Earthquake's effect on multistory buildings: a simple model

Kalin Ivanov

ČVUT FSv

Mathematics 2 Semester Project

15 June 2017

supervised by Ing. Michal Beneš, Ph.D

Model Description

- Motion of earth caused by earthquake puts buildings into motion
- Multistory buildings modelled as point-masses connected via springs and dashpots

Figure: Reality

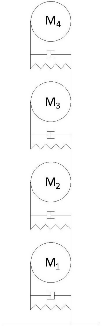


Figure: Simplification

- Consider mass m_i, spring constant k_i, damping constant c_i, displacement x_i, and velocity v_i of the i^{th} floor

Physical Description

- Forces on each floor depend on the floors above and below

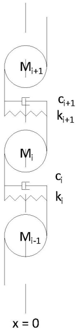


Figure: i^{th} floor statics

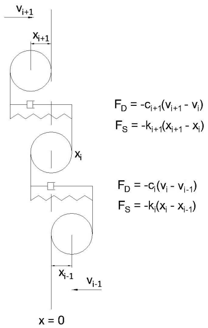


Figure: i^{th} floor dynamics

- Hooke's law describes the spring force $F_S = -kx$
- A linear damping force is considered in a similar manner $F_D = -c\dot{x}$

- Define system of equations for n floors

$$\mathbf{M}\ddot{X} + \mathbf{C}\dot{X} + \mathbf{K}X = F \tag{1}$$
$$X(0) = D_0 \;,\; \dot{X}(0) = V_0$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_n & k_n \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -c_2 & c_2 + c_3 & -c_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -c_n & c_n \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_n \end{bmatrix}, \quad F = \begin{bmatrix} f_1(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad X = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Newmark Method

- Discretization of position and velocity using truncated Taylor series expansions that have defined constants β and γ

$$D_{n+1} = D_n + V_n \Delta t + \frac{A_n \Delta t^2}{2} + 2\beta \dot{A}_n \Delta t^3$$
$$V_{n+1} = V_n + A_n \Delta t + \gamma \dot{A}_n \Delta t^2$$

- Definition of derivative approximation

$$\dot{A}_n = \frac{A_{n+1} - A_n}{\Delta t}$$

- The Newmark method is defined by the following 3 equations

$$\mathbf{M} A_{n+1} + \mathbf{C} V_{n+1} + \mathbf{K} X_{n+1} = F_{n+1} \tag{2}$$

$$D_{n+1} = D_n + V_n \Delta t + \frac{\Delta t^2}{2}[(1-2\beta)A_n + 2\beta A_{n+1}] \tag{3}$$

$$V_{n+1} = V_n + \Delta t[(1-\gamma)A_n + \gamma A_{n+1}] \tag{4}$$

Implementation of Newmark Method

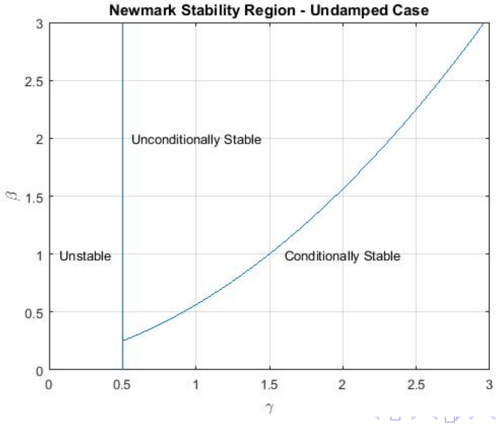
1. Using initial conditions D_0 and V_0 calculate A_0
$$A_0 = \mathbf{M}^{-1}(F_0 - \mathbf{C}V_0 - \mathbf{K}D_0)$$
2. Define constants
$$c_1 = \frac{1}{\beta \Delta t^2} \qquad c_2 = \frac{1}{\beta \Delta t} \qquad c_3 = \frac{1}{2\beta} - 1$$
$$c_4 = \frac{\gamma}{\beta \Delta t} \qquad c_5 = 1 - \frac{\gamma}{\beta} \qquad c_6 = \Delta t \left(1 - \frac{\gamma}{2\beta}\right)$$
3. Define inverted matrix allowing for explicit calculation of D_{n+1}
$$\left(\frac{\mathbf{M}}{\Delta t^2 \beta} + \frac{\mathbf{C}\gamma}{\Delta t\, \beta} + \mathbf{K}\right) D_{n+1} = F_{n+1} + \mathbf{M}\left(\frac{\widetilde{D}_{n+1}}{\Delta t^2 \beta}\right) - \mathbf{C}\left(\widetilde{V}_{n+1} - \frac{\gamma \widetilde{D}_{n+1}}{\Delta t\, \beta}\right)$$
4. For each time step calculate D_{n+1}, V_{n+1} and A_{n+1}
$$\widetilde{A}_{n+1} = -c_1 D_n + c_2 V_n + c_3 A_n$$
$$\widetilde{V}_{n+1} = c_4 D_n + c_5 V_n + c_6 A_n$$
$$D_{n+1} = \mathbf{W}[F_{n+1} + \mathbf{M}\widetilde{A}_{n+1} - \mathbf{C}\widetilde{V}_{n+1}]$$
$$V_{n+1} = c_4 D_{n+1} + \widetilde{V}_{n+1}$$
$$A_{n+1} = c_1 D_{n+1} - \widetilde{A}_{n+1}$$

Stability of Undamped Motion

- Stability condition $\rho \leq 1 \Rightarrow \gamma \geq \frac{1}{2}$ and $(\gamma + \frac{1}{2})^2 - 4\beta \leq \frac{4}{\Omega_i^2}$
- Unconditionally stable condition

$$\beta \geq \frac{1}{4}\left(\gamma + \frac{1}{2}\right)^2 \tag{5}$$

- Average constant acceleration method: $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$



- Consider $\ddot{x} + x = 0$ with initial conditions $x(0) = 1$, $\dot{x}(0) = 0$ and solution $x = \cos(t)$

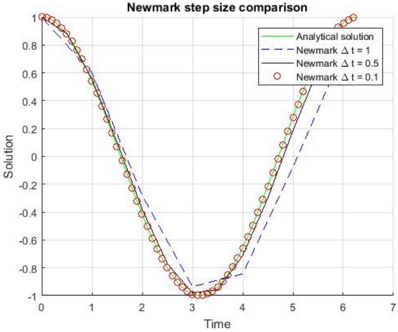


Figure: Newmark method solution with different step sizes

- Amplitude error

$$\rho - 1 = -\frac{1}{2}\left(\gamma - \frac{1}{2}\right)\omega^2 \Delta t^2 + O(\Delta t^4)$$

- Periodicity error

$$\frac{\Delta T}{T} = \frac{\omega \Delta t}{\phi} - 1 = \frac{1}{2}\left(\beta - \frac{1}{12}\right)\omega^2 \Delta t^2 + O(\Delta t^3)$$

Algorithm	γ	β	Ω	$\rho - 1$	$\frac{\Delta T}{T}$
Unconditionally stable	1	$\frac{3}{2}$	∞	$-\frac{\Omega^2}{4}$	$\frac{17\Omega^2}{24}$
Conditionally stable	2	$\frac{1}{2}$	0.97	$-\frac{3\Omega^2}{4}$	$\frac{5\Omega^2}{24}$
Unstable	1	$\frac{1}{4}$	0	$-\frac{\Omega^2}{4}$	$\frac{\Omega^2}{12}$
Constant Acceleration	$\frac{1}{2}$	$\frac{1}{4}$	∞	$O(\Delta t^4)$	$\frac{\Omega^2}{12}$
Purely Explicit	0	0	0	$\frac{\Omega^2}{4}$	$-\frac{\Omega^2}{24}$
Linear Acceleration	$\frac{1}{2}$	$\frac{1}{6}$	3.46	$O(\Delta t^4)$	$\frac{\Omega^2}{24}$

Order of Newmark error

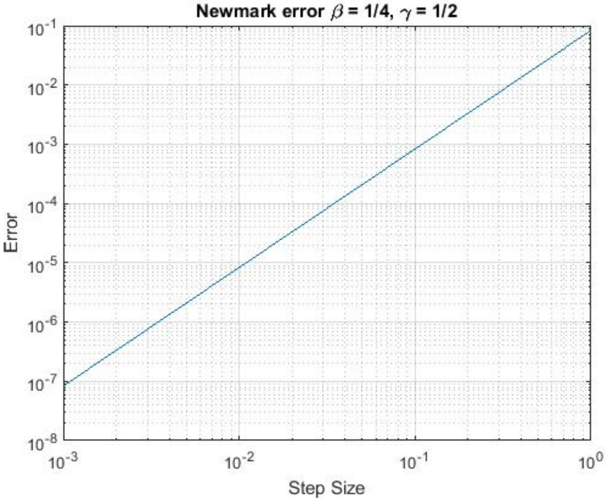


Figure: Newmark method error with different step sizes

- Earthquake: 1940, Imperial Valley in California, 6.9 on the Richter Scale
- 2-story house: $m_i = 100,000\ kg$, $k_i = 12\text{x}10^5\ \frac{kg}{s^2}$, $c_i = 10^5\ \frac{kg}{s}$

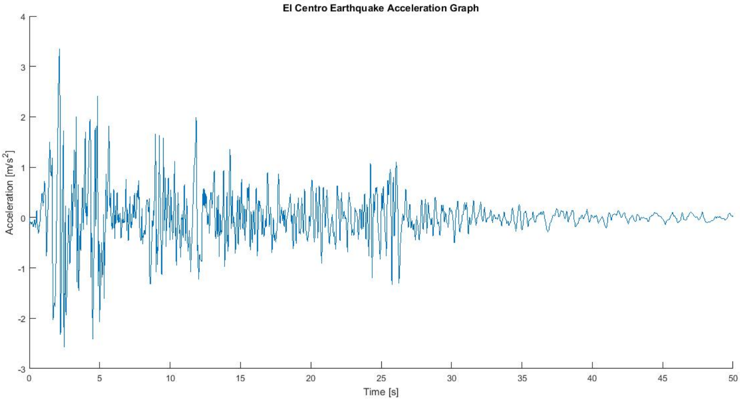


Figure: Acceleration data and 1st floor behavior

El Centro earthquake effect on 2-story house

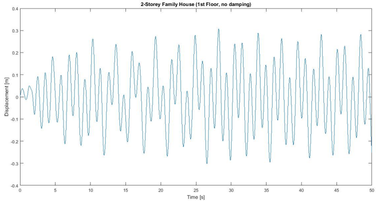


Figure: 1^{st} floor without damping

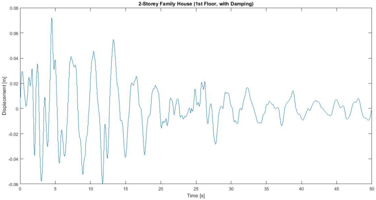


Figure: 1^{st} floor with damping

Bibliography

- Dennis G. Zill *Differential Equations and Their Applications*. Tenth Edition, Loyola Marymount University, Cengage Learning, 2012.
- Thomas J. R. Hughes. *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*. Chapter 9 Algorithms for Hyperbolic and Parabolic-Hyperbolic Problems, Dover Publications Inc., Mineola, New York, 2000.
- Nathan M. Newmark. *A Method of Computation for Structural Dynamics*. Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, 1959.
- Michel Gérardin, Daniel Rixen *Mechanical Vibrations Theory and Application to Structural Dynamics*. Wiley, New York, 1994.
- Laurene V. Fausett *Applied Numerical Analysis Using MATLAB*. Second Edition, Pearson Education Inc., Upper Saddle River, New Jersey, 2008.
- Martin Braun *Differential Equations and Their Applications*. Fourth Edition, Springer-Verlag New York Inc., New York, 1993.
- Carlos Felippa *Intoduction to Stability Analysis*. Fluid Structure Interaction Chapter 3, University of Colorado at Boulder, 2004.
- *El Centro Earthquake Page*. Vibration Data.

Thank You For Your Attention!