# Complexity vs. Optimality: Unraveling Source-Destination Connection in Uncertain Graphs

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# Outline

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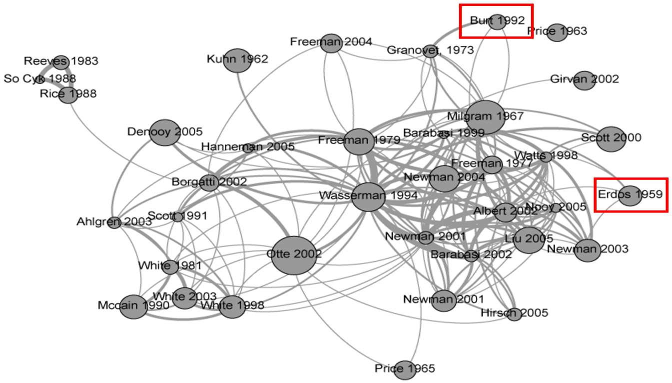
# Motivating Examples



Can the two nodes communicate with each other?

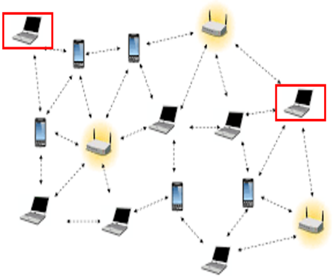
# Motivating Examples



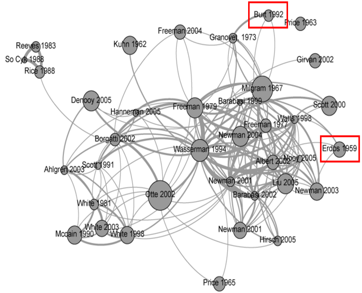



Is the two papers related with each other?

# Motivating Examples



Communication Network




Citation Network

- Link Failure, "Gift Citation"
- **Solution:** Link Probing, Text Mining

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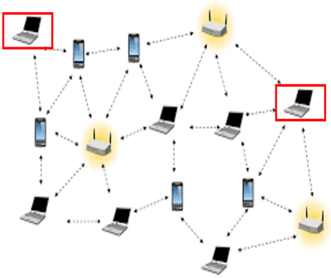

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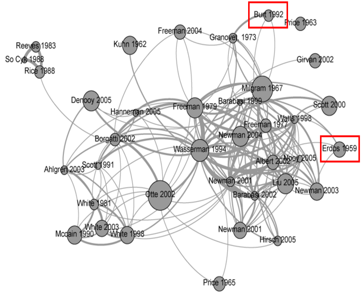


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# Motivating Examples

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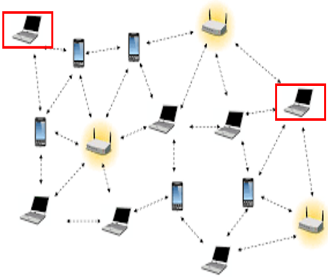

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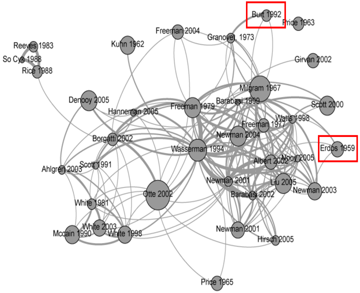


Citation Network

**How to perform tests cost-effectively?**

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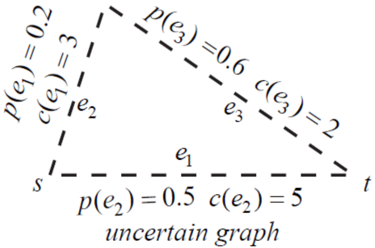
# Uncertain Graph

- Network topology (Prior,$(V, E)$)
- Prior existence probability (Edges, $p$)
- Testing cost (Edges, $c$)
- Composition of realizations

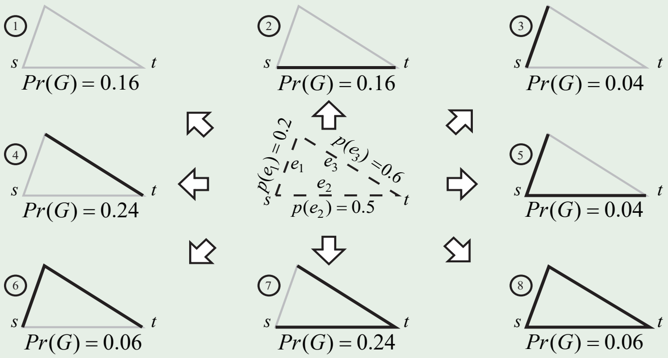


# Uncertain Graph and Underlying Graph



For an uncertain graph $\mathcal{G}$, denote $G$ as its underlying realization. $\mathcal{G}$ can be interpreted as a product distribution over all its possible realizations.

## Example (Uncertain graph and its underlying realizations)



# Problem Definition



**Definition (Connectivity Determination Problem)**

Given an uncertain graph $\mathcal{G}(V, E, p, c)$ and two nodes $s, t \in V$ designated as source and destination, find a testing strategy to determine the $s$-$t$ connectivity while incurring the minimum expected cost.

- The results of tests are dictated by the (priorly unknown) underlying graph.
- The expectation of cost is taken over all possible realizations of $\mathcal{G}$.
- The testing strategy can be adaptive.

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- The testing strategy can be adaptive.

# Adaptive Testing Strategy

How to properly define the testing strategy?

- A strategy decides the next edge to test based on the previous results.
- A strategy terminates by verifying the existence of an *s*-*t* path or an *s*-*t* cut.

**Definition (Temporary State)**

A temporary state s of an uncertain graph $\mathcal{G}(V, E, p, c)$ is an $|E|$-dimension vector with elements "0", "1" and "*". Define $\mathcal{S} = \{0, 1, *\}^{|E|}$ to be the set of temporary states associated with $\mathcal{G}$.

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## Definition (Adaptive Testing Strategy)

An adaptive testing strategy is a mapping $\pi : \mathcal{S} \mapsto E \cup \{\perp\}$.

- Define $E_\pi(G)$ as the set of edges strategy $\pi$ on the underlying graph $G$.
- The expected cost of $\pi$ is given as $Cost(\pi) = \sum_{G\in\mathcal{G}}[Pr(G)\sum_{e\in E_\pi}(G)c(e)]$.

# Outline

# Two Variants of the Problem



1. Compute the whole strategy (P1)

**Definition (Decision Version of P1)**

Given an uncertain graph $\mathcal{G}(V, E, p, c)$ and two nodes $s, t \in V$ designated as source and destination, is there a testing strategy that determines the $s$-$t$ connectivity with expected cost less than $k$.

2. Compute the strategy sequentially (P2)

**Definition (Decision Version of P2)**

Given an uncertain graph $\mathcal{G}(V, E, p, c)$, two nodes $s, t \in V$ designated as source and destination and the current temporary state, decide the optimal next edge to test.

# Two Variants of the Problem



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# Complexity-theoretic Results on P1



**Theorem**

*Computing the expected cost of the optimal strategy is #P-hard [1]. (The decision version of a #P-hard problem is NP-hard)*

**Proof**

- By reduction from the network ($s$-$t$) reliability problem.

**Definition (The Network Reliability Problem)**

Given a network $G(V, E)$ with two nodes in $V$ designated as source and destination, assuming that each edge fail with probability $\frac{1}{2}$, compute the probability of $s$ being connected to $t$.

[1] L. G. Valiant, "The complexity of enumeration and reliability problems"

# Complexity-theoretic Results on P2



**Theorem**

*Deciding the optimal first edge to test is NP-hard.*

**Proof.**

The proof is done by reduction from the set cover problem.

**Definition (The Set Cover Problem)**

Given a universe $\mathcal{U}$ of elements, a family $\mathcal{S}$ of subsets of the universe and a predefined integer $k$, does there exist a subfamily $\mathcal{C} \subseteq \mathcal{S}$ such that $\bigcup_{C \in \mathcal{C}} = \mathcal{U}$ and $|\mathcal{C}| \leq k$.

□

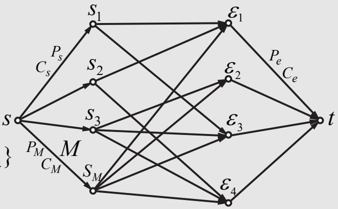
# Complexity-theoretic Results on P2

**Theorem**

*Deciding the optimal first edge to test is NP-hard.*

**Proof.**

$s_1 = \{\varepsilon_1, \varepsilon_3\}$
$s_2 = \{\varepsilon_1, \varepsilon_4\}$
$s_3 = \{\varepsilon_2, \varepsilon_3, \varepsilon_4\}$




Through appropriately assigning the value of $P_s$, $C_s$, $P_M$, $C_M$, $P_e$, $C_e$, we can show that the optimal first edge to test is $M$ if and only if there does not exist a set cover $\mathcal{C}$ with $|\mathcal{C}| \leq k$. $\square$

# Outline

# Markov Decision Process Framework

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## Definition (Markov Decision Process)

A mathematical model for modeling decision making under uncertain situations. A Markov Decision Process (MDP) model contains:

- State Space
- Decision Epochs
- Action Sets
- Transition Probabilities and Rewards
- Strategy (with certain expected total reward)

Markov Property: the effects of an action only depends on the current state and the action itself.

# Mapping the Problem into MDP



The correspondence between the key elements:

- State Space: the set of temporary states
  - $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \ldots \mathcal{S}_{|E|}$
- Decision Epochs
- Action Sets: testing edges or terminating
  - $A_{\mathbf{s}}$, $A = \bigcup_{s \in \mathcal{S}} A_{\mathbf{s}} = E \cup \{\perp\}$
- Transition Probabilities and Rewards
  - From $\mathbf{s}$ to $\mathbf{s} \cdot e$
  - From $\mathbf{s}$ to $\mathbf{s} \backslash e$
- Strategy

# Exact Algorithm Based on Dynamic Programming

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**Algorithm 1** The MDP-based Exact Algorithm

**Input:** Uncertain graph $\mathcal{G}(V, E, p, c)$, source $s$, destination $t$
**Output:** The optimal testing strategy $\pi$

```
Initialize:  u_π(s) = 0, for all s ∈ S_|E|
for i = |E| to 0 do
    for All s in S_i do
        if s is a terminating state then
            u_π(s) := 0, π(s) := ⊥.
        else
            e* := argmax_{e∈A_s} {−c(e) + p(e)u_π(s·e)
                  +(1 − p(e))u_π(s\e)},
            u_π(s) := −c(e*) + p(e*)u_π(s·e*)
                  +(1 − p(e*))u_π(s\e*),
            π(s) := e*.
return π
```

Backward Induction

# A Simple Greedy Approach



A strategy that tests the edges following the ascending order of costs is an $O(|E|)$-approximation.

**Theorem**

*Given an uncertain graph $\mathcal{G}(V, E, p, c)$ and two nodes $s$ and $t$ as source and destination, let $\pi$ be a strategy that tests the edges in $E$ according to their costs sorted in an increasing order. Then, $Cost(\pi) \leq |E| \cdot Cost(\pi^*)$, where $\pi^*$ is the optimal strategy*

# Adaptive Submodular Algorithm – Preliminaries

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**Definition (Extension)**

For two temporary states $\mathbf{a}, \mathbf{b} \in \mathcal{S}$, we say $\mathbf{a}$ is an extension of $\mathbf{b}$, written as $\mathbf{a} \sim \mathbf{b}$ if $\mathbf{a}_i = \mathbf{b}_i$ for all $\mathbf{b}_i \neq *$.

**Definition (Function on Temporary States)**

Let $g : \mathcal{S} \mapsto \mathbb{N}$ be a utility function on temporary states.

- $g$ is *monotonically increasing* if $g(\mathbf{s}') - g(\mathbf{s}) \geq 0$ for all $s \in \mathcal{S}, \mathbf{s}' \sim \mathbf{s}$.
- $g$ is *adaptive submodular* if $g(\mathbf{s} \cdot e) - g(\mathbf{s}) \geq g(\mathbf{s}' \cdot e) - g(\mathbf{s}')$ and $g(\mathbf{s} \backslash e) - g(\mathbf{s}) \geq g(\mathbf{s}' \backslash e) - g(\mathbf{s}')$ whenever $\mathbf{s}' \sim \mathbf{s}$ and $\mathbf{s}_e = \mathbf{s}'_e = *$.

# Adaptive Submodular Algorithm – Utility Function



The utility function $g$ should be *assignment feasible* in the sense that:

- $g(*, *, \ldots, *) = 0$.
- $g(\mathsf{s}) = Q$ iff $\mathsf{s}$ is a terminating state, where $Q$ is the target value.

**Lemma (The Adaptive Submodular Framework [2])**

*Each time choosing (testing) the edge with the maximum expected gain:*

$$\frac{p(e)g(\mathsf{s} \cdot e) + (1 - p(e))g(\mathsf{s} \backslash e) - g(\mathsf{s})}{c(e)}$$

*yields an $O(\ln |Q|)$-approximation.*

[2] D. Golovin and A. Krause, "Adaptive Submodularity: Theory and Applications in Active Learning and Stochastic Optimization"

# Adaptive Submodular Algorithm – Utility Function

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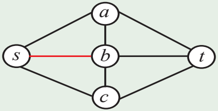
## Definition (The Design of Utility Function $g$)

Define $\mathcal{P}$ and $\mathcal{C}$ as the collection of $s$-$t$ paths and $s$-$t$ cuts in $\mathcal{G}$ respectively. Define $\mathcal{P}_e$ and $\mathcal{C}_e$ as the collection of $s$-$t$ paths and $s$-$t$ cuts that edge $e$ lies on in $\mathcal{G}$ respectively. We have,

$$g_p(\mathbf{s}) = |\bigcup_{e:\mathbf{s}_e=0} \mathcal{P}_e|, \quad g_c(\mathbf{s}) = |\bigcup_{e:\mathbf{s}_e=1} \mathcal{C}_e|,$$

$$g(\mathbf{s}) = |\mathcal{P}||\mathcal{C}| - (|\mathcal{P}| - g_p(\mathbf{s}))(|\mathcal{C}| - g_c(\mathbf{s})).$$

## Example




- Set of paths
- Set of cuts

# Adaptive Submodular Algorithm

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Determine the strategy sequentially

**Algorithm 2** The Adaptive Submodular Algorithm

**Input:** Uncertain graph $\mathcal{G}(V, E, p, c)$, source and destination.

**Output:** Testing strategy $\pi$

**Initialize:** Current state $\mathbf{s} := (*, *, \ldots, *)$, The set of tested edges $E_\pi$ as an empty set.

**Repeat** until $s$ becomes a terminating state.

$e^* := \arg\max_{e \in E \setminus E_\pi} \{ \frac{p(e)g(\mathbf{s} \cdot e) + (1-p(e))g(\mathbf{s} \setminus e) - g(\mathbf{s})}{c(e)} \}$.

$E_\pi := E_\pi \cup \{e^*\}$, *test* $e^*$ and observe the outcome.

**if** edge $e^*$ exists **then**

$\quad \mathbf{s}_{e^*} := 1$

**else**

$\quad \mathbf{s}_{e^*} := 0$

Test the edge with the maximum marginal gain

# Adaptive Submodular Framework – Performance



**Theorem**

*$g$ is monotonically increasing, adaptive submodular, and assignment feasible.*

Proof:

By the definition of $g$.

**Theorem**

*The Adaptive Submodular Algorithm yields an $O(\ln |Q|) = O(\ln |P||C|)$-approximation.*

Proof:

By the Adaptive Submodular framework proposed by Golovin and Krause [3].

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# Experiment Settings



**Experiment Datasets:**

- Citation Networks (273751 nodes, 993025 edges)
- Internet Peer to Peer Networks (5000 nodes, 16469 edges)
- Twitter Ego Networks (213 nodes, 17930 edges)

**Parameters Assignments:**

- Costs: drawn from $\mathcal{N}(50, 100)$
- Probabilities: $p(e = (x, y)) = |\Gamma(x) \cap \Gamma(y)|/|\Gamma(x) \cup \Gamma(y)|$[4]

**Performance Metric:**

- The expected costs are approximated by the averaged costs on 1000 underlying graphs.

[4] $\Gamma(x)$ denotes the set of neighbors of $x$.

# Algorithms Involved in Comparisons

- Greedy Algorithm (Greedy)
- Adaptive Submodular Algorithm (AdaSub)
- Optimistic Sort Algorithm (OpSort): follow an increasing order of $c/p$
- Pessimistic Sort Algorithm (PeSort): follow an increasing order of $c/(1-p)$
- Intersection Sort Algorithm (IntSort): test the edge with the minimum cost that lies on the intersection of a shortest $s$-$t$ path and a minimum $s$-$t$ cut.
- MDP-based Algorithm (MDP): only on a sequence of small subnetworks.

# Experiment Results

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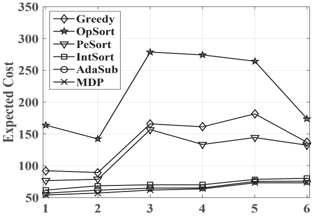


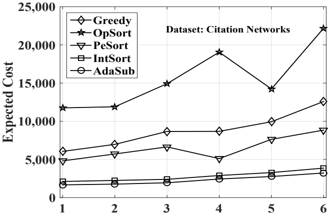


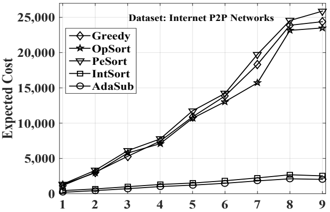


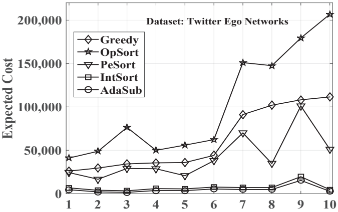

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# Conclusion



- Considering the prevalent uncertainties in networks, we formally define the source-destination connectivity determination problem in uncertain networks.
- By proving the related hardness results, we demonstrate the computational complexity of the problem.
- We propose exact algorithm and efficient approximation algorithms for solving the problem.
- Extensive experiment results demonstrate the effectiveness of our proposed algorithms and the superiority over other heuristics.

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# Thank You!