Introduction to Linear System
with continuous time

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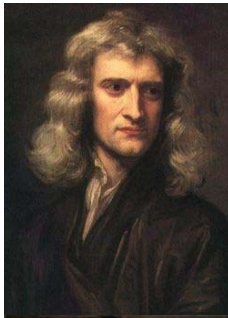
Outline of the lecture

1. Introduction
2. System described by one equation
   1. Static system
   2. First order
   3. Second order

Introduction to dynamic systems

- history:
  - Newton, Kepler, Tycho de Brahe
  - motivation: celestial mechanics

- dynamic system
  = actual states + rules to obtain following states
- system
  - deterministic vs. stochastic
  - linear vs. nonlinear
  - continuous time vs. discrete time
- consider further
  linear deterministic system with continuous time




Static (zero order) system

$$a_0 q(t) = u$$

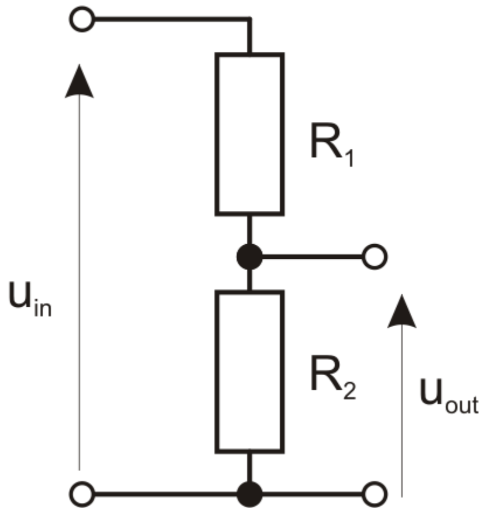
- static = special case of dynamic system

$$q(t) = \frac{1}{a_0} u$$

- model of the problem
  where the inertia of system can be neglected

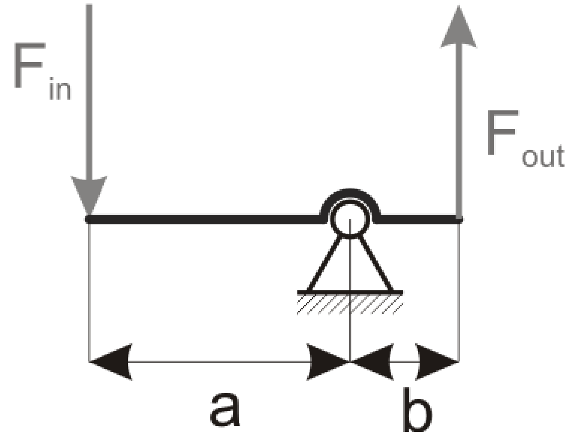
- example: Resistance voltage divider

$$u_{out} = \frac{R_2}{R_1 + R_2} u_{in}$$



Static (zero order) system

- example: Double arm lever



$$F_1 a = F_2 b$$

$$F_2 = \frac{a}{b} F_2$$

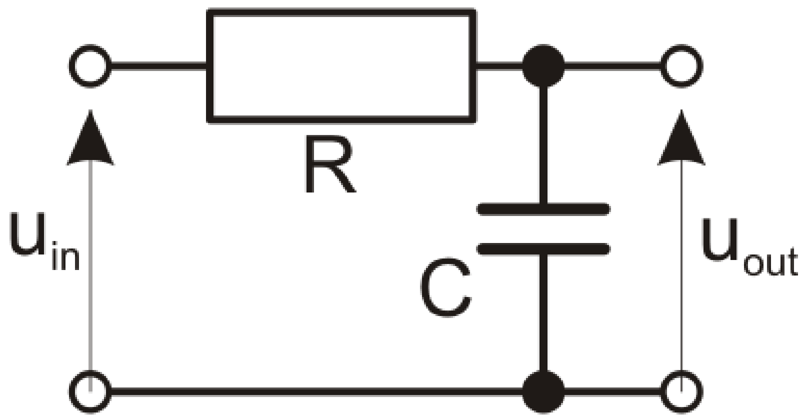
- note: this model can be used if the inertial moment of lever can be neglected

First order system

- mathematical model (linear ordinary differential equation) $$a_1\dot{q}(t)+a_0q(t)=u$$
- often in this form: $$\tau\dot{q}(t)+q(t)=ku$$
- solution for u=const: $$q(t)=ku(1-e^{-\frac{1}{\tau}t})$$
- => system has exponential response to step on input

First order system :: Example – RC circuit

- resistor and capacitor



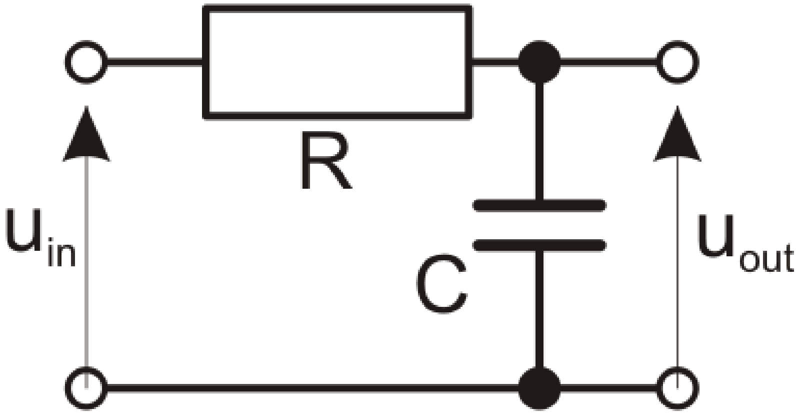
- equation:

$$u = u_{\text{in}} = u_{\text{C}} + u_{\text{R}} = \frac{1}{C} q + R\dot{q}$$

$$u_{\text{out}} = \frac{1}{C} q = y$$

$$RC\dot{y} + y = u$$

First order system :: Example – RC circuit



- solution of
- equation:

$$RC\dot{y} + y = u$$

$$y = u_{\mathrm{out}} = u_{\mathrm{in}}(1 - e^{-\frac{1}{RC}t})$$

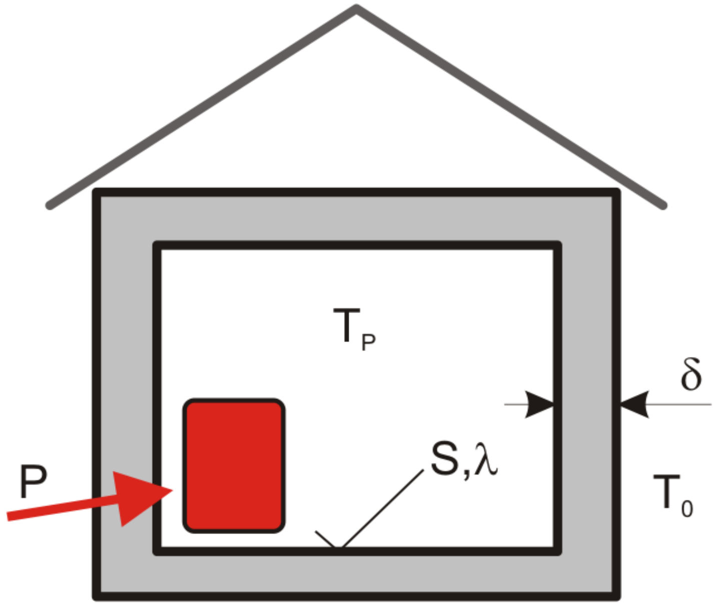
- consequence: this circuit works as low pass filter

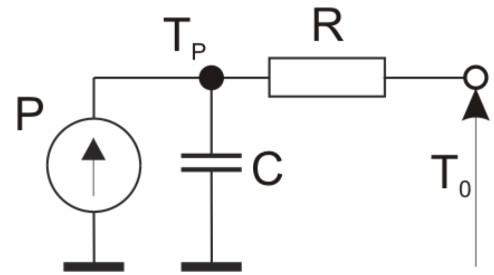
First order system :: House heating

- ideal house
- heating source of P [W]
- thermal conductance λ = 0.16 (polystyrene foam)
- thermal capacity
  C = 1000 Ws/kgK (air)
- equation:

$$mC\dot{y} + \frac{1}{R}(y - T_0) = u$$

$$R = \frac{\delta}{\lambda S}, u = P, y = T$$





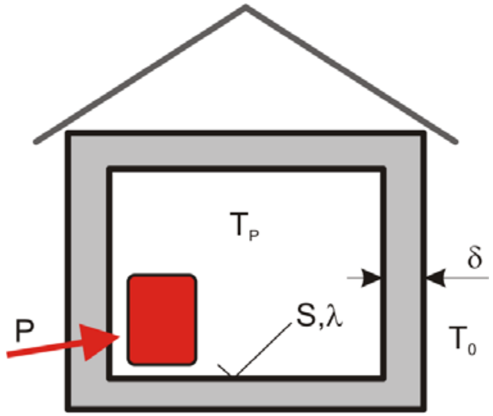
First order system :: House heating

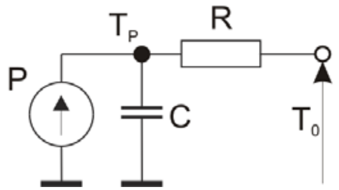
- solution of eq.

$$\frac{\delta}{\lambda S} mC\dot{y} + y = Ru + T_0$$

- time constant $\tau = \dfrac{\delta}{\lambda S} mC$

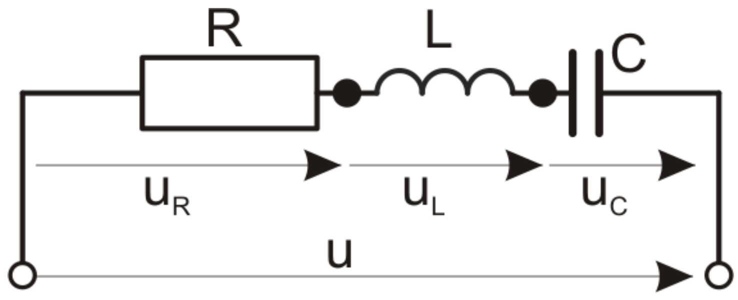
$$y = T = RP(1 - e^{-\frac{1}{\tau}t})$$





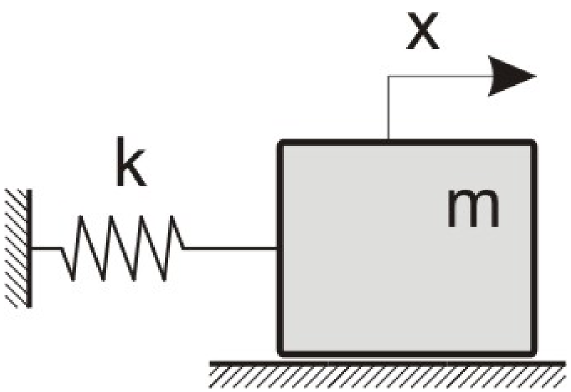
Second order system

- equation $$a_2\ddot{q}(t) + a_1\dot{q}(t) + a_0 q(t) = u$$
- typical examples:
  - harmonics oscilator
  - RLC circuit



- behaviour
  - much more complex compare to 1-st order system
  - sub-/ super- critical damping
  - resonance
  - …

Spring – Mass system

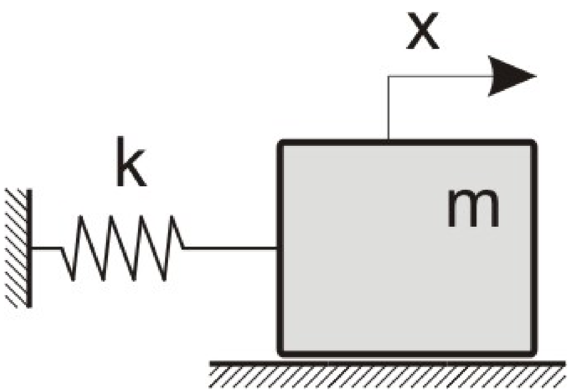


- consider 1dof spring mass system (no friction, no external force, no damping)
- problem = How will this system behave from any initial condition?
- Equation of motion: $$m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0$$

$$\ddot{x} + \Omega^2 x = 0$$

Spring – Mass system :: Analytical solution



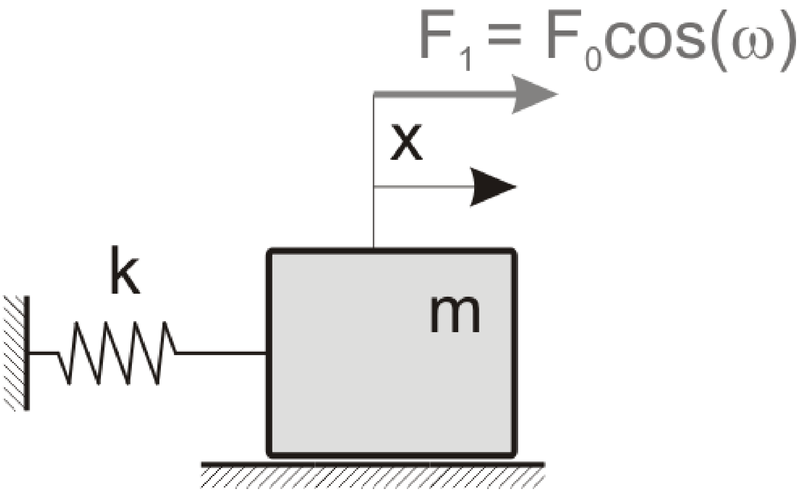
- solution of the eq. of motion

$$\ddot{x} + \Omega^2 x = 0$$

- is of the form

$$x(t) = x_0 \sin \Omega t + \frac{\dot{x}_0}{\Omega} \cos \Omega t$$

Spring – Mass system with External force

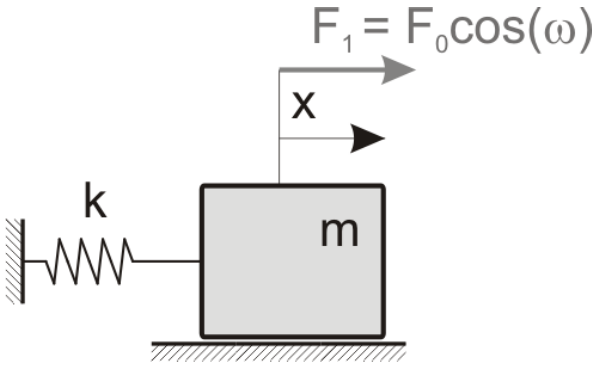


- consider mass-spring system with external harmonic force
- equation of motion:

$$\ddot{x} + \Omega^2 x = \frac{F_0}{m} \cos \omega t$$

- solution of eq. of motion:

$$\ddot{x}+\Omega^2 x=\frac{F_0}{m}\cos\omega t$$



- is of the form

$$x(t)=x_{\mathrm{hom}}(t)+x_{\mathrm{p}}(t)$$

$$x_{\mathrm{hom}}(t)=C_1\sin\Omega t+C_2\cos\Omega t$$

$$x_{\mathrm{p}}(t)=C_3\cos\omega t$$

$$C_3=\frac{F_0}{\left(\Omega^2-\omega^2\right)m}$$

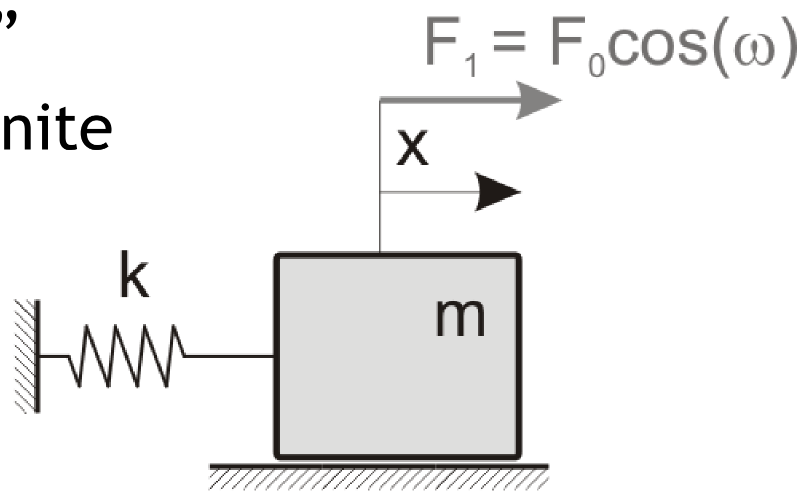
Spring – Mass system with External force :: Resonance

- solution:

$$x(t) = \ldots + \frac{F_0}{\left(\Omega^2 - \omega^2\right)m} \cos \omega t$$

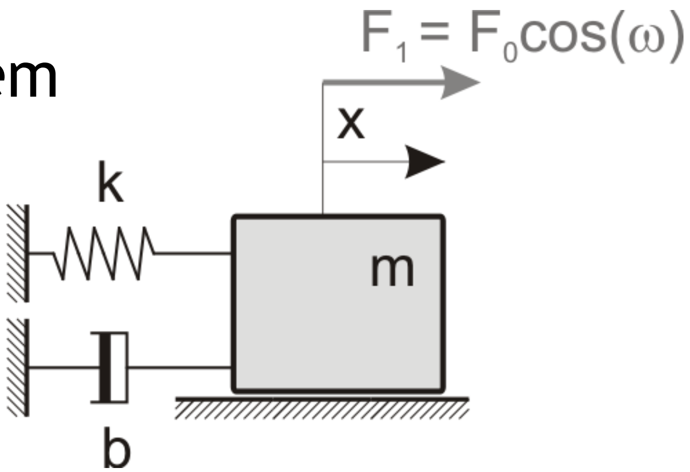
- what happened if $\Omega = \omega$?

- we call this effect “resonance”
- theoretically: amplitude is infinite



Mass-Damper-Spring with External force

- consider spring-damper-mass system
- with external harmonic force

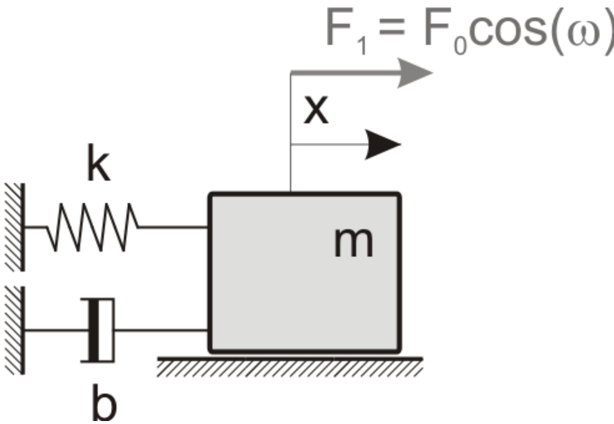


- eq. of motion:

$$m\ddot{x} = -kx - b\dot{x} + F_0 \cos \omega t$$

$$m\ddot{x} + \Omega^2 x + 2\delta \dot{x} = \frac{F_0}{m} \cos \omega t$$

- solution of eq. of motion



$$m\ddot{x} + \Omega^2 x + 2\delta\dot{x} = \frac{F_0}{m}\cos\omega t$$

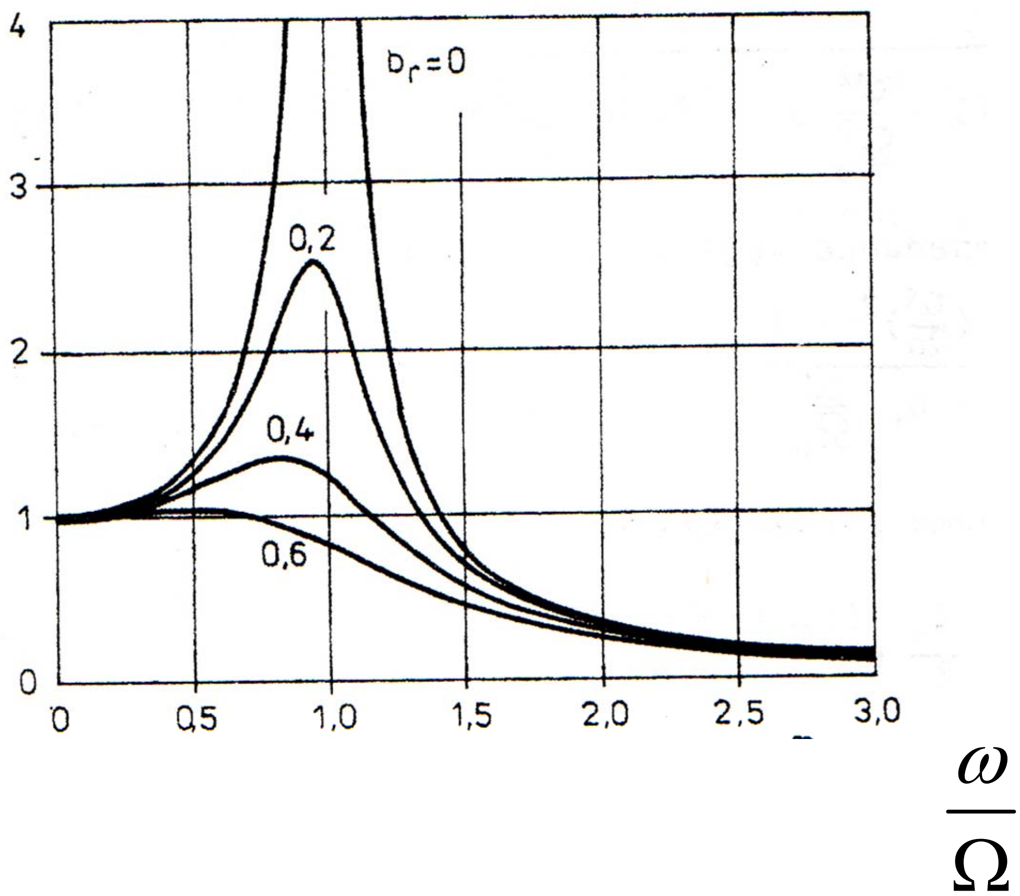
- … is more complex (see textbook for complete der.):
  - homogenous solution disappear due to damping
    = transient process
  - particular solution is of the form

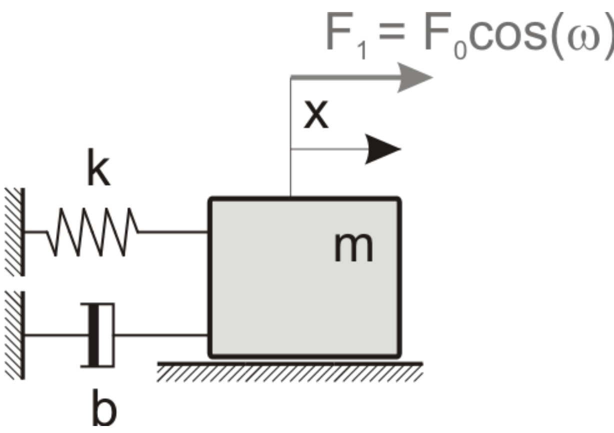
$$x(t) = \ldots + \frac{F_0}{m\sqrt{\left(\Omega^2 - \omega^2\right)^2 + 4\delta^2\omega^2}}\cos(\omega t + \varphi)$$

$$\varphi = \mathrm{atan}\left(-\frac{2\delta\,\Omega}{\Omega^2 - \omega^2}\right)$$

 Mass-Damper-Spring with External force

- main result:





Exercise 1: First order system

Consider one of the described examples of first order system, e.g. House heating.

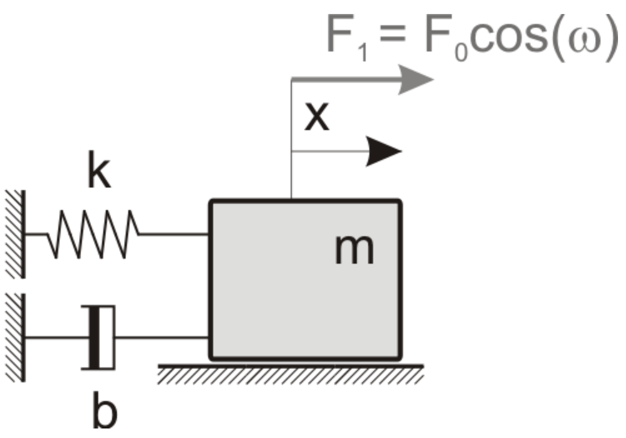
- Define parameters in Matlab script (m-file).
- Create simulation model of dynamic system in Simulink.

$$\frac{\delta}{\lambda S} m C\dot{y} + y = Ru + T_0$$

- Start Simulation model from Matlab and store results in Matlab Workspace.
- Compare numerical solution of Simulink to analytical:

$$y = T = RP(1 - e^{-\frac{1}{\tau}t})$$

Exercise 3: Model of mechanical harmonic oscilator



- Define system parameters in Matlab script.
- Create Simulink model of the system.
- Test using simulation:
  - system behavior without external force
  - static external force
  - harmonic external force
  - simulate resonance of the system.

Exercise 4: Speed control of DC motor

Goal of the exercise:

Create simulation model of speed control of DC motor with following properties:

- DC motor with permanent magnets
- with planetary gearbox
- mechanical load of form

$$\tau_{load} = 1 + 0.35\omega$$

- P (PI, PID) controller
- pedal sensor for definition of desired speed

Exercise 4: Speed control of DC motor – Step 1

Step 1:

- Recall model from Exercise 2.
- Add gearbox to the system. Select gearbox ration to have nominal output torque approx. 3Nm
- Use input voltage 0-24 V and find steady-state velocity.

Exercise 4: Speed control of DC motor – Step 2

Step 2:

- Add simple P controller of the velocity to the model.
- Use Step as input command (desired velocity in range of 0-24V).
- Observe:
  - influence of P value
  - how the current is influenced?

Exercise 4: Speed control of DC motor – Step 3

Step 3:

- Add the load (connected machine) to the output shaft of gearbox in the form

$$\tau_{load} = 1 + 0.35\omega$$

- Test the P controller behaviour.

Exercise 4: Speed control of DC motor – Step 4

Step 4:

- Can we somehow improve system performance? Consider PI, PID controller.

- Use e.g. random stairs as input command (desired velocity).