PEP of Bregman Halpern

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September 5, 2021

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Progress

Finished:

- (last time) Complete proof for BPG
- Recover Euclidean case of Bregman Halpern [Lieder, 2021]

In progress:

- Bregman Halpern iteration for non-euclidean kernel

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Bregman Nonexpansive

Let $T: C \to \mathcal{H}$ where $\mathcal{H}$ is an Hilbert space, $C \subseteq \mathcal{H}$.
Let f be a Legendre kernel funciton.
T is *Bregman Nonexpansive* if for any $x, y \in C$,

$$D_f(Tx, Ty) \leq D_f(x, y)$$

Bregman Halpern

Problem:

$$\text{Find} \quad x = T(x) \tag{P}$$

Method: Bregman Halpern's iteration

$$x_{k+1} = \nabla f^{*}[\lambda_k \nabla f(x_0) + (1 - \lambda_k)\nabla f(T(x_k))] \tag{BH}$$

$\lambda_k = \frac{1}{k+2}$

PEP Formulation

Let $y_k = T(x_k)$, $fx_k = f(x_k)$, $fy_k = f(y_k)$, $g_k = \nabla f(x_k)$, $h_k = \nabla f(y_k)$.

$$\max \quad D_f(y_N, x_N) \tag{PEP}$$
$$s.t. \begin{cases} \text{constraints on } f \\ D_f(y_i, y_j) - D_f(x_i, x_j) \leq 0 \\ G^{gx}_{i+1,j} - \lambda_i G^{gx}_{0,j} - (1-\lambda_i) G^{hx}_{i,j} = 0 \\ G^{gy}_{i+1,j} - \lambda_i G^{gy}_{0,j} - (1-\lambda_i) G^{hy}_{i,j} = 0 \end{cases}$$

$P = \left(P_x, P_y, P_g, P_h\right) =$
$\left(x_0, \ldots, x_N, y_*, y_0, \ldots, y_N, g_*, g_0, \ldots, g_N, h_*, h_0, \ldots, h_N\right)$,
$G = P^T P$.

Euclidean Case

$$\max \quad D_f(y_N, x_N) \qquad \text{(PEP)}$$

$$s.t. \begin{cases} G^{xx}_{i,i} - 2 * fx_i = 0, G^{yy}_{i,i} - 2 * fy_i = 0, i \in I \\ G^{gx}_{i,j} - G^{xx}_{i,j} = 0, G^{gy}_{i,j} - G^{xy}_{i,j} = 0, \\ G^{hx}_{i,j} - G^{yx}_{i,j} = 0, G^{hy}_{i,j} - G^{xx}_{i,j} = 0, i, j \in I \\ D_f(y_i, y_j) - D_f(x_i, x_j) \leq 0, (i + 1 = j \text{ or } i = *, j = N) \\ G^{gx}_{i+1,j} - \lambda_i G^{gx}_{0,j} - (1 - \lambda_i) G^{hx}_{i,j} = 0, \\ G^{gy}_{i+1,j} - \lambda_i G^{gy}_{0,j} - (1 - \lambda_i) G^{hy}_{i,j} = 0, j \in I \\ G^{xx}_{j,*} - G^{xy}_{j,*} = 0, \\ G^{yx}_{j,*} - G^{yy}_{j,*} = 0, j \in I \\ D_f(x_*, x_0) \leq R. \end{cases}$$

Results

We recover the result in [Lieder, 2021]:

$$D_f(T(x_N), x_N) \leq \frac{4D_f(x_*, x_0)}{(N+1)^2},$$

that is

$$\|T(x_N) - x_N\| \leq \frac{2\|x_* - x_0\|}{N+1}.$$

Other metrics like $\min D_f(x_{i-1}, i)$ is also bounded.

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Other Kernel?

In Euclidean case, we force $fx_i = \frac{1}{2}\|x_i\|^2$, $g_i = x_i$. Here we do not need the convexity of f.

When extended to other kernel, we cannot control $f(x_N)$. The convexity of f alone does not lead to boundness of PEP.

Have tried some (in)equalities discussed last time, but failed.

Also tried adding $\langle x_i, e\rangle = 1$, but failed.
It is also flawed that the inner product $\langle e, e\rangle$ is not dimensional independent!

Bibliography I

Lieder, F. (2021).
On the convergence rate of the halpern-iteration.
Optimization Letters, pages 1–14.