Bregman Method from PEP Perspective

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Progress

Finished:

- evidence of improvement (numerical)

In progress:

- optimal fixed step size (dual matrix)
- valid one-step analysis
- toolbox for explicit dual matrix (in Python)

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Formulation

Problem:

$$\min_{x \in \mathbb{R}^d} \quad F(x) \triangleq f(x) + \phi(x) \tag{P}$$

Assumptions:

- f is *convex*, *proper*, *lsc* and continuously differentiable.
- ϕ is *convex*, *proper*, *lsc* (possibly nonsmooth).
- h is a Legendre kernel function.
- f is L-smooth relative to h, i.e., $Lh - f$ convex.[Bauschke et al., 2017]
- f, h more restricted.
- bounded initial distance and well-posedness of problem and method.

Formulation

Problem:

$$\min_{x \in \mathbb{R}^d} \quad F(x) \triangleq f(x) + \phi(x) \tag{P}$$

Method: Bregman Proximal Gradient

$$x_{k+1} = \operatorname*{arg\,min}_{x \in \mathbb{R}^d} \{ f(x_k) + \langle \nabla f(x_k), x - x_k \rangle + \phi(x) + LD_h(x, x_k) \} \tag{BPG}$$

Case 1: h - SC & Smooth

Fix $\phi(x) \equiv 0$. Assume: h is σ_h-strongly convex and L_h-smooth. Through PEP we obtain following result:

$$(f_1 - f_*) + \frac{L \left\| \nabla h(x_1) - \nabla h(x_*) - \sqrt{L_h \sigma_h}(x_1 - x_*) \right\|^2}{2(L_h - \sigma_h)}$$

$$\leq (1 - \frac{1}{\sqrt{L_h/\sigma_h} + 1}) L D_h(x_*, x_0).$$

Upper bound can be achieved when $\nabla h(x_1) - \nabla h(x_*) = \sqrt{L_h \sigma_h}(x_1 - x_*)$.

$$f_1 - f_* \leq (1 - \frac{1}{\sqrt{L_h/\sigma_h} + 1}) L D_h(x_*, x_0).$$

Numerically verified! Worst case (right hand side) can be reached.

Case 1: h - SC & Smooth

How is it derived?

- PEP gives
  - convexity of f between x_* and x_0: $1-c$;
  - S & SC of h between x_* and x_1: L;
  - convexity of $Lh - f$ between x_1 and x_0: $1-c$;
  - convexity of $Lh - f$ between x_1 and x_*: c.
- Summing up inequalities and we have

$$
\begin{aligned}
&(f_1 - f_*) + \frac{L\left\|\nabla h(x_1) - \nabla h(x_*) - \sqrt{L_h\sigma_h}(x_1 - x_*)\right\|^2}{2(L_h - \sigma_h)} \\
&- \left(\frac{L(cL_h + (1-c)\sigma_h - \sqrt{L_h\sigma_h})}{L_h - \sigma_h}\right)(D_h(x_*, x_1) + D_h(x_1, x_*)) \\
&\quad \leq (1-c)LD_h(x_*, x_0).
\end{aligned}
$$

Derivation

$$
\begin{aligned}
&(f_1 - f_*) + \frac{L \left\| \nabla h(x_1) - \nabla h(x_*) - \sqrt{L_h \sigma_h}(x_1 - x_*) \right\|^2}{2(L_h - \sigma_h)} \\
&- \left(\frac{L(cL_h + (1-c)\sigma_h - \sqrt{L_h \sigma_h})}{L_h - \sigma_h} \right) (D_h(x_*, x_1) + D_h(x_1, x_*)) \\
&\quad \leq (1-c) L D_h(x_*, x_0).
\end{aligned}
$$

To eliminate the negative term on LHS, solve

$$
\frac{L(cL_h + (1-c)\sigma_h - \sqrt{L_h \sigma_h})}{L_h - \sigma_h} = 0,
$$

which gives

$$
c = \frac{1}{\sqrt{L_h/\sigma_h} + 1}.
$$

Attempt 0

So we hope that given $\{c_i\}_{i=1}^m$ and corresponding inequalities from N step PEP, we are able to establish some kind of result in the form

$$\begin{aligned} & f_N - f_* + \underbrace{\sum \|\dots\|^2}_{I} - \underbrace{\sum E_i(c_1,\dots,c_m)(D_h(\dots))}_{II} \\ & \leq C(c_1,\dots,c_m)\frac{LD_h(x_*,x_0)}{N}, \end{aligned}$$

so that by solving equations $\{E_i(c_1,\dots,c_m)=0\}$ we get the $\{c_i\}$ and also the upper bound.

But seems failed!

- might have multiple ways to express I and II.
- attempted ways gives unimproved result. $(C(c_1,\dots,c_m)=1)$

Attempt 1

Attempt 1: generalize the one-step worst case
$(\nabla h(x_1) - \nabla h(x_*) = \sqrt{L_h \sigma_h}(x_1 - x_*))$ to every step.

Theorecitally it provides a lower bound on the upper bound (if feasible).

Not plausible! - numerical result with constraints

$$\begin{cases} \nabla h(x_i) - \nabla h(x_*) = \sqrt{L_h \sigma_h}(x_i - x_*), & \forall i = 1, \ldots, N, \\ \nabla h(x_i) - \nabla h(x_{i-1}) = \sqrt{L_h \sigma_h}(x_i - x_{i-1}), & \forall i = 1, \ldots, N, \end{cases}$$

is near to optimal.
But gap still exists (about 1e-2, relatively significant).

Attempt 2

Attempt 2: find potential-like inequality in the form of

$$f(x_{k+1}) - f_* \le a_k(f(x_k) - f_*) + b_k D_h(x_*, x_k).$$

or equivalently,

$$f(x_1) - f_* \le a(f(x_0) - f_*) + bD_h(x_*, x_0).$$

The bounded initial divergence constraint in PEP is replaced by

$$f(x_0) - f_* + \frac{b}{a} D_h(x_*, x_0) \le R.$$

Uncertain! - there's always terms like $D_h(x_0, x_1)$ and can't get the worst case solution like in $N = 1$ case.

Attempt 3

Attempt 3: add the symmetric coefficient constraints on D_h. That is,

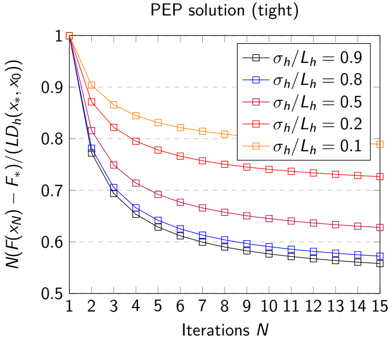
$$D_h(x, y) \leq \alpha D_h(y, x).$$

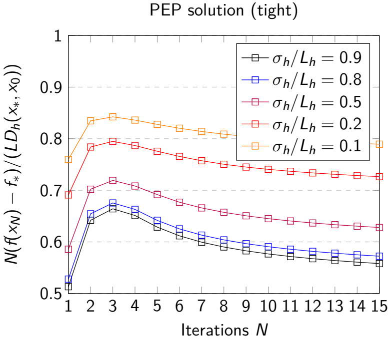
Uncertain!

- when α is large, no effect on result;
- when α is close to 1, it dominates the result, regardless of L_h/σ_h, the curve is similar.

Numerical Observation

Compare proximal case ($\phi \not\equiv 0$) and smooth case ($\phi \equiv 0$)





So far we only proved the smooth case when $N = 1$.

BPG and BG does not share the same behavior at the beginning.

Numerical Observation

Some results related to PEP formulation are weird.

Supposedly, if f is L-smooth relative to h, and h is L_h-smooth, then f should be LL_h-smooth.
But adding LL_h-smooth interpolation condition directly on constraints of f gives better (smaller) numerical result of PEP.

Theoretically the combination of former two conditions are stronger, but seemingly weaker direct interpolation condition is more "restrictive".

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Acceleration

Acceleration in [Lin et al., 2020, Lin et al., 2019] is not that clear.

Especially the relationship with estimate sequence? (Lack of intuition.)

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