Linear Time Algorithm for Maximum Weighted Independent Set on Probe Interval Graphs

Pritish Chamania, Ross Mcconnell, Zhisheng Xu

Abstract

In this paper, we linear time algorithm for maximum weighted independent set problem on probe interval graphs.

Keywords:
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1. Introduction

2. Preliminary

Suppose $G = (P, N, E)$ is a probe interval graph. P is the set of Probe vertices, N is the set of Non-Probes and $V = P \cup N$. Every vertice i is labelled with weight w_i. McConnell[]'s recoginition algorithm gave an layout of the intervals in linear time. In the layout, each vertice i will be labelled as a Non-probe or Probe. Let (l_i, r_i) be the left endpoint and right endpoint for each vertice V_i in the layout. We can assume that all starting and ending points are unique and ranged from 1 to 2n.

A clique is a set of intervals that intersect pairwise. A maximal clique is one that is not contained in any clique. If a clique C is maximal, then there exists a interval intersects C and C is its leftmost intersected clique, similarily, there exists a interval intersects C and C is its rightmost intersected clique. The number of maximal cliques in P is O(n). Let S be the set of all maximal cliques in P.

3. Algorithm

We compute non-negative values w_{ij} for every edge (i,j) and (i,i), such that

$$\sum_j w_{ij} = w_i$$

We can seperate interval [1,2n] into segments by start points. There are n starting points, so we can seperate [1,2n] into n segments(1 must be a left endpoint). Let's define $seg(i)$ to be segment starts from l_i. Also for each $seg(i)$, we get a value y_i such that

$$y_i = \max(\sum_{j \in N} w_{ji}, \max_{k \in P} w_{ki})$$

We'll show in next section that $\sum_i y_i$ is the solution. The method for assigning w_{ij} and y_i as follows:

Result: Assign w_{ij} and y_i values
Make a copy $w_i^{'}$ for each w_i, $w_i^{'} = w_i$ at the beginning
for *$Seg(i)$ from right to left* **do**
  Let $S_P(i)$ be the set of all Probes that intersects i.
  Let $S_N(i)$ be the set of all Non-Probes that intersects i.
  $y_i \leftarrow w_i^{'}$
  $w_{ii} \leftarrow y_i$
  for *j in $S_P(i)$* **do**
    $w_{ji} \leftarrow min(w_j^{'}, y_i)$
    $w_j^{'} \leftarrow w_j^{'} - w_{ji}$
  end
  if *I is a Probe* **then**
    $y_i^{'} \leftarrow y_i$
    for *k in $S_N(i)$ from right to left by starting point* **do**
      $w_{ki} \leftarrow min(w_k^{'}, y_i^{'})$
      $w_k^{'} \leftarrow w_k^{'} - w_{ki}$
      $y_i^{'} \leftarrow y_i^{'} - min(w_{ki}, y_i^{'})$
    end
  end
end

Algorithm 1: Assigning w_{ij} and y_i values

The process goes from right to left, w_{ij} can be seen as the weight interval i unloaded into $seg(j)$. Every time we see a starting point of an interval i. $w_i^{'}$ is the remaining part of original w_i, we set $y_i = w_i^{'}$, unloading . No matter whether i is a probe or a non-probe, every probe j that intersects i will spend y_i weight on i, thus $w_j^{'}$ will be decre

4. Analysis

Lemma 4.1. *For any independent set I:* $\sum\limits_{i \in I} w_i \leq \sum\limits_{C} y_C$

Proof

$$\sum_{i \in I} w_i = \sum_{i \in I} \sum_{C: i \in C} w_{iC} = \sum_{C} \sum_{i \in I \cap C} w_{iC} \leq \sum_{C} \max(\sum_{i \in N} w_{iC}, \max_{i \in P} w_{iC}) = \sum_{C} y_C$$

The third inequality follows from the fact that I can only intersects clique C with a Probe or a set of Non-probes, which has less weight than sum of all weights of Non-probes or the Probe with largest weight in that clique.