Divide and Conquer Algorithms

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Divide and Conquer Algorithms

- Study three divide and conquer algorithms:
  - Counting inversions.
  - Finding the closest pair of points.
  - Integer multiplication.
- First two problems use clever conquer strategies.
- Third problem uses a clever divide strategy.

Motivation

- Collaborative filtering: match one user's preferences to those of other users.
- Meta-search engines: merge results of multiple search engines to into a better search result.

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- Fundamental question: how do we compare a pair of rankings?
- Suggestion: two rankings are very similar if they have few inversions.
  - Assume one ranking is the ordered list of integers from 1 to n.
  - The other ranking is a permutation $a_1, a_2, \ldots, a_n$ of the integers from 1 to n.
  - The second ranking has an *inversion* if there exist i, j such that $i < j$ but $a_i > a_j$.
  - The number of inversions s is a measure of the difference between the rankings.
- Question also arises in statistics: *Kendall's rank correlation* of two lists of numbers is $1 - 2s/\left(n(n-1)\right)$.

Counting Inversions

COUNT INVERSIONS
INSTANCE: A list $L = x_1, x_2, \ldots, x_n$ of distinct integers between 1 and n.
SOLUTION: The number of pairs $(i, j), 1 \leq i < j \leq n$ such $x_i > x_j$.

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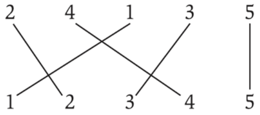

Figure 5.4 Counting the number of inversions in the sequence 2, 4, 1, 3, 5. Each crossing pair of line segments corresponds to one pair that is in the opposite order in the input list and the ascending list—in other words, an inversion.

Counting Inversions: Algorithm

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- Sorting removes all inversions in $O(n \log n)$ time. Can we modify the Mergesort algorithm to count inversions?
- Candidate algorithm:
  1. Partition L into two lists A and B of size $n/2$ each.
  2. Recursively count the number of inversions in A.
  3. Recursively count the number of inversions in B.
  4. Count the number of inversions involving one element in A and one element in B.

Counting Inversions: Conquer Step

- Given lists $A = a_1, a_2, \ldots, a_m$ and $B = b_1, b_2, \ldots b_m$, compute the number of pairs a_i and b_j such $a_i > b_j$.

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- Key idea: problem is much easier if A and B are sorted!
- MERGE-AND-COUNT procedure:

  Maintain a *current* pointer for each list.

  Initialise each pointer to the front of the list.
  While both lists are nonempty:
    Let a_i and b_j be the elements pointed to by the *current* pointers.
    Append the smaller of the two to the output list.

    Advance the current pointer in the list that the smaller element belonged to.
  EndWhile
  Append the rest of the non-empty list to the output.
  Return *count* and the merged list.

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- Running time of this algorithm is $O(m)$.

Counting Inversions: Final Algorithm

```
Sort-and-Count(L)
  If the list has one element then
      there are no inversions
  Else
      Divide the list into two halves:
         A contains the first ⌈n/2⌉  elements
         B contains the remaining ⌊n/2⌋ elements
      (r_A, A) = Sort-and-Count(A)
      (r_B, B) = Sort-and-Count(B)
      (r, L) = Merge-and-Count(A, B)
   Endif
   Return r = r_A + r_B + r, and the sorted list L
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- Running time $T(n)$ of the algorithm is $O(n \log n)$ because $T(n) \leq 2T(n/2) + O(n)$.

Counting Inversions: Correctness

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- Prove by induction.

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SOLUTION: The number of pairs $(i, j), 1 \leq i < j \leq n$ such $x_i > x_j$.

- Prove by induction.
- Base case: $n = 1$.
- Inductive hypothesis: Algorithm counts number of inversions correctly for all sets of $n - 1$ or fewer numbers.

Counting Inversions: Correctness continued

- Inductive step: Pick an arbitrary k and l such that $k < l$ but $x_k > x_j$. When is the inversion counted?

Counting Inversions: Correctness continued

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  - $i, j \leq \lfloor n/2 \rfloor$:
  - $i, j \geq \lceil n/2 \rceil$:
  - $i \leq \lfloor n/2 \rfloor, j \geq \lceil n/2 \rceil$:

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  - $i, j \leq \lfloor n/2 \rfloor$: $x_i, x_j \in A$, counted in r_A.
  - $i, j \geq \lceil n/2 \rceil$: $x_i, x_j \in B$, counted in r_B.
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  - $i \leq \lfloor n/2 \rfloor, j \geq \lceil n/2 \rceil$: $x_i \in A, x_j \in B$, counted by Merge-and-Count?

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- Establish correctness of Merge-and-Count.

Computational Geometry

- Algorithms for geometric objects: points, lines, segments, triangles, spheres, polyhedra, ldots.
- Started in 1975 by Shamos and Hoey.
- Problems studied have applications in a vast number of fields: ecology, molecular biology, statistics, computational finance, computer graphics, computer vision, ...

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- At first glance, it seems any algorithm must take $\Omega(n^2)$ time.
- Shamos and Hoey figured out an ingenious $O(n \log n)$ divide and conquer algorithm.

Closest Pair: Set-up

- Let $P = \{p_1, p_2, \ldots, p_n\}$ with $p_i = (x_i, y_i)$.
- Use $d(p_i, p_j)$ to denote the Euclidean distance between p_i and p_j.
- Goal: find the pair of points p_i and p_j that minimise $d(p_i, p_j)$.

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- How do we solve the problem in 1D?
  - Sort: closest pair must be adjacent in the sorted order.
  - Divide and conquer after sorting: closest pair must be closest of
    1. closest pair in left half.
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    3. closest among pairs that span the left and right halves. How many such pairs do we need to consider?

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- These ideas do not work directly in 2D.

Closest Pair: Algorithm Skeleton

1. Divide P into two sets Q and R of $n/2$ points such that each point in Q has x-coordinate less than any point in R.
2. Recursively compute closest pair in Q and in R, respectively.

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3. Let δ_1 be the distance computed for Q, δ_2 be the distance computed for R, and $\delta = \min(\delta_1, \delta 2)$.
4. Compute pair (q, r) of points such that $q \in Q$, $r \in R$, $d(q, r) < \delta$ and $d(q, r)$ is the smallest possible.

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   - How do we implement this step in $O(n)$ time?

Closest Pair: Conquer Step

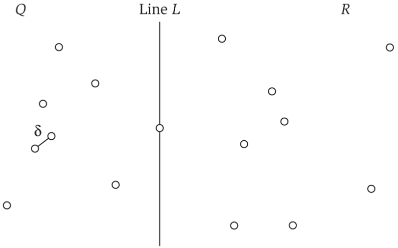

Figure 5.6 The first level of recursion: The point set P is divided evenly into Q and R by the line L, and the closest pair is found on each side recursively.

- Line L passes through right-most point in Q.

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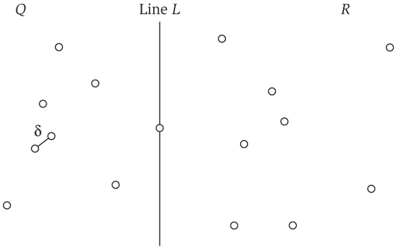

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- Claim: If there exist $q \in Q$, $r \in R$ such that $d(q, r) < \delta$, then q and r are both within distance δ of L.

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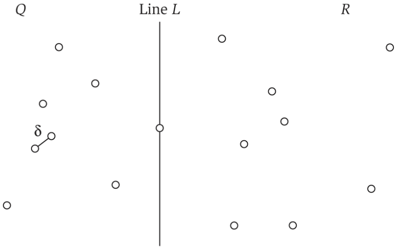

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- Line L passes through right-most point in Q.
- Claim: If there exist $q \in Q$, $r \in R$ such that $d(q, r) < \delta$, then q and r are both within distance δ of L.
- Let S be the set of points within distance δ of L and let S_y denote these points sorted by increasing y-coordinate.
- Claim: There exist $q \in Q$, $r \in R$ such that $d(q, r) < \delta$ if and only if there exist $s, s' \in S$ such that $d(s, s') < \delta$.

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- Intuition: if there are "too many" points in S that are closer than δ to each other, then there must be a pair in Q or in R that are less than δ apart.

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- For a point $s \in S$, let s_y denote its y-coordinate.
- Converse of the claim: If there exist $s, s' \in S$ such that s' appears 16 or more indices after s in S_y, then $s'_y - s_y \geq \delta$.

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- Converse of the claim: If there exist $s, s' \in S$ such that s' appears 16 or more indices after s in S_y, then $s'_y - s_y \geq \delta$.
- Idea behind the proof: pack the plane with squares, argue that each square contains at most one point.

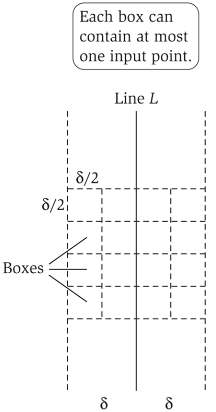

Figure 5.7 The portion of the plane close to the dividing line L, as analyzed in the proof of (5.10).

Closest Pair: Final Algorithm

```
Closest-Pair(P)
  Construct P_x and P_y   (O(n log n) time)
  (p*_0, p*_1) = Closest-Pair-Rec(P_x,P_y)

Closest-Pair-Rec(P_x, P_y)
  If |P| ≤ 3 then
    find closest pair by measuring all pairwise distances
  Endif

  Construct Q_x, Q_y, R_x, R_y (O(n) time)
  (q*_0,q*_1) = Closest-Pair-Rec(Q_x, Q_y)
  (r*_0,r*_1) = Closest-Pair-Rec(R_x, R_y)

  δ = min(d(q*_0,q*_1), d(r*_0,r*_1))
  x* = maximum x-coordinate of a point in set Q
  L = {(x,y) : x = x*}
  S = points in P within distance δ of L.

  Construct S_y (O(n) time)
  For each point s ∈ S_y, compute distance from s
     to each of next 15 points in S_y
     Let s, s' be pair achieving minimum of these distances
     (O(n) time)

  If d(s,s') < δ then
     Return (s,s')
  Else if d(q*_0,q*_1) < d(r*_0,r*_1) then
     Return (q*_0,q*_1)
  Else
     Return (r*_0,r*_1)
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Closest-Pair-Rec(P_x, P_y)

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  (q_0^*,q_1^*) = Closest-Pair-Rec(Q_x, Q_y)

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Construct S_y ($O(n)$ time)
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Integer Multiplication

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SOLUTION: The product xy

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- Algorithm we learnt in school takes

$$
\begin{array}{r}
12 \\
\times\, 13 \\ \hline
36 \\
12 \\ \hline
156
\end{array}
\qquad\qquad
\begin{array}{r}
1100 \\
\times\, 1101 \\ \hline
1100 \\
0000 \\
1100 \\
1100 \\ \hline
10011100
\end{array}
$$

(a) (b)

Figure 5.8 The elementary-school algorithm for multiplying two integers, in (a) decimal and (b) binary representation.

Integer Multiplication

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- Multiply two n-digit integers.
- Result has at most $2n$ digits.
- Algorithm we learnt in school takes $O(n^2)$ operations. Size of the input is not 2 but $2n$,

$$\begin{array}{r} 12 \\ \times\, 13 \\ \hline 36 \\ 12 \\ \hline 156 \end{array} \qquad \begin{array}{r} 1100 \\ \times\, 1101 \\ \hline 1100 \\ 0000 \\ 1100 \\ 1100 \\ \hline 10011100 \end{array}$$

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- Assume integers are binary.
- Let us use divide and conquer

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- Let us use divide and conquer by splitting each number into first $n/2$ bits and last $n/2$ bits.
- Let x be split into x_0 (lower-order bits) and x_1 (higher-order bits) and y into y_0 (lower-order bits) and y_1 (higher-order bits).

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$$
\begin{aligned}
T(n) &\leq 3T(n/2) + cn \\
&\leq O(n^{\log_2 3}) = O(n^{1.59})
\end{aligned}
$$

Final Algorithm

```
Recursive-Multiply(x,y):
  Write x = x_1 · 2^{n/2} + x_0
        y = y_1 · 2^{n/2} + y_0
  Compute x_1 + x_0 and y_1 + y_0
  p = Recursive-Multiply(x_1 + x_0, y_1 + y_0)
  x_1y_1 = Recursive-Multiply(x_1, y_1)
  x_0y_0 = Recursive-Multiply(x_0, y_0)
  Return x_1y_1 · 2^n + (p − x_1y_1 − x_0y_0) · 2^{n/2} + x_0y_0
```

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Write $x = x_1 \cdot 2^{n/2} + x_0$

$y = y_1 \cdot 2^{n/2} + y_0$

Compute $x_1 + x_0$ and $y_1 + y_0$

p = Recursive-Multiply($x_1 + x_0$, $y_1 + y_0$)

$x_1 y_1$ = Recursive-Multiply(x_1, y_1)

$x_0 y_0$ = Recursive-Multiply(x_0, y_0)

Return $x_1 y_1 \cdot 2^n + (p - x_1 y_1 - x_0 y_0) \cdot 2^{n/2} + x_0 y_0$