_t(v)}{\mathcal{X}^j_s(v)}\frac{x^j(t,u,v)}{x^j(s,u,v)}\left(\frac{1+\delta F^D_X(t,u,v)}{1+\delta F^D_X(s,u,v)}\right)^{a^j(v)}.$$

In the Gaussian HJM framework, the risk-free forward rates are exponential martingales in the $P^D(.,v)$ numeraire as described in Lemma A.2. Using that result we have

$$\begin{aligned}
&\mathrm{E}^v\left[1+\delta F^{\mathrm{CPN},j}_X(t,u,v)\middle|\mathcal{F}_s\right]\\
&=\frac{x^j(t,u,v)}{x^j(s,u,v)}\beta^j_s(v)(1+\delta F^D_X(s,u,v))\exp\left(-\frac{1}{2}(1+a^j)a^j\alpha^2(s,t,u,v)\right)\\
&\quad\mathrm{E}^v\left[\frac{\mathcal{X}^j_t}{\mathcal{X}^j_s}\exp\left(-\alpha(s,t)(1+a^j)X^v_{s,t}-\frac{1}{2}\alpha^2(s,t,u,v)(1+a^j)^2\right)\middle|\mathcal{F}_s\right]\\
&=\frac{x^j(t,u,v)}{x^j(s,u,v)}(1+\delta F^{\mathrm{CPN},j}_X(s,u,v))\exp\left(-\frac{1}{2}(1+a^j)a^j\alpha^2(s,t,u,v)\right).
\end{aligned}$$

The remaining expected value in the second line is equal to 1. To obtain the equality, note that both $\mathcal{X}^j_t/\mathcal{X}^j_s$ and $X^v_{s,t}$ are independent of $\mathcal{F}_s$ and the conditional expected value becomes an expected value. Moreover due to the independence of $\mathcal{G}^1_t$ and $\mathcal{G}^2_t$, the expected value of the product is the product of the expected values. Both terms are martingales of initial value 1.

The quantity $1+\delta F^{\mathrm{CPN},j}_X(t,u,v)$ is a martingale only if

$$x^j(t,u,v)=\exp\left(\frac{1}{2}(1+a^j)a^j\alpha^2(0,t,u,v)\right). \tag{7.20}$$

Note that x^j is linked to the dependency parameter $a^j(v)$ and the dynamic of F^D_t through the volatility $\alpha(0,t,u,v)$.

7.8.1 Spread rate dependency

The general description in Equation (7.19) of multiplicative spread is split into two parts. The first, comprising the part with an exponent a^j, is the part of the spread that depends on the level of risk-free rates. This is the systematic part, similar to the part multiplied by a coefficient α in the previous section.

The results of this section can be compared to those in the previous section by linearising Equations (7.18) and (7.19). The approximation gives

$$1+\delta F^j(t)\simeq 1+\delta\left(F^j(0)-F^j(0)+a^j(F^D(t)-F^D(0))+\bar{\mathcal{X}}+F^D(t)\right)$$

for $1+\delta\bar{\mathcal{X}}=\mathcal{X}$. This means that the hypothesis **SMS** is similar to the one in the previous section for the a^j here equal to the α there and the $\bar{\mathcal{X}}$ here equal to the $\beta\mathcal{X}$ there. As δF^D and $\bar{\mathcal{X}}$ are small with respect to 1, the linearisation is fairly precise.

We repeat the graph on dependency of spread and rate level in Figure 7.4. We use the same initial levels $F^D(0)=2.00\%$, $F^j(0)=2.20\%$ and the horizon of one year and values a and σ in line with that of the previous section. The figures look remarkably similar. The dynamic between the two approaches are the same to the first order. For some products, the multiplicative spread approach facilitates obtaining explicit formulas as described below.

When the coefficient a^j is 0, there is no dependency of the spread on the rate. There is no systematic change of the spread level with the rate level. When $a>0$, the spread increases when the rate increases above its original level. When $a<0$, the opposite behaviour appears. The framework can cope with periods where the spread is positively correlated to the level of rates or periods where the spread is negatively correlated to the level of rates. Moreover, we have two independent processes X and $\mathcal{X}$ that drive the level of risk-free rate and the non-dependent part of the spread. The total volatility of the forward rate $\alpha_{Y,j}$ described below can be

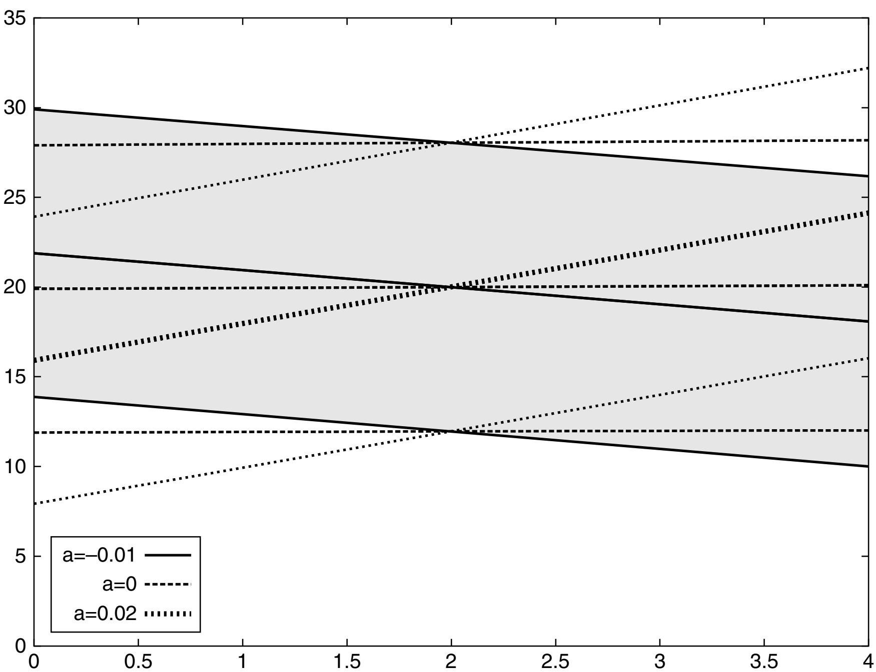


Figure 7.4 The additive spread $F_X^{\mathrm{CPN},j}(t,u,v)-F_X^D(t,u,v)$ for different levels of risk-free rate $F_X^D(t,u,v)$ in the multiplicative stochastic spread framework. The rate is in percent and the spread in basis points.

split into risk-free rate and the credit-related spread in any way required to fit the market historical data.

Note that the exponent a is on the quantity $1+\delta F_t^D$. This quantity is always positive as it is a ratio of assets. The quantity β^j is thus well defined for any value of a (positive or negative) and any financially possible value of F_X^D.

The independent stochastic part creates more possibility of changes in the spreads. This is why it was introduced. In the parsimonious HJM framework of Section 7.6, the spread is only a function of the level of rate.

7.8.2 Ibor rate dynamic

Using the spread dependency functions (7.18) and (7.19), the form of the independent martingale (7.20) and the dynamic of the forward risk-free rate described in Lemma A.2, the solution for the forward Ibor rate can be written in the $P_X^D(.,v)$ numeraire as

$$\begin{aligned}1+\delta F_X^{\mathrm{CPN},j}(\theta,u,v) &= \beta_0^j \mathcal{X}_\theta^j \frac{x^j(\theta)}{(1+\delta F_X^D(0,u,v))^a}(1+\delta F_X^D(\theta,u,v))^{1+a}\\ &= (1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\exp\left(-Y_\theta^{v,j}-\frac{1}{2}\alpha_j^2\right)\end{aligned}$$

with $Y_\theta^{v,j}=X_\theta^{\mathcal{X},j}+(1+a)X_{0,\theta,u,v}^v$ and $\alpha_j^2(\theta)=\sigma_{\mathcal{X},j}^2(\theta)+(1+a)\alpha^2(0,\theta)$. The definition of $X_{0,\theta,u,v}^v$ is given in Appendix A. The random variable $Y_\theta^{v,j}$ is normally distributed with mean 0 and variance $\alpha_{Y,j}^2$ in the $P^D(.,v)$ numeraire.

A similar equation can be written in the cash account numeraire using Lemma A.1:

$$1+\delta F_X^{\mathrm{CPN},j}(\theta,u,v)=(1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\exp\left(-Y_\theta^{j}-\frac{1}{2}\alpha_j^2\right)\gamma(0,\theta,u,v)^{1+a^j}$$

with $Y_\theta^j=X_\theta^{\mathcal{X},j}+(1+a)X_{0,\theta,u,v}$.

The Ibor forward dynamic is similar to the forward risk-free dynamic but with a different volatility. The prices of cap/floor in the framework have a form very similar to the prices of cap/floor in the standard one-curve Gaussian HJM framework.

Theorem 7.9 (Cap/floor prices) *In the Gaussian HJM with multiplicative stochastic spread model in the multi-curve framework, the price of a cap of strike K and expiry θ and accrual factor δ is given by*

$$C_0=P^D(0,v)\left((1+\delta F_X^{\mathrm{CPN},j}(0,u,v))N(\kappa+\alpha_j(\theta))-(1+\delta K)N(\kappa)\right)$$

with

$$\kappa=\frac{1}{\alpha_j(\theta)}\left(\ln\left(\frac{1+\delta F_X^{\mathrm{CPN},j}(0,u,v)}{1+\delta K}\right)-\frac{1}{2}\alpha_j^2(\theta)\right).$$

7.8.3 STIR futures

The pricing of STIR futures in the multiplicative stochastic spread framework is given in the following theorem.

Theorem 7.10 (STIR futures pricing in the multiplicative spread framework). *Let $0 \le t \le t_0 \le u < v$. In the Gaussian HJM with multiplicative stochastic spread multi-curve framework (hypotheses* ***D, I*** *and* ***SMS****) the price of the futures fixing in t_0 for the period $[u, v]$ with accrual factor δ is given at t by*

$$\Phi_t^j = 1 + \frac{1}{\delta} - \frac{1}{\delta}(1 + \delta F_X^{\mathrm{CPN},j}(t, u, v))\gamma^{1+a}(t, t_0, u, v). \tag{7.21}$$

Proof: Using the generic pricing futures price process theorem

$$\Phi_t^j = \mathrm{E}^N\left[1 - F_X^{\mathrm{CPN},j}(t_0, u, v)\middle| \mathcal{F}_t\right]$$

where $\mathrm{E}^N[]$ is the cash account numeraire expectation, the future price can be written as

$$1 - F_X^{\mathrm{CPN},j}(t_0, u, v) = 1 + \frac{1}{\delta} - \frac{1}{\delta}\frac{\mathcal{X}_{t_0}^j}{\mathcal{X}_t^j}\frac{x(t_0)}{x(t)}\beta_t^j\frac{(1 + \delta F_X^D(t_0, u, v))^{1+a}}{(1 + \delta F_X^D(t, u, v))^a}$$

The important part in the expected value is, using Lemma A.1,

$$\begin{aligned}(1 + \delta F_X^D(t_0, u, v))^{1+a} &= (1 + \delta F_X^D(t, u, v))^{1+a}\\ &\times \exp\left(-(1+\alpha)X_{t,t_0} - \frac{1}{2}(1+a)^2\alpha^2(t, t_0)\right)\frac{x(t)}{x(t_0)}\gamma^{1+a}(t, t_0, u, v)\end{aligned}$$

with the exponential term having an expected value of 1. This gives

$$\begin{aligned}&\mathrm{E}^{\mathbb{N}}\left[1 - F_X^{\mathrm{CPN},j}(t_0, u, v)\middle| \mathcal{F}_t\right]\\ &= 1 + \frac{1}{\delta} - \frac{1}{\delta}\beta_t^j\frac{(1 + \delta F_X^D(t, u, v)(v))^{1+a}}{(1 + \delta F_X^D(t, u, v))^a}\gamma^{1+a}\\ &= 1 + \frac{1}{\delta} - \frac{1}{\delta}(1 + \delta F_X^{\mathrm{CPN},j}(t, u, v))\gamma^{1+a}\end{aligned}$$

where we have used that $\mathcal{X}_{t_0}/\mathcal{X}_t$ and X_{t,t_0} are $\mathcal{F}_t$ independent and independent of each other. □

Note that the pricing formula reduces to the one proposed in Section 2.6 in the deterministic spread hypothesis when $a = 0$. The volatility of the independent part of the spread $\mathcal{X}^j$ has no impact on the futures price.

The price of a futures on the Ibor rate is reduced to the price of a forward adjusted by the convexity adjustment on the risk-free rate and by the parameter describing the spread dependency on risk-free rates.

The futures will have different convexity adjustments for the same forward rate dynamic, this is the same volatility α_Y, but different split between the risk-free part and the independent part. At the extreme, when $a = -1$, the Ibor rate is independent of the risk-free rate and there is no convexity adjustment.

The price formula also allows us to write the forward rate as a function of the futures price

$$F_X^{\text{CPN},j}(t,u,v) = \gamma(t,t_0,u,v)^{-1-a}\left(\frac{1}{\delta} - (1-\Phi_t^j)\right) - \frac{1}{\delta}.$$

7.8.4 STIR futures options – margin

In this section we analyse the options on STIR futures with continuous margining. The margined futures options are traded on LIFFE (USD, EUR, GBP, CHF) and Eurex (EUR). There is a margining process on the option itself similar to the margining process on the underlying futures. Let θ be the option expiry date and K its strike price. We also use the notation $\tilde{K} = 1 - K$, which we call strike rate. For the futures itself, we use the same notation as in the previous section.

Due to the margining process on the option, the price of the European option with margining is

$$\mathrm{E}^N\left[(\Phi_\theta - K)^+\right].$$

The futures options usually have an American feature. Due to the margining process, the American options have the same price as the European options. This general observation for options with continuous margining can be found in Chen and Scott (1993). The proof of the equality is very short and is repeated below. A lower bound for the European call can be obtained using the Jensen inequality on the convex function $(.)^+$:

$$C_0 = \mathrm{E}^N\left[(\Phi_\theta - K)^+\right] \geq \left(\mathrm{E}^N\left[\Phi_\theta - K\right]\right)^+ = (\Phi_0 - K)^+.$$

The value of the European call is always above the intrinsic value. It is never optimal to exercise the option early.

Theorem 7.11 (Option with continuous margining in the stochastic multiplicative spread framework) *Let $0 \leq \theta \leq t_0 \leq u < v$. The price of a STIR futures call option (European or American) of expiry θ and strike K with continuous margining in the Gaussian HJM with multiplicative stochastic spread multi-curve framework is given in 0 by*

$$C_0 = \frac{1}{\delta}\left(\left(1+\delta\tilde{K}\right)N(-\kappa_\gamma) - \left(1+\delta F_X^{\text{CPN},j}(0,u,v)\right)\right.$$
$$\left.\times\gamma(0,t_0)^{1+a}N(-\kappa_\gamma - \alpha_{Y,j}(\theta))\right)$$

where κ_Y is defined by

$$\kappa_Y = \frac{1}{\alpha_{Y,j}(\theta)}\left(\ln\left(\frac{1+\delta F_X^{\mathrm{CPN},j}(0,u,v)}{1+\delta\tilde{K}}\gamma(0,t_0)^{1+a}\right) - \frac{1}{2}\alpha_{Y,j}^2(\theta)\right).$$

The price of a STIR futures put option is given by

$$P_0 = \frac{1}{\delta}\left((1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\gamma(0,t_0)^{1+a}N(\kappa_Y+\alpha_Y) - (1+\delta\tilde{K})N(\kappa_Y)\right)$$

Proof: The futures price is given in Theorem 7.10. Using the generic pricing theorem we have

$$\begin{aligned}
C_0 &= \mathrm{E}^N\left[((\Phi_\theta - K)^+)\right]\\
&= \frac{1}{\delta}\mathrm{E}^N\Bigg[\Bigg((1+\delta\tilde{K}) - \gamma(\theta,t_0)^{1+a}\beta_0\mathcal{X}_\theta(1+\delta F_X^D(0,u,v))\gamma(0,\theta)^{1+a}\\
&\qquad\times\ \exp\left(-(1+a)X_{0,\theta} - \frac{1}{2}(1+a)^2\alpha(0,\theta)^2\right)\Bigg)^+\Bigg].
\end{aligned}$$

Using the form (7.20) for $\mathcal{X}^j$ and the definition of Y_θ^j in Section 7.8.2, the quantity in the parenthesis is positive when

$$\gamma(0,t_0)^{1+a}(1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\exp\left(-Y_\theta^j - \frac{1}{2}\alpha_{Y,j}^2\right) < 1+\delta\tilde{K},$$

i.e. it is positive when $Y_\theta^j > \alpha_j\kappa_Y$.

The price is then given by

$$\begin{aligned}
C_0 &= \frac{1}{\delta}\mathrm{E}^{\mathbb{N}}\Bigg[\mathbb{1}_{\{Y^j>\alpha_{Y,j}\kappa_Y\}}\Bigg((1+\delta\tilde{K}) - (1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\gamma(0,t_0)^{1+a}\\
&\qquad\times\ \exp\left(-Y_\theta^j - \frac{1}{2}\alpha_j^2\right)\Bigg)\Bigg]\\
&= \frac{1}{\delta}\frac{1}{\sqrt{2\pi}}\int_{y>\kappa_Y}\Bigg((1+\delta\tilde{K})\\
&\quad -(1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\gamma(0,t_0)^{1+a}\exp\left(-\alpha_{Y,j}y - \frac{1}{2}\alpha_j^2\right)\Bigg)\exp\left(-\frac{1}{2}y^2\right)dy\\
&= \frac{1}{\delta}\left((1+\delta\tilde{K})N(-\kappa_Y) - (1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\gamma(0,t_0)^{1+a}N(-\kappa_Y-\alpha_j)\right)
\end{aligned}$$

□

The pricing formula is similar to a Black formula with a price $1+\delta\tilde{K}$, a forward $1+\delta F^j$ and an adjustment on the forward given by the factor γ^{1+a}. The structure of the option formula is not very different from the one of the cap/floor in Theorem 7.9. The difference being the adjustment and the exercise boundary κ_γ.

Note that the adjustment factor $\gamma(0,t_0)^{1+\alpha}$ which appears in the price formula and in the formula for κ_γ is the same as the one used in the futures price. This factor can be obtained directly from the swap curve, the market price of futures and Equation (7.22) without calibrating the model itself.

Similarly the volatility parameter $\alpha_j(\theta)$ is the same as the one used in the cap/floor formula of Theorem 7.9. If the price of the cap/floor is available, the total volatility quantity α_j can be obtained directly from it. Even if a multi-factor model with time-dependent parameters is used, there is no need to calibrate each parameter individually to price the options on futures; it is enough to obtain the total volatility quantity $\alpha_j(\theta)$. Note that in practice this simplified calibration approach is valid only if the corresponding cap/floor is traded. That would not be the case for mid-curve options as 'mid-curve' cap/floors are not traded.

A pricing of options on futures coherent with swaps, futures and cap/floor can be obtained from the above instrument quotations in the proposed framework with very light calibration. The required calibration is not to fit all the model parameters but only to compute quantities deduced from the model parameters ($\gamma^{1+\alpha}$ and $\alpha_j(\theta)$) which can be read almost directly from the market quotes.

The formula in Theorem 7.11 can be written in term of futures price and strike as

$$C_0 = \left(1 - K + \frac{1}{\delta}\right) N(-\kappa_\gamma) - \left(1 - \Phi_t + \frac{1}{\delta}\right) N(-\kappa_\gamma - \alpha_j(\theta)).$$

In this way the futures options price is directly written in terms of its underlying futures.

Example

Using the above formulas we provide a sensitivity comparison between put/call on futures and cap/floor on Ibor rates. The calibration was done as described above using EUR market data from 9 August 2013. The quantity γ^{1+a} is obtained from the forward/futures difference and the quantity α_j is obtained from the cap/floor prices.

The underlying futures is the June 14 Euribor futures, this is ERM4. The futures expires in slightly more than ten months and its underlying rate has a maturity in slightly more than one year and one month.

The options are striked at 99.50%. This is close to at-the-money (ATM) with the forward rate for the same period being approximatively 0.455%. The results are displayed in Table 7.1 for ten contracts on futures options and ten million notional for the cap/floor. The table represents the market quote sensitivities, also called key rate sensitivities, in the multi-curve framework. The discounting curve – denoted DSC in the table – is calibrated using OIS quotes and the forward curve – denoted FWD in the table – is calibrated using Euribor three months fixing and fixed versus Euribor three months swaps. The call/put on futures have a sensitivity to the discounting market quotes, even if only the forward rates appear in the formulas of Theorem 7.11. This dependency comes from the curve calibration as the forward curve depends on OIS and Euribor swaps market quotes.

Even if the two options are exercised in exactly the same conditions, that is the call is exercised if and only if the floor is exercised, the current moneyness is not exactly the same. One strike should be compared to the forward rate and the other to the futures price, which is not exactly one minus the forward rate. The difference comes from the different contractual obligations between the two instruments: one is margined, like the futures itself, and the other is not.

7.8.5 STIR futures options – premium

In this section we analyse the price of STIR futures options with up-front premium payment. Those options are traded for Eurodollar on CME and SGX and for JPY Libor on SGX. Let θ be the option expiry date and K its strike price. For the futures itself, we use the same notation as in the previous sections.

The price of options on futures with up-front premium payment in the Gaussian HJM model and the one-curve framework was first proposed in Henrard (2005). The extension to the multi-curve framework with deterministic spread is described in Quantitative Research (2012b). The results described here were first proposed in Henrard (2013c).

The premium is paid up-front and the value of the European call option is

$$C_0 = N_0 \, \mathrm{E}^{\mathbb{N}} \left[N_\theta^{-1} (\Phi_\theta - K)^+ \right].$$

The main stochastic processes and constant of the HJM model are described in Appendix A. The interaction between the stochastic parts of $X_{0,\theta}^N$ and Y_θ (see

Table 7.1 Call/Put and Floor/Cap comparison

	Call		Floor		Put		Cap	
Tenor	**DSC**	**FDW**	**DSC**	**FDW**	**DSC**	**FDW**	**DSC**	**FDW**
3M	0.00	0.10	0.00	0.10	0.00	−0.16	0.00	−0.16
6M	0.00	0.00	0.00	0.00	-0.01	0.00	0.00	0.00
9M	0.37	-191.89	0.38	−199.56	−0.59	312.18	−0.58	305.42
1Y	−0.57	244.34	−0.49	254.11	0.55	−397.51	0.75	−388.90
2Y	−0.05	40.47	−0.04	42.09	0.04	−65.84	0.06	−64.42

The sensitivities are to the market quotes and scaled to one basis point.

Section 7.8.2 for its definition) is given by

$$\alpha_{Nj}(t) = (1+a^j)\int_0^t (\nu(\tau,u)-\nu(\tau,v))\cdot\nu(\tau,t)d\tau.$$

There is no part dependent on $\sigma_{\mathcal{X}}$ as $\mathcal{X}_t^j$ is independent of W_t.

The variance/covariance matrix of the random variable $(X_{0,\theta}^N, Y_\theta^j)$ is given by

$$\Sigma = \begin{pmatrix} \alpha_N^2(\theta) & \alpha_{Nj}(\theta) \\ \alpha_{Nj}(\theta) & \alpha_j^2(\theta) \end{pmatrix}.$$

Theorem 7.12 (Options on STIR futures with premium) *Let $0 \le \theta \le t_0 \le u < v$. The value of a STIR futures call (European) option of expiry θ and strike K with up-front premium payment in the stochastic multiplicative spread framework with Gaussian HJM model on the risk-free curve is given in 0 by*

$$\begin{aligned} C_0 = \frac{1}{\delta}P^D(0,\theta)\Bigg((1+\delta\tilde{K})N\left(-\kappa_\gamma+\frac{\alpha_{Nj}}{\alpha_j}\right) \\ -(1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\gamma(0,t_0)^{1+\alpha}\exp(-\alpha_{Nj})N\left(-\kappa_\gamma-\alpha_j+\frac{\alpha_{Nj}}{\alpha_j}\right)\Bigg)\end{aligned}$$

where κ_γ is defined in Theorem 7.11.

The price of a STIR futures put option is given by

$$\begin{aligned} P_0 = \frac{1}{\delta}P^D(0,\theta)\Bigg((1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\gamma(0,t_0)^{1+\alpha}\exp(-\alpha_{Nj})N\left(\kappa+\alpha_j-\frac{\alpha_{Nj}}{\alpha_j}\right) \\ -(1+\delta\tilde{K})N\left(\kappa-\frac{\alpha_{Nj}}{\alpha_j}\right)\Bigg)\end{aligned}$$

Proof: Using the same formulas as in the previous section,

$$\begin{aligned} \delta C_0 = P^D(0,\theta)\,\mathrm{E}^{\mathbb{N}}\Bigg[\exp\left(X_t^N-\frac{1}{2}\alpha_N^2(\theta)\right) \\ \times \mathbb{1}_{\{Y>\sigma_Y\kappa_\gamma\}}\Bigg((1+\delta\tilde{K})-(1+\delta F_X^{\mathrm{CPN},j}(0,u,v))\gamma(0,t_0)^{1+\alpha} \\ \times \exp\left(-Y_\theta-\frac{1}{2}\alpha_j^2(\theta)\right)\Bigg)\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= P^D(0,\theta)\frac{1}{2\pi\sqrt{|\Sigma|}}\int_{x_2>\alpha_{Y,j}\kappa_\gamma}\quad\int_{\mathbb{R}}\exp\left(x_1-\frac{1}{2}\alpha_N^2(\theta)\right)\\
&\quad\times\exp\left(-\frac{1}{2}x^T\Sigma^{-1}x\right)dx_1\\
&\quad\times\left((1+\delta\tilde{K})-(1+\delta F_X^{\text{CPN},j}(0,u,v))\gamma(0,t_0)^{1+\alpha}\exp\left(-x_2-\frac{1}{2}\alpha_j^2(\theta)\right)\right)dx_2\\
&= P^D(0,\theta)\left((1+\delta\tilde{K})N\left(-\kappa+\frac{\alpha_{Nj}}{\alpha_j}\right)\right.\\
&\quad\left.-(1+\delta F_X^{\text{CPN},j}(0,u,v))\gamma(0,t_0)^{1+a}\exp(-\alpha_{Nj})N\left(-\kappa-\alpha_j+\frac{\alpha_{Nj}}{\alpha_j}\right)\right)
\end{aligned}
$$

The computation of the inner integral is done using Lemma A.4. □

As for the options with margin, most of the model constants can be deduced from futures and cap/floors; the exception in this case is α_{Nj}. The convexity adjustment γ^{1+a} is given by the futures and used in the computation of κ_γ. The cap/floor volatility α_j is used directly and in κ_γ computation.

The cap formula can be written directly as a function of the futures price with

$$
\begin{aligned}
C_0 = P^D(0,\theta)&\left(\left(1-K+\frac{1}{\delta}\right)N\left(-\kappa_\gamma+\frac{\alpha_{Nj}}{\alpha_j}\right)\right.\\
&\left.-\left(1-\Phi_0^j+\frac{1}{\delta}\right)\exp(-\alpha_{Nj})N\left(-\kappa_\gamma-\alpha_j+\frac{\alpha_{Nj}}{\alpha_j}\right)\right).
\end{aligned}
$$

7.8.6 Ceci n'est pas une option (This is not an option)

Ceci n'est pas une pipe. (This is not a pipe)

La trahison des images, René Magritte, 1928.

Even if new options on STIR futures have been added recently to the exchange traded offering, the offer is still in some way incomplete as described earlier. The equivalent of swaption is certainly a missing instrument in the exchange traded offering[1]. It is not an option for traders to trade the equivalent of swaptions in the futures world, hence the title of this section.

The filling of the missing offering could be achieved through options on deliverable swap futures or through the options on packs and bundles. To our knowledge none of these products are offered on any exchange. Given the push for more standardisation and exchange traded products, the question is which exchange will be the first to offer such a product?

[1] Some options on swap futures exists, but they are mostly illiquid.

Messieurs les Anglais, tirez les premiers!

Attributed by Voltaire to the (French) Comte d'Auteroche at the battle of Fontenoy, 1745

The XXIst century corporate warfare is probably not as courteous as the XVIIIth century real warfare and there will probably be no similar offer, but the question is open: who will shoot first?

On the other side the exchange traded offering is in some sense larger than the OTC one. There is no equivalent in the OTC world to the mid-curve options. These options are interesting from a calibration perspective as they give the view of the market on the short term volatility of forward rates further down the curve. The availability of even a more complete offer would allow a larger set of financial risks to be hedged, and enable quantitative analysts to better calibrate their models.

To obtain an explicit formulas for options on packs and bundles, we add an *separability hypothesis* on the stochastic part of the rate evolution $Y^j_\theta(v)$.

Sep: The random variables $Y^j_\theta(v)$ satisfy

$$Y^j_\theta(v) = \alpha_j(\theta, v)\tilde{Y}^j_\theta \tag{7.22}$$

with $\tilde{Y}^j_\theta$ standard normally distributed and identical for all v.

We will give below financial circumstances where this hypothesis is satisfied. The condition is inspired by the separability condition described in Appendix A. With this hypothesis all periods randomness behave similarly and the model is essentially one-factor for each forward curve. There is still a multi-curve framework with stochastic spread, but each forward curve made of the base discounting curve and its own spread acts like in a one-factor model, with all the stochasticity summarised by only one random variable: $\tilde{Y}^j_\theta$.

The options we want to price are margined options on packs or on bundles. The packs and bundles are described in Section B.4. Both results can be written in the same way. Let n be the number of futures in the pack or bundle, that is $n = 4$ for a pack and $n = 4 \times k$ for a k-year bundle. The option expiration date is denoted θ and the strike is denoted $n \times K$. We write the strike in the form $n \times K$ to emphasise that there are n futures composing the pack or bundle and they have an average strike of K. In this way, the final result is easier to compare to the one futures result. The dates related to the n underlying futures are denoted t_i for the last trading dates and $[u_i, v_i]$ for the corresponding Ibor periods with accrual factors δ_i.

For a call, the quantity we want to compute is

$$C_0 = \mathrm{E}^N\left[\left(\sum_{i=1}^n \Phi^j_\theta(v_i) - nK\right)^+\right].$$

Using hypothesis **Sep**, the result on the futures price and on the dynamic of $1 + \delta F_X^{\mathrm{CPN},j}(\theta, u, v_i)$ in the cash account numeraire, the quantity in the parentheses is equal to

$$\sum_{i=1}^{n} \frac{1}{\delta_i}(1+\delta_i \tilde{K}) - \sum_{i=1}^{n} \frac{1}{\delta_i}(1+\delta_i F_X^{\mathrm{CPN},j}(\theta, u, v_i))\gamma^{1+a}(\theta, t_i, u_i, v_i)$$
$$= \sum_{i=1}^{n} \frac{1}{\delta_i}(1+\delta_i \tilde{K}) - \sum_{i=1}^{n} \frac{1}{\delta_i}(1+\delta_i F_X^{\mathrm{CPN},j}(0, u, v_i))$$
$$\times \exp\left(-\alpha_{Y,j}(\theta, v_i)\tilde{Y}_\theta^j - \frac{1}{2}\alpha_{Y,j}^2(\theta, v_i)\right)\gamma^{1+a}(0, t_i, u_i, v_i)$$

In a way very similar to what is done for swaptions in Henrard (2003) and in Section 7.3.3, we define κ as the solution of

$$\sum_{i=1}^{n} \frac{1}{\delta_i}(1+\delta_i \tilde{K}) - \sum_{i=1}^{n} \frac{1}{\delta_i}(1+\delta_i F_X^{\mathrm{CPN},j}(0, u, v_i))$$
$$\times \exp\left(-\alpha_{Y,j}(\theta, v_i)\kappa - \frac{1}{2}\alpha_{Y,j}^2(\theta, v_i)\right)\gamma^{1+a}(0, t_i, u_i, v_i) = 0$$

Like in the above-mentioned result it can be proved that the solution of the equation is unique and non singular. Any numerical method will solve it very efficiently.

The quantity in the expected value above is positive when $\tilde{Y}_\theta^j > \kappa$. Using that result and computing the expected value of the quantity dependent on a normal standard random variable using the usual integral approach, one gets the following theorem.

Theorem 7.13 *The price of the option on pack or bundle, in the multi-curve framework* ***D–I**$^{\mathbf{CPN}}$ *with multiplicative stochastic spread* ***SMS*** *under the separability hypothesis* ***Sep*** *is given by*

$$\sum_{i=1}^{n} \frac{1}{\delta_i}(1+\delta_i \tilde{K})N(-\kappa)$$
$$- \sum_{i=1}^{n} \frac{1}{\delta_i}(1+\delta_i F_X^{\mathrm{CPN},j}(0, u, v_i))\gamma^{1+a}(0, t_i, u_i, v_i)N(-\kappa - \alpha_{Y,j}(\theta, v_i)).$$

where κ is the solution of Equation (7.24).

The ingredients of this formula are, like for the options on futures, the convexity coefficient γ^{1+a} and the volatilities $\alpha_{Y,j}$. Here we require several of them. Several futures and (mid-curve) cap/floors are required to calibrate the different parts of the formula.

To obtain a particular case of the separability condition **Sep**, we base the hypothesis on the features of the one-factor extended Vasicek (or Hull-White) model.

SepV: The HJM volatility ν satisfies

$$\nu(\tau,u)-\nu(\tau,v)=H(u,v)g(\tau) \tag{7.23}$$

where $H:\mathbb{R}^2\to\mathbb{R}$ and the spread random variable $X^{\mathcal{X}}$ satisfies

$$X_\theta^{\mathcal{X}}(u,v)=H(u,v)\int_0^\theta g^{\mathcal{X}}(\tau)\cdot dW_\tau^{\mathcal{X}} \tag{7.24}$$

where H is the same function as above and $W_\tau^{\mathcal{X}}$ is a Brownian motion independent of W_τ.

7.9 Libor market model on multiple curves

The results described in this section were originally presented in Mercurio (2010c). The starting point of the approach is that, as described in Theorem 2.1, the forward rates $F_X^{\mathrm{CPN},j}(.,u,v)$ are martingales in the $P_X^D(.,v)$-numeraire. The modelling of the forward rates using an LMM approach can be done like in the one-curve approach, except that on top of the Libor forward rates, one also needs to model the risk-free curve.

7.9.1 The model

We describe the model for a unique index j. As for any LMM like approach, one selects a set of dates $0<t_0<\cdots<t_M$ which are j-period apart, that is $t_{k+1}=t_k+j$.

Note that in Mercurio (2010c) the quantity he calls $L_k^i(t)$ corresponds to our $F_X^{\mathrm{CPN},i}(t,t_{k-1},t_k)$. The notation is different but the meaning is the same. Similarly what he denotes $F_k^D(t)$ we denote $F_X^D(t,t_{k-1},t_k)$.

The equation for the forward (Ibor) rates is, in the $P^D(.,t_k)$ numeraire,

$$dF_X^{\mathrm{CPN},j}(t,t_{k-1},t_k)=\sigma_k(t)F_X^{\mathrm{CPN},j}(t,t_{k-1},t_k)dZ_k^k(t)$$

where Z_k^k is a Brownian motion associated to the relevant numeraire.

Similarly each forward risk-free rate follows a geometric Brownian motion in the associated $P^D(.,t_k)$ numeraire,

$$dF_X^D(t,t_{k-1},t_k)=\sigma_k^D(t)F_X^D(t,t_{k-1},t_k)dZ_k^{k,D}(t)$$

where $Z_k^{k,D}$ is a Brownian motion associated to the relevant numeraire.

The correlations between the different Brownian motions are given by

$$dZ_k^k(t)dZ_l^l(t) = \rho_{k,l}dt$$
$$dZ_k^{k,D}(t)dZ_l^k(t) = \rho_{k,l}^D dt$$
$$dZ_k^{k,D}(t)dZ_l^{l,D}(t) = \rho_{k,l}^{D,D} dt$$

Like in the standard LMM, it is useful to write all the equations under the same numeraire. The change of numeraire is similar to the one-curve one. We give the equations only for the case where the last time, t_M, is used as numeraire:

$$dF_X^{\text{CPN},j}(t, t_{k-1}, t_k) = \sigma_k(t)F_X^{\text{CPN},j}(t, t_{k-1}, t_k)$$
$$\left(-\sum_{h=k+1}^{m} \frac{\rho_{k,h}^D \delta_h \sigma_h^D(t) F_X^D(t, t_{k-1}, t_k)}{1 + \delta_h F_X^D(t, t_{k-1}, t_k)} dt + dZ_k^m(t)\right)$$
$$dF_X^D(t, t_{k-1}, t_k) = \sigma_k^D(t)F_X^D(t, t_{k-1}, t_k)$$
$$\left(-\sum_{h=k+1}^{m} \frac{\rho_{k,h}^{D,D} \delta_h \sigma_h^D(t) F_X^D(t, t_{k-1}, t_k)}{1 + \delta_h F_X^D(t, t_{k-1}, t_k)} dt + dZ_k^{m,D}(t)\right).$$

7.9.2 Pricing caplets and floorlets

As the main hypothesis is a log-normal equation for the forward rates in the forward numeraire, the pricing of caplet and floorlet is straightforward and corresponds, like in the one-curve framework, to the Black price. The present value formulas are presented in Section 7.4.

Like most of the term structure models in the multi-curve framework, this approach has many parameters and most of them cannot be calibrated using standard market instruments. There are no optional instruments on risk-free rates to calibrate σ_k^D and no real hybrid instruments to estimate the correlations ρ^D and ρ^{DD}.

7.9.3 Pricing swaptions

Like in the one-curve LMM, analytical approximation for the implied volatility of swaptions can be derived in this approach. To describe the swap underlying the swaption, we use the same notation as in Section 2.4.

The forward rate S_t^j can be written as a linear combination of the forward rates with

$$S_t^j = \sum_{i=1}^{n} \omega_i(t) F_X^{\text{CPN},j}(t, t_{i-1}, t_i)$$

and

$$\omega_i(t) = \frac{P_X^D(t, t_i)\delta_i}{\sum_{i=1}^{\tilde{n}} P_X^D(t, \tilde{t}_i)\tilde{\delta}_i}.$$

The weights ω are not constants but depend on the risk-free discount factors. The standard initial freeze approximation technique is used. The weights are replaced by their initial values $\omega_i(0)$. The approximation of the swap rate is

$$S_t^j \simeq \sum_{i=1}^{n} \omega_i(0) F_X^{\mathrm{CPN},j}(t, t_{i-1}, t_i).$$

This approximation leads to the equation for the swap rate

$$dS_t^j \simeq \sum_{i=1}^{n} \omega_i(0)\sigma_i(t) F_X^{\mathrm{CPN},j}(t, t_{i-1}, t_i) dZ_k^m(t) + \cdots dt.$$

We know that in the present value of a basis point A_t numeraire the swap rate is a martingale, so we don't worry about the drift term at this point.

We would like to obtain a log-normal equation for the swap rate of the type

$$dS_t^j = S_t^j \sigma_S(t) dZ_S(t). \tag{7.25}$$

We equate the instantaneous quadratic variations in the previous two equations and we obtain

$$(S_t^j \sigma_S(t))^2 = \sum_{i,k=1}^{n} \omega_i(0)\omega_k(0)\sigma_i(t)\sigma_k(t) F_X^{\mathrm{CPN},j}(t, t_{i-1}, t_i) F_X^{\mathrm{CPN},j}(t, t_{k-1}, t_k)\rho_{i,k}.$$

Using the initial freeze technique again, this time on $F_X^{\mathrm{CPN},j}$ and S_t^j, we obtain the following volatility

$$\sigma_S(t) = \frac{\sqrt{\sum_{i,k=1}^{n} \omega_i(0)\omega_k(0)\sigma_i(t)\sigma_k(t) F_X^{\mathrm{CPN},j}(0, t_{i-1}, t_i) F_X^{\mathrm{CPN},j}(0, t_{k-1}, t_k)\rho_{i,k}}}{S_0^j}.$$

From Equation (7.26), the price of the swaption is obtained with the usual Black formula as

$$P = A_t \mathrm{Black}(S_0^j, K, \sigma_S(t)).$$

8 Collateral and Funding

8.1 Introduction

The post-crisis multi-curve framework, as described in this book, relies on credit risk-free discounting curves and does not incorporate collateral and funding.

The goal in this chapter is to extend the multi-curve framework to incorporate collateral and funding. We prove generic collateral results that extend those presented in Kijima et al. (2009), Macey (2011), Pallavicini et al. (2012) and Piterbarg (2012) and specialise them to the multi-curve framework. Using techniques similar to the ones developed in Henrard (2013c) for the multi-curve framework with stochastic spread we propose explicit formulas to evaluate the impact of the collateral corrections or convexity adjustments. Our analysis of collateral framework has also benefited from the insights of McCloud (2013b), Ametrano and Bianchetti (2013), Pallavicini and Brigo (2013) and Fujii and Takahashi (2013).

The question of the impact of collateral on pricing of derivatives products has certainly been analysed in the literature, especially since the beginning of the 2007 crisis. Some research predates the crisis and we cite in that category Johannes and Sundaresan (2007). Obviously with the increasing importance of collateral and the widening of spreads between different rates, the literature on the subject has boomed since the start of the crisis and we cite as a short list (Fujii et al. 2010a, Fujii et al. 2010b, Fujii and Takahashi 2010, Brigo et al. 2011, Piterbarg 2010, Pallavicini et al. 2011, Brigo et al. 2012a, Brigo et al. 2012b, Castagna 2012a, McCloud 2013a, Crépey 2012 and Han et al. 2013).

Our first goal is to compute the theoretically consistent amount to be transferred for a given collateral agreement. By theoretically consistent amount we mean that the amount transferred is the exact compensation for the rest of the contract, including the impact of the collateral itself. If at any date a party breaks the contract, the contract ceases to exist and the collateral is kept by the other party; the two parties do not occur any financial loss. There is a feedback loop between the collateral contract details and the value computed. The collateral contract creates extra cash-flows with respect to the stand alone derivative contract. The daily payments of collateral and potentially of interest have to be taken into account when computing the 'value'. The term of 'value' has to be reviewed in line with the above description. As explained at the end of the introduction, we will use the term 'quote' to describe the theoretical number computed.

Some of the above-mentioned articles and books propose frameworks significantly more general than the one we describe below, in particular regarding funding and credit risk. We concentrate mainly on the collateral impact. On the other hand our definition of collateral situation is *more general* than the ones used in the above literature as it incorporates not only collateral, repo, no-collateral and futures but also proportional under- and over-collateral, collateral by other assets, with and without interest, collateral in foreign currency and collateral with haircut. Our definition of collateral is given in the first section. The specificity of our approach is also to describe the impacts of those generic results for the multi-curve framework and to provide explicit results on the convexity adjustments on the Ibor forward rates.

The starting point of the multi-curve framework is the existence of (credit) risk-free fixed cash-flows (and the associated rates). Part of the collateral approach can be described in theory without supposing the existence of such rates; see for example Macey (2011) and Piterbarg (2012). Pallavicini and Brigo (2013) suppose the existence of a risk-free rate but obtain pricing formulas based on the market quoted instruments only. It is probably debatable from a philosophical point of view if risk-free rates exist and if we need them at all. Nevertheless from a practical point of view, in some cases it is easier to suppose they exist, even if we don't suppose they are used in actual market contracts. We will do so when it simplifies the approach.

The multi-curve framework is the foundation for pricing Ibor related instruments. Most of the liquid instruments are nowadays collateralised in one way or another. It is thus paramount to have a detailed framework that combines collateral and multi-curve frameworks. We provide such a framework here. Similar definitions related to a multi-curve framework with collateral can be found in Fries (2013).

The combination of collateral and multi-curve framework has often been reduced to an 'OIS discounting' formula. I will not use that term here as it is, in my opinion, a misleading term. A collateral at overnight is not enough to justify the use of the technique. It requires several other conditions, including OISs themselves to be collateralised with the same rule, a condition not always satisfied in practice. The exact requirements for 'OIS discounting' are detailed later in the chapter.

It is important to insist that our approach to collateralised instrument valuation supposes a feedback loop. The quote computed from the perfect collateral framework should be used to transfer the collateral amount. If the collateral transferred is computed using a different rule, like an old one-curve framework valuation or an accounting rule, then it does not fit our definition of collateral. We are describing a world where the computations of quantitative finance are imposed on the rest of the process, and in particular on back-offices and accounting departments. In other words, our collateral world is ‘perfect’ if quants are ruling the world!

An approach with only one collateral rule for each currency is not general enough to cover all practical cases. Collateral with foreign currencies, often described in the literature, is only one special case. Collateral with bonds, leading to ‘collateral square’-type results, is often used; some contracts impose an haircut. Modified collateral rates, like overnight plus a fixed spread, are also found in practice. This chapter provides answers for those different cases.

The collateral practice leads to a somewhat confusing terminology. In this chapter we use a term which has to be has explained in the collateral framework. Due to collateral rules, when an asset is purchased with perfect collateral, the amount obtained from the market quote is paid but returned immediately through the collateral, resulting in a net zero payment at the start of the trade. Only the change of the figure and interests on the collateral account are exchanged (daily). We will use the more neutral term *quote* or *collateral quote* to designate that quantity. Personally I prefer the term *reference number* or *reference index* but it is a little bit too long to use. The quote could be shifted by an arbitrary figure without impact on the economy if the reference amount for the collateral was similarly adapted. This consideration on prices was originally proposed in Henrard (2006b) for futures and later adapted for collateral in Piterbarg (2012).

STIR futures are an important instrument to include in any multi-curve and collateral framework discussion. Not only is it a very liquid instrument and an important source of market information, but from a technical point of view its pricing can be viewed as the pricing with a very specific collateral rule paying a rate of 0. Understanding the pricing of those instruments in a multi-curve and collateral framework is the first step to a more general understanding of change of collateral. We dedicate one part of the ‘Modelling with collateral’ section to those instruments.

8.2 Collateral: rate, asset or both

In general, we are interested in analysing the collateral impact not only in one currency but in a multi-currency framework. The currencies will be denoted X and Y. In the first part of this section we restrict ourself to a unique currency and will extend the results to collateral with assets and more than one currency later.

8.2.1 Definition of collateral rate

We need to define *collateral rate c for a quote V*. In this section, all the cash-flows are expressed in the same currency, which we call the domestic currency. If a cash-flow is paid in another currency, it is supposed to be converted into the domestic currency at the exchange rate prevailing at the cash-flow time.

Definition 8.1 Collateral rate *The quote V_t has a collateral rate c_t if a continuous dividend*

$$dD_t = dV_t - c_t V_t dt \tag{8.1}$$

without any other payments (neither up-front nor final payment) can be created.

In the naming convention of Duffie (2001), which is also used in Brigo et al. (2012b), this means that it is possible to create an asset package with price process constant at 0 and a dividend process D_t given above.

The dividend payment does not necessarily come from one asset on its own; it can be provided by a package of instruments, including collateral contracts, which, combined together, create the above dividend cash-flow. This is the case for example for uncollateralised transactions where the dividend is created by a combination of an asset and its funding.

This may look like a convoluted way to describe collateral but it covers several financial realities and allows us to have only one treatment for all of them. Several financial realities are covered, as described below; nevertheless by abuse of language, we refer to it everywhere with the name *collateral rate*. To our knowledge, this generalised definition of collateral has not been used before, except in Crépey (2012) in a different setting. We will create even more general definitions of collateral in the next sections to include collateral by assets, including foreign currencies, with or without interest payments. In some sense, through that definition we bundle together the instrument and its financing (collateral in this case).

> La mathématique est l'art de donner le même nom à des choses différentes.
>
> *Henri Poincaré.*
>
> Meta translation: Mathematics is the art of giving twice the same quote in different chapters.

In case of perfect cash collateral in domestic currency, if one purchases an asset, he pays V_t but the amount is immediately paid back to him as collateral. So there is no cash-flow at the start. At the end of the interval of time dt, he pays the interest $c_t V_t dt$ on the collateral and receives at the same time as new collateral the change of value of the asset dV_t. At any stage the contract can be cancelled without further cash-flow; the collateral compensates for the instrument value due to the perfect collateral. The resulting cash-flows satisfy our definition above.

The situation of an asset for which a special repo rate is available is also covered by the definition. One can purchase the asset paying the amount V_t, immediately repoing it out, receiving the same[1] cash-flow V_t. No net cash-flow occurs for the

[1] Here we ignore haircut. See Section 52.5 for the haircut treatment.

buying party. At the end, the buyer receives the asset back and pays the repo (principal plus interest), $-V_t(1 + c_t dt)$. He can sell the asset and receive the proceeds $V_t + dV_t$. In total the continuous cash-flow is $dV_t - c_t V_t dt$. This situation also falls within our general collateral definition.

The situation where there is no special repo for that asset and the asset can be financed only by a naked borrowing at rate c_t is also covered. The rate is the risk-free rate if we suppose that it is possible to borrow at that rate or the funding rate if not. Here we ignore the possibility of our own default. The funding rate is becoming the equivalent of the risk-free rate, the one at which one can borrow or lend. This equivalence was one of the starting arguments in Henrard (2007). One purchases the asset at V_t, borrowing the cash. No net cash-flow is paid at the start by the purchasing party. The payment at the end of the dt period is the borrowing reimbursement, $-V_t(1 + c_t dt)$, and the simultaneous sell of the asset for the price $V_t + dV_t$. Again the total net payment is $dV_t - c_t V_t dt$. This situation also falls in our general definition. Note that the funding cash account itself is an asset with collateral rate equal to the funding rate. The account satisfies $dV_t = c_t V_t dt$, which means that the continuous payment $dV_t - c_t V_t dt$ is a continuous payment of 0.

Our definition also covers the case of proportional cash collateral (under-collateral or over-collateral). Suppose that the collateral value in cash is αV_t. If $\alpha = 1$ we have the perfect collateral case, if $\alpha = 0$ we have the no-collateral case, if $0 < \alpha < 1$ it is a proportional under-collateral, and if $\alpha > 1$ it is a proportional over-collateral. The situation is covered by the above definition with an equivalent collateral rate of $\alpha c + (1 - \alpha) f$, where c is the collateral rate paid on the cash and f is the risk-free/funding rate for the extra amount borrowed or invested to fill the collateral account. The proportional over-/under-collateral is discussed in Pallavicini and Brigo (2013) in the Central Counterpart (CCP)s framework. Our approach does not cover the case where the over-/under-collateral is not proportional. This would be the case for a CCP where an initial margin is required which is risk proportional, not value proportional. Pallavicini (2013) described initial amount corresponding to initial margins in a very general setting including credit risk.

Finally we mention the margined futures and futures options. In a future, one can enter into a contract without any up-front payment. There is an initial margin, but we ignore it here; it corresponds to an imperfect collateral which is beyond the scope of our analysis. The next futures payment is done at each margining time, where the amount dV_t is paid. The futures situation enters into our collateral definition with $c_t = 0$. This equivalence between futures and zero rate collateral is mentioned in Johannes and Sundaresan (2007), Henrard (2012d) and Piterbarg (2012).

Obviously in the five cases, the collateral rate, denoted c_t in all cases, will not be the same. In the first case, it is the collateral rate as agreed in the contract terms and conditions (often through a CSA); in the second case, it is the market repo rate for the particular asset; in the third case, it is the risk-free rate or funding rate specific to the institution; and in the futures case it is 0. The unique notation covers different situations, allowing us to deal with all of them in the same unified framework. A unified approach between repo and funding rate is implicit in (Han et al., 2013, Section 3); we prefer to have all our hypotheses and choices explicit.

One asset can have a collateral rate that changes from one category to another through time. A fully collateralised swap at overnight rate will have a collateral rate equal to the overnight rate up to its payment. After that, the payment exits the legal framework of the collateral agreement and becomes a non-collateral cash, with funding rate. With our extended definition of collateral, an instrument can still be modelled after a coupon payment. With a restricted version of collateral, at the first coupon payment, the instrument exits the framework and the theoretical world ends; the modelling is not possible anymore.

Without our generalised definition of collateral, it is difficult to deal with instruments paying an actual cash-flow. For example the generic dynamic given by Equation (8.2) is written as a diffusion and would not be valid at a coupon payment time. At the payment time, there would be a jump in the quote. For the framework to be coherent, the amount paid needs to stay in the framework and an uncollateralised amount needs to satisfy the collateral definition. Otherwise, after the end of the contract there is no collateral in place and no equation left. The time horizon for each instrument would be its payment date. This means also that the pricing would not be truly additive; the equations would be meaningful only to the first payment date in the package. But if the equations are meaningful only to the first payment date, what is the value of the instrument at that date? The value cannot be computed by discounting from the end date of the other payments, as our economy would not exist any more after the first payment date. The pricing of a swap as the sum of its coupons would then be debatable. The additive property can be brought in at the cost of extending the valuation beyond the maturity and this can be done by reintroducing the funding rate and using a generalised definition of collateral.

Note also that we use the term cash-flow to denote any change of value. It can be achieved by an actual cash-flow, a cash-flow in another currency converted in the domestic currency or a change of value which could be realised by selling part of the asset. All the values are no-arbitrage values, not accounting figures. We do not allow hiding profit, losses or future cash-flows in banking book like places or historical cost accounting.

The first assumption of the multi-curve framework concerns risk-free discounting. Here we do not work with a risk-free rate but with collateral. Our hypothesis is the existence of collateralised assets.

A: There exist packages in our economy satisfying Definition 8.1.

At this stage this hypothesis is very vague, but we will give more precision later.

The general filtration of the economy is denoted $(\mathcal{F}_t)_{t\in[0,T]}$. Later we will introduce other filtrations to describe the randomness of some specific components. When we do so we will do it through sub-filtrations $\mathcal{G}_t^i$ $(i=1,\ldots)$.

Suppose that the continuously compounded rate paid on the collateral is the rate c_t and let

$$N_t^c = \exp\left(\int_0^t c_\tau \, d\tau\right)$$

be the collateral cash account. The rate c can be stochastic (and in most cases will be). Even if N_t^c is called the collateral cash account, we do not suppose that we can invest or borrow through that account freely. Only the cash used for collateral will be in that account, according to the quotes associated with the collateralised trades.

Results similar to those described below can be found, when applied to a restricted definition of collateral, in Macey (2011), Crépey (2012), Piterbarg (2012), Pallavicini et al. (2012) and Fujii and Takahashi (2013). The proof approaches in those papers are each quite different to the others, but the results specialised to a common framework are similar. Note that Macey (2011) restricts the collateral to be the overnight index, a restriction we do not impose. In our proof, we follow the general techniques used in Macey (2011) and later in Piterbarg (2012) even if we add some ingredients missing in the original proofs' sketches and apply them to the more general definitions of collateral introduced here. Crépey (2012) on the other hand proposes a more general equation than a diffusion equation on the underlying asset.

Let $V_u^{X,c,Y}$ be the *quote* in currency X at time u of an asset with collateral in currency Y at rate c. At some stage, the value $V_u^{X,c,Y}$ will be independent of the collateral. For example a fixed cash-flow has a value on its payment date independent of the collateral currency and rate and equal to the amount paid. In that case we don't indicate the currency and the collateral rate and denote the price $V_u^{X,c,Y} = V_u$. When the asset and collateral currencies are the same and the context is clear, we write $V_u^{X,c,X} = V_u^c$.

We now develop the generic pricing formula under collateral. The proof is based on portfolio hedging, not dissimilar to the one used in standard Black-Scholes pricing formula. In this case we hedge one package generating the dividend (8.1) with n other packages as a source of randomness. The dimension n is the dimension of the Brownian motion underlying the economy. This extra asset is important to create the hedging procedure. In the Black-Scholes proof the underlying asset of the option and a cash account are used; here in general we don't have the same flexibility to access the collateralised account as the instrument and the collateral account are tied together.

8.2.2 Single currency cash collateral

Suppose that we have $n+1$ collateralised assets with collateral rates c. By rate c, we mean a vector c with one rate for each asset. The rates are potentially different for each asset. The collateral cash account for multiple assets and rates has also to be understood as a multidimensional value with each element corresponding to one asset and given by the above defined cash account. When a multiplication of vector is done element by element, like a multidimensional rate multiplied by a multidimensional quote, we use the notation $*$, like $c_t * V_t^c$.

The quotes of the assets depend on an n-dimensional Brownian motion W_t through the equation

$$dV_t^c = \mu_t dt + \Sigma_t dW_t. \tag{8.2}$$

The quotes V and the drift μ have dimension $n+1$, the volatility matrix has dimension $n+1 \times n$. Suppose moreover that the matrix composed of the n first lines of Σ_t is invertible for all t. This means that we really have n assets and not a lower number with some of them recombined in a different way. We denote the square matrix with the first n lines by $\bar{\Sigma}_t$. Similarly, for any vector x of dimension $n+1$, we denote by $\bar{x}$ its first n elements. The i-th line of Σ_t is denoted Σ_t^i. The drift and the volatility are potentially stochastic (and potentially dependent on V itself). We suppose also that $\mathcal{F}_t$ is the natural filtration of W_t and μ and Σ are 'sufficiently regular' and in particular that they are $\mathcal{F}_t$-adapted. The measure in which those equations are valid can be the physical measure or any other measure; we don't restrict the result to the physical measure like Macey (2011) and Piterbarg (2012). The only important point at this stage is that the measure is the same for all instruments.

Like in the Black-Scholes approach, the instrument valuation is based on a replicating portfolio approach by eliminating risk. The portfolios used for hedging will be self-financing. The intuition of self-financing is that there is no cash appearing or disappearing from the portfolio. The change of value of the portfolio is triggered only by the change of value of the assets in the portfolios and the dividends.

Definition 8.2 *The portfolio with $\mathcal{F}_t$-adapted quantities w_t in each package, where the packages have only a dividend process D_t and a zero price process, is said to be* self-financing *when*

$$w_t^T dD_t = 0.$$

The self-financing property is not strictly speaking required for some of the computations in the proof, but without the property or something equivalent there is not actual proof. The strategy described is a valid replication strategy only if it is self-financing at the portfolio level. It is paramount to verify this requirement to obtain a replicable price.

Several different proof strategies can be used for the result below. I like the one used below because the hedging strategy to eliminate the risk is explicit. Given the volatility matrix Σ_t, the hedging strategy is explicit.

The self-financing definition above is the one used in Duffie (2001) and Brigo et al. (2012b) adapted to our situation. Our description starts from (asset) packages with zero price process and a dividend process. The equality between the gain process and the price process reduces in our case to zero dividend process as defined above.

The main result for pricing under collateral is provided below. The result is written as a generic formula for one asset, even if the proof requires several assets for hedging purposes.

We want to prove that any $\mathcal{F}_t$-measurable quote V_u^{n+1} is replicable and we want to describe the formula for the quote at earlier times.

In our framework, replicating the quote V_u^{n+1} means finding $\mathcal{F}_t$-measurable quantities $\bar{w}_t$ and a random variable Z_t such that the portfolio $(\bar{w}_t, 1)$ is self-financing and $Z_u = V_u^{n+1}$.

Theorem 8.1 (Cash collateral formula). *Under the Brownian motion hypothesis (8.2), any $\mathcal{F}_u$-measurable quote V_u with collateral rate c is replicable and the quote at time t satisfies*

$$V_t = N_t^c \, \mathrm{E}^{\mathbb{X}}\left[(N_u^c)^{-1} V_u \middle| \mathcal{F}_t \right] \tag{8.3}$$

for some measure $\mathbb{X}$ (identical for all assets, but potentially currency dependent).

Proof: Suppose that V_u^{n+1} is replicable. Then there exists $\mathcal{F}_t$-measurable quantities $\bar{w}_t$ such that the strategy $(\bar{w}_t, 1)$ is self-financing. This means

$$0 = w_t^T dD_t = w_t^T (dV_t - c_t * V_t dt) = w^T(\mu_t - c_t * V_t) dt + w_t^T \Sigma_t dW_t.$$

We can deduce from there a property of w and the fact that $w^T(\mu_t - c_t * V_t)$ =0. Let $x = \mu_t - c_t * V_t$. As $\bar{\Sigma}_t$ is invertible, there exists λ_t with values in $\mathbb{R}^n$ such that $\bar{x} = \bar{\Sigma}_t \lambda_t$. Using the definition of w and λ, we have $0 = w_t^T x = -\Sigma_t^{n+1} \lambda_t + x^{n+1}$. This shows that $\mu_t - c_t * V_t = \Sigma_t \lambda_t$. The equations for V_t become

$$\begin{aligned} dV_t &= (\Sigma_t \lambda_t + c_t * V_t) dt + \Sigma_t dW_t = c_t * V_t dt + \Sigma_t (dW_t + \lambda_t dt) \\ &= c_t * V_t dt + \Sigma_t dW_t^{\mathbb{X}} \end{aligned} \tag{8.4}$$

where $dW_t^{\mathbb{X}}$ is a new Brownian motion defined by $dW_t + \lambda_t dt$. In the measure associated to that Brownian motion, each quote V_t^i has a drift c_t^i. In that measure, each price rebased by its collateral account $N_t^{c^i}$ is a martingale and the result on the quote formula follows.

We still need to prove that any $\mathcal{F}_u$-measurable quote V_u^{n+1} can be replicated. Let $\tilde{Z}_t = \mathrm{E}^{\mathbb{X}}\left[(N_u^{c^{n+1}})^{-1} V_u^{n+1} \middle| \mathcal{F}_t \right]$. The random variable $\tilde{Z}_t$ is an $\mathbb{X}$-martingale.

By the martingale representation theorem, there exists K_t, $\mathcal{F}_t$-adapted, such that $d\tilde{Z}_t = K_t dW_t^X$. We define $\bar{w}_t = -N_t^{c^{n+1}} K_t(\bar{Z}_t)^{-1}$. The quantity is such that $\bar{w}_t d\bar{D}_t = \bar{w}_t \bar{\Sigma}_t dW_t^{\mathbb{X}} = -N_t^{c^{n+1}} K_t dW_t^{\mathbb{X}}$. Let $Z_t = \tilde{Z}_t N_t^{c^{n+1}}$. Then $dZ_t = c_t^{n+1} Z_t dt + N_t^{c^{n+1}} K_t dW_t^{\mathbb{X}}$. The portfolio $(\bar{w}_t, 1)$ with the last component in the package of quote Z_t, is self-financing. The final quote Z_u is $N_u^{c^{n+1}} \mathrm{E}^{\mathbb{X}}\left[(N_u^{c^{n+1}})^{-1} V_u^{n+1} \middle| \mathcal{F}_u \right] = V_u^{n+1}$. This proves that V_u^{n+1} is replicable. □

At this stage the question is: what is this still mysterious measure $\mathbb{X}$? To analyse it, remember that we used $n+1$ assets to create a hedged portfolio but we have only a dimension n source of randomness. Remember that λ_t is defined by

$$\lambda_t = (\bar{\Sigma}_t)^{-1}(\bar{\mu}_t - \bar{c}_t * \bar{V}_t).$$

The quantity λ_t that defines the new Brownian motion and the measure is uniquely defined by any n assets with volatility matrix of full rank. It is possible to remove the last asset and replace it by any other asset (with its own collateral rule). If the asset is such that another sub-matrix of full rank n including that asset exists, we can change the order of the lines and have the same resulting λ_t from another set of assets. We can do this for any number of assets and replace all the original assets by new assets, one by one, without changing λ_t. The λ_t is intrinsic to the economy and the starting measure. If the starting measure is the real economy one, the quantity λ_t is usually called the price of risk.

We can describe the elements of the proof using an economic explanation and we do so in the following lines. The assets depend on several underlying 'economic' factors W_t. Each asset depends on the economic factors with different weights, represented by the matrix Σ_t.

In absence of arbitrage, the growth rate of the assets is necessarily linked to those risk weights. Each risk has an intrinsic growth rate, called the price of risk and denoted λ_t. The intrinsic growth rates associated to the risks are added to the risk-less growth represented by the rate c. In the collateral case the risk-less growth is the collateral rate paid for that particular asset; in absence of collateral, the growth is at risk-free rate. If the intrinsic growth rate of each risk factor was not respected, it would be possible to combine the assets to eliminate the risk and keep the growth mismatch. This would create a risk-less portfolio 'printing' money.

Note also that there is no relation imposed between the different growth rates λ associated to the different risk factors. All the risk factors are represented by standard Brownian motions with the same volatility $\sqrt{t}$. The risks, measured by their volatility, are of the same size, but the returns are (potentially) very different.

A result similar to Theorem 8.1 can be obtained starting from different hypotheses. This is for example the approach chosen in (Crépey, 2012, Section 4.1). The assumption is to suppose the existence of a measure $\mathbb{X}$ equivalent to the economic probability such that the gain process (denoted dD_t here and $d\mathcal{M}_t$ in the above article) is an $\mathbb{X}$-martingale. Such a hypothesis guarantees that there is no arbitrage opportunity. If the dividend process is a martingale, the quote process, which is

$$dV_t = c_t V_t dt + dD_t$$

has the same form as described above and the result is proved like in the last paragraph of our proof.

The approach is more generic as it only supposes a martingale property, and not a special form of martingale as with our $\Sigma_t dW_t$ involving a Brownian motion, and infers the arbitrage-free property from it. On the other hand our approach proves the existence of such a martingale (under some conditions) and gives an explicit hedging strategy through the quantities w_t.

There is one type of collateral situation for which we know the formula: when there exists a risk-free rate and uncollateralised asset. In that case, the formula above is valid for $\mathbb{X}$, the measure associated to the risk-free cash account numeraire, and c, the risk-free rate. The measure $\mathbb{X}$ is the one associated to the risk-free cash account if we suppose that such a concept exists.

Here we don't explicitly need this existence, so we keep the vague name, using only the currency as indication. Later in specific cases, for the valuation of futures in particular, we may reintroduce the risk-free rate existence.

It is also important to notice that the quantity N_t^c is not the numeraire associated to the standard measure $\mathbb{X}$. If the measure were the one associated to the numeraire N_t^c, Formula (8.3) would be the standard formula without consideration of collateral. Moreover there are infinitely many N_t^c, one for each rate c, and only one measure $\mathbb{X}$.

Also the quantities V_t^c analysed and that we call *quotes* are not 'present values' in the traditional sense of quantitative finance. They are quantities used in the collateral framework but cannot be used outside their context. In particular a quote rescaled by a numeraire will never be a martingale (except in the particular case where the collateral is the risk-free rate). What we have is that the quote rebased by the collateral cash account associated to that particular quote is a martingale. The pricing situation is more complex than without collateral and always involves three elements: the quote, the collateral discounting and a numeraire, as opposed to only two, the price and the numeraire, in standard quantitative finance.

From the collateral account, we define the collateral pseudo-discount factors.

Definition 8.3 (Collateral pseudo-discount factors). *The collateral (pseudo-) discount factors for collateral rate c paid in currency X are defined by*

$$P_X^c(t,u) = N_t^c \, \mathrm{E}^{\mathbb{X}} \left[(N_u^c)^{-1} \middle| \mathcal{F}_t \right]. \tag{8.5}$$

The quantity $P_X^c(t,u)$ is a positive random variable. The quantity is the quote at time t for paying a cash-flow 1 (in currency X) at time u collateralised in the currency of the cash-flow at rate c_t. Note that N_t^c is not currency dependent but P_X^c is currency dependent. The currency dependency appears through the expected value $\mathrm{E}^{\mathbb{X}}[.]$.

Let $\mathrm{E}^{c,v}[]$ be the expectation associated to the numeraire $P^c_X(.,v)N^{\mathbb{X}}_.(N^c_.)^{-1}$. This is the numeraire such that

$$N^c_t \mathrm{E}^{\mathbb{X}}\left[(N^c_u)^{-1} Y_u \middle| \mathcal{F}_t\right] = P^c_X(t,v)\,\mathrm{E}^{c,v}\left[(P^c_X(u,v))^{-1} Y_u \middle| \mathcal{F}_t\right] \tag{8.6}$$

for any random variable Y. The associated measure is called a c-collateralised v-forward measure by Pallavicini and Brigo (2013). We use the same denomination here. In particular we have that any quote with collateral c scaled by $P^c_X(t,v)$ is a martingale in that measure. Moreover a random variable is a martingale in the $N^{\mathbb{X}}$ numeraire measure when rescaled by the collateral account if and only if it is a martingale in the c-collateralised v-forward measure when rescaled by the collateral pseudo-discount factor. This equivalence will be used when we model the collateral rates later.

The change of numeraire for a time u has density L_u given by

$$L^{c,v}_u = \frac{P^c_X(u,v)(N^c_u)^{-1}}{P^c_X(0,v)(N^c_0)^{-1}}, \tag{8.7}$$

that is, the change of numeraire measure can be written as

$$\mathrm{E}^{\mathbb{X}}[Z_u] = \mathrm{E}^{c,v}\left[L^{c,v}_u Z_u\right].$$

This density, together with Girsanov's theorem, will be used when we obtain explicit change of collateral adjustments in Section 8.4.3.

When the collateral rate is the overnight index in the currency of the payment, the valuation formula (8.6) is often called 'OIS discounting'. This terminology is correct in the sense that an overnight-related discounting is used to discount the pay-off inside the expected value. Nevertheless the terminology is hiding several facts. The first is that the expectation is done under a measure for which the discounting may *not be the numeraire*. The result is not the same as replacing the risk-free rate of the no-collateral approach by the overnight rate. Moreover, for the term to be right one needs not only the instrument priced to be collateral at overnight, but also to link the OIS quotes to overnight collateral account. This last part requires the OISs themselves to be collateral with the same rule and a formula that links collateral pseudo-discount factors to OIS instruments. We provide the latter in Theorem 8.7. The 'OIS discounting' does not depend only on the collateral rule of the instrument priced, but also on the OIS collateral rule, the OIS pricing formula and the pricing in a specific numeraire. For those reasons, the term 'OIS discounting' is misleading. We will refrain from using the term further.

To have symmetrical results later, we define forward rates for the collateral curve.

Definition 8.4 (Forward collateral rate). *The* collateral forward *rate over the period* $[u, v]$ *is given at time t by*

$$F_X^c(t, u, v) = \frac{1}{\delta}\left(\frac{P_X^c(t, u)}{P_X^c(t, v)} - 1\right). \tag{8.8}$$

This means that

$$1 + \delta F_X^c(t, u, v) = \frac{P_X^c(t, u)}{P_X^c(t, v)}.$$

We call the above quantity the 'investment factor', as opposed to the discount factor; it represents the amount obtained at the end of the period if 1 is invested at the start with the rate.

Using the definition of c-collateral v-forward measure and the remark above, the investment factor $1 + \delta F_X^c(t, u, v)$ is a martingale in the c-collateral v-forward measure. Even if this result seems trivial at this stage, because of its importance for rate modelling, we write it as a theorem.

Theorem 8.2 (Martingale property of forward collateral rate). *In the collateral framework, the collateral forward rate* $F_X^c(t, u, v)$ *is a martingale in the* c*-collateral* v*-forward measure.*

8.2.3 Change of collateral: independent spread

We have the general formula for computing the price in a framework with collateral. A natural question is to ask what is the relation between the prices of assets with same final price V_u but different collateral rules.

Suppose that we have two collateral rates with the relation

$$c_t^2 = c_t^1 + s_t$$

with s_t a $\mathcal{G}_t^2$-adapted random variable and V and c_1 both $\mathcal{G}_t^1$-adapted with $\mathcal{G}_t^2$ independent of $\mathcal{G}_t^1$. Often the spread will be constant or deterministic. The situation can be, for example, that the rate paid on the collateral is the overnight index plus a fixed spread.

By definition of the spread, the collateral accounts are such that $N_t^{c_2} = N_t^{c_1} N_t^s$. Suppose that the price at time u is collateral independent (for example it is the cash-flow date). With this simple relation between the rates, the relation between the quotes is

$$\begin{aligned} V_0^{c^2} &= N_0^{c_1} N_0^s \,\mathrm{E}^X\left[(N_u^{c_1} N_u^s)^{-1} V_u\right] \\ &= N_0^s \,\mathrm{E}^X\left[(N_u^s)^{-1}\middle|\mathcal{F}_t\right] N_0^{c_1}\,\mathrm{E}^X\left[(N_u^{c_1})^{-1} V_u\right] \\ &= P_X^s(0, u) V_0^{c_1} \end{aligned} \tag{8.9}$$

where for the second equality we have used the independence property of the variables.

This can be interpreted as, when there is a spread paid on the collateral and the spread is independent, the quote is the original quote discounted by the spread.

Note also that we have

$$P_X^{c^2}(0,u) = P_X^s(0,u)P_X^{c^1}(0,u) \tag{8.10}$$

as a special case of the above result.

8.2.4 Simplified example of hedging

Suppose that we use two deterministic collateral rates $c_i(t)$ $(i = 1,2)$ and that the quotes of the related assets can be written as

$$V_t^{c_i} = P_X^{c_i}(t,v)F_t(v).$$

The quantity $F_t(v)$ can be interpreted as a forward. This situation will be the one we will deal with in Section 8.3 when we analyse the multi-curve framework with collateral. To fit in our framework, the quantity F_t is supposed to follow a one-dimensional diffusion equation

$$dF_t(v) = \cdots dt + \sigma_t dW_t^{\mathbb{X}}.$$

The value of the drift is not important for what we want to show here. Due to the deterministic nature of the collateral rates, it can be proved that the drift is 0. The pseudo-discount factors satisfy the equations $dP_X^{c_i}(t,v) = c_i(t)P_X^{c_i}(t,v)dt$. The quotes satisfy

$$dV_t^{c_i} = c_i(t)P_X^{c_i}(t,v)F_t(v)dt + P_X^{c_i}(t,v)\sigma_t dW_t^{\mathbb{X}}.$$

In this setting, we can describe the hedging quantities w_t explicitly. To hedge one unit of $V_t^{c_2}$, we need

$$\bar{w}_t = -\frac{P_X^{c_2}(t,v)}{P_X^{c_1}(t,v)}$$

units of $V_t^{c_1}$. The quantity depends in theory on σ, but as the same σ is used in both equations, they simplify. If we buy one unit of $V_t^{c_2}$, we hedge it with $\bar{w}_t$ units of $V_t^{c_1}$.

We look at two scenarios around this hedging. First suppose that the value F changes instantaneously (no interest payment involved). The change of the second quote is $P^{c_2}(t,v)dF_t$ while the first is $P^{c_1}(t,v)dF_t$ to be multiplied by the quantity $\bar{w}_t$ and the two changes cancel. As expected from a perfect hedging, no profit is incurred when the underlying changes.

In the second scenario, the value F does not change over a certain period but the time is passing and collateral interest should be paid. For the second instrument, the interest on the cash account is $P^{c_2}(t,v)F_t(v)c_2 dt$ and the change of quote is $dV_t^{c_2} = c_2(t)P^{c_2}(t,v)F_t(v)dt$. The two compensate exactly. The interest payment on the collateral account is compensated by the carry on the instrument.

In summary, we have two assets with different collateral, the change of value of one of them is exactly the change of the other and the supplementary cash required by one collateral account is provided by the cash paid by the other one. The interest payment on each account is compensated by the carry on the instrument itself. This is due to the pricing formula used, not to an external financial mechanism that provides the funding.

The hedging, even in this simplistic case of linear products and deterministic collateral rates, is dynamic. The hedging of one unit of the second product is not done with one unit of the first one, but by the quantity $\bar{w}_t$, which is changing through time.

8.2.5 Collateral with asset

In some cases the collateral posted is not cash but a different asset. The asset itself is subject to collateral (remember that no collateral is considered as a form of collateral in our generalised definition). This can be described as *collateral square*[2].

This is the case with interest rate swaps. The standard is cash collateral at overnight rate but in some cases the collateral consists of treasuries. This is often the case with asset managers who prefer to deposit assets instead of cash. This section describes how to coherently price assets collateralised with other assets.

We need to define asset collateral. As before, we do it through a definition of continuous dividends. We denote by V_t^C the quote for an asset with collateral with quote C.

Definition 8.5 (Collateral with asset). *The quote V_t has an asset with quote C_t as collateral if a continuous dividend*

$$dD_t = dV_t - \frac{V_t}{C_t} dC_t$$

without any other payments (neither up-front nor final payment) can be created.

The explanation of the definition is the following. If the collateral is an asset with associated quote C_t, the collateral cash-flow is the change of asset dV_t. There is no payment of interest per se but the change of the value of the collateral asset needs to be compensated. The quantity of asset with value C_t to cover the value V_t is V_t/C_t. The change of value of the asset is dC_t. When the collateral asset increases in value we have to release the extra value to the counterpart, hence a negative sign in front of the change of value. In the analysis, we always suppose that the collateral can

2 The name was suggested by [Brigo 2013].

be rehypothecated. In particular, if you buy the asset and receive the other asset as collateral in guarantee, you can sell the collateral to cancel the initial cash-flows.

With that definition, a quote has a collateral rate c_t if the quote has the collateral account as a collateral asset. The new definition is a natural extension of our general definition of collateral with cash paying a given rate.

Like before, we suppose we have $n+1$ assets depending on an n-dimensional Brownian motion W_t. The equations for the assets are the same as before and given by Equation (8.2). We suppose that the first n assets have collateral rates c_t. We also keep the hypothesis that the volatility matrix $\bar{\Sigma}_t$ for the first n assets is invertible. The difference is for the last asset; we suppose that the last asset is collateralised by one of the previous assets. Without loss of generality, we suppose the collateral is the first asset. The continuous cash-flow that can be created by the last asset with collateral the first asset is

$$dV_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} dV_t^1.$$

Theorem 8.3 (Asset collateral formula). *In presence of collateral with asset of quote C_t which has itself a collateral rate c_t, the quote at time t for an asset with quote V_u^C at time u is*

$$V_t^C = N_t^c \,\mathrm{E}^{\mathbb{X}}\left[(N_u^c)^{-1} V_u^C \middle| \mathcal{F}_t \right]. \tag{8.11}$$

Proof: Let

$$\bar{w}_t^T = \left(\frac{V_t^{n+1}}{V_t^1} \Sigma^1 - \Sigma_t^{n+1} \right) (\bar{\Sigma}_t)^{-1}$$

and $w_t^T = (\bar{w}_t^T, 1)$.

We create a portfolio $w_t^T V_t$. From the equations of V,

$$\begin{aligned}
\bar{w}_t^T dD_t &= \bar{w}_t^T (d\bar{V}_t - \bar{c}_t * \bar{V}_t dt) + dV_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} dV_t^1 \\
&= \left(\bar{w}_t^T (\bar{\mu}_t - \bar{c}_t * \bar{V}_t) + \left(\mu_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} \mu_t^1 \right) \right) dt \\
&\quad + \left(\bar{w}_t \bar{\Sigma}_t + \Sigma_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} \Sigma_t^1 \right) dW_t
\end{aligned}$$

By definition of w, the coefficient in front of the Brownian motion is equal to 0. In absence of arbitrage, the dividend cannot have only drift and be non-zero. Consequently we get the drift $w_t^T x = 0$ with

$$x = \begin{pmatrix} \bar{\mu}_t - \bar{c} * \bar{V} \\ \mu_t^{n+1} - \frac{V_t^{n+1}}{V_t^1} \mu_t^1 \end{pmatrix}.$$

With that choice, the portfolio is self-financing.

As $\bar{\Sigma}_t$ is invertible, there exists λ_t with values in $\mathbb{R}^n$ such that $\bar{x} = \bar{\Sigma}_t \lambda_t$. Using the definitions of w and λ, we have

$$0 = w_t^T x = \bar{w}_t^T \bar{\Sigma}_t \lambda_t + x^{n+1} = \left(\frac{V_t^{n+1}}{V_t^1} \Sigma_t^1 - \Sigma_t^{n+1} \right) \lambda_t + x^{n+1}$$

and so $x^{n+1} = \left(\Sigma^{n+1} - V_t^{n+1} / V_t^1 \Sigma^1 \right) \lambda_t$. From the definition of λ we have $\bar{\mu}_t = \bar{\Sigma}_t \lambda_t + \bar{c} * \bar{V}$. Using the result for x^{n+1} and the above result for μ_1, we obtain $\mu_t^{n+1} = \Sigma_t^{n+1} \lambda_t + c_t^1 * V_t^{n+1}$.

Define $c_t^{n+1} = c_t^1$. The equations for V_t become

$$dV_t = c_t * V_t dt + \Sigma_t (dW_t + \lambda_t dt) = c_t * V_t dt + \Sigma_t dW_t^X. \tag{8.12}$$

In the measure associated to the $W_t^{\mathbb{X}}$ Brownian motion, each asset has a drift c^i equal to its collateral rate. The drift for the $n+1$-th asset is c_t^1, that is, the collateral rate of the collateral asset. □

Note that the definition of λ_t depends only on $\bar{\Sigma}_t$, $\bar{\mu}$ and $\bar{c}$, that is, only on the first n instruments equation, and collateral is used to define the change of measure. The new Brownian motion $W^{\mathbb{X}}$ and the associated measure $\mathbb{X}$ are the same as those obtained in the cash only collateral case. Both formulas use the same expectation.

The result could be described as the quote of an asset collateralised by another asset is obtained by discounting the quote at the rate of the collateral collateral, that is discounting by the collateral rate of the collateral. Hence the name ‘collateral square’ mentioned earlier.

8.2.6 Collateral with asset and rate payment

In this section we extend the analysis to the combination of the two previous cases and add a gearing factor. The collateral is composed of another asset with gearing and at the same time there is the payment of some rate on the collateral value. The gearing factor is used to model haircut as described later. For a collateral asset αC and a collateral rate c, we denote the quote by $V_t^{\alpha,C,c}$.

The payment of interest on the collateral is required for foreign currency collateral. The currency itself changes value and collateral has to be added or removed with that change of value. The foreign currency is an asset when viewed from the domestic currency and not simple cash. At the same time a conventional interest has to be paid on the amount. Note that even if the rate is paid in a foreign currency it is still paid on the full collateral value, which is V_t. There is no need to know the exchange rate to know the domestic currency value of interest paid; it will be $c_t V_t dt$.

To our knowledge, this generalised definition of collateral has not been used before.

Definition 8.6 (Collateral with asset and rate). *The asset with quote V_t has a collateral asset C_t and a collateral rate c_t if a continuous dividend*

$$dD_t = dV_t - \alpha \frac{V_t}{C_t} dC_t - c_t V_t dt$$

without any other payments (neither up-front nor final payment) can be created.

As hinted above, the definition covers collateral with haircut. Suppose that collateral asset C is accepted with a haircut h, for a value V_t; one has to post assets for a total value of $V_t/(1-h)$ in asset C. To use the notation of the definition, a gearing factor of $\alpha = 1/(1-h)$ is required. What is the dividend of such a collateral process? At t, one party pays the instrument V_t, receives the collateral for a total value αV_t, sells it and invests the difference $(\alpha - 1)V_t$ in cash at the funding rate f. After the time dt, he sells the instrument for $V_t + dV_t$, repays to collateral for $\alpha V_t / C_t (C_t + dC_t)$ and receives from the cash investment $(\alpha - 1)V_t(1 + fdt)$. The total dividend created is $dV_t - \alpha V_t / C_t dC_t + (\alpha - 1) f V_t dt$. The definition is satisfied with $\alpha = 1/(1-h)$ and $c = (\alpha - 1)f$.

The same result can be obtained for foreign currency for which haircut is required and a conventional rate $\bar{c}$ is paid. The definition is satisfied with the same gearing factor $\alpha = 1/(1-h)$ and $c = \alpha\bar{c} + (1-\alpha)f$. Obviously, if there is no haircut, $\alpha = 1$ and the implied collateral rate is simply the actual collateral rate.

Like before, we suppose we have $n+1$ assets depending on an n-dimensional Brownian motion W_t. The equations for the assets are the same as before and given by Equation (8.2). We suppose that the first n assets have collateral rates c_t. We also keep the hypothesis that the volatility matrix $\bar{\Sigma}_t$ for the first n assets is invertible. The difference is for the last asset. We suppose that it is collateralised by an asset and the rate c_t^{n+1}. Without loss of generality, we suppose that the collateral asset is the first one. So the continuous payment that can be created by the last asset is

$$dV_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} dV_t^1 - c_t^{n+1} V_t^{n+1} dt.$$

Theorem 8.4 (Geared asset and rate collateral formula). *Suppose the asset has collateral with asset C_t, gearing α and rate c_t^{n+1} and the collateral asset itself has a collateral rate c_t^1. Then the quote at time t of an asset with quote $V_u^{\alpha,C,c^{n+1}}$ at time u is*

$$V_t^{\alpha,C,c^{n+1}} = N_t^{\alpha(c^1+c^{n+1})} \mathrm{E}^{\mathbb{X}}\left[(N_u^{\alpha(c^1+c^{n+1})})^{-1} V_u^{\alpha,C,c^{n+1}} \middle| \mathcal{F}_t \right]. \tag{8.13}$$

Proof: Let

$$\bar{w}_t^T = \left(\alpha \frac{V_t^{n+1}}{V_t^1} \Sigma_t^1 - \Sigma_t^{n+1} \right) (\bar{\Sigma}_t)^{-1}$$

and $w_t^T = (\bar{w}_t^T, 1)$.

We create a portfolio $w_t^T V_t$. From the equation of the assets,

$$
\begin{aligned}
\bar{w}_t^T dD_t &= \bar{w}_t^T (d\bar{V}_t - \bar{c}_t * \bar{V}_t dt) + dV_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} dV_t^1 - c_t^{n+1} V_t^{n+1} dt \\
&= \left(\bar{w}_t^T (\bar{\mu}_t - \bar{c}_t * \bar{V}_t) + \left(\mu_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} \mu_t^1 - c_t^{n+1} V_t^{n+1} \right) \right) dt \\
&\quad + \left(\bar{w}_t \bar{\Sigma}_t + \Sigma_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} \Sigma_t^1 \right) dW_t
\end{aligned}
$$

By the choice of $\bar{w}_t$, the coefficient in front of the Brownian motion is equal to 0. In absence of arbitrage, the dividend cannot have only drift and be non-zero. Consequently we get the drift $w_t^T x = 0$ with

$$
x = \begin{pmatrix} \bar{\mu}_t - \bar{c}_t * \bar{V}_t \\ \mu_t^{n+1} - \alpha \frac{V_t^{n+1}}{V_t^1} \mu_t^1 - c_t^{n+1} V_t^{n+1} \end{pmatrix}.
$$

With that choice, the portfolio is self-financing.

As $\bar{\Sigma}_t$ is invertible, there exists λ_t with values in $\mathbb{R}^n$ such that $\bar{x} = \bar{\Sigma}_t \lambda_t$. Using the definition of w and λ, we have

$$
0 = w_t^T x = \bar{w}_t^T \bar{\Sigma}_t \lambda_t + x_{n+1} = \left(\alpha \frac{V_t^{n+1}}{V_t^1} \Sigma_t^1 - \Sigma_t^{n+1} \right) \lambda_t + x_{n+1}
$$

and so $x^{n+1} = \left(\Sigma^{n+1} - \alpha V_t^{n+1} / V_t^1 \Sigma^1 \right) \lambda_t$. By construction we have $\bar{\mu}_t = \bar{\Sigma}_t \lambda_t + \bar{c} * \bar{V}$. Using the result for x^{n+1} and the above result for μ_t^1, we obtain $\mu_t^{n+1} = \Sigma_t^{n+1} \lambda_t + (c_t^1 + c_t^{n+1}) V_t^{n+1}$.

Define $\tilde{c} = (\bar{c}_t, \alpha(c_t^1 + c_t^{n+1}))$. The equations for V_t become

$$
dV_t = \tilde{c}_t * V_t dt + \Sigma_t (dW_t + \lambda_t dt) = \tilde{c}_t * V_t dt + \Sigma_t dW_t^{\mathbb{X}}. \tag{8.14}
$$

In the numeraire associated to that Brownian motion $W^{\mathbb{X}}$, each asset has a drift $\tilde{c}_t^i$ corresponding to its collateral rate. The drift for the $n+1$-th asset is $\alpha(c_t^1 + c_t^{n+1})$, that is, the collateral rate of the collateral asset plus the collateral rate itself multiplied by the gearing factor. □

An approach covering collateral by other assets is described in McCloud (2013c), where he defines a generalised collateral rate similar to our $c^1 + c^{n+1}$. The result he proposes does not cover haircut.

The above result can be applied to collateral in foreign currencies. For a foreign currency, the asset collateral rate in the sense of Definition 8.1 is the forex swap annualised points. For a domestic currency X and a foreign currency Y, we denote the points by $p_t^{X,Y}$. Note that the market quoted forex forward points are not annualised and not relative (they are absolute numbers to be added to the current exchange rate).

The collateral rate for foreign currency is obtained in the following way. Buying one unit of foreign currency can be covered by an instantaneous forex swap, selling the foreign currency in the near leg and buying the same quantity in the far leg. If the forward points are annualised and relative, the swap far leg will pay the exchange rate f_t times $1 + p_t^{X,Y} dt$. At the end of the period, one can sell the foreign currency unit of the far leg at $f_t + df_t$. The total cash-flow is then $df_t - f_t p_t^{X,Y} dt$. By Definition 8.1, $p_t^{X,Y}$ is the collateral rate of the foreign currency asset Y.

When risk-free rates exist in both currencies, one has $p_t^{X,Y} = r_t^X - r_t^Y$.

If we define the mixed currency account for the collateral of an asset in currency X with a rate c in currency Y by

$$N_t^{X,c,Y} = \exp\left(\int_0^t c_\tau + p_\tau^{X,Y} d\tau\right).$$

our result specialised to the currency case without haircut reads like (Pallavicini et al., 2012, Proposition 3.7) and (Fujii and Takahashi, 2013, Theorem 6.1).

Theorem 8.5 (Foreign currency collateral formula). *In presence of collateral in currency Y with rate c_t, the quote in currency X at time t of an instrument with quote $V_u^{X,c,Y}$ at time u is*

$$V_t^{X,c,Y} = N_t^{X,c,Y}\, \mathrm{E}^{\mathbb{X}}\left[(N_u^{X,c,Y})^{-1} V_u^{X,c,Y} \middle| \mathcal{F}_t \right] \tag{8.15}$$

Note that this is simply the result for single-currency collateral with the collateral being another currency. The cross-currency value $V_t^{X,c,Y}$ is simply the single-currency value $V_t^{X,c+p^{X,Y},X}$.

Like in Definition 8.3, we can define a pseudo-discount factor associated to the foreign currency cash collateral.

Definition 8.7 (Foreign currency collateral pseudo-discount factors). *The foreign currency collateral (pseudo-)discount factors for collateral rate c paid in currency Y for an instrument in currency X are defined by*

$$P^{X,c,Y}(t,u) = N_t^{X,c,Y}\, \mathrm{E}^{\mathbb{X}}\left[(N_u^{X,c,Y})^{-1} \middle| \mathcal{F}_t \right]. \tag{8.16}$$

Suppose that forex swaps between currency X and Y are quoted in the market with the characteristics that forex swaps are for the period $[0, t_0]$ with points $p^{Y,X,c,X}(t_0)$ paid in currency Y and the collateral on the swap is paid in currency X at the rate c. The leg in currency X has cash-flows -1 in 0 and 1 in t_0, its value is $-1 + P_X^c(0, t_0)$. The cash-flows in currency Y are 1 in 0 and $-(1 + p^{Y,X,c,X}(t_0) t_0)$ paid in t_0. Because the cash-flows are fixed, the can be valued using pseudo-discount factors and the value of the Y leg converted into currency X is $1-(1+p^{Y,X,c,X}(t_0)t_0)P^{Y,c,X}(0,t_0)$. If we suppose that the pseudo-discount factor $P^c_X(0,t_0)$ is known, we obtain one (linear) equation with one unknown to solve to obtain $P^{Y,c,X}(0,t_0)$: $P^c_X(0,t_0)-(1+p^{Y,X,c,X}(t_0)t_0)P^{Y,c,X}(0,t_0)=0$. A similar result can be obtained if the forex points are paid in currency X: $(1+p^{X,Y,c,X}(t_0)t_0)P^c_X(0,t_0)-P^{Y,c,X}(t_0)=0$.

This is exactly the same equation as needs to be solved in the multi-curve framework without collateral to compute one discounting curve from the discounting curve in one currency to the discounting curve in another currency through forex swaps (see for example Henrard (2010)).

With this result, from EUR/USD forex swaps with points in USD and collateralised in USD at Fed Funds rates, and from the pseudo-discount factors for USD collateralised at Fed Funds, we can obtain the pseudo-discount factors for EUR collateralised in USD at Fed Funds.

In McCloud (2013a) and McCloud (2013c), a quantification of the convexity adjustments for forex forwards is provided. For cases when forex swaps with collateral in one of the two currencies are not quoted but other forex swaps are quoted, one may want to estimate (in our notation) $P^{X,c^Y,Y}$ from $P^{X,c^Z,Z}$ and hypothesise on the different spreads, rate and forex dynamics. The paper above provides such a formula using Hull-White like dynamics for the rates.

8.3 Multi-curve framework with collateral

8.3.1 Forward index rate and spread

We apply the general collateral results to the multi-curve framework. Like in Chapter 2, the multi-curve framework with collateral supposes the existence of a specific set of assets, the Ibor coupons.

I: A j floating coupon in currency X with collateral in currency Y at rate c is an asset for each tenor j, each fixing date t_0, each collateral rate c and each currency X and Y.

The curve description approach is based on the following definitions.

Definition 8.8 (Forward index rate with collateral). *The* forward curve $F_t^{X,c,Y,j}(\theta,u,v)$ *is the continuous function of t such that,*

$$P^{X,c,Y}(t,v)\delta F^{X,c,Y,j}(t,u,v) \tag{8.17}$$

is the quote in t of the j-Ibor coupon in currency X with fixing date t_0, start date u, maturity date v ($t\le t_0\le u=\text{Spot}(t_0)<v$) and accrual factor δ collateralised at rate c in currency Y.

Let $I_X^j(t_0)$ denote the fixing rate published on t_0 for the index in currency X for the period $[u, v]$ of length j. The fixing rate is independent of the collateral. On the fixing date we have

$$F^{X,c,Y,j}(t_0, u, v) = I_X^j(t_0)$$

for any collateral rate c and currency Y.

An IRS is described by a set of fixed coupons or cash-flows c_i at dates $\tilde{t}_i$ $(1 \leq i \leq \tilde{n})$. For those flows, the collateral pseudo-discounting curve is used. The IRS also contains a set of floating coupons over the periods $[t_{i-1}, t_i]$ with $t_i = t_{i-1} + j$ $(1 \leq i \leq n)$. The accrual factors for the periods $[t_{i-1}, t_i]$ are denoted δ_i. The value of a (fixed rate) receiver IRS in $t < t_0$ is

$$\sum_{i=1}^{\tilde{n}} c_i P^{X,c,Y}(t, \tilde{t}_i) - \sum_{i=1}^{n} P_X^{X,c,Y}(t, t_i)\delta_i F^{X,c,Y,j}(t, t_{i-1}, t_i). \tag{8.18}$$

In the textbook one curve pricing approach, the IRSs are usually priced through either the *discounting forward rate* approach or the *cash-flow equivalent* approach. The *discounting forward rate* approach is similar to the above formula.

As with the one-curve framework, we can define a forward swap rate. This is the fixed rate for which the vanilla IRS price is 0:

$$S_t^{X,c,Y,j} = \frac{\sum_{i=1}^{n} P^{X,c,Y}(t, t_i)\delta_i F^{X,c,Y,j}(t, t_{i-1}, t_i)}{\sum_{i=1}^{\tilde{n}} P^{X,c,Y}(t, \tilde{t}_i)\tilde{\delta}_i}. \tag{8.19}$$

8.3.2 Same currency collateral

Suppose that the coupon currency X is the same as the collateral currency Y. We use the short notation for the forward: $F^{X,c,X,j}(t, u, v) = F_X^{c,j}(t, u, v)$. This last notation is closer to the multi-curve framework notation.

In the c-collateral v-forward numeraire described after Definition 8.3, the collateralised forward satisfies

$$F_X^{c,j}(t, u, v) = \mathrm{E}^{c,v}\left[F_X^{c,j}(s, u, v) \middle| \mathcal{F}_t \right], \tag{8.20}$$

this is the forward rate and is a martingale.

Like in Henrard (2010) and Henrard (2013a), we can define the multiplicative spreads between the different curves.

Definition 8.9 (Multiplicative spread). *The* multiplicative spread $\beta^{A|B}$ *between the forward rate* $A \in \{D; c; c, j\}$ *and the forward rate* B *is defined by*

$$(1 + \delta F^A(t, u, v))\beta_t^{A|B}(u, v) = 1 + \delta F^B(t, u, v). \tag{8.21}$$

Like in the standard multi-curve framework, we write the spread in a multiplicative way. The multiplicative spread is natural in interest rate modelling as it corresponds to the composition of rates.

Using that notation the quote of a collateralised coupon becomes

$$P^c_X(t,v)(1+\delta F^c_X(t,u,v))\beta^{c|c,j}_t(u,v) - P^c_X(t,v) = P^c_X(t,u)\beta^{c|c,j}_t(u,v) - P^c_X(t,v).$$

We have a formula similar to the one-curve framework where the value of a coupon is obtained by discounting the notional at start date and at end date. Here the start date amount is adjusted (by the ratio β) to take into account the spread between the discounting curve and the index projection curve.

8.3.3 Overnight indexed swaps

Up to now we have defined the collateral discount factors, but have not described how to obtain them from the market. The most commonly used collateral rates are overnight rates. There exist liquid market instruments with pay-off linked to those rates; they are called *overnight indexed swaps* (OIS). We show how to price OISs in an overnight collateral framework.

The proof below is adapted from Quantitative Research (2012a), where a similar proof is provided for compounded coupons with deterministic spread between the discounting and forward curves. In Fujii et al. (2010a) a similar result is provided for continuously compounded rates (not periodically compounded) and no spread. A similar-looking result is also proposed in (Macey, 2011, Section 3.3.1) but from undefined notation.

The overnight rate for the period $[u,v]$ is fixed in u and is denoted $I^O_X(u,v)$. The date v is u plus one business day. The associated accrual factor is δ.

Suppose the rate paid on the collateral account is the overnight rate plus a deterministic spread, which means that N_v is $\mathcal{F}_u$-adapted and

$$N^c_v(N^c_u)^{-1}\beta^{c|O}_u(u,v) = 1+\delta I^O_X(u,v)$$

with $\beta^{c|O}_u$ deterministic. Usually β will come from a constant spread like $\beta_u = 1+\delta s$ or $\beta_u(u,v) = \exp(s(v-u))$ with s a constant.

Using Equation (8.5) defining the collateral discount factor and the above equality, we have

$$P^c_X(u,v)(1+\delta F^{c,O}_X(u,u,v)) = P^c_X(u,v)(1+\delta I^O_X(u)) = \beta^{c|O}_u(u,v).$$

By Definition 8.8 with the overnight index O, the above quantity in u is the price of an asset (fixed amount plus overnight coupon). Its price satisfies the generic collateral pricing and

$$P^c_X(s,v)(1+\delta F^{c,O}(s,u,v)) = N^c_s \mathrm{E}^X\left[(N^c_u)^{-1}\middle|\mathcal{F}_s\right]\beta^{c|O}_u(u,v) = P^c_X(s,u)\beta^{c|O}_u(u,v).$$

The forward index rate collateralised by the same rate satisfies the familiar looking equation as follows.

Theorem 8.6 (One period overnight forward rate value). *The rate of a one period (that is with $v = u+1$ day) overnight coupon, when the collateral rate is the overnight rate multiplied by a deterministic spread, is*

$$1+\delta F_X^{c,O}(s,u,v) = \frac{P_X^c(s,u)}{P_X^c(s,v)}\beta_u^{c|O}(u,v).$$

The description of an overnight indexed (compounded) coupon is as follows. The times associated are denoted $(t_i)_{i=0,\dots,n}$. The rate for the period $[t_{i-1},t_i]$ is fixed in t_{i-1} $(i=1,\dots,n)$. The accrual factor in the index convention for the period $[t_{i-1},t_i]$ is δ_i.

The overnight coupon pays the amount

$$\left(\prod_{i=1}^{n}(1+\delta_i I_X^O(t_{i-1}))\right)-1$$

in t_p. In this section we suppose that $t_p = t_n$.

Using the result above, we have, for the expected value of the product in the numeraire defined by Equation 8.6),

$$\begin{aligned}
&\mathrm{E}^{c,t_n}\left[\left(\prod_{i=1}^{n}(1+\delta_i F^{c,O}(t_{i-1},t_{i-1},t_i))\right)\right]\\
&=\mathrm{E}^{c,t_n}\left[\left(\prod_{i=1}^{n-1}\frac{P_X^c(t_{i-1},t_{i-1})}{P_X^c(t_{i-1},t_i)}\mathrm{E}^{c,t_n}\left[\frac{P_X^c(t_{n-1},t_{n-1})}{P_X^c(t_{n-1},t_n)}\middle|\mathcal{F}_{t_{n-2}}\right]\right)\right]\prod_{i=1}^{n-1}\beta_{t_{i-1}}^{c|O}(t_{i-1},t_i)\\
&=\mathrm{E}^{c,t_n}\left[\left(\prod_{i=1}^{n-1}\frac{P_X^c(t_{i-1},t_{i-1})}{P_X^c(t_{i-1},t_i)}\frac{P_X^c(t_{n-2},t_{n-1})}{P_X^c(t_{n-2},t_n)}\right)\right]\prod_{i=1}^{n-1}\beta_{t_{i-1}}^{c|O}(t_{i-1},t_i)\\
&=\mathrm{E}^{c,t_n}\left[\left(\prod_{i=1}^{n-2}\frac{P_X^c(t_{i-1},t_{i-1})}{P_X^c(t_{i-1},t_i)}\frac{P_X^c(t_{n-2},t_{n-2})}{P_X^c(t_{n-2},t_n)}\right)\right]\prod_{i=1}^{n-1}\beta_{t_{i-1}}^{c|O}(t_{i-1},t_i)\\
&=\mathrm{E}^{c,t_n}\left[\left(\prod_{i=1}^{n-2}\frac{P_X^c(t_{i-1},t_{i-1})}{P_X^c(t_{i-1},t_i)}\frac{P_X^c(t_{n-3},t_{n-2})}{P_X^c(t_{n-3},t_n)}\right)\right]\prod_{i=1}^{n-1}\beta_{t_{i-1}}^{c|O}(t_{i-1},t_i).
\end{aligned}$$

In the first equality we use Theorem 8.6, the fact that F_s^O is $\mathcal{F}_{t_{n-2}}$-measurable for $s \leq t_{n-2}$ and the tower property. In the second equality, we use the martingale property of the c-quotes rescaled by the c-pseudo-discount factors with maturity t_n in the $\mathrm{E}^{c,t_n}[]$ measure. The third equality is simply the simplification of fractions. In the fourth equality we use the same technique as in the first and second but with $\mathcal{F}_{t_{n-3}}$. The same procedure can be repeated down for all factors in the product and we obtain

$$\mathrm{E}^{c,t_n}\left[\frac{P_X^c(t_0,t_0)}{P_X^c(t_0,t_n)}\right]\prod_{i=1}^{n-1}\beta_{t_{i-1}}^{c|O}(t_{i-1},t_i).$$

After applying once more the martingale property and the definition of the numeraire, we obtain the following result.

Theorem 8.7 *Under the hypotheses **A** and **I**, the price of the overnight indexed (compounded) coupon described above in the overnight collateral framework is given by*

$$P^c_X(0,t_0)\prod_{i=1}^{n-1}\beta^{c|O}_{t_{i-1}}(t_{i-1},t_i) - P^c_X(0,t_n).$$

The result means that for collateral equal to the overnight index plus a deterministic spread, the OIS curve can be built like in the one-curve framework (taking the spread into account). The formula is very simple and resembles the one for a deposit. Once we have this first building block, we can, like in the multi-curve framework without collateral, start to build index curves with collateral. The index (forward) curves are the curves as described in Definition 8.8 for which the model quote is equal to its market quote (or for which the model par rate is equal to the market par rate).

In this situation the tools developed for curve calibration in the multi-curve framework without collateral can still be used. We refer in particular to Ametrano and Bianchetti (2013) and Henrard (2013a) for more details on challenges of the curve calibration.

As mentioned earlier the 'OIS discounting' framework requires not only that the instrument analysed is collateralised at overnight, but also that the OISs themselves are collateralised at overnight and that the above formula is correct. Not all those conditions are satisfied in all markets. For example in SGD, there is a (quite illiquid) market on OIS with fixing on SONAR. But most of the cash collateral pays an interest indexed on overnight SIBOR, a different index (Tee (2013)). The 'OIS discounting' rule cannot be applied sensu stricto in the SGD market.

8.3.4 Change of collateral: independent spread

Suppose that, like in Section 8.2.3, the spread between two collateral rates is independent of the initial rate, that is

$$c^2_t = c^1_t + s_t$$

with s_t a $\mathcal{G}^2_t$-adapted random variable and I^j_X and c_1 are $\mathcal{G}^1_t$ adapted with $\mathcal{G}^2_t$ independent of $\mathcal{G}^1_t$. The forward rates are defined through the value of coupons and are, by Equation (8.9) and (8.10), linked by

$$P^{c_2}_X(0,v)\delta F^{c_2,j}(0,u,v) = P^s_X(0,v)P^{c_1}_X(0,v)\delta F^{c_1,j}(0,u,v) = P^{c_2}_X(0,v)\delta F^{c_1,j}(0,u,v)$$

This means that the forward rates are the same:

$$F^{c_2,j}(0,u,v) = F^{c_1,j}(0,u,v)$$

If a forward curve has been calibrated for a given collateral rate it can be used for other collateral rates if the spread is independent. Obviously the equality does not extend to quotes, which are discounting dependent. In particular it does not extend to swap rates. The swap rates do not depend only on forward rates but also on discounting. We will show in Section 8.3.8 that the dependency is actually quite mild. We will come back to not independent spreads later and introduce the convexity adjustment required in those cases.

8.3.5 Collateral in foreign currency and independent spread

In this section we want to price interest rate products linear on Ibor rates but with collateral in a foreign currency. At a later stage we want to price cross-currency swaps between the domestic currency and a foreign currency with the collateral in the domestic currency (a foreign currency for the foreign currency leg).

Using the previous notation we want to price $V^{Y,c,X}$. Let the standard collateral used in Y be denoted c^Y. In particular the currency Y IRS quoted in the market uses that collateral rate. From the previous development one can obtain the curve $F^{Y,c^Y,Y,j}$ and $P_Y^{c^Y}$ from the market. Suppose that the standard collateral rule in currency X refers to the rate c^X. In particular we can obtain the discounting curve $P_X^{c^X}$ from the market. Remember that the discounting in currency Y for collateral in a currency X at rate c^X is done with the rate $c_t^X + p_t^{Y,X}$. Suppose that the collateral and forward points are, like in Section 8.2.3, such that

$$c_t^X + p_t^{Y,X} = c_t^Y + s_t$$

with s_t a $\mathcal{G}_t^2$-adapted random variable and $I^{Y,j}$ and $c^X + p^{Y,X}$ $\mathcal{G}_t^1$-adapted with $\mathcal{G}_t^2$ independent of $\mathcal{G}_t^1$. We don't know s yet (nor $p^{X,Y}$) but we suppose that the rates have the above relationship. For a price with common value V_u we have, using the remark after Theorem 8.5 and Equation (8.9),

$$\begin{aligned} V_0^{Y,c^X,X} &= P_Y^s(0,v) N_0^{c^Y} \mathrm{E}^Y\left[(N_v^{c^Y})^{-1} I_Y^j(t_0)\right] \\ &= P_Y^s(0,v) P_Y^{c^Y}(0,v) F^{Y,c^Y,Y}(0,u,v) \\ &= P_Y^{s+c^Y}(0,v) F^{Y,c^Y,Y}(0,u,v). \end{aligned}$$

The collapse of $P_Y^s P_Y^{c^Y}$ into $P_Y^{s+c^Y}$ is due to the independence hypothesis as noted in Equation (8.10).

The value of a fixed collateralised cash-flow N at time v can be written, by definition, as

$$V_0^{Y,c^X,X} = P_Y^{s+c^Y}(0,v) N.$$

In cross-currency swaps, the pricing of fixed coupons is required for the potential spreads and for the notional payments.

Suppose that cross-currency swaps are collateralised in domestic currency. By providing the spread in the cross-currency swap, the market is providing the price of the foreign leg with collateral in the domestic currency. On the theoretical side, the price can be obtained from two curves: the new discounting curve including the standard collateral and the unknown spread $P^{Y,s+c^Y}$ and the forward rate curve $F^{Y,c^Y,Y}$. The second curve is known from the foreign currency swap market. We are thus faced with solving a 'one curve and one set of constraints' problem. This new curve calibration can be done using the same techniques used for standard multi-curve framework calibration. The inputs are different but the numerical techniques are the same.

The main challenge is probably to keep track of the numerous inputs and relations between them to create a large Jacobian/transition matrix which covers all the inputs and outputs. Such a mechanism was described in Quantitative Research (2013) for the multi-curve framework. We extend it below to the collateral multi-currency framework.

8.3.6 Keeping track of transition matrices

To obtain the transition matrices between the market quotes and the curve parameters, one can directly compute a huge matrix with all the market quotes as input and all the curve parameters as the output of a very large root-solving problem.

This is possible, but with many currencies, many collaterals and many ways to combine them, one easily obtains 10 curves with 20 parameters each. Solving a 200 by 200 non-linear system of equations and computing the associated Jacobian matrix is probably not the most efficient way to achieve the result. As described above the calibration can be done in an inductive way for most of the curves. The last curves calibrated depend on all the previous curves, but at each step only a reduced system has to be solved. Usually the sub-system, which we call unit in the sequel, contains only one to three curves and not ten or more like the full system.

Suppose that the curves are obtained as a multidimensional root-finding process in an inductive way. The market quotes for instrument n are denoted m_n. The parameters to describe the calibrated curves are denoted p_n. There is the same number of market instruments as number of parameters for the curves. The parameters are grouped in units with indices $[n_{i-1}, n_i)$ where $n_{-1} = n_0 = 0$ and there are $n_i - n_{i-1}$ instruments and parameters in each unit.

The goal is to obtain the calibrated curves $(p_n)_{n\in[0,N)}$ from the market rates $(m_n)_{n\in[0,N)}$ but also the generalised transition matrix

$$\left(D_{m_l} p_n\right)_{n\in[0,N),l\in[0,N)}.$$

Due to the way the curves are built, we know that a good part of the matrix is full of zeros; the non-zero part, which we are interested in, is made of the sub-matrices

$$\left(D_{m_l} p_n\right)_{n \in [n_{i-1}, n_i), l \in [0, n_i)}.$$

The equations solved to obtain the parameters at the i-th step are:

$$s[n_{i-1}, n_i)(p[0, n_{i-1}), p[n_{i-1}, n_i)) = 0$$

where the notation $x[i,j)$ represents the vector with values $(x_k)_{k \in [i,j)}$. The functions s_i are the spreads between the market quotes and the quotes computed from the same instrument implied by the curves with parameters $p[0, n_i)$.

Suppose that at each step, the previous step transition matrices $D_{m_l} p_n$ are available for $n \in [n_{i-2}, n_{i-1})$ and $l \in [0, n_{i-1})$. We can compute $D_{p_n} m_l = D_{p_n} s_l$ for $n, l \in [n_{i-1}, n_i)$ directly when solving the root-finding for the current unit. Moreover

$$\left(D_{m_l} p_n\right)_{n,l \in [n_{i-1}, n_i)} = \left(\left(D_{p_n} m_l\right)_{n,l \in [n_{i-1}, n_i)}\right)^{-1}.$$

So we have the derivative with respect to the market quotes of the current unit. For the previous market quotes, we use composition. Suppose that we have $D_{p_l} p_n$ for $n \in [n_{i-1}, n_i)$ and $l \in [0, n_{i-1})$. Then

$$\left(D_{m_l} p_n\right)_{n \in [n_{i-1}, n_i), l \in [0, n_{i-1})} = \left(D_{p_l} p_n\right)_{n \in [n_{i-1}, n_i), l \in [0, n_{i-1})} \left(D_{m_k} p_l\right)_{l \in [0, n_{i-1}), k \in [0, n_{i-1})}$$

The second factor is provided by the previous steps; we still have to obtain the first factor. Using the implicit function theorem, we have

$$\left(D_{p_l} p_n\right)_{n \in [n_{i-1}, n_i), l \in [0, n_{i-1})} = -\left((D_{p_n} s_k)_{k \in [n_{i-1}, n_i), n \in [n_{i-1}, n_i)}\right)^{-1} (D_{p_l} s_k)_{k \in [n_{i-1}, n_i), l \in [0, n_{i-1})}.$$

We now have all the required results to keep track of the full transition matrix.

In the following example we use an implementation[3] of the above construction. We build the USD Fed Fund, USD Libor 3M, EUR Eonia and EUR Euribor 3M as in the standard multi-curve framework. The USD Libor 3M is the one for collateral at Fed Fund and the EUR Euribor 3M is the one for the collateral at Eonia. We now want to construct the EUR discounting curve for collateral in USD at Fed Fund. We use the above developments with the hypothesis of independence to calibrate the curve itself and to build the associated extended transition matrix.

The transition matrix will depend on all the previous curves. The EUR Dsc USD FedFund parameters depend on five curves' market rates. Using the notation of our implementation, we have the summary dependency in Table 8.1. It is interpreted

[3] The implementation we used is the OpenGamma OG-Analytics library. It is open source and available at developers.opengamma.com/downloads.

Table 8.1 Summarised representation of the dependency of the collateral curve to the other curves of the example

USD Dsc FedFund=[0, 13],
EUR Dsc Eonia=[13, 12],
USD Fwd Libor3M FedFund=[25, 8],
EUR Fwd Euribor3M USD FedFund=[33, 8],
EUR Dsc USD FedFund=[41, 13]

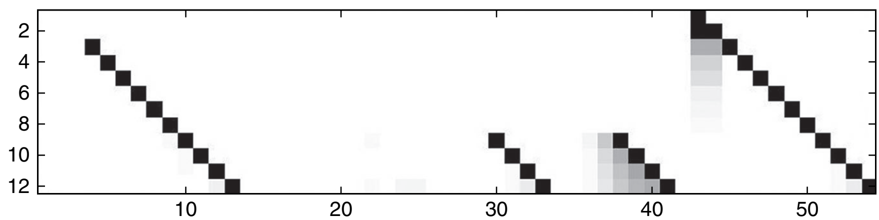


Figure 8.1 Visual representation of the transition matrix of the EUR discounting for collateral in USD with Fed Fund rates.

as follows: the generalised transition matrix depends on five curves (with the listed name), for each curve the numbers are the index of the first parameter of the curve and the number of parameters in the curve. Note that even if we have reused the Euribor 3M curves built with Eonia collateral for the forwards, we give to the copy a different name (EUR Fwd Euribor3M USD FedFund), even if we do not compute convexity adjustment for that curve. In this simplified example we have 54 parameters.

The total transition matrix for the last curve has 13 rows and 54 columns and is too large to be represented directly in this document. Instead we give a graphical representation of it in Figure 8.1. The dark squares indicate a strong dependency (absolute value above 0.10) and the grey squares a low dependency down to the white squares where the dependency is 0.

The discounting curve in EUR for collateral in USD obviously depends on the cross-currency instruments (FX swaps and cross-currency swaps), but also on the USD discounting (OIS), USD single-currency swaps and EUR single-currency swaps.

Using this huge quantity of information and managing the dependencies is probably the main challenge in collateralised cross-currency trading and risk management.

8.3.7 Collateral in foreign currency: cross-currency swaps

At the end of Section 8.2.6, we described how forex swap can be used to transfer information on collateral discounting from one currency to another. When the forex swaps between currencies X and Y are collateralised in X cash at rate c, their information can be used to obtain the pseudo-discount factors $P^{Y,c,X}$. In the multi-curve framework with collateral, this is only one small part of the requirement, as we also need the forward rates $F^{Y,c,X,j}$ to be able to price Ibor coupon swaps.

There are several ways to obtain these curves. One is to suppose some independence hypothesis like in Section 8.3.5 and use single-currency curve $F_X^{c,j}$. This is a hypothesis based approach without guarantee it will work in practice. A second approach is to compute the convexity adjustment like we will do in Section 8.4.3. This is a model based approach. The most direct approach is to use market quoted instruments to calibrate the figures. This approach works for a limited list of indexes and currencies for which such information is available.

This is the case when cross-currency swaps between two currencies with cash collateral in one of them are quoted on top of the forex swaps described above. Suppose cross-currency swaps between index j^X in currency X and index j^Y in currency Y (plus spread S) with collateral at rate c in currency X are quoted in the market. This would be the case for example for USD/EUR cross-currency swaps collateralised in USD. Using forward notation, the price of such a cross-currency swap is

$$\sum_{i=1}^{n} P_X^c(0,t_i)\delta_i F_X^{c,j^X}(0,t_{i-1},t_i) - \sum_{i=1}^{n} P^{Y,c,X}(0,t_i)\delta_i(F^{Y,c,X,j^Y}(0,t_{i-1},t_i)+S).$$

From the c-OIS and j^X-IRS c-collateralised in X, we can obtain P_X^c and F_X^{c,j^X}. From the forex swaps we can obtain the pseudo-discount factors $P^{Y,c,X}$. We are left with one unknown, F^{Y,c,X,j^Y}, and one constraint, which is the present value using market quote spread S should be 0. We write the explanation as a consecutive curve calibration exercise, but it can obviously be solved as a simultaneous curve calibration process if required.

This approach does not require any hypothesis but requires instruments quoted in the market. In practice those instruments are available only for a small set of currencies and usually only for one index by currency. The most common cross-currency swaps are based on three month tenors.

To come back to our USD/EUR example, this approach would be suitable to obtain the forward curve for Euribor three months collateralised at Federal Funds in USD, if we consider that the cross-currency market quotes are valid for USD Federal Funds collateral. The other indexes, like Euribor six months or Euribor one month, need to be obtained in a different way.

8.3.8 Collateral in foreign currency: correct curves with wrong instruments

We work in a set-up similar to the conclusion of Section 8.3.5 and foreign currency collateral. Suppose that cross-currency swaps are collateralised in domestic currency and the foreign currency swaps are collateralised in foreign currency. Can we calibrate the discount factors and the forward rates directly and not by reusing the forward rates of a first calibration like before?

A priori we can not combine the domestic currency collateralised leg of the cross-currency swaps with foreign currency collateralised instruments to coherently calibrate the curves.

The start of the answer, on the short part of the curve, is yes. If the forward rates are by hypothesis the same, the quotes of the FRAs collateralised in foreign currency are the same as the quote of the FRAs collateralised in domestic currency. The present value is collateral dependent, but the quote, which is the K such that the present value is zero:

$$P_X^c(0,v)\frac{\delta}{\beta}(F_0^{c,j}(0,u,v)-K)=0$$

is the same for any P_X^c. The part of the curve where FRAs are used to build the foreign currency forward curve can be combined with forex swaps and cross-currency swaps in a coherent way. The quotes are the same for any collateralisation (under the 'same forward' hypothesis).

When we move to the part of the curve calibrated to vanilla swaps, the same is not true. The swap rate is a weighted average of the forward rates (this is good) but the weights are discount factor dependent (this is bad). Using the notation of Equation (8.19),

$$S_t^{X,c,Y,j}=\frac{\sum_{i=1}^{n} P_X^{c+p^{X,Y}}(t,t_i)\delta_i F^{X,c,Y,j}(t,t_{i-1},t_i)}{\sum_{i=1}^{\tilde{n}} P_X^{c+p^{X,Y}}(t,\tilde{t}_i)\tilde{\delta}_i}.$$

This weight is independent of the discounting in the very particular case where the forward curve is constant and floating and fixed payments have the same frequency and convention (like GBP and AUD). When the forward rates are not constant, is the difference acceptable for typical change of collateral rate and a typical curve shape? To answer that question, we have to define 'acceptable' and 'typical'. Here we loosely define 'totally acceptable' as within the tick value of the instruments and 'acceptable' as within the bid-offer of the instrument. For the graphs, we set the tick value at 0.1 basis points and the bid-offer at one basis point.

For the typical value of the spread, we have to remember what the spread is. It is not the difference between the interest rates in the two currencies but the difference between the collateral in one currency plus the forex implied forward points and the collateral in the other currency. If the risk-free rates exists and collateral is done

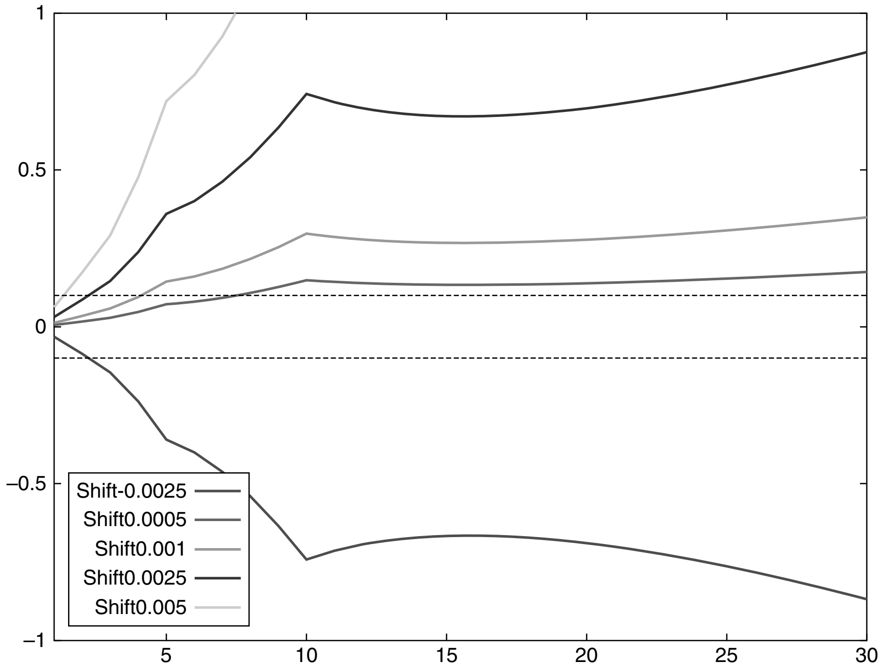


Figure 8.2 Differences in swap rates for different collateral discounting curves. Swap with tenors between 1 and 30 years. Figures for GBP swaps.

at risk-free rate, the spread is 0. The spread can be estimated by the cross-currency OIS spread. The market where the spread is the most liquid is EUR/USD and the spread is currently between −7 bps (short term) and −27.5 bps (long term). The spreads were generally lower, that is, more negative, by as much as 20 bp over last year. For the GBP/USD the spread is currently between −2 and −15 basis points. We can think of 'typical' as anything between −30 and +30 bps. Before the crisis, those spreads were typically −5 to +5 basis points.

In Figure 8.2, the difference for swap rates for different collateral discounting curves are displayed. The figures are computed for GBP swaps (with standard GBP conventions). The initial forward pseudo-discount factors curve is very steep with short rates at 1.00% and long term rates at 4.00%. The difference is very acceptable for spreads up to 5 bps. It is also acceptable for spreads below 25 bps.

The lower difference for shorter swaps is understandable as for shorter swaps the forward rates have less variability and the approximation is better. This leads to a possible strategy to reduce the difference: using shorter swaps for all the curves. This can be achieved by using forward swaps. In Figure 8.3(a) the difference is displayed for forward swaps of tenor one year (the X-axis represents the start tenor of the swap). For those short swaps, the difference is totally acceptable. Even if the

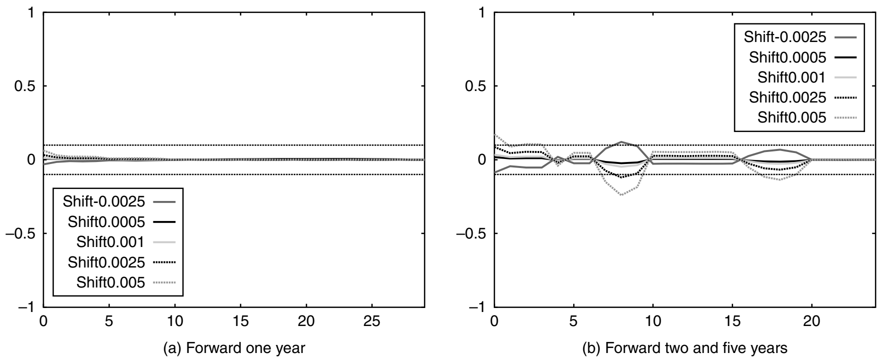


Figure 8.3 Differences in swap rates for different collateral discounting curves. Forward swaps with start year given by the X-axis and tenor given in the subfigure. Figures for GBP swaps.

instruments are not correct (they are collateralised in the domestic currency) they produce the correct information (the correct swap rates for swap collateralised in foreign currency) for any collateralisation spread.

Using one year swaps is maybe too much too ask. In Figure 8.3 we have displayed the difference for swaps with tenor 2 years for the starting years 0 to 7 and forward swaps with tenor 5 years for longer starting periods. The differences are totally acceptable for most of the starting points and tenors. The domestic currency collateral pseudo-discount factors in foreign currency can be computed from the foreign currency (forward) swap rates and forex or cross-currency swaps collateralised in domestic currency with small differences.

The same technique can be used, even more efficiently, for basis swap. The spread in a basis swap (quoted on leg j^1) is computed from the curves by

$$S_t^{X,c,Y,j^1,j^2} = \frac{\sum_{i=1}^{n} P_X^{c+p^{X,Y}}(t,t_i)\delta_i F^{X,c,Y,j^2}(t,t_{i-1},t_i) - \sum_{i=1}^{\tilde{n}} P_X^{c+p^{X,Y}}(t,\tilde{t}_i)\tilde{\delta}_i F^{X,c,Y,j^1}(t,\tilde{t}_{i-1},\tilde{t}_i)}{\sum_{i=1}^{\tilde{n}} P_X^{c+p^{X,Y}}(t,\tilde{t}_i)\tilde{\delta}_i}.$$

This time, the results will be independent of the level of rate if the spread between the two forward curves is relatively constant. The shape of the curve is not the most important feature; the shape of the spread curve is more important. The spread curve is usually more stable than the forward curve itself.

In Figure 8.4 we display the impact of change of discounting curve on computing the basis swap spread for GBP Libor three months plus spread versus GBP Libor six months swaps. The initial spread between the curves is 15 basis points at the front end and 10 basis points at the long end of the curve. The difference between the quotes with discounting curve as far as 50 basis points apart is still acceptable, with the maximum difference being 0.2 bp for a 50 basis points change. When

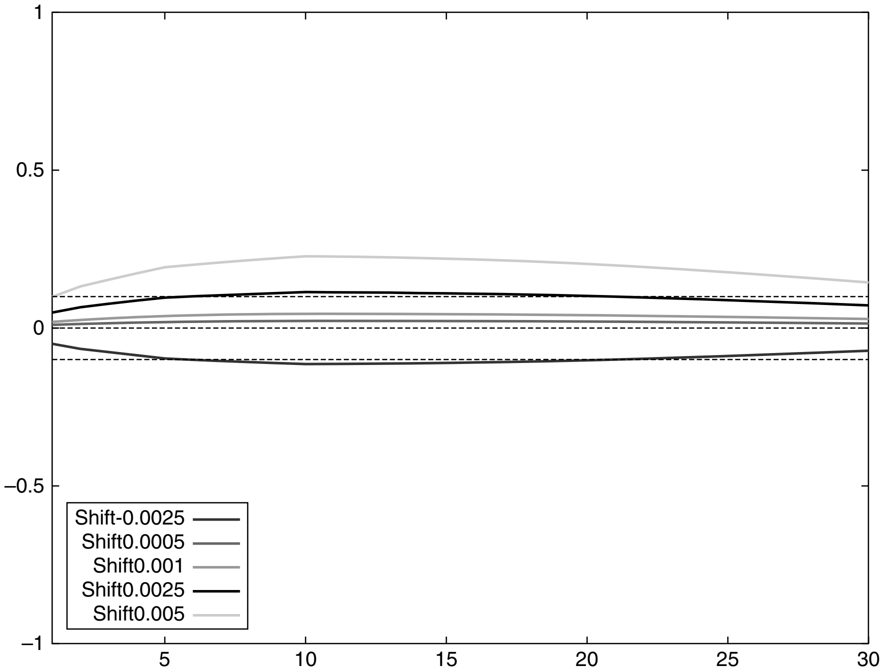


Figure 8.4 Differences in basis swap spreads for different collateral discounting curves. Swap with tenors between 1 and 30 years. Figures for GBP swaps.

the discounting curves are less than 25 basis points apart, the difference is very acceptable.

In general, as a first good approximation, when curves have been built for one tenor and collateral in another currency, it is relatively easy to build the curves with other tenors using the same currency basis swaps. Any adjustment required on one leg is also required on the other leg; the adjustments tend to cancel. The at-the-money market quotes for basis swaps with one collateral are very likely to be close to the at-the-money quote with the other collateral. The remark is valid only of at-the-money quotes, not for out-of-the-money collateral quotes, as described in Equation (8.13) which are strongly discounting curve dependent.

8.3.9 Change of collateral: general case

In this section we give a general description of the pricing of swap-related products for general change of collateral (including foreign currency collateral). The result is written as a general *convexity adjustment* result. To be really useful in practice, the result should be made more explicit with a modelling of the different components and an estimate of the adjustment value. This will be done for an HJM model on the collateral curve and stochastic basis in Section 8.4.3.

Theorem 8.8 *The forward rate with collateral $c+s$ can be written as a function of the forward rate with collateral c through*

$$F_X^{c+s,j}(t,u,v) = F_X^{c,j}(t,u,v)\left(1+\gamma^{c,v,s}(t,u,v)\right)$$

with

$$\gamma^{c,v,s}(t,u,v) = \frac{P_X^c(t,v)}{P_X^{c+s}(t,v)} \frac{\operatorname{cov}^{c,v}\left[N_t^s(N_v^s)^{-1}, I_X^j(t_0)\middle|\mathcal{F}_t\right]}{F_X^{c,j}(t,u,v)}.$$

Proof: The value of a coupon with collateral c_t+s_t is given, by the definition of $F_X^{c+s,j}$, by

$$P_X^{c+s}(t,v)F_X^{c+s,j}(t,u,v).$$

Using Theorem 8.1, the price is also

$$\begin{aligned}
V_t^{c+s} &= N_t^{c+s}\mathrm{E}^{\mathbb{X}}\left[(N_v^{c+s})^{-1}I_X^j(t_0)\middle|\mathcal{F}_t\right]\\
&= N_t^s P_X^c(t,v)\,\mathrm{E}^{c,v}\left[(N_v^s)^{-1}I_X^j(t_0)\middle|\mathcal{F}_t\right]\\
&= N_t^s\,\mathrm{E}^{c,v}\left[(N_v^s)^{-1}\middle|\mathcal{F}_t\right]P_X^c(t,v)F_X^{c,j}(t,u,v)\\
&\quad\left(1+\frac{\operatorname{cov}^{c,v}\left[(N_v^s)^{-1}, I_X^j(t_0)\middle|\mathcal{F}_t\right]}{\mathrm{E}^{c,v}\left[(N_v^s)^{-1}\middle|\mathcal{F}_t\right]F_X^{c,j}(t,u,v)}\right).
\end{aligned}$$

From the definition of $\mathrm{E}^{c,v}[]$, we have

$$\begin{aligned}
N_t^s\,\mathrm{E}^{c,v}\left[(N_v^s)^{-1}\middle|\mathcal{F}_t\right] &= N_t^s(P_X^c(t,v))^{-1}P_X^c(t,v)\,\mathrm{E}^{c,v}\left[(P_X^c(v,v))^{-1}(N_v^s)^{-1}\middle|\mathcal{F}_t\right]\\
&= N_t^s(P_X^c(t,v))^{-1}N_t^c\,\mathrm{E}^X\left[(N_v^cN_v^s)^{-1}\middle|\mathcal{F}_t\right]\\
&= P_X^{c+s}(t,v)(P_X^c(t,v))^{-1}.
\end{aligned}$$

Combining the two results, we have

$$F_X^{c+s,j}(t,u,v) = F_X^c(t,u,v)\left(1+\gamma^{c,v,s}(t,u,v)\right)$$

as announced. □

8.4 Modelling with collateral: collateral HJM model

8.4.1 Collateral curve

The results described in this section are inspired by Fujii et al. (2011), Fujii and Takahashi (2013) and Pallavicini and Brigo (2013). From the definition of $P^c_X(t,v)$, the ratio $P^c_X(t,v)/N^c_t$ is a martingale in the $\mathbb{X}$ measure. If we model the ratio by a diffusion equation, we need an equation without drift. We write it, in a way reminiscent of Heath et al. (1992) modelling of the risk-free curve, as

$$d\left(\frac{P^c_X(t,v)}{N^c_t}\right) = -\frac{P^c_X(t,v)}{N^c_t}\nu^c(t,v)\cdot dW^{\mathbb{X}}_t \tag{8.22}$$

where $W^{\mathbb{X}}_t$ is an n-dimensional Brownian motion in the standard measure $\mathbb{X}$. The volatility ν^c can be stochastic. In all cases, we have $\nu^c(t,t) = 0$. If we write the equation for $P^c_X(t,v)$, taking into account the definition of N^c, we have

$$dP^c_X(t,v) = P^c_X(t,v)c_t\, dt - P^c_X(t,v)\nu^c(t,v)\cdot dW^{\mathbb{X}}_t. \tag{8.23}$$

Using the above equation, the definition of the collateral forward rate, Definition 8.4, and Ito's lemma, we obtain the following equation for the collateral investment factor $1+\delta F^c_t(u,v)$:

$$\begin{aligned}
&d(1+\delta F^c_t(u,v))\\
&\quad= \frac{dP^c(t,u)}{P^c(t,v)} - \frac{P^c(t,u)}{(P^c(t,v))^2}dP^c(t,v)\\
&\qquad+\left(-\frac{P^c(t,u)}{P^c(t,v)}\nu^c(t,u).\nu^c(t,v) + \frac{P^c(t,u)}{P^c(t,v)}|\nu^c(t,v)|^2\right)dt\\
&\quad= (1+\delta F^c_t(u,v))\big((\nu^c(t,v)-\nu^c(t,u))\cdot dW^{\mathbb{X}}_t + \nu^c(t,v)\cdot(\nu^c(t,v)-\nu^c(t,u))dt\big).
\end{aligned}$$

This leads to the following solution for the collateral forward rates.

Lemma 8.1. (HJM dynamic of collateral forward rates). *In the collateral HJM model, the collateral forward rates satisfy in the standard measure $\mathbb{X}$, for $t \geq s$,*

$$\begin{aligned}
1+\delta F^c_t(u,v) = (1+\delta F^c_s(u,v))\exp\left(-\alpha_c(s,t,u,v)X^{c,\mathbb{X}}_{s,t} - \tfrac{1}{2}\alpha^2_c(s,t,u,v)\right)\\
\gamma_c(s,t,u,v)
\end{aligned} \tag{8.24}$$

with

$$\alpha_c(s,t,u,v)X^{c,\mathbb{X}}_{s,t} = \int_s^t (\nu^c(\tau,u)-\nu^c(\tau,v))\cdot dW^{\mathbb{X}}_\tau,$$

$$\alpha^2_c(s,t,u,v) = \int_s^t |\nu^c(\tau,u)-\nu^c(\tau,v)|^2 d\tau$$

and

$$\gamma_c(s,t,u,v) = \exp\left(\int_s^t \nu^c(\tau,v)\cdot(\nu^c(\tau,v)-\nu^c(\tau,u))d\tau\right).$$

The result above is in the standard measure $\mathbb{X}$. The same result could be written in the measure associated to $\mathrm{E}^{c,v}[]$ in which case there would be no adjustment γ_c as the investment factor is a martingale in that measure.

By Equation (8.22), the quantity $P^c_X(.,v)/N^c_.$ is an exponential martingale and the change of numeraire density defined by Equation (8.7) is

$$L^{c,v}_u = \exp\left(-\int_0^u \nu^c(\tau,v)\cdot dW^{\mathbb{X}}_\tau - \frac{1}{2}\int_0^u |\nu^c(\tau,v)|^2 d\tau\right).$$

By Girsanov's theorem (see for example (Lamberton and Lapeyre, 1997, Théorème 2.2)), the Brownian motion for the measure associated to $\mathrm{E}^{c,v}[]$ is

$$W^{c,v}_t = W^{\mathbb{X}}_t + \int_0^t \nu^c(\tau,v)\, d\tau.$$

The change of numeraire between $W^{c_1,v}$ and $W^{c_2,v}$ for two collateral rates c_1 and c_2 is

$$W^{c_1,v}_t = W^{c_2,v}_t + \int_0^t \left(\nu^{c_1}(\tau,v)-\nu^{c_2}(\tau,v)\right) d\tau.$$

The same lemma can be written in terms of the discount factors, instead of the investment factor $1+\delta F^c$. By inverting the above lemma, we have the following.

Lemma 8.2. (HJM dynamic of collateral discount factors). *In the collateral HJM model, the collateral discount factors satisfy, for $t \geq s$,*

$$\frac{P^c_X(t,v)}{P^c_X(t,u)} = \frac{P^c_X(s,v)}{P^c_X(s,u)}\exp\left(\alpha_c(s,t,u,v)X^{c,\mathbb{X}}_{s,t} - \frac{1}{2}\alpha^2_c(s,t,u,v)\right)\gamma_c(s,t,v,u). \tag{8.25}$$

We now detail a specific model where the above equation for P^c_X is satisfied.

When the discount curve $P^c_X(t,.)$ is absolutely continuous, which is something that is always the case in practice as the curve is constructed by some kind of interpolation, the equivalent of the instantaneous forward rate can be defined for the collateral pseudo-discount factors.

Definition 8.10 *Suppose that the pseudo-discount factor curve $P^c_X(t,.)$ is absolutely continuous. The* instantaneous forward collateral rate *$f^c_X(t,u)$ is defined by*

$$P^c_X(t,u) = \exp\left(-\int_t^u f^c_X(\tau,u)d\tau\right).$$

This is equivalent to, for a weak differentiation,

$$f_X^c(t,u) = -D_u \ln(P_X^c(t,u)).$$

Let $\sigma^c : \mathbb{R}^2 \to \mathbb{R}^n$ and

$$\nu^c(t,u) = \int_t^u \sigma^c(t,\tau)d\tau.$$

From Equation (8.23) and Ito's lemma, we have

$$\ln(P_X^c(t,u)) - \ln(P_X^c(0,u)) = \int_0^t \left(c_\tau - \frac{1}{2}\nu^2(\tau,u) \right) d\tau + \nu(\tau,u)\cdot dW_\tau^{\mathbb{X}}.$$

Taking the opposite of the derivative with respect to u on both sides, and exchanging integral and derivative in the right-hand side term, we obtain

$$df_X^c(t,u) = \sigma^c(t,u)\cdot\nu^c(t,u)dt + \sigma^c(t,u)\cdot dW_t^{\mathbb{X}}.$$

in the standard measure.

We defined the short rate associated with the instantaneous forward rate by $r_t^c = f_X^c(t,t)$. We will show below that the short rate r^c is equal to the collateral rate c.

By definition of r^c,

$$\begin{aligned} r_\tau^c = f_X^c(\tau,\tau) &= f_X^c(u,\tau) + \int_u^\tau df_X^c(s,\tau)ds \\ &= f_X^c(u,\tau) + \int_u^\tau \nu^c(s,\tau)\cdot D_2\nu^c(s,\tau)ds + \int_u^\tau D_2\nu^c(s,\tau)\cdot dW_s^{\mathbb{X}}. \end{aligned}$$

Then using Fubini, we have

$$\int_u^v r_\tau^c d\tau = \int_u^v f_X^c(u,\tau)d\tau + \frac{1}{2}\int_u^v |\nu^c(s,v)|^2 ds + \int_u^v \nu^c(s,v)\cdot dW_s^{\mathbb{X}}$$

where we have used the fact that $\nu^c(s,s) = 0$.

The result is obtained by taking the exponential of the last equality and using the definition of f_X^c:

$$\begin{aligned} N_u^{r^c}(N_v^{r^c})^{-1} = \exp\left(-\int_u^v r_\tau^c d\tau\right) &= P_X^c(u,v) \\ &\times \exp\left(-\int_u^v \nu^c(s,v)\cdot dW_s^{\mathbb{X}} - \frac{1}{2}\int_u^v |\nu^c(s,v)|^2 ds\right). \end{aligned}$$

Using the above equality and the above lemma for pseudo-discount factors, we have that

$$\frac{P_X^c(t,v)}{N_t^{r^c}} = \frac{P_X^c(0,v)}{N_0^{r^c}} \exp\left(-\int_0^t \nu^c(s,v)\cdot dW_s^{\mathbb{X}} - \frac{1}{2}\int_0^t |\nu^c(s,v)|^2 ds\right).$$

This implies that

$$dP_X^c(t,v) = P_X^c(t,v)r_t^c - P_X^c(t,v)\nu^c(t,v)\cdot dW_t^{\mathbb{X}}.$$

Combined with Equation (8.23), it proves that $r_t^c = c_t$.

Lemma 8.3. (HJM dynamic of collateral cash account). *In the collateral HJM model, the collateral cash account $N_.^c$ satisfies the equation*

$$N_u^c(N_v^c)^{-1} = P_X^c(u,v)\exp\left(-\int_u^v \nu^c(s,v)\cdot dW_s^{\mathbb{X}} - \frac{1}{2}\int_u^v |\nu^c(s,v)|^2 ds\right). \quad (8.26)$$

The equivalence between the collateral rate c and the short term rate r^c can be obtained more directly (Fujii 2013). Using the definition of f_X^c, of P_X^c and exchanging differentiation and expectation, we have

$$\begin{aligned} f_X^c(t,u) &= -D_u \ln\left(P_X^c(t,u)\right) \\ &= \frac{1}{P_X^c(t,u)} \mathrm{E}^{\mathbb{X}}\left[\exp\left(-\int_t^u c_\tau\, d\tau\right) c_u \middle| \mathcal{F}_t\right] \end{aligned}$$

If one takes $u = t$ in the above equation, one obtains

$$r^c(t) = f_X^c(t,t) = \mathrm{E}^{\mathbb{X}}\left[\, c(t)|\mathcal{F}_t\right] = c(t).$$

Using the above expression of $f_X^c(t,u)$ as an expectation and changing the measure, one has

$$f_X^c(t,u) = \mathrm{E}^{c,u}\left[\, c(u)|\mathcal{F}_t\right].$$

The instantaneous forward collateral rate is the expected value in the c-collateral u-forward measure of the collateral rate.

Spread description

Suppose that $c_2 = c_1 + s$ and c_1 and c_2 follow collateral HJM models with volatilities ν^{c_1} and ν^{c_2} respectively. Then the spread s satisfies

$$\begin{aligned} (N_t^s)^{-1} = (N_t^{c_2})^{-1}(N_t^{c_1}) = \frac{P_X^{c_2}(0,t)}{P_X^{c_1}(0,t)} \exp\Bigg(-\int_0^t (\nu^{c_2}(\tau,t) - \nu^{c_1}(\tau,t))\cdot dW_\tau^{\mathbb{X}} \\ -\frac{1}{2}\int_0^t |\nu^{c_2}(\tau,t)|^2 - |\nu^{c_1}(\tau,t)|^2 ds\Bigg). \end{aligned}$$

The dynamic of the spread $s = c_2 - c_1$ is linked to the volatility

$$\nu^s(\tau,v) = \nu^{c_2}(\tau,t) - \nu^{c_1}(\tau,t).$$

Nevertheless note that in general

$$P^s(0,t) = \mathrm{E}^{\mathbb{X}}\left[(N_t^s)^{-1}\right] \neq \frac{P^{c_2}(0,t)}{P^{c_1}(0,t)}$$

as the second part of the exponential is not the variance of the first. The equality is satisfied for all t only if $(\nu^{c_2}(\tau,t) - \nu^{c_1}(\tau,t)) \cdot \nu^{c_1}(\tau,t) = 0$ for almost all τ, that is, ν^s is orthogonal to ν^{c_1}. In some senses we have obtained the converse of Equation (8.10).

Theorem 8.9 [Orthogonality of volatilities]. *In the HJM collateral framework, the pseudo-discount factor of the spread satisfies $P_X^{c_2}(0,t) = P_X^{c_1}(0,t)P_X^s(0,t)$ for all t if and only if the volatility of the spread $\nu^s(\tau,t)$ is orthogonal to the volatility $\nu^{c_1}(\tau,t)$ for almost all (τ,t) with $\tau \leq t$.*

8.4.2 STIR futures

STIR futures are an important instrument to include in any multi-curve and collateral framework discussion. Not only is it a very liquid instrument and an important source of market information, but from a technical point of view it involves a mechanism of margin payment equivalent to a zero rate collateral. An understanding of the collateral impact would not be complete without a detailed analysis of those instruments. Technically, this is a simplified version of the change of collateral presented later.

In this section we provide an explicit pricing formula for STIR futures in a Gaussian HJM model using as data the collateral curve and the collateralised forward curve. By Gaussian HJM, we mean we use the curve modelling described in the previous section with ν^c deterministic. The results described are very close to those described in Henrard (2013c) and in large part based on Henrard (2005).

For this section, we denote by $\mathcal{G}_t^1$ the filtration generated by the collateral rate and the N^X numeraire. That is N_t^c, $P_x^c(t,c)$ and N_t^X are $\mathcal{G}_t^1$-adapted. As F^c is a martingale in the $E^{c,v}$ numeraire,

$$1 + \delta F_t^c(u,v) = (1 + \delta F_s^c(u,v)) \exp\left(-\alpha_c(s,t,u,v)X_{s,t}^{c,v} - \tfrac{1}{2}\alpha_c^2(s,t,u,v)\right) \quad (8.27)$$

with

$$\alpha_c(s,t,u,v)X_{s,t}^{c,v} = \int_s^t (\nu^c(\tau,u) - \nu^c(\tau,v))dW_\tau^{c,v}.$$

and $W^{c,v}$ the Brownian motion associated to the $\mathrm{E}^{c,v}\,[]$ measure.

The main hypothesis on the index basis is as follows.

SjS: The multiplicative spread $\beta_t^{c|c,j}$ is written as

$$\beta_t^{c|c,j}(u,v) = \beta_0^{c|c,j}\mathcal{X}_t^{c|c,j}\left(\frac{1+\delta F_t^c}{1+\delta F_0^c}\right)^a x^{c|c,j}(t,u,v)$$

with $\mathcal{X}_t$ $\mathcal{G}_t^2$-adapted with $\mathcal{G}_t^2$ independent of $\mathcal{G}_t^1$ and $\mathcal{X}_t$ a $\mathbb{X}$-martingale with independent relative increments.

By independent relative increments, we mean that $\mathcal{X}_t/\mathcal{X}_s$ is independent of $\mathcal{F}_s$. In particular, this will be the case if $\mathcal{X}_t$ is a geometric Brownian motion. The deterministic function $x^{c|c,j}(t)$ is selected such that the rate $F^{c,j}$ is a martingale in $\mathrm{E}^{c,v}[]$ as required.

In our modelling assumptions, we do not allow the spread β to depend on the risk-free rate, if such a rate exists. The spread depends only on the collateral rate c and a random variable $\mathcal{X}$ independent of c and of the standard numeraire N^X. We restrict the dependency on quantities provided by the market.

Note that as $\mathcal{X}_t$ is a martingale in $\mathrm{E}^X[]$ and $\mathcal{X}_t$ is $\mathcal{G}_t^2$-adapted and the numeraire of $\mathbb{X}$ and $\mathrm{E}^{c,v}[]$ are $\mathcal{G}_t^1$ adapted, then $\mathcal{X}_t$ is also a martingale in $\mathrm{E}^{c,v}[]$.

The choice of $x^{c|c,j}$ is similar to the one in Henrard (2013c) and is obtained, for $t < s$, through

$$\begin{aligned}
1+\delta F^{c,j}(t) &= \mathrm{E}^{c,v}\left[1+\delta F^{c,j}(s)\middle|\mathcal{F}_t\right]\\
&= \mathrm{E}^{c,v}\left[\frac{(1+\delta F^c(s))^{1+a}}{(1+\delta F^c(t))^a}\beta_t\frac{\mathcal{X}_s}{\mathcal{X}_t}\frac{x(s)}{x(t)}\middle|\mathcal{F}_t\right]\\
&= \mathrm{E}^{c,v}\left[\exp\left(-\frac{1}{2}(1+a)\alpha_c(t,s)X_{t,s}^{c,v}-\frac{1}{2}(1+a)^2\alpha_c^2(t,s)\right)\frac{\mathcal{X}_s}{\mathcal{X}_t}\middle|\mathcal{F}_t\right]\\
&\quad\times(1+\delta F^{c,j}(t))\frac{x^{c|c,j}(s)}{x^{c|c,j}(t)}\exp\left(\frac{1}{2}(1+a)a\alpha_c^2(t,s)\right)
\end{aligned}$$

Given that the exponential is a martingale in the $\mathrm{E}^{c,v}[]$ numeraire and $X_{s,t}$ and $\mathcal{X}_t/\mathcal{X}_s$ are independent of $\mathcal{F}_s$, the expected value is 1. The equality is true only if

$$x^{c|c,j}(t,u,v)=\exp\left(-\frac{1}{2}(1+a)a\alpha_c^2(0,t,u,v)\right).$$

Theorem 8.10 (Futures price in collateral framework with stochastic spread). *Let $0\le t\le t_0\le u\le v$. In the multi-curve and collateral framework with the collateral HJM model and* ***SjS*** *hypothesis on the spread, the price of the futures fixing in t_0 for the period $[u,v]$ with accrual factor δ is given in t by*

$$\Phi_t^j = 1+\frac{1}{\delta}-\frac{1}{\delta}(1+\delta F_X^{c,j}(t,u,v))\gamma_c^{1+a}(t,t_0,u,v). \tag{8.28}$$

Proof: We use the generic pricing futures price process theorem which is equivalent to the collateral pricing formula with 0 collateral rate,

$$\Phi_t^j=\mathrm{E}^{\mathbb{X}}\left[1-I_X^j(t_0)\middle|\mathcal{F}_t\right].$$

The Ibor rate can be written as

$$1-I_X^j(t_0)=1+\frac{1}{\delta}-\frac{1}{\delta}\beta_t^{c|c,j}(u,v)\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\frac{\left(1+\delta F_X^c(t_0,u,v)\right)^{1+a}}{(1+\delta F_X^c(t,u,v)^a}\frac{x^{c|c,j}(t_0)}{x^{c|c,j}(t)}.$$

Using Lemma 8.1 and the definition of $x^{c|c,j}$ we have

$$1 - I_X^j(t_0) = 1 + \frac{1}{\delta} - \frac{1}{\delta}\beta_t^{c|c,j}(u,v)\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\left(1+\delta F_X^c(t,u,v)\right)$$
$$\times \exp\left(-(1+a)\alpha_c(t,t_0)X_{t_0}^c - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0)\right)\gamma_c^{1+a}(t,t_0,u,v).$$

The variables X_{t,t_0}^c and $\mathcal{X}_{t_0}/\mathcal{X}_t$ are independent of $\mathcal{F}_t$ and independent of each other, so

$$\begin{aligned}
&\mathrm{E}^N\left[\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\exp\left(-(1+a)\alpha_c(t,t_0)X_{t_0}^c - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0)\right)\middle|\mathcal{F}_t\right]\\
&= \mathrm{E}^N\left[\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\exp\left(-(1+a)\alpha_c(t,t_0)X_{t_0}^c - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0)\right)\right]\\
&= \mathrm{E}^N\left[\frac{\mathcal{X}_{t_0}}{\mathcal{X}_t}\right]\mathrm{E}^N\left[\exp\left(-(1+a)\alpha_c(t,t_0)X_{t_0}^c - \frac{1}{2}(1+a)^2\alpha_c^2(t,t_0)\right)\right] = 1
\end{aligned}$$

From there the result is direct. □

As both collateralised Ibor coupon based instruments, like swaps, and futures are liquid in most markets, the above result can be used to calibrate part of the model. With the market providing information on both the collateralised forward rates $F_X^{c,j}$ and the futures prices Φ_t^j, one can obtain the market implied quantity γ_c^{1+a}. The quantity can be used as an input to the pricing of other instruments. In particular Henrard (2013c) shows how to use that information (together with cap/floor prices) to obtain the price of futures options coherent with both futures and cap/floor market and very light calibration.

8.4.3 Change of collateral: explicit formulas in collateral HJM

Suppose that we have two collateral rates c_1 and c_2. We suppose that both those rates satisfy the HJM model described in Section 8.4.1 with volatilities ν^{c_1} and ν^{c_2}. Let $\mathcal{G}_t^1$ be a filtration such that N^X and the rates c_1, c_2, F^{c^1} and F^{c^2} are $\mathcal{G}_t^1$-adapted.

Our goal is to write the forward rate with respect to one collateral $F^{c_2,j}$ as a function of the known rate with the other collateral $F^{c_1,j}$ and some explicit adjustments. In practice this would be used in particular for computing the forward in one currency for collateralisation in a foreign currency from the forwards with the natural collateralisation in the domestic currency.

To achieve that result, we use a modelling framework of stochastic spreads very similar to the one of the previous section for STIR futures. We work directly with the forward rate $F^{c_1,j}$ and write it through its dependent and independent parts.

We use the following hypothesis with $\mathcal{G}_t^2$ a filtration independent of $\mathcal{G}_t^1$

SjS2: The forward rate $F^{c_1,j}$ is written as

$$\frac{1+\delta F^{c_1,j}(t,u,v)}{1+\delta F^{c_1,j}(0,u,v)} = \mathcal{X}_t^{c_1,j,c_2} x^{c_1,j,c_2}(t,u,v) \left(\frac{1+\delta F^{c_1}(t,u,v)}{1+\delta F^{c_1}(0,u,v)}\right)^{a_1} \left(\frac{1+\delta F^{c_2}(t,u,v)}{1+\delta F^{c_2}(0,u,v)}\right)^{a_2}$$

with $\mathcal{X}_t^{c_1,j,c_2}$ a $\mathcal{G}_t^2$-adapted random variable and a martingale in the measure $\mathbb{X}$. The deterministic function x^{c_1,j,c_2} will be selected to ensure that the forward rate $F^{c_1,j}$ is a martingale in $\mathrm{E}^{c,v}[]$. Note that, like in the previous section, due to the fact that $\mathcal{X}^{c_1,j,c_2}$ is an $\mathbb{X}$-martingale and due to the independence, $\mathcal{X}^{c_1,j,c_2}$ is also a $\mathrm{E}^{c_1,v}[]$-martingale.

We could have equivalently, like in the previous section, written the hypothesis on $\beta^{c_1|c_1,j}$. We prefer the more direct hypothesis here as we are looking at results on $F^{c,j}$.

We need to indicate F^{c_2} explicitly in the dependency and not hide it partly in $\mathcal{X}$ like in the previous section. The reason is that the adjustment will depend on the co-movement of Ibor rates and the two pseudo-discount curves, as described in Section 8.3.9. Even if the rate $F^{c_1,j}$ can be written as function of F^{c_1} and an independent part, we need to extract from that independent part what is related to the new collateral c_2.

The choice of a_1 and a_2 will never be unique. If F^{c_1} and F^{c_2} are correlated, we can transfer some dependency from a_1 to a_2 and stuff the remaining independent part in $\mathcal{X}$.

The developments are based on the two following results. The first one is equivalent to Equation (8.24) and the second one describes the dynamic of the collateral account given by Lemma 8.3 for each rate in the standard expectation:

$$1+\delta F^{c_i}(t,u,v) = (1+\delta F^{c_i}(s,u,v)) \exp\left(-\int_s^t (\nu^{c_i}(\tau,u)-\nu^{c_i}(\tau,v))dW_\tau^{\mathbb{X}} \right.$$
$$\left. -\frac{1}{2}\int_s^t (|\nu^{c_i}(\tau,u)-\nu^{c_i}(\tau,v)|^2)d\tau\right)\gamma_{c_i}(s,t)$$

and

$$(N_v^{c_i})^{-1} = P_X^{c_i}(0,v)\exp\left(-\int_0^v \nu^{c_i}(\tau,v)\cdot dW_\tau^{\mathbb{X}} - \frac{1}{2}\int_0^v |\nu^{c_i}(\tau,v)|^2 d\tau\right).$$

We first obtain the value of x^{c_1,j,c_2} like in the previous section. We know that $1+\delta F^{c_1,j}$ is a martingale in $\mathrm{E}^{c_1,v}[]$. Using exactly the same technique as in the previous section, we obtain, after a lengthy but relatively straightforward computation

$$
\begin{aligned}
x^{c_1,j,c_2}(t) = \exp\Bigg(-\frac{1}{2}\Bigg(\int_0^t & a_1(a_1-1)|\nu^{c_1}(\tau,u)-\nu^{c_1}(\tau,v)|^2 + a_2(a_2-1) \\
& \times |\nu^{c_2}(\tau,u)-\nu^{c_2}(\tau,v)|^2 \\
& +2a_1a_2(\nu^{c_1}(\tau,u)-\nu^{c_1}(\tau,v))\cdot(\nu^{c_2}(\tau,u)-\nu^{c_2}(\tau,v)) \\
& -2a_2(\nu^{c_2}(\tau,u)-\nu^{c_2}(\tau,v))\cdot(\nu^{c_2}(\tau,v)-\nu^{c_1}(\tau,v))d\tau\Bigg)\Bigg)
\end{aligned}
$$

If we take $a_2=0$ and $a_1=a+1$, we obtain $x^{c_1,j,c_2}(t)=x^{c|c,j}(t)$ for the function defined in the previous section.

To obtain the relation between $F_X^{c_1,j}$ and $F_X^{c_2,j}$ we use the definition of forward rate and the fact that $F_X^{c_2,j}(t_0)=F_X^{c_1,j}(t_0)$ as both are fixing on the same index. We have

$$
P^{c_2}(0,v)(1+\delta F^{c_2,j}(0)) = N_0^{c_2}\,\mathrm{E}^{\mathbb{X}}\left[(N_v^{c_2})^{-1}(1+\delta F^{c_1,j}(t_0))\right].
$$

We can then replace $N_v^{c_2}$ using the result above and develop $F^{c_1,j}$ using the hypothesis **SjS2** and writing the value $F^{c_i}(t_0)$ as a function of $F^{c_i}(0)$ using the rate dynamic from above.

We obtain

$$
P^{c_2}(0,v)(1+\delta F^{c_2,j}(0)) = P^{c_2}(0,v)(1+\delta F^{c_1,j}(0))\xi(0,t_0,a_1,a_2)
$$

with

$$
\begin{aligned}
\xi(0,t_0,a_1,a_2) = \exp\Bigg(\int_0^{t_0} & (\nu^{c_2}(\tau,v)-\nu^{c_1}(\tau,v)) \\
& \cdot(a_1(\nu^{c_1}(\tau,u)-\nu^{c_1}(\tau,v))+a_2(\nu^{c_2}(\tau,u)-\nu^{c_2}(\tau,v)))d\tau\Bigg).
\end{aligned}
$$

Theorem 8.11 (Change of collateral in HJM and stochastic spread). *In the HJM model for both collateral, if the forward rate spread satisfies the hypothesis* ***SjS2****, the forward rates in the different collaterals are linked by*

$$
(1+\delta F_0^{c_2,j}(t_0,u,v)) = (1+\delta F_0^{c_1,j}(t_0,u,v))\xi(0,t_0,a_1,a_2)
$$

From a practical point of view, the above result is as explicit as the computation of ξ.

Special cases

If $\nu^{c_2}=0$, that is, c_2 is deterministic, then $\xi=\gamma_{c_1}^{a_1}$ as described in the STIR futures section. When one collateral rate is deterministic or 0, the problem is reduced to the same as the futures margining, which is equivalent to a collateral of 0.

If $\nu^{c_2}=\nu^{c_1}$, $\xi=1$ and there is no convexity adjustment. If the rates are moving together, then we are in a case similar to a deterministic spread.

Suppose that ν^{c_2} has two components associated to independent motions: $\nu^{c_2} = (\tilde{\nu}^{c_1}, \tilde{\nu}^{s})$ with $\nu^{c_1} = (\tilde{\nu}^{c_1}, 0)$, with s interpreted as a spread (see the end of Section 8.4.1 for more on spread in the collateral HJM framework). This means that the quantities associated to F^{c_1} evolve with two independent parts. Suppose also that, like in Section 8.3.4, the filtration underlying the spread is independent of the forward rate $F^{c_1,j}$. If there is independence, it is natural to have $a_2 = 0$, that is, no dependent part between $F^{c_1,j}$ and s through F^{c_2}. In the description of ξ, the first factor is the difference between ν^{c_2} and ν^{c_1}, which is ν^{s} here. The scalar product in the integral of ξ is thus the scalar product between two orthogonal vectors. This proves that there is no adjustment in that situation, as already proved in Section 8.3.4.

Appendix A
Gaussian HJM

In this appendix, we describe a generic multi-factor Gaussian HJM model on the discounting curve. The model describes only the risk-free discounting curve. Additional hypotheses on the forward curves or on the relationship between the multiple curves will be required when the model is used in a multi-curve framework.

A.1 Model

A term structure model describes the behaviour of $P^D_X(t,u)$, the price in t of the risk-free zero-coupon bond paying 1 in u $(0 \leq t, u \leq T)$. To be able to model curves with the usual tools of quantitative finance one needs to fix an upper bound on the time frame on which the modelling is done. This is for example described in (Hunt and Kennedy, 2004, Section 7.4.4). This is why we impose an upper bound T. If the bound T is taken sufficiently large, it will have no impact in practice.

We suppose that the discount curve $P^D_X(t,.)$ is absolutely continuous. This is always the case in practice as the curve is constructed by some kind of regular interpolation or through parameterised functions as described in Chapter 5. When the curve is absolutely continuous, there exists $f(t,u)$, called the *instantaneous forward rate*, such that

$$P^D_X(t,u) = \exp\left(-\int_t^u f(t,\tau)d\tau\right). \tag{A.1}$$

The short rate associated to the curve is denoted $(r_t)_{0\leq t\leq T}$ with $r_t = f(t,t)$. The cash account numeraire is

$$N_t = \exp\left(\int_0^t r_\tau \, d\tau\right).$$

The idea of Heath et al. (1992) is to model f with a stochastic differential equation

$$df(t,u) = \mu(t,u)dt + \sigma(t,u)\cdot dW_t$$

for some suitable μ and σ and to deduce the behaviour of P^D_X from there. The stochastic variable W_t is an n – dimensional standard Brownian motion. The volatility σ is a function $\sigma : \mathbb{R}^2 \to \mathbb{R}^n$. The function is defined on $\{(t,u) \in \mathbb{R}^2 : t \geq 0$ and $u \geq t\}$.

To ensure the arbitrage-free property of the model, a relationship between the drift and the volatility is required. The model technical details can be found in the original paper or in the chapter *Dynamical term structure model* of Hunt and Kennedy (2004).

To simplify the writing, the notation

$$\nu(t,u) = \int_t^u \sigma(t,\tau)d\tau$$

is used throughout the book. The integrated volatility ν also has values in $\mathbb{R}^n$. Note that $\nu(t,t) = 0$.

The arbitrage-free equations of the model in the cash account numeraire measure associated to N_t are

$$df(t,u) = \sigma(t,u)\cdot\nu(t,u)dt + \sigma(t,u)\cdot dW_t. \tag{A.2}$$

Let W_t^v be the Brownian motion associated to the measure for the $P_X^D(.,v)$ numeraire. The relation between dW_t and dW_t^v is

$$dW_t^v = dW_t + \nu(t,v)dt.$$

In this book we restrict ourselves to the case where the following hypothesis is true.

G: The function $\sigma(t,u)$ (and ν) is a deterministic function of the current time and the maturity u.

This case is generally called the *Gaussian HJM* model, a name used in the Appendix title. With this hypothesis the model is very tractable, as will be seen in the next sections.

A.2 Generic results

The forward volatility of a re-based zero-coupon bond is the positive number defined by

$$\alpha(\theta_0,\theta_1,u,v)^2 = \int_{\theta_0}^{\theta_1} |(\nu(s,v) - \nu(s,u))|^2\, ds.$$

The description of how we arrived at that number is provided below.

The value of the zero-coupon bonds rebased by another bond can be evolved explicitly in the numeraire associated to the second bond. As an asset rebased by the numeraire is a martingale, it is not surprising that the zero-coupon bond solution is the easiest to write in that context.

Formulas similar to the one described below can be found in different articles, in particular in Henrard (2006a) for the Gaussian one-factor HJM and in (Brody and Hughston, 2004, (3.3),(3.4)) in the framework of coherent interest-rate models.

Note the following equality that will be used several times:

$$\int_{\theta_0}^{\theta_1} \left(|\nu(s,v)|^2 - |\nu(s,u)|^2\right) ds = \alpha^2(\theta_1,\theta_2,u,v) + 2\int_{\theta_0}^{\theta_1} \nu(s,u)\cdot(\nu(s,v)-\nu(s,u))ds.$$

The quantity

$$\gamma(\theta_0,\theta_1,u,v) = \exp\left(\int_{\theta_0}^{\theta_1} \nu(s,u)\cdot(\nu(s,v)-\nu(s,u))ds\right) \tag{A.3}$$

will also be used on several occasions, in particular for convexity adjustments, and deserves a name of its own.

Lemma A.1. (Discount factors HJM dynamic). *Let $0 \le t \le s \le u, v$. In an HJM multi-factor model, the ratio of zero-coupon bonds can be written in the cash account numeraire as*

$$\frac{P_X^D(s,v)}{P_X^D(s,u)} = \frac{P_X^D(t,v)}{P_X^D(t,u)} \exp\left(-\alpha(t,s,u,v)X_{t,s,u,v} - \frac{1}{2}\alpha^2(t,s,u,v)\right)\gamma(t,s,u,v) \tag{A.4}$$

for a standard normally distributed random variable $X_{t,s,u,v}$ independent of $\mathcal{F}_t$.

Proof: By definition of the forward rate and its equation,

$$\begin{aligned}
P_X^D(t,u) &= \exp\left(-\int_t^u f(t,\tau)d\tau\right) \\
&= \exp\left(-\int_t^u \left[f(0,\tau) + \int_0^t \nu(s,\tau)\cdot D_2\nu(s,\tau)ds + \int_0^t D_2\nu(s,\tau)\cdot dW_s\right] d\tau\right).
\end{aligned}$$

Then using again the definition of forward rates and the Fubini theorem on inversion of iterated integrals, we have

$$\begin{aligned}
P_X^D(t,u) = & \frac{P_X^D(0,u)}{P_X^D(0,t)} \exp\left(-\int_0^t\int_t^u \nu(s,\tau)\cdot D_2\nu(s,\tau)d\tau\, ds\right. \\
& \left. - \int_0^t\int_t^u D_2\nu(s,\tau)d\tau\cdot dW_s\right) \\
= & \frac{P_X^D(0,u)}{P_X^D(0,t)} \exp\left(-\frac{1}{2}\int_0^t \left(|\nu(s,u)|^2 - |\nu(s,t)|^2\right) ds\right. \\
& \left. - \int_0^t (\nu(s,u)-\nu(s,t))\cdot dW_s\right).
\end{aligned}$$

The result is obtained using the equality mentioned before the theorem and the definition of α and γ. □

It is also useful to write the same result in the $P_X^D(.,u)$ numeraire.

Lemma A.2. (Discount factors HJM dynamic). *Let $0 \leq t \leq s \leq u, v$. In an HJM multi-factor model, the ratio of zero-coupon bonds can be written in the $P_X^D(., u)$ numeraire as*

$$\frac{P_X^D(s,v)}{P_X^D(s,u)} = \frac{P_X^D(t,v)}{P_X^D(t,u)} \exp\left(-\alpha(t,s,u,v)X^u_{t,s,u,v} - \frac{1}{2}\alpha^2(t,s,u,v)\right) \tag{A.5}$$

for a standard normally distributed random variable $X^u_{t,s,u,v}$ independent of $\mathcal{F}_t$.

The result can be obtained using the change of numeraire described above. Alternatively one can notice that the ratio of discount factors should be a martingale as it is the ratio of an asset by the numeraire. The volatility part should be the same as in the previous result, only the mean is changed.

In the general HJM framework the volatility term is not deterministic and can depend on P_X^D. For deterministic volatility the lemmas above provide a solution to the present value problem. When the volatility is state dependent saying that the price *is equal* to the formula presented is a misuse of language. The result is another equation satisfied by the price; the differential equation is replaced by an integral equation.

The cash account value itself can be written explicitly as a function of the Brownian motion.

Lemma A.3. (Cash account HJM dynamic). *Let $0 \leq u \leq v$. In the HJM framework*

$$\begin{aligned} N_u N_v^{-1} &= \exp\left(-\int_u^v r_\tau \, d\tau\right) \\ &= P_X^D(u,v) \times \exp\left(-\int_u^v \nu(s,v)\cdot dW_s - \frac{1}{2}\int_u^v |\nu(s,v)|^2 ds\right). \end{aligned}$$

Proof: The computation of this lemma is similar to that of Lemma A.1. By definition of r,

$$\begin{aligned} r_\tau &= f(\tau,\tau) = f(u,\tau) + \int_u^\tau df(s,\tau)ds \\ &= f(u,\tau) + \int_u^\tau \nu(s,\tau)\cdot D_2\nu(s,\tau)ds + \int_u^\tau D_2\nu(s,\tau)\cdot dW_s. \end{aligned}$$

Then using Fubini, we have

$$\int_u^v r_\tau \, d\tau = \int_u^v f(u,\tau)d\tau + \frac{1}{2}\int_u^v |\nu(s,v)|^2 ds + \int_u^v \nu(s,v)\cdot dW_s$$

where we have used the fact that $\nu(s,s) = 0$.

The result is obtained by taking the exponential of the last equality and using the definition of f. □

In some circumstances, we will use the following notation for the cash account variables:

$$X^N_{u,v} = \int_u^v \nu(\tau, v) \cdot dW_\tau$$

and

$$\alpha_N^2(u,v) = \int_u^v |\nu(\tau, v)|^2 d\tau.$$

A.3 Special cases

A.3.1 Separability

The separability hypothesis on the HJM volatility is defined as follows.

Definition A.1 Separability hypothesis. *The function σ is said to be* separable *if $\sigma(t,u) = g(t)h(u)$ for some positive functions g and h.*

When σ is separable, the zero-coupon volatility ν can be written as

$$\nu(t,u) = g(t)(H(u) - H(t))$$

for H a primitive of h. It means also that in Lemma A.1 and A.2, the random variables $X_{t,s,u,v}$ are the same for all u and v, that is $X_{t,s,u,v} = X_{t,s}$. This will be important when analysing instruments dependent on more than one date, like swaptions and coupons with date mismatches. The random variables depend on the starting and end date of the randomness period but not on the maturity dates of the cash flow payments.

A.3.2 Hull–White

The Hull and White (1990) model, also called extended Vasicek model, is a special case of a one-factor Gaussian HJM model. It is a one-factor model with $\sigma(s,t) = \eta(s)\exp(-a(t-s))$ and $\nu(s,t) = (1-\exp(-a(t-s)))\eta(s)/a$. The model satisfies the separability hypothesis.

The model is often used with a piecewise constant volatility. By this we mean that there exists $0 = \tau_0 < \tau_1 < \cdots < \tau_n = +\infty$ such that $\eta(s) = \eta_i$ for $\tau_{i-1} \leq s < \tau_i$.

With the piecewise constant volatility, the value of α and γ can be written explicitly. The expiry dates θ_1 and θ_2 are between some of the dates defining the piecewise constant function. The dates are denoted $\tau_p \leq \theta_0 < \theta_1 \leq \tau_q$. To shorten the notation an intermediary notation is used: $r_p = \theta_0 < r_l = \tau_l < r_q = \theta_1$. With this notation, one has

$$\alpha^2(\theta_0,\theta_1,u,v) = \left(\exp(-au) - \exp(-av)\right)^2$$
$$\frac{1}{2a^3}\sum_{l=p}^{q-1}\eta_l^2\left(\exp(2ar_{l+1}) - \exp(2ar_l)\right).$$

The value of γ, the convexity adjustment, can also be written explicitly for that special case. With the same notation as above, we have

$$\ln\gamma(\theta_0,\theta_1,u,v) = \frac{1}{2a^3}(\exp(-au) - \exp(-av))$$
$$\sum_{l=p}^{q-1}\eta_l^2(\exp(ar_{l+1}) - \exp(ar_l))(2 - \exp(-a(u-r_{l+1})) - \exp(-a(u-r_l))).$$

The Hull–White model can also be written as a short rate model. A good reference for the description of the short rate approach is (Brigo and Mercurio, 2006, Section 3.3). The book description refers to the case of the constant volatility model. The stochastic (one-factor) equation for the short rate is, in the cash account numeraire,

$$dr_t = (\theta(t) - ar_t)\,dt + \eta(t)dW_t. \tag{A.6}$$

For a constant volatility η, the mean reversion level is given by

$$\theta(t) = D_2 f(0,t) + af(0,t) + \frac{\eta^2}{2a}(1 - \exp(-2at)).$$

A.3.3 G2++

The term G2++ stands for Gaussian two factors additive model. A description of the model can be found in Brigo and Mercurio (2006).

The model has two mean reversions a_1 and a_2 and a correlation ρ. The stochastic two factors equations for the short rate are, in the cash-account numeraire,

$$dx_t^i = -a_i x_t^i dt + \eta_i(t)d\bar{W}_t^i$$
$$r_t = x_t^1 + x_t^2 + \varphi(t), \qquad r(0) = r_0 \tag{A.7}$$

where $\left(\bar{W}_t^1, \bar{W}_t^2\right)$ is a two-dimensional Brownian motion with instantaneous correlation ρ.

The G2++ can be written as an HJM model. It can be written as a model with correlated Brownian motions $\bar{W}_t = \left(\bar{W}_t^1, \bar{W}_t^2\right)$. The correlation matrix between the two motions is denoted

$$R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

and can be decomposed as $R = A^T A$ with

$$A = \begin{pmatrix} \sqrt{1-\rho^2} & 0 \\ \rho & 1 \end{pmatrix}$$

The equations in the correlated Brownian motions are

$$df(t,u) = \bar{\sigma}^T(t,u)\rho\bar{\nu}(t,u)dt + \bar{\sigma}^T(t,u)d\bar{W}_t.$$

If we take $\bar{\sigma}^T = \sigma^T A^{-1}$ and $\bar{\nu}^T = \nu^T A^{-1}$, the equation above reduces to the standard HJM Equation (A.2) .

For the G2++ model the volatility functions are

$$\bar{\sigma}_i(s,u) = \eta_i(s)\exp(-a_i(u-s)).$$

The equivalence of G2++ to the special HJM case described above is analysed in (Brigo and Mercurio, 2006, Section 5.2). The bond volatilities are

$$\bar{\nu}_i(s,u) = \frac{\eta_i(s)}{a_i}\left(1 - \exp(-a_i(u-s))\right).$$

A.4 Monte Carlo (Hull–White)

To speed-up Monte-Carlo simulations on several periods, it is useful to decompose the volatility into a maturity dependent part and an expiry part. Let $g(s) = \eta(s)\exp(as)$ and $h(t) = \exp(at)$. We define $H(u) = \int_0^u h(t)dt = \exp(au)/a$. The expiry dependent part is

$$\bar{g}(\theta_0,\theta_1) = \int_{\theta_0}^{\theta_1} g^2(s)ds = \frac{1}{2a}\sum_{l=p}^{q-1}\eta_l^2\left(\exp(2ar_{l+1}) - \exp(2ar_l)\right).$$

The volatility is

$$\alpha(\theta_0,\theta_1,u,v) = \sqrt{\bar{g}(\theta_0,\theta_1)}(H(v) - H(u)).$$

Let $s_0 = 0 < s_1 \cdots < s_i < \cdots$ be a set of times and $Z_i = \int_{s_{i-1}}^{s_i} g(s)dW_s$. The stochastic variables can be written as

$$\alpha(0,s_i,u,v)X_{0,s_i} = (H(v) - H(u))Y_i$$

where $Y_i = \sum_{j=1}^{i} Z_j$. The random variables Y_i are such that $Y_i = Y_{i-1} + Z_i$ with Z_i independent normally distributed with variance $\bar{g}(\tau_{i-1},\tau_i)$.

The variables Y_i are not independent; they can be viewed as equal on some time interval with one of them continuing after. The covariance between Y_i and Y_j (with $i \le j$) is the variance of Y_i and is

$$\sum_{k=1}^{i}\int_{s_{k-1}}^{s_k} g^2(s)ds = \int_0^{s_i} g^2(s)ds = \gamma(0,s_i).$$

As the variables are not independent anymore, we simulate independent variables and multiply them by the Cholesky decomposition of the covariance matrix.

It is therefore very easy to generate Monte Carlo simulations at important dates. There is no need to generate intermediary points as the solutions are explicit. One can also generate the values at the different dates directly from 0 without looking at the intermediary values. The dependence between the different times is provided by the correlation structure.

A.5 Miscellaneous

The following technical lemma is not particularly relevant to the other information in this chapter, but has been included here for the lack of a better place.

Lemma A.4. (Normal integral).

$$\frac{1}{\sqrt{2\pi}\sqrt{|\Sigma|}}\int_{\mathbb{R}} \exp\left(x_2 - \frac{1}{2}\sigma_2^2 - \frac{1}{2}x^T\Sigma^{-1}x\right) dx_2 = \frac{1}{\sigma_1}\exp\left(-\frac{1}{2\sigma_1^2}(x_1 - \sigma_{12})^2\right).$$

Appendix B
Conventions

In the multi-curve framework, more curves are constructed and more types of instruments are required to obtain those curves. The interactions between the different instruments also plays an important role. It is therefore paramount to incorporate all the instruments with their exact conventions. Some of these details are neglected in textbooks, rendering them almost useless in practice.

This chapter describes the detailed conventions for the instruments used in curve calibration. It is a description of the term and conditions of the instrument, at least those important for the pricing, but not a description of the way to price them. The pricing of the instruments is discussed in other chapters and in particular in Chapter 2 and in Chapter 6.

An important part of this chapter content is borrowed from the OpenGamma *Interest Rate Instruments and Market Conventions Guide* (Quantitative-Research (2012c)). The booklet, which is now at its second version, covers a lot more instruments and currencies than those described in this appendix. Here we restrict ourselves to the instruments used in curve calibration and to the main currencies (in alphabetical order: CHF, EUR, GBP, JPY, USD).

In this appendix we do not describe some of the building blocks on which the conventions are based: day-count conventions, business day conventions, exchanges on which some of the instruments are traded and bodies involved in setting the fundamental indexes used through this book. For those details, we refer to the above-mentioned guide.

B.1 Ibor indexes

Ibor-like indexes are often related to interbank lending with maturities between one day and one year. It is usually computed as the trimmed average between rates contributed by participating banks. The rates are bank's estimates but usually do not refer to actual transactions.

The most common usages of these indexes in interest rate derivatives are in swaps and STIR futures fixing.

The main currencies' Ibor-like indexes and their main characteristics are summarised in Table B.1.

Table B.1 Ibor-like indexes

Currency	Name	Maturities	Convention	Spot lag	Bbg
CHF	LIBOR	O/N–12M	ACT/360	2	SF00xxx
EUR	EURIBOR	1W–12M	ACT/360	2	EUR0xxx
EUR	LIBOR	O/N–12M	ACT/360	2	EU00xxx
GBP	LIBOR	O/N–12M	ACT/365	0	BP00xxx
JPY	LIBOR	O/N–12M	ACT/360	2	JY00xxx
JPY	Japan TIBOR	1W–12M	ACT/365	2	
JPY	Euroyen TIBOR	1W–12M	ACT/360	2	
USD	LIBOR	O/N–12M	ACT/360	2	US00xxx

In the Bloomberg code, the `xxx` should be replaced by the tenor (T/N, 01W, 11M, and so on) and followed by `␣Index`.

B.1.1 LIBOR

LIBOR is the acronym for London Interbank Offered Rate. Up to 2013, LIBOR was calculated on behalf of the British Bankers' Association by Reuters. Major banks submit their estimated cost of borrowing unsecured funds for several tenors and currencies.

Up to 2013, there were 15 tenors in 10 currencies (AUD, CAD, DKK, EUR, JPY, NZD, GBP, SEK, CHF, USD). Some of them were phased out in the first half of 2013. By May, BBA had cut back to 42 LIBOR rates. Rates are published for six currencies (EUR, EUR same day, JPY, GBP, CHF, USD). Only seven tenors are covered: overnight/spot-next, one-week, and one, two, three, six and 12 months. In July 2013 it was announced that the NYSE Euronext will be in charge of the administration through it's subsidiary *NYSE Euronext Rates Administration Limited.* In the mean time NYSE Euronext has been bought by ICE and the LIBOR is now administrated by ICE.

The conventions are the same for most of the currencies. *For all currencies other than EUR and GBP the period between fixing date and value date will be two London business days. However, if that day is not both a London business day and a business day in the principal financial centre of the currency concerned, the next following day that is a business day in both centres shall be the Value Date.* The business day convention is *modified following* and the *end-of-month* rule applies. The dates for the deposit underlying the index are calculated in the calendar of the principal financial centre of the currency concerned. For all currencies except GBP, the day-count convention is ACT/360.
Reference: http://www.bbalibor.com/technical-aspects/fixing-value-and-maturity.
Reference: http://www.nyx.com/libor

B.1.2 GBP-LIBOR

The fixing date and value date are the same (0 day spot lag). The day-count convention is ACT/365.

B.1.3 EUR-LIBOR

The value date is two TARGET business days after the fixing date. The dates of the deposit underlying the index are calculated according to the TARGET calendar.

B.1.4 EURIBOR

EURIBOR is an acronym for EURo InterBank Offered Rate. The day-count convention is ACT/360 and the spot lag is two days. The business day convention is *modified following* and the *end-of-month* rule applies. There are 43 contributor banks. The rates are published at 11:00 a.m. (CET). The dates of the deposit underlying the index are calculated according to the TARGET calendar.
Reference: http://www.euribor-ebf.eu/euribor-org/about-euribor.html

B.1.5 JPY-TIBOR

TIBOR is the acronym for Tokyo Interbank Offered Rate. It is published by the Japanese Bankers Association. There are two types of TIBOR: The 'Japanese Yen TIBOR' rates reflect prevailing rates on the unsecured Japanese market; the 'Euroyen TIBOR' rates reflect the Japan offshore market. The JBA TIBOR is calculated by JBA as a prevailing market rate based on quotes for 13 different maturities (1 week, 1-12 months) provided by reference banks as of 11:00 a.m. Tokyo time each Tokyo business day. The day-count convention is ACT/365 for the domestic market and ACT/360 for the Euroyen market.
Reference: http://www.zenginkyo.or.jp/en/tibor/the_jba_tibor/

B.2 Overnight indexes

Overnight indexes are indexes related to interbank lending on a one day horizon. Most indexes are for overnight loans and some for tomorrow/next loans. The rates are computed as a weighted average of actual transactions.

The most common usage of these indexes in interest rate derivatives is in overnight indexed swaps (see Section B.9) and in collateral interest payment (see Chapter 8).

The standard overnight indexes for the main currencies and their characteristics are summarised in Table B.2.

Table B.2 Overnight indexes for the main currencies

Currency	Name	Reference	Convention	Publication lag
CHF	TOIS	TN	ACT/360	0
EUR	EONIA	ON	ACT/360	0
GBP	SONIA	ON	ACT/365	0
JPY	TONAR	ON	ACT/365	1
USD	Effective Federal Funds	ON	ACT/360	1

Publication lag is the number of days between the start date of the period and the rate publication. A lag of 0 means on the start date, a lag of 1 means on the period end date.

B.2.1 USD-Effective Federal Funds Rate

The daily effective Federal Funds rate is a volume-weighted average of rates on trades arranged by major brokers. The effective rate is calculated by the Federal Reserve Bank of New York using data provided by the brokers and is subject to revision. The rate is published in the morning (between 7:00 and 8:30) of the period end date. The day-count convention is ACT/360.

Reference: http://www.newyorkfed.org/markets/omo/dmm/fedfundsdata.cfm

B.2.2 EUR-EONIA

EONIA is the acronym of *Euro OverNight Index Average*. It is computed as a weighted average of all overnight unsecured lending transactions undertaken in the interbank market, initiated within the Euro area by the contributing banks. It is calculated by the European Central Bank. The rate is published in the evening (around 19:00 CET) of the period start date. The day-count convention is ACT/360.

Reference: http://www.euribor-ebf.eu/euribor-eonia-org/about-eonia.html

B.2.3 GBP-SONIA

SONIA is the acronym of *Sterling OverNight Index Average*. It is the weighted average rate of all unsecured sterling overnight cash transactions brokered in London by WMBA member firms between midnight and 16:15 CET with all counterparts in a minimum deal size of GBP 25 million. The rate is published in the evening (around 17:00 CET) of the period start date. The day count convention is ACT/365.
Reference: http://www.bba.org.uk/policy/article/
sterling-overnight-index-average-sonia-a-guide/benchmarks/

B.2.4 JPY-TONAR-Uncollateralised overnight call rate

TONAR is the acronym of *Tokyo OverNight Average Rate*. It is the weighted average rate of all unsecured overnight cash transactions between financial institutions. The rate is published by the Bank of Japan (BOJ). The day-count convention is ACT/365.

A provisional result is published on the evening (at 17:15 JST except on the last business day of the month where it is 18:15 JST) of the period start. The final result is published in the morning (10:00 JST) of the end date.

Reference: http://www.boj.or.jp/en/statistics/market/short/mutan/

B.3 Forward rate agreement

Forward rate agreements (FRA) are over-the-counter (OTC) contracts linked to an Ibor-like index. At the trade date a *reference rate* (R), a *start period*, and a *reference index* are agreed. The *end period* is equal to the start period plus the index tenor (a six month start period and a three month tenor give a nine month end period). The instrument *reference dates* are computed in the following way. Its *start date* is computed from today by adding the index spot lag and then the start period (using the business day convention and calendar of the index). Its *end date* is computed from today by adding the index spot lag and then the end period. The *fixing date* (or exercise date) is the spot lag before the start date. The accrual factor between the start date and the end date (in the index day count) is denoted δ. In some (rare) cases the dates described above are not computed but decided arbitrarily by the counterparties (usually changing the dates by one day or two for convenience reasons).

The FRA *settlement date* is the start date (not the end date). On the settlement date the payoff is, for the FRA buyer,

$$\delta \frac{L_\theta - R}{1 + \delta L_\theta}$$

where L_θ is the value of the reference index on the fixing date. The payoff for the FRA seller is obviously the same amount with an opposite sign. The term to describe such a settlement method is often referred to as *FRA discounting*.

The term *FRA buyer* can be interpreted in the following way: the buyer pays a known fixed price R and in exchange receives a good (the index rate L_θ). The *FRA seller* obviously has the opposite exposure.

Note that all the cash-flows are settled on the start date and not on the end date. In some accounting schemes, the payment is accrued between the start date and the end date. The instrument stays 'alive' from an accounting point of view even if it has already fully settled.

Note also that the FRA's end date may be (slightly) different from the end date of the theoretical deposit underlying the Ibor rate. This potential mismatch comes from a difference in adjustment of the non-good business days between the different ways to compute the periods. To use the example above, a period of six months followed by a period of three months is not always equal to a period of nine months.

Table B.3 FRA dates with differences between end of the accrual period and end of the underlying fixing deposit period

FRA	Trade date	Spot date	Fixing	Start accr.	End accr.	End fixing
1Mx4M	9-Sep-13	11-Sep-13	9-Oct-13	11-Oct-13	13-Jan-14	13-Jan-14
1Mx4M	10-Sep-13	12-Sep-13	10-Oct-13	14-Oct-13	13-Jan-14	14-Jan-14
1Mx2M	10-Sep-13	12-Sep-13	10-Oct-13	14-Oct-13	12-Nov-13	14-Nov-13
1Mx4M	14-Mar-14	18-Mar-14	16-Apr-14	22-Apr-14	18-Jul-14	22-Jul-14
1Mx3M	14-Mar-14	18-Mar-14	16-Apr-14	22-Apr-14	18-Jun-14	23-Jun-14

Dates in TARGET calendar.

Several cases of dates mismatch are proposed in Table B.3. The difference in the table is up to five days.

FRA can also be traded as International Money Market (IMM) FRA, that is, FRA with accrual dates equal to consecutive IMM dates as used in STIR futures (see Section B.4). The underlying Ibor rate has as tenor the one relevant for the IMM dates frequency (three months Ibor for quarterly dates and one month Ibor for monthly dates).

B.4 STIR futures

The futures type described in this chapter are the Ibor-based futures, also called Short Term Interest Rate (STIR) Futures. They all have the same settlement mechanism but differ on the notional, the underlying STIR fixing and exchange on which they are traded.

The dates related to these futures are based on the third Wednesday of the month[1], which is the *start date* of the Ibor rate underlying the future. The rate is fixed at a *spot lag* prior to that date (see Table B.1 for the different conventions); the fixing usually takes place on the Monday (or on the Wednesday itself for GBP). The fixing date is also the *last trading date* for the future. The index underlying the futures is a three or one month index. The three month futures are the most liquid.

The margining process works in the following way. For a given *closing price* or *settlement price* (as published by the exchange), the daily margin paid is that price minus the *reference price* multiplied by the notional and by the accrual factor of the future. Equivalently it is the price difference multiplied by one hundred and by the *point value*, the point value being the margin associated with a one (percentage) point change in the price. The reference price is the trade price on the trade date and the previous closing price on the subsequent dates. The *tick value* is the value of

[1] When the day is a non-good business day, it is adjusted to the following day.

the smallest increment in price. The price usually changes in 1.0 or 0.5 basis point increments.

We denote the futures price in t by Φ_t^j with j the tenor of the underlying index. On the fixing date at the moment of the publication of the underlying Ibor rates $I_X^j(t_0)$, the future price is $\Phi_{t_0}^j = 1 - I_X^j(t_0)$. Before that moment, the price evolves with demand and offer.

The three month STIR futures are in some places described as '90 day deposit futures'. The origin of the 90 days can be traced back to the futures accrual factor which is 0.25 and correspond to a 90 day period in the ACT/360 convention, the most used day-count convention for money market instruments. But there is no actual reference to a 90 day period in the contract itself. It appears to be an apocryphal description of the instrument trying to invent a explanation behind a conventional number. Moreover that description would be wrong for the GBP futures for which the day-count convention is ACT/365. To avoid introducing an unnecessary confusion for no benefit, we will refrain from using that terminology.

The futures are designated by character codes. The first part depends on the data provider and is usually two to four characters. The main codes are given in Table B.4. The second part describes the month, with the codes given in Table B.5, and the year, with its last digit. For example the March 2013 three month USD LIBOR futures is called `EDH4`; the June 2014 Eurex Euribor futures is called `FPM5`. As interest rate futures are quoted for fixing dates up to 10 years in the future only, there is no ambiguity by using only one figure for the year. Note also that it means that when a future reaches its last trading date, a new one is created a couple of days later with the same name but for a maturity 10 years in the future.

Table B.4 Interest rate futures on Ibor details and codes

Currency	Tenor	Exchange	Underlying	Notional	Bloomberg
CHF	3M	Liffe	LIBOR	1,000,000	ES
EUR	3M	Eurex	EURIBOR	1,000,000	FP
EUR	3M	Liffe	EURIBOR	1,000,000	ER
GBP	3M	Liffe	LIBOR	500,000	L_
JPY	3M	SGX/CME	TIBOR	100,000,000	EY
JPY	3M	SGX	LIBOR	100,000,000	EF
USD	3M	CME	LIBOR	1,000,000	ED
USD	1M	CME	LIBOR	3,000,000	EM
USD	3M	SGX	LIBOR	1,000,000	DE
USD	3M	Liffe	LIBOR	1,000,000	LEDA

Table B.5 Rate futures month codes

Month	Code	Month	Code	Month	Code
January	F	February	G	March	H
April	J	May	K	June	M
July	N	August	Q	September	U
October	V	November	X	December	Z

The futures are not only traded individually but also by *packs*, which are groups of four consecutive futures. The packs are referred to by colours. The colours are white (first four futures), red (next four), green, blue, gold, purple, orange, pink, silver and copper.

The futures are also traded as *n-year bundles*, which are groups of the futures up to n years from now. The 1-year bundle contains the first four futures, the 2-year bundle the first eight futures, and so on.

If a pack or bundle is traded, the resulting position is a position in all the futures composing the pack or bundle. The futures traded through pack and bundle are fungible with the futures traded individually. Packs and bundles are not quoted directly in price but in average change of price since the last settlement price. A pack or bundle is quote as '$\pm p$' ticks. The trade price of each of the futures of the pack or bundle will be the previous settlement price of that futures plus or minus the quoted p ticks.

B.4.1 USD

USD interest rate futures are traded on CME, on SGX and on Liffe. For three months futures, the nominal is USD 1,000,000 and the accrual factor is 1/4. The fixing index is LIBOR. For one month futures, the nominal is USD 3,000,000 and the accrual factor is 1/12. In both cases, the nominal multiplied by the accrual factor, called the *unit amount*, is USD 250,000.

B.4.2 EUR

The EUR three month interest rate futures are traded on Liffe and on Eurex. The nominal is EUR 1,000,000 and the accrual factor is 1/4. The fixing index is Euribor.

B.4.3 GBP

The GBP three month interest rate futures are traded on Liffe. The nominal is GBP 500,000 and the accrual factor is 1/4. The fixing index is LIBOR.

B.4.4 JPY

The JPY three month interest rate futures are traded on CME and on SGX for Tibor-based futures and on SGX for LIBOR-based futures. The nominal is JPY 100,000,000 and the accrual factor is 1/4.

B.4.5 CHF

CHF interest rate futures are traded on Liffe. The nominal is CHF 1,000,000 and the accrual factor is 1/4. The fixing index is LIBOR.

B.5 Coupons

The coupons are the building blocks of the legs and swaps described in the next sections. We use five main types of coupon: fixed, simple Ibor-linked, compounded Ibor-linked, overnight compounded and overnight arithmetic average –a USD specific coupon used in Federal Funds swaps. The ways to compute the start and end dates of the coupons are described in the appropriate leg or swap sections.

B.5.1 Fixed coupons

A fixed coupon is simply the payment of a fixed amount at a fixed date. The amount is usually written as a multiple of an accrual factor. The accrual factor for the period in the appropriate day-count convention is denoted δ. The amount paid at the payment date for a fixed rate of R and a notional of N is $N\delta R$. Most of the time, the payment date is the end accrual date, but not in all cases. The most frequent case with a difference between these two dates that of the overnight indexed swaps described later.

B.5.2 Simple Ibor floating coupons

An Ibor coupon or Ibor floating coupon is the payment of an amount linked to an Ibor index fixed at a given date t_0 and paid at another given date t_p. The amount is written as a multiple of an accrual factor and the index fixing. Let δ denote the accrual factor for the period in the payment day-count convention . The amount paid at the payment date for a fixing rate $I_X^j(t_0)$ and a notional of N is $N\delta I_X^j(t_0)$.

In the most standard coupons, the fixing date is the spot lag associated to the index before the start accrual date, the payment date is the end accrual date and the accrual period is equal to the index tenor. There can be up to a couple of days difference, similarly to the difference described for FRAs; this will be the case for all Ibor floating coupons used in the curves construction.

Some Ibor floating coupons also have a *spread*. This means that a fixed payment is added to the variable amount described above. The total payment is, for a spread s, $N\delta(I_X^j(t_0)+s)$.

B.5.3 Compounded Ibor floating coupons

The Ibor floating coupons described above have one payment for one fixing. Other types of Ibor coupons compound several fixings for one payment.

The accrual period is divided into several sub-periods with one fixing associated to each sub-period. The associated dates are denoted $(t_i)_{i=0,\dots,n}$. The fixing rate for each sub-period $[t_{i-1},t_i]$ with fixing in t_{t-1}^f is denoted $I_X^j(t_{t-1}^f)$ $(i=1,\dots,n)$ and the accrual factors in the index convention δ_i. The total accrual factor for the full period $[t_0,t_n]$ is denoted δ and is equal to $\sum_{i=1}^n \delta_i$.

For a notional of N, the coupon pays in t_p the amount

$$N\left(\left(\prod_{i=1}^n (1+\delta_i I_X^j(t_{t-1}^f))\right)-1\right).$$

Compounded floating coupons can also have spread. The International Swap and Derivative Association (ISDA) definition references three different ways to compound spread (see Mengle (2009)). They are called *Compounding*, *Flat Compounding* and *Compounding Treating Spread as Simple Interest*. We describe only two of them.

B.5.3.1 Compounding

In this method, both the rate and the spread are compounded. For a spread s the amount paid is

$$N\left(\left(\prod_{i=1}^n (1+\delta_i (I_X^j(t_{t-1}^f)+s))\right)-1\right).$$

B.5.3.2 Compounding treating spread as simple interest

In this method, the spread is added at the end using a simple interest method:

$$N\left(\left(\prod_{i=1}^n (1+\delta_i I_X^j(t_{t-1}^f))\right)-1+\delta s\right).$$

Compound interests are the most powerful force in the universe.

Attributed to Albert Einstein – 1879–1955

B.5.4 Compounded overnight indexed coupons

Let t_0 be the coupon fixing period start date and t_n be its fixing period end date. Let $t_0 < t_1 < \cdots < t_n$ be all the successive good business days in the coupon fixing period. Let δ_i be the accrual factor between t_{i-1} and t_i ($1 \leq i \leq n$) and δ the accrual factor for the total period $[t_0, t_n]$. The overnight rates between t_{i-1} and t_i are given in t_{i-1} by $I_X^j(t_{t-1}^f)$. The paid amount is

$$\left(\prod_{i=1}^{n}(1+\delta_i I_X^j(t_{t-1}^f))\right) - 1$$

multiplied by the notional. This corresponds to the composition of the overnight rates over the period.

The payment is usually not done on the fixing period end date t_n, but at a certain lag after the last fixing publication date. The reason for the lag is that the actual amount is only known at the very end of the period; the payment lag allows for a smooth settlement. The standard conventions for the main currencies are given in Table B.7.

The description of compounded overnight coupons are in some places described as *interest accrued through geometric averaging of the floating index rate*. This is for example the case on Wikipedia (as of December 2012), other web sites and in a Suisse (2001) note Wikepedia refers to. This description should be understood as a *poetic licence* on the meaning of geometric average and not as a precise description. The computation of the rate involves a multiplication, like the computation of a geometric average. But the multiplication is not applied on the rates themselves and no root of the result is taken to obtain the final rate. The licence with the mathematical definition of geometric average is also pointed out in West (2011).

B.6 Legs

The dates on different instruments are spaced by a given payment period. Due to holidays, conventions and broken periods, the way to compute these dates should be detailed. The description below refers to the usual method; as the products are OTC, any variant is possible if agreed by the parties.

The dates are computed from the start (or settlement) date. The last date will be the start date plus the total length (tenor) of the leg. The intermediary dates are spaced by the given payment period, except potentially one when the total tenor is not a multiple of the payment period. The non-standard period is usually the first one. For example a 15 month leg with a 6 month payment period will pay after $3(15 - 2 \times 6)$, 9 and 15 months. The dates will be adjusted by the business day convention and the end-of-month rule. All the dates are first computed without adjustment and then all the dates are adjusted. This means that if a swap starts on the 5th of the month and its maturity is on a Saturday and adjusted to the Monday (the 7th), the intermediary payments take place on the 5th (potentially adjusted) of the intermediary months, not on the 7th.

The non-standard period is called the *stub*. It can be *short* (shorter than one period) or *long* (between one and two periods). The reason the non-standard period is usually the first one is that once that period is finished, the instrument has the same date as a standard one. The non-standard instrument remains non-standard for the shortest period of time as possible. If the stub was the last period, it would never become a standard one.

The term *roll* (like in '*29 roll*') is also used. It means that the (unadjusted) dates will be on the given day. When it is used, it is often around the month end to clarify the payment schedule.

Ibor legs

Due to the way the payment dates of legs are computed, for Ibor legs there can be a mismatch between the payment dates and the dates for which the fixing is valid. This type of mismatch is the same as the one described for FRAs.

B.7 Swaps

The *start*, *effective* or *settlement date*, of a swap is usually a certain lag (called spot lag) after the trade date. Usually the lag for the swaps is equal to the lag for the related Ibor index. The start date can also be forward. In that case the start date is the trade date, plus the forward period plus the spot lag. The forward period is a given number of months or of years.

B.8 Interest rate swaps: fixed for Ibor

The payments on the fixed leg are regularly spaced by a given period, most of them with a six month or twelve month period.

In fixed for floating swaps, the terms *payer* and *receiver* refer to the fixed leg. A swap is a payer for one party if that party pays the fixed leg (and receives the floating leg). A payer swap for one party is a receiver swap for the other party.

Like for FRAs, the terms *buyer* and *seller* are also used. The swap buyer buys the floating leg for a given fixed price; he is the fixed leg (and swap) payer.

In a vanilla IRS, all the coupons have the same notional and all the coupons on the fixed leg have the same rate.

Table B.6 Vanilla swap conventions

		Fixed Leg		Floating Leg		
Currency	**Spot Lag**	**Period**	**Convention**	**Reference**	**Period**	**Convention**
CHF: 1Y	2	1Y	30/360	LIBOR	3M	ACT/360
CHF: >1Y	2	1Y	30/360	LIBOR	6M	ACT/360
EUR: 1Y	2	1Y	30/360	Euribor	3M	ACT/360
EUR: >1Y	2	1Y	30/360	Euribor	6M	ACT/360
GBP: 1Y	0	1Y	ACT/365	LIBOR	3M	ACT/365
GBP: >1Y	0	6M	ACT/365	LIBOR	6M	ACT/365
JPY	2	6M	ACT/365	Tibor	3M	ACT/365
JPY	2	6M	ACT/365	LIBOR	6M	ACT/360
USD	2	6M	30/360	LIBOR	3M	ACT/360
USD (London)	2	1Y	ACT/360	LIBOR	3M	ACT/360

The spot lag is the lag in days between the trade date and the swap start or effective date.

The payments on the floating leg are also regularly spaced, most of them with a three month or six month period. The period between the payments is equal to the Ibor index tenor. The fixing date for floating payment is the index spot lag before the period start date. The lag is the one given by the index and is usually the same as the swap spot lag, which means that the first fixing takes place on the trade date.

Standard conventions for vanilla swaps in different currencies are provided in Table B.6.

In USD, the swaps are also traded as spread over the treasuries. This quotation mechanism is mainly used for 2, 3, 5, 10 and 30 year swaps where the on-the-run treasuries are the most liquid. The spread is the difference between the conventional yield of the corresponding on-the-run treasury and the swap rate. No adjustment is made for the difference in convention or difference in maturity.

B.9 Overnight indexed swaps

The overnight indexed swaps (OIS) exchange a leg of fixed payments for a leg of floating payments linked to an overnight index.

The start (or settlement date) of the swap is a certain lag (called spot lag) after the trade date. The most common lag is two business days.

The payments on the fixed leg are regularly spaced by a given period. Most of the OISs have one payment if shorter than one year and a twelve month period for longer swaps. The payments on the floating leg are also regularly spaced, usually on

Table B.7 Overnight indexed swap conventions

		Fixed Leg		Floating Leg		
Currency	**Spot**	**Period**	**Convention**	**Reference**	**Convention**	**Pay lag**
CHF $\leq$ 1Y	2	tenor	ACT/360	TOIS	ACT/360	2
CHF $>$ 1Y	2	1Y	ACT/360	TOIS	ACT/360	2
EUR $\leq$ 1Y	2	tenor	ACT/360	EONIA	ACT/360	2
EUR $>$ 1Y	2	1Y	ACT/360	EONIA	ACT/360	2
GBP $\leq$ 1Y	0	tenor	ACT/365	SONIA	ACT/365	1
GBP $>$ 1Y	0	1Y	ACT/365	SONIA	ACT/365	1
JPY $\leq$ 1Y	2	tenor	ACT/365	TONAR	ACT/365	1
JPY $>$ 1Y	2	1Y	ACT/365	TONAR	ACT/365	1
USD $\leq$ 1Y	2	tenor	ACT/360	Fed Fund	ACT/360	2
USD $>$ 1Y	2	1Y	ACT/360	Fed Fund	ACT/360	2

The spot lag is the lag in days between the trade date and the swap start date. The pay lag is the lag in days between the last fixing publication and the payment.

the same date as the fixed leg. The amount paid on the floating leg is computed by composing the rates.

The standard conventions for OIS are provided in Table B.7.

B.9.1 CHF

In CHF the payment is one day after the end of the fixing period. This one day is computed as the last publication date, which is at the start of the last period and one day before the end of the last period, plus two lag days. Note that in CHF, the reference index is a tomorrow/next index.

B.9.2 EUR

In EUR the payment is one day after the end of the fixing period. This one day is computed as the last publication date, which is at the start of the last period and one day before the end of the last period, plus two lag days.

B.9.3 GBP

In GBP the payment is on the end of the fixing period, that is, on the fixing period end date.

B.9.4 JPY

In JPY the payment is two days after the end of the fixing period which is also two days after the last index publication.

B.9.5 USD

In USD the payment is two days after the end of the fixing period. These two days are computed from the last publication date, which is at the end of the last period. In USD above two or three years, the OISs are not the most liquid instruments related to the overnight index. The Federal Fund swaps (see Section B.12) are usually more liquid.

B.9.6 Committee meetings

A somehow popular choice of start or end date for OIS swaps are the dates of the relevant central bank committee meetings. The links to pages with the dates of the main currencies' central bank meetings are provided below.
ECB: http://www.ecb.int/events/calendar/mgcgc/html/index.en.html
BoE: http://www.bankofengland.co.uk/monetarypolicy/Pages/decisions.aspx
Fed: http://www.federalreserve.gov/monetarypolicy/fomccalendars.htm

B.10 Basis swap: Ibor for Ibor

In a basis swap, both legs are floating legs and depend on an Ibor index in the same currency (see Section B.15 for swaps with legs in different currencies). In most cases, the indexes have different tenors. A spread above the Ibor index is paid on one of the legs. The quoting convention is to quote the spread on the shorter tenor leg.

Suppose you trade a swap USD LIBOR 3M vs USD LIBOR 6M quoted at 12 (bps) for ten million paying the three months. You will pay on a quarterly basis the USD LIBOR three month rate plus the spread of 12 bps and receive on a semi-annual basis the USD LIBOR six month rate.

This is the convention for almost all currencies, with the notable exception of EUR. In EUR, the basis swaps are conventionally quoted as two swaps. A quote of EURIBOR 3M vs EURIBOR 6M quoted at 12 (bps) for ten million paying the three months has the following meaning. You enter with the counterpart into two swaps fixed against Ibor. In the first swap you receive a fixed rate and pay the three month EURIBOR. In the second swap, you pay the same fixed rate plus the spread of 12 bps and receive the EURIBOR six months. Note that with that convention the spread is paid on an annual basis, like the standard fixed leg of a fixed versus Ibor swap. Even if the quote refers to the spread of a three months versus six months swap, the actual spread is paid annually with the fixed leg convention.

The composition of Ibor index described in Section B.5.3 is not restricted to fixed for Ibor swaps. Some basis swaps are also traded on a compounded basis to align the payment of both legs. For example a basis swap LIBOR one month versus LIBOR three months can be quoted with the one month LIBOR compounded over three periods and paid quarterly in line with the three months period. Note that the exact convention on the spread compounding needs to be made precise for the trade.

B.11 Basis swap: Ibor for overnight

The OISs are very liquid for the short term, and certainly up to one year or two years. For longer maturities, these swaps are less liquid. In some currencies for longer maturities the basis swaps Ibor for overnight are more liquid. These swaps have two floating legs and exchange Ibor indexed payments for an overnight index composed over the same period. As with the fixed for overnight swaps, a payment lag in the payment is applied on each leg. Both payments (or in practice the netted amount) are paid on the same day.

One exception to this mechanism is the USD where the most liquid medium term OISs are the Federal Fund swaps, which have a different payment mechanism and are described in the next section.

B.12 Federal Funds swaps

The Federal Fund swaps, or simply *Feds* or *Fed swaps*, like the basis swaps Ibor for overnight, exchange an Ibor payment for an overnight index based payment. The difference comes from the way the payment on the overnight leg is computed. In this case the payment is computed as the arithmetic sum of the overnight interest (USD-Effective Federal Funds rate) on the notional (also described as the arithmetic average of the rate). This payment mechanism is similar to the one for the Federal Fund futures. The quarterly coupon payment on the overnight leg is not equal to a three month OIS. In some cases the Federal Funds swaps are traded versus on-month LIBOR. This type of swap is less liquid.

Let $0 \leq t_0 < t_1 < \cdots < t_n < t_n$ be the relevant dates (all good business dates) in the overnight coupon period. Let δ_i be the accrual factor between t_{i-1} and t_i $(1 \leq i \leq n)$ and δ the total accrual factor for the total period $[t_0, t_n]$. The overnight rates between t_{i-1} and t_i are given in t_{i-1} by $I_X^O(t_{i-1})$. The paid amount is $\sum_{i=1}^{n} \delta_i I_X^O(t_{i-1})$ multiplied by the notional.

A special final Federal Funds effective fixing or rate cut-off is applied to the last two fixing days. In formula it means that the amount paid is

$$\sum_{i=1}^{n-1} \delta_i I_X^O(t_{i-1}) + \delta_n I_X^O(t_{n-1}).$$

It is possible to trade absent the rate cut-off, but this requires the counterpart to make the payment on the same day as the last fixing information is published.

The swaps are quoted with a spread on the overnight leg. A quote of x – often in basis points – means the swap exchanges LIBOR for overnight average plus a spread of x. The spread is usually positive. The computation of the interest on the floating overnight leg is additive with simple compounding and the spread is also additive with simple compounding. There is no alternative for the composition technique like in the Ibor compounding case. The payment, up to the final day repeated fixing option, is

$$\left(\sum_{i=1}^{n} \delta_i I_X^O(t_{i-1})\right) + \delta s.$$

B.13 Present value quoted deliverable swap futures

The futures nominal is USD 100,000 per contract; the notional is denoted N in this section. The margining feature is the future-type daily margin on the quoted price[2]. The underlying swap has the standard conventions for USD swaps: semi-annual bond basis versus LIBOR 3M. The futures are quoted for swaps with tenors 2, 5, 10 and 30 years. The underlying swap has a fixed rate as decided by the exchange on the first trading date of the contract. The rate is changed in increments of 25 basis points. Note, the rate is fixed at a predefined value, like the reference coupon of bond futures.

The *delivery dates* follow the quarterly cycle standard to interest rate futures. The *delivery date* is the third Wednesday of the quarterly months (March, June, September, December). The *last trading date* or *expiry date* is two business days prior to that date, usually on the Monday. The expiry date is denoted θ here.

On the expiry date, the parties agree to enter into a swap where the party long the futures receives fixed on the swap and the party short the futures pays fixed. The *effective date* of the swap, denoted t_0, is the delivery date. The fixed rate of the swap is the one attached to the swap futures. The swap also has an up-front payment on the delivery date. The up-front payment is obtained from the futures settlement price on the last trading date, denoted F_θ. The amount received by the long party is $(1 - F_\theta) * N$[3].

The 1 in the formula is a cosmetic feature. One could not add it to compute the up-front amount and keep the same economic reality. The only thing that would change is that the future *price* would be around 0 and not around 1 and the range

[2] Note that the price is quoted in (percentage) points and 32nd of points, like the bond futures contracts.
[3] If the amount is negative, it should be interpreted as the absolute value is paid by the long party.

of price would be different from the other futures. Personally I would have preferred it without the 1, but it is a personal taste.

Regarding the long/short choice in the contract definition (long is receiving fixed), it can be related to bond futures. The interest rate sensitivity is similar between the two types of futures. Being long the future is similar to being long a bond and short the rate/yield.

B.14 Forex swaps

Among foreign exchange products, the most liquid are certainly the spot forex transactions. But they contain no direct information about interest rate and are not used directly to build curves; we do not analyse them further.

The second next most liquid forex products are the forex swaps. They exchange notional amounts in two currencies at two different dates, receiving one currency at one date and paying it at the other date. The forex swaps are more liquid than the forex forwards. Even if from a theoretical point of view the forex swaps can be viewed as the sum of a forex spot and a forex forward, from a practical point of view, the forex forward is viewed as the difference of a forex swap and a forex spot. The reason is the following. The forex spot deal bears a forex risk (and provides forex information) and no interest rate risk. The forex swaps bear interest rate risk (see below) and almost no forex risk. The forex forwards are hybrid products with forex and interest rate risks. From a business perspective, the desks dealing with interest rate risks and forex risks are often split. When dealing with forex forward, the resulting risk has to be split between two centres of expertise, hence the reduced liquidity.

The quotation of forex swaps is done through *swap points*, also called *forward points*. For a typical forex swap, a notional N is agreed in one currency and is exchanged at the spot date for a notional in the other currency at the prevailing spot exchange rate x. That exchange is called the *near leg*. In some cases the agreed exchange rate is not in line with the current exchange rate, but this is usually done for accounting purposes or to hide some worrying economic reality. A second exchange is done at the forward date. The opposite of the notional (if you are receiving in the near leg, you pay in this one) is exchange at a rate computed as the initial rate plus the forward points p. The forward points are not annualised or relative to the exchange rate. This exchange is called the *far leg*. The cash-flows are described in Table B.8.

If the interest rate is neglected, one can see that the total cash-flow in the first currency is zero. There is very little total exposure to the first currency. The total cash-flow in the second currency is given by Np and is small. There is also very little total exposure to the second currency. The present value is how much value there

Table B.8 Vanilla swap conventions

Date	Currency 1	Currency 2
Near date	N	$-N \times x$
Far date	$-N$	$N \times (x+p)$

The near date is usually the spot date and the exchange rate the one prevailing at the trade moment.

Table B.9 Conventional currency strength

Strength 1	Strength 2	Strength 3	Strength 4	Strength 5
EUR	GBP	USD	CHF	JPY

is in receiving and paying the same amount at different dates in the first currency with respect to receiving and paying the very similar amounts at different dates in the second currency. The answer lies in the difference in interest rate between the two currencies. From a pure etymological perspective, these instruments would be better called *cross-currency interest rate swaps*, but this is not the terminology used in the market.

For this reason forex swaps are an important source of information for curve calibration in a multi-currency setting. They allow us to transfer interest rate information from one currency to another.

Another important practical question is in which order the forex products should be quoted. Do you quote EUR/USD or USD/EUR? Do you have to declare it for each transaction? For a forex spot trade, an EUR/USD transaction can be written as a USD/EUR transaction easily, but for forex swaps, where one currency has the same cash-flow amounts in both legs, an EUR/USD forex swap can not be transformed into a USD/EUR forex swap. The market has adopted a standard quotation order based on the *strength* of currencies. The strength is not an economic assessment of the current situation but a conventional one. The order, from the strongest to the weakest, of the five main currencies is given in Table B.9.

B.15 Cross-currency swaps: Ibor for Ibor

In this section we restrict ourselves to cross-currency swaps for which both legs are floating legs linked to Ibor indexes.

The notional is not the same on both legs as they are in different currencies. The notional on one leg is usually the notional in the other leg translated into the other currency through the exchange rate. The rate is often the exchange rate at the moment of the trade as agreed between the parties. The notional is paid on both legs, at the start and at the end of the swap.

In some cases the FX rates used are not in line with the market rate. Usually this is to disguise some debts from accounting rules. This type of cross-currency swap at non-market exchange rates were famously used by Greece to hide some of its debt when it entered the Euro. The swaps used for curve construction are done at the market exchange rate.

There also exists some cross-currency swaps with forex rate reset, called *market-to-market cross-currency swap*. At each payment date, the exchange rate between the legs is reset to the exchange rate at the time of reset and full notionals are exchanged. This feature is created to reduce the credit risk created by the movement of exchange rates.

Both legs of the swap are linked to an Ibor-like index. In the standard swaps, the Ibor tenor on both legs is the same. The payments are done on the same days for both legs to reduce the credit risk. It means that the payment calendar is the joint calendar of both currencies involved in the swap.

The most liquid cross-currency swaps exchange three month payments. Even if one currency has a six month index as its most used index, the cross-currency swap use three month payments. This is in particular the case of USD/JPY and USD/EUR swaps that use three month payments, even if the six month EUR Euribor and six month JPY LIBOR are the standard floating references for the IRS in those currencies.

The cross-currency swaps also pay a spread on one of the legs. In which currency the spread is paid depends on the currency pairs. For almost all swaps involving USD, the spread is paid in the other currency.

Appendix C
Implementation in a library

The third subtitle of this book is 'implementation'. Throughout this book we have focused on the practical aspects of the framework, how to deal with the market idiosyncrasies of each instrument and currency. In this section we describe some of the most important concepts that need to be put in place in an IT library that implements the multi-curve framework.

The description is done using the Java terminology and pseudo-code. The concepts extend to any programming language. Note that the code is always 'pseudo-code', this is code that gives all the important information but that may in some cases not compile or work. The reason for presenting only pseudo-code is to avoid filling the book with listings of redundant information. Also, in some cases to fit the code on the page, some semantic simplifications are introduced.

A full implementation is available in an open source version to which I'm an important contributor. The framework is the OpenGamma platform and the project containing the quantitative finance specific code is OG-Analytics. The code is available at http://developers.opengamma.com/downloads.

C.1 Curve universe object

The set of curves used in the multi-curve framework is not simply a set of independent curves, there is an important conceptual link between the curves and other financial concepts. These financial links should be represented into the code structure.

The other financial objects closely related to curves are:

Currency: For each currency there is one discounting curve. The link between the currency and its discounting curve should be explicit.

Ibor Index: For each Ibor index there is one curve. The same curve can be potentially shared by several indexes, such as between EURIBOR and EUR LIBOR. From a purist point of view this may not be desirable, as each index is different and has its own curve, but from a practical point of view the market information may be limited and some simplifications may be necessary.

Overnight Index: For each overnight index there is one curve. Note that in some cases, like for OIS discounting, the same curve can be used by a currency and an index. This flexibility should be available.

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Listing C.1 Multi-curve framework provider

```
public interface MulticurveProviderInterface{
  double getDiscountFactor(Currency ccy, double time);
  double getForwardRate(IndexIbor index, double startTime,
    double endTime, double accrualFactor);
  double getForwardRate(IndexON index, double startTime,
    double endTime, double accrualFactor);
  double getFxRate(Currency ccy1, Currency ccy2);
}
```

Forex rate: When the curves are used in a multi-currency environment it is important that the forex rates used for valuation are the same as those used in the curve construction.

The multi-curve object should provide the financially meaningful numbers. For a discounting curve associated to a currency, the object should provide the discount factor at each time. For a forward curve associated to an index, the object should provide the forward rate between two dates (maybe dependent on the accrual factor between them). The access to these numbers should be independent of the way the data is stored or the curve implemented. The different implementations described in Chapter 3 should have no impact on the user. For example the link between pseudo-discount factor and forward rate (Equation 3.1) should not be repeated in the valuation code each time a forward is required. The method needing the forward should simply request the forward. The model knows that a forward is required and asks for it, but it has no knowledge of how the curve is stored. The two should be kept separated.

This notion of a 'service' being available independently of the internal representation of the data is well represented in Java by an interface. The interface to multi-curve frameworks have been implemented under different names in different libraries. I have encountered the names *universe*, *market* and *provider*. In the code below we use the term provider. The simplified code of a multi-curve provider is in Listing C.1.

One single payment may depend on several indexes, depending on the contract description and the collateral rules. It is not the user's job to provide a restricted set of curves for a specific valuation. He should give the provider and the exact description of the instrument. It is to the pricing method to chose the correct curves for a given financial reality. Before the crisis, a lot of implementations were limited to one or two curves (usually one for discounting and one for forward). The description of an instrument would explicitly include the link to the two curves. Such an approach

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Listing C.2 Present value method for an Ibor floating coupon

```
public MultipleCurrencyAmount presentValue(CouponIbor cpn,
    MulticurveProviderInterface multicurve) {
  double forward = multicurve.getForwardRate(cpn.getIndex(),
    cpn.getFixingPeriodStartTime(),
    cpn.getFixingPeriodEndTime(),
    cpn.getFixingAccrualFactor());
  double df = multicurve.getDiscountFactor(cpn.getCurrency(),
    cpn.getPaymentTime());
  double pv = cpn.getNotional() * cpn.getPaymentAccrualFactor()
    * forward * df;
  return MultipleCurrencyAmount.of(cpn.getCurrency(), pv);
}
```

considerably limits what can be modelled correctly. All multi-Ibor coupons and collaterallised instruments are mis-priced because the instrument description and its modelling are not separated cleanly. In such a framework where the instrument itself is aware of the modelling through the names of the curves, a model change means from a database/code perspective that everything needs to be changed, even objects not related to the model.

With the separated approach we describe the pricing code becomes very short and clear. For example the present value of an Ibor coupon is proposed in Listing C.2

Note that the result of the present value computation is a Multiple Currency Amount and not a simple double. In general we work with multiple currencies, in particular for cross-currency swaps. It is important to explicitly state for each number to which currency it is associated.

C.2 Curve calibration

Given the requirement of multiple curves for different purposes, there is the parallel requirement of more instruments to build the curves. One cannot restrict the instrument used to build the curves to a small set, like deposit, FRA, futures and IRS. All the instruments that convey information on the market should be available for curve construction purposes.

The instruments I have used to build curves recently are: deposit, Ibor fixing, OIS (fixed tenor and central bank meetings dates), FRA, IRS (fixed versus Ibor, fixed tenor and IMM dates), basis swaps (Ibor versus Ibor, overnight versus Ibor), Fed Fund swaps, FX swaps, cross-currency swaps, STIR futures, Fed Fund futures and swap futures. The interaction between instruments and curves may be complex. One instrument often depends on three curves. The idiosyncrasies of some markets may be that different products depending on different indexes are liquid

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for different tenors. Those particularities require a very flexible curve calibration mechanism.

The flexibility means that one should be allowed to chose any set of instrument with any dependency on index and use them for curve calibration. The only mechanism that appears to satisfy these requirements is a large root-finding procedure. The root-finding is not for one individual curve but for a full universe of curves. It is in some cases possible to split the large root-finding into sub-problems solved separately. This simplification is described in Section 5.3. This is more a particular case than the general approach. Splitting the problem into sub-problems usually provides computation time improvement. But it introduces some extra house keeping problems. As mentioned in the curve calibration chapter, calibrating the curves to compute instruments present values is only a small part of the problem. A more important one, and usually more computationally intensive, is to compute the derivatives or sensitivities of the present value to all the market data input used in the curve construction.

The way to do this is usually by implementing some version of algorithmic differentiation (see next section). The first building block of algorithmic differentiation is the ability to compute the derivative of any sub-process used in the computation of the final numbers. In the curve construction case, one has to compute the Jacobian matrices of sensitivities of the curve internal parameters to the market quotes. Given the complexity of the interactions of all the curves, the matrix is usually not stored as a large unique matrix. That would mean storing a square matrix of dimension several hundred and carrying it around, with the indication of the order of the market quotes, even if only one curve is used for a particular computation. Our suggested implementation is to compute the partial matrix related to each curve and store one such matrix for each curve.

The information required for one curve is the list of curves on which its parameters depend and the number of those parameters. We call it a CurveBlock. Its code is provided in Listing C.3. An example of content for such an object is proposed in Table 8.1. We describe the curves simply by their name (String). It would be better to describe them by a proper identifier; this is not analysed here.

Once we have the list of curves a curve relates to, we need the bundle of those relationships for all curves and the actual Jacobian matrices. We propose to store this information in a bundle we call CurveBlockBundle. Its code is provided in Listing C.4. An example of the type of matrices stored can be found in Table 5.1 and Table 5.2.

A theoretical description of how to produce the partial Jacobian matrices without recomputing the full Jacobian matrix for each curve is proposed in Section 8.3.6.

Listing C.3 Curve building block with reference to other curves and their market quote sizes label

```
public class CurveBlock {
  /**
   * List of curves in the block as a map:
   *   Curve name to a pair of integers:
   *   1) Start index of the curve in the list of all
        parameters
   *   2) Number of parameters in the curve
   */
  private Map<String, Pair<Integer, Integer>> _unit;

  public CurveBlock(Map<String, Pair<Integer,
     Integer>> unit) {
    _unit = new Map<>();
    _unit.putAll(unit);
  }

  public Integer getStart(String name) {
    return _unit.get(name).getFirst();
  }

  public Integer getNbParameters(String name) {
    return _unit.get(name).getSecond();
  }
}
```

Listing C.4 Bundle of all curve building blocks and their Jacobian matrices

```
public class CurveBlockBundle {

  private Map<String, Pair<CurveBlock, Matrix2D>> _bundle;

  public CurveBlockBundle(Map<String, Pair<CurveBlock,
     Matrix2D>> bundle) {
    _bundle = new Map<>();
    _bundle.putAll(bundle);
  }

  public void add(String name, CurveBlock block,
     Matrix2D mat)
    _bundle.put(name, Pair.of(block, mat));
  }

  public Pair<CurveBlock, Matrix2D> getBlock(String name)
    return _bundle.get(name);
  }
}
```


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C.3 Algorithmic differentiation

Computing the first order derivative of results with respect to the input is an important task of any financial software. In finance, these derivatives are known as *the greeks*. In the multi-curve framework this is an even more important task as there are more inputs. As described in the curve construction chapter, the number of inputs can be large and often the different curves are entangled and cannot be split. Even the simplest instruments depend on many inputs. A standard IRS, priced using two curves, can depend on more than 40 points. When curves are built through cross-currency instruments, a simple cross-currency swap can depend on 100 inputs. In the simplified example described in Section 5.7, we used 50 points. This multiplication of input is a feature of the multi-curve framework and should certainly not be overlooked. The number of required outputs in the computations increases with the complexity of the curve construction.

A traditional approach to first order derivative computation is to use the *bump and recompute* approach, also called *finite difference*. It is based on the definition of derivative through the limit of differentiation ratio.

Definition C.1. (Differentiation ratio). *The derivative of a function* $f : \mathbb{R}^n \to \mathbb{R}$ *in the direction* e_i *is defined, when it exists, by the limit*

$$D_{e_i} f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon e_i) - f(x)}{\epsilon}.$$

By definition the above limiting process is true in theory. In practice, in a finite precision arithmetic, the above equality is not true. When decreasing the bump size, the machine precision is reached and the results diverge. This comes from the fact that the limit is of the style '0/0'. For standard implementations, the level at which the divergence starts is around 1.0×10^{-10}. This is the best case, when the process to compute f does not contain heavy numerical procedure. The user is squeezed between taking a small bump to obtain the theoretical convergence and a larger one in order to avoid the numerical instability problem.

An example of instability for small increments is proposed in Figure C.1. This particular example is a swaption present value in the Hull-White model. Any pricing will present a similar picture. The error decreases when the bump size is decreased, up to a certain point where the numerical instability kicks in and the number become meaningless. The level where the numerical instability kicks in will depend on the algorithm used in the computation.

The *bump and recompute* approach is the standard approach in most implementations. As it is the dominant approach, often the numerical method is confused with its goal. There is a confusion between the cause and the consequence. For example, (Traven 2010, Section 4) presents the the bump and recompute method as the definition of delta risk. (Sadr 2009, Chapter 3) also uses the bump and

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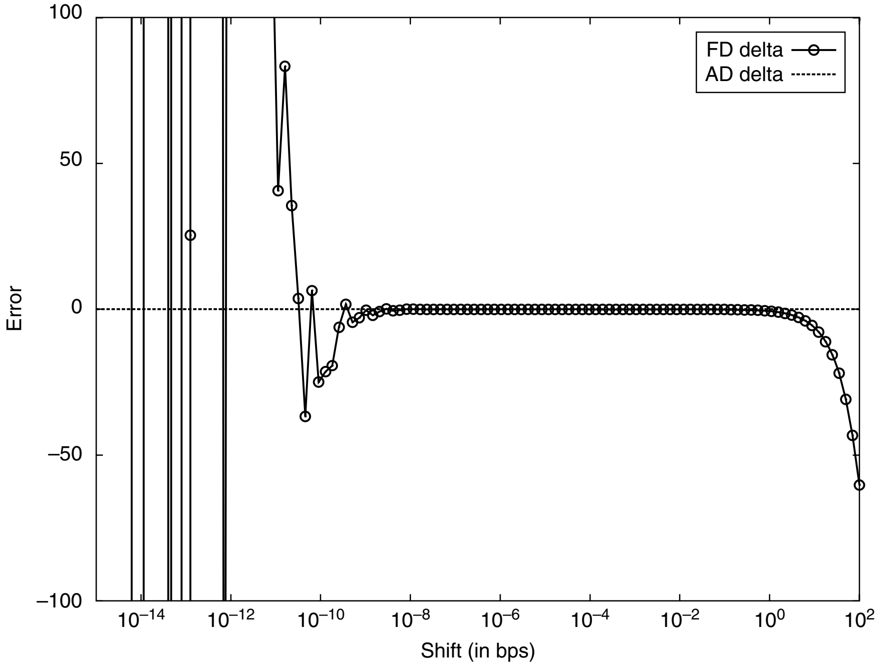

Figure C.1 Instability of finite difference computation of derivative. The horizontal axis represents the bump size ϵ in logarithmic scale. The vertical axis is the error. The true value is represented by the horizontal line.

recompute method as the definition of PV01. This last reference even includes in the definition the fact that the bump should be one sided and of particular size (one basis point). Under that definition, partial PV01 does not add to the total PV01 as the definition explicitly includes some convexity and higher order effects. Le Floc'h (2012) analyses the quality of interpolation methods for curve construction through their stability in the above numerical procedure. The stability question appears only if a poor procedure is used in the computation.

The bump and recompute approach becomes more and more expensive when one implements a full scope multi-curve framework and even more so in its collateral version. Not only because several curves are used for one instrument but also because of the dependency between them. An efficient implementation of the derivatives computation is not a nice-to-have anymore, but a fundamental requirement for a near real time system.

The algorithmic differentiation approach appears as the only viable approach. The theoretical foundation of the method can be found in the classical book Griewank and Walther (2008). The approach, including outside the multi-curve framework, has taken on more importance in finance. One of the articles that

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relaunched interest in the approach in finance was probably Giles and Glasserman (2006). More recent descriptions of implementations in finance can be found in Capriotti (2011) and Capriotti and Giles (2011).

One of the main results of algorithmic differentiation is that in its *adjoint* or *reverse* mode, the relative cost between the computation of the function and its derivatives (F+D) and the function (F) alone is (see (Griewank and Walther, 2008, Section 4.6))

$$\frac{\text{Cost}(F+D)}{\text{Cost}(F)} \leq \omega_A$$

with $\omega_A \in [3,4]$ for a function with an output of dimension 1 (like a present value).

The theoretical performance can be achieved in practice when a library is built with the algorithmic differentiation requirement in mind. When the performance is achieved in practice, this represents in the multi-curve framework an improvement of a factor between 5 and 25. For very large portfolios, it can be the difference between a one hour computation time and a one day computation time.

With an efficient implementation of algorithmic differentiation, the time to compute the present value and the curve sensitivities should not be above four times the time of computing the present value alone. The sensitivities include risk figures like the bucketed deltas, even if there are several curves and a lot of instruments used to calibrate the curves. The factor four is obviously not equal to one, which would be a free lunch, but is close enough that if the real-time pricing is achievable, the real-time sensitivities computation is also possible.

C.3.1 Principles of algorithmic differentiation

The starting point is the algorithm to compute a value:

$$z = f(a).$$

We consider only the case where f has its values in $\mathbb{R}$. This is the traditional case in quantitative finance, where the first goal is to compute the present value of a financial instrument. The function inputs are $a = a[0 : p_a]$, a vector of dimension $p_a + 1$. The function output is z, a single double, that is z is of dimension 1.

The code is summarised in Table C.1. The code starts by initialising the internal variables with the inputs. Then the actual algorithm take place. We restrict ourself to the case where the algorithm is composed of a list of consecutive assignments. Each line of code is potentially using all the previous variables, those coming from the inputs and the intermediary variables. This algorithm is suppose to be implemented in the code.

The goal is to compute the derivatives of z with respect to the different inputs a_i:

$$\frac{\partial}{\partial a_i} f(a) = \frac{\partial}{\partial a_i} z.$$

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Table C.1 The generic code for a function computing a single value

Initialisation	$[j=-p_a:0]$	$b[j]$	$=a[j+p_a]$
Algorithm	$[j=1:p_b]$	$b[j]$	$=g_j(b[-p_a:j-1])$
Value		z	$=b[p_b]$

The result of this computation is a vector of size p_a+1, like the input vector a.

There are two modes to viewing this problem. The first mode puts emphasis on 'with respect to a_i'; this is called the *standard* or *forward* mode. One starts from the inputs a_i and at each line of code in the value computation associates p_a+1 lines of code which compute the different derivatives of each intermediary variable with respect to the different inputs.

The second mode put emphasis on 'of z'; this is called the *adjoint* of *reverse* mode. One starts from the output and reads the code in the reverse order. At each line of code we associate a new line of code which computes the derivative of the output with respect to the intermediary variables of that line.

In each case, to be able to compute the derivative of each line of code, we need to have the derivatives $\partial g_j/\partial b_i$ $(-p_a \leq i < j)$ available. In practice, this means that the function g_j is a known mathematical function, like exp, or a function previously built in our library for which we have already developed an algorithmic differentiation version. The algorithmic differentiation works satisfactorily only when it is built in the library from scratch. All the building blocks, like interpolators and Black formula, need to have an algorithmic differentiation version before starting with the more complex algorithms and models. There is no point in trying to implement the approach for the most time consuming algorithm if one has not already spent the time implementing the approach for all the fundamental blocks. This is a bottom-up approach.

The final result is the same in both cases: the derivatives of the output with respect to all the inputs. From the brief description above, one can guess the efficiency of the two procedures. In the forward mode the computation time is proportional to the number of inputs while in the reverse mode the computation time is proportional to the number of outputs, which is 1 in our problem. In general in finance, where one present value is computed from numerous inputs, the adjoint mode is significantly more efficient. We do not discuss the forward mode further here.

Let's look at the adjoint mode in more detail. Our goal is to compute $\partial z/\partial a_i$. We achieve that by computing for each j $(-p_a \leq j \leq p)$ the value

$$\bar{b}[j] = \frac{\partial}{\partial b[j]} z.$$

Table C.2 The generic code for a function computing a single value and its algorithmic differentiation code

Init	$[j = -p_a : 0]$	$b[j]$	$= a[j + p_a]$
Algorithm	$[j = 1 : p_b]$	$b[j]$	$= g_j(b[-p_a : j-1])$
Value		z	$= b[p_b]$
Value		$\bar{z}$	$= 1.0$
Value		$\bar{b}[p_b]$	$= 1.0$
Algorithm	$[j = p_b - 1 : -1 : -p_a]$	$\bar{b}[j]$	$= \sum_{k=j+1}^{p_b} \frac{\partial}{\partial b_k} z \frac{\partial}{\partial b_j} b_k$
			$= \sum_{k=j+1}^{p_b} \bar{b}[k] \frac{\partial}{\partial b_j} g_k$
Init	$[i = 0 : p_a]$	$\frac{\partial}{\partial a_i} z$	$= \bar{b}[i - p_a + 1]$

The summary of the algorithm is presented in Table C.2. We start with the easy part. For $j = p_b$, the derivative of z with respect to b_j is simply the derivative of z with respect to itself, which is 1. This is the starting point of a recursive algorithm.

From there we read the code in the reverse order and use the derivative composition. Each intermediary variable $b[j]$ is used only in the lines that follow in the computation. The derivative $\bar{b}[j]$ is given by

$$\bar{b}[j] = \frac{\partial}{\partial b_j} z = \sum_{k=j+1}^{p_b} \frac{\partial}{\partial b_k} z \cdot \frac{\partial}{\partial b_j} b_k = \sum_{k=j+1}^{p_b} \bar{b}[k] \cdot \frac{\partial}{\partial b_j} g_k.$$

From the recursive approach, the values $\bar{b}[k]$ for $k > j$ are already known. The derivatives of the functions g_k can be obtained from the implementation of algorithmic differentiation for building blocks or as known mathematical functions. Using that approach one can recursively obtain the values $\bar{b}[j]$ down to $\bar{b}[j]$ for $j = -p_a, \ldots, 0$. These numbers are equal to the derivatives of z with respect to a_i $(i = 0, \ldots, p_a; j = i - p_a + 1)$. This concludes the algorithm.

C.3.2 Example: curve calibration

The first example of computation efficiency concerns an indirect application of algorithmic differentiation. In the curve calibration, we use a multidimensional root-finding algorithm. The implementation uses the derivative of the function to iterate the point in the root-finding algorithm. The derivative can be computed through finite difference or explicitly by algorithmic differentiation. We compare the finite difference performance to the adjoint algorithmic differentiation performance.

The example refers to the building of three curves in EUR (discounting, forward three months and forward six months). We use OIS, fixed versus three months

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Table C.3 Time to build curves with and without algorithmic differentiation

	Number of units		
Algorithmic differentiation	**1**	**3**	**Ratio**
No	109.1	41.5	2.63
Yes	33.3	20.2	1.65
Ratio	3.28	2.05	5.40

Time in milliseconds.

IRS and fixed versus six months IRS. The choice of instruments is done in such a way that the curves are not entangled and can be built one by one; no simultaneous building is required. With that choice, we can analyse two items: the interest of having a flexible implementation allowing for simultaneous or successive curve building and the algorithmic differentiation impact. Note that the algorithmic differentiation impact is indirect as the derivative of the objective function is only a small part of the computation.

The example uses three curves with 20 instruments each. The results are presented in Table C.3. When three units are used the ratio between the version without algorithmic differentiation and the version with algorithmic differentiation is 2.05; when the three curves are built simultaneously, the ratio is 3.28. In the most complex problems, the algorithmic differentiation, even if used indirectly, has a real impact.

The possibility of splitting the problem into several units is assessed by the last column. The ratio between one unit and three is 1.65 with algorithmic differentiation and 2.63 without. The computation of the derivative being very efficient and only slightly impacted by the number of inputs, the difference is not huge but far from trivial. In the absence of efficient derivative computation, the ratio is significantly larger (2.63). Finally, the last ratio is between the number with one unit and no algorithmic differentiation and with three units and algorithmic differentiation. The combination of these two numerical improvements reduces the computation time more than fivefold.

C.3.3 Bucketed delta computation

As described earlier, computing the risk of a book with respect to all the market quotes is a requirement of any implementation. The next item is to apply algorithmic differentiation to the multi-curve risk computation. We suggest doing so in three steps.

The first step is to move from the market quotes to the parameterised curves. The output of that step is obviously the parameterised curves, but also the Jacobian or transition matrices, as explained in Section C.2.

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Listing C.5 Multicurve sensitivity implementation

```
public class MulticurveSensi {

  private Map<String, List<DoublesPair>> _sensiYieldDisc;

  private Map<String, List<ForwardSensi>> _sensiForward;

  MulticurveSensi(Map<String, List<DoublesPair>>
    sensiYieldDisc,
    Map<String, List<ForwardSensi>> sensiForward) {
    _sensiYieldDisc = sensiYieldDisc;
    _sensiForward = sensiForward;
  }

  public MulticurveSensi plus(MulticurveSensi other) {
    Map<String, List<DoublesPair>> resultDsc
      = SensiUtils.plus(_sensiYieldDisc, other._sensiYieldDisc);
    Map<String, List<ForwardSensi>> resultFwd
      = SensiUtils.plusFwd(_sensiForward, other._sensiForward);
    return new MulticurveSensi(resultDsc, resultFwd);
  }

  public MulticurveSensi multipliedBy(double factor) {
    final Map<String, List<DoublesPair>> resultDsc
      = SensiUtils.multipliedBy(_sensiYieldDisc, factor);
    final Map<String, List<ForwardSensitivity>> resultFwd
      = SensiUtils.multipliedByFwd(_sensiForward, factor);
    return new MulticurveSensi(resultDsc, resultFwd);
  }
}
```

The last step concerns the computation of the sensitivities of a specific instrument with respect to some curve information. The way we suggest to implement it is the following. For the discounting curve, the sensitivity is computed for each payment date (not only the dates used as node points if the curve is interpolated). This ensure that no information is lost at this stage. We compute the sensitivity with respect to each payment. Similarly, for each forward rate, the sensitivity to that forward rate is computed, again without any projection to any curve node. The information so obtained is larger than the information coming from the sensitivity to the curve parameters, so can be viewed as too large. But at this stage there is no reason to lose information on the exact dates of the sensitivity. Producing that information is important in some circumstances; one may want to check the sensitivity to a rate on a particular date.

In the implementation we create an object called MulticurveSensi that contains these sensitivities. The pseudo-code to the class is given in Listing C.5.

Listing C.6 Forward rate sensitivity object implementation

```
public class ForwardSensi {
  private double_startTime;
  private double_endTime;
  private double_accrualFactor;
  private double_value;
}
```

The sensitivity to the discounting curves is stored, for each curve represented by its name, as a List of DoublesPair. For each cash-flow date, represented by the first double, the sensitivity, represented by the second double, is stored. One needs a convention on how the sensitivity is represented. In our implementation, we decided to describe the sensitivity as the sensitivity with respect to the zero-coupon continuously compounded rate equivalent to the discount factor. To be fully compliant with the theory, we should have represented it as the sensitivity to the discount factor itself. But there, like in many other places, the *evolution* was stronger than the *foundation* and we kept the rate (or yield) as the storing base.

The sensitivity to the forward curve is stored, for each curve represented by its name, as a List of ForwardSensi. The last object is provided in Listing C.6. It simply collects the sensitivity to a particular forward rate with the rate itself described by its start time, end time and accrual factor. Again to be in line with the theory, it would be enough to store only the start time, as the end time and accrual factor are implicit from the description of the relevant index. But it is a lot more efficient to compute those numbers only once and store them than to have the curve recomputing them time and time again when required.

An example of the implementation of the curve sensitivity computation for an Ibor coupon is proposed in Listing C.7. Note that, like for the present value, the output is a multiple currency object, as the sensitivities can be with respect to several currencies. The sensitivities can also be in a currency different to the curve currency as described in Section 5.7.

In between these two steps lies the missing link. From the sensitivity to each cash-flow, and the curve description, we need to be able to compute the sensitivity to the curve parameters. This is a job for the curve itself to provide that sensitivity. The implementation will be strongly curve representation dependent. The sensitivity to a interpolated curve on continuously compounded rates will be different from a parameterised curve like Nelson-Siegel or a spread curve built from two curves interpolated on discount factors.

To achieve that result, the curve interface described in Listing C.1 is completed by the methods described in Listing C.8. For a given sensitivity (to discounting yield or forward rates), the method returns the sensitivity to the curve parameters. In this

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Listing C.7 Curve sensitivity computation for a Ibor coupon

```
public
MultipleCurrencyMulticurveSensipresentValueCurveSensitivity(
    CouponIbor cpn, MulticurveProviderInterface multicurve) {
  double forward = multicurve.getForwardRate(cpn.getIndex(),
    cpn.getFixingPeriodStartTime(),
    cpn.getFixingPeriodEndTime(),
    cpn.getFixingAccrualFactor());
  double df = multicurve.getDiscountFactor(cpn.getCurrency(),
    cpn.getPaymentTime());
  double pv = cpn.getNotional() * cpn.getPaymentAccrualFactor()
    forward * df;
  // Backward sweep
  double pvBar = 1.0;
  double forwardBar = coupon.getNotional()
    * coupon.getPaymentYearFraction() * df * pvBar;
  double dfBar = coupon.getNotional()
    * coupon.getPaymentYearFraction() * forward * pvBar;
  Map<String, List<DoublesPair>> mapDsc = new Map<>();
  List<DoublesPair> listDiscounting = new List<>();
  listDiscounting.add(Pair.of(coupon.getPaymentTime(),
    -coupon.getPaymentTime() * df * dfBar));
  mapDsc.put(multicurve.getName(coupon.getCurrency()), listDiscounting);
  Map<String, List<ForwardSensitivity>> mapFwd = new Map<>();
  List<ForwardSensitivity> listFwd = new List<>();
  listFwd.add(new ForwardSensitivity(coupon.getFixingPeriodStartTime(),
    coupon.getFixingPeriodEndTime(), coupon.getFixingAccrualFactor(),
    forwardBar));
  mapFwd.put(multicurve.getName(coupon.getIndex()),
    listForward);
  return MultipleCurrencyMulticurveSensitivity.of(coupon.getCurrency(),
  MulticurveSensitivity.of(mapDsc, mapFwd));
  }
```

Listing C.8 Methods for the computation of parameters sensitivities

```
double[] parameterSensi(String name,
  List<DoublesPair> pointSensi);

double[] parameterForwardSensi(String name,
  List<ForwardSensi> pointSensi);
```

case, for each curve, the number of parameters is known and independent of the instrument. It can be returned as a double array with a known size.

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