

# 1 Pneumatic system modeling

## 1.1 Pneumatic system overview

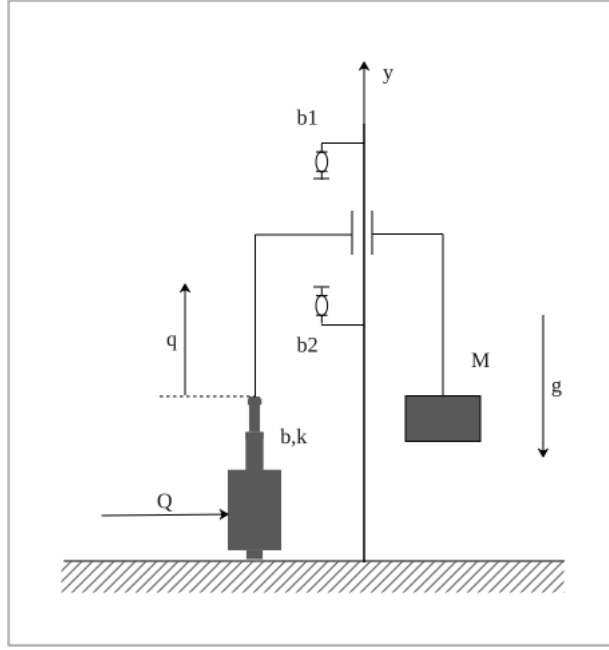


Figure 1: Schematic model



## 1.2 General 1DOF model dynamic

The motion of the pneumatic piston mechanism describes in terms of the general 1dof dynamical equation 1.

$$m\ddot{x} + b\dot{x} + kx = u \quad (1)$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}u$$

$$\ddot{x} + 2\delta\dot{x} + \Omega^2x = \frac{1}{m}u$$

where  $\delta = \frac{b}{2m}$  is damping and  $\Omega = \sqrt{\frac{k}{m}}$  is natural frequency of system.

$$\ddot{x} + 2\zeta\Omega\dot{x} + \Omega^2x = \frac{1}{m}u$$

where  $\zeta = \frac{b}{2\sqrt{km}}$  is damping ratio.

In the case of the pneumatic piston, the equation 1 transforms into an equation ??.

$$(M + M_L)\ddot{x} + F_{damp} + F_g + F_{hs} = F_p \quad (2)$$

Where  $M$  represents a mass of the all moveable part of the piston,  $M_L$  is load mass,  $F_g$  gravity force acting to mechanical moving assembly,  $F_{hs}$  - models endpoints (hard stop),  $F_{damp}$  represents shock absorbers acted at endpoints,  $F_p$  is a force produced by the pneumatic piston ??.

$$F_p = P_A S_A - P_B S_B - P_0 S_0 \quad (3)$$

### 1.3 Hard stop

Hard stop can be represented as spring and dumps:

$$F_{HS} = \begin{cases} K_p(x - g_p) + D_p v & \text{for } x \geq g_p \\ 0 & \text{for } g_n < x < g_p \\ K_n(x - g_n) + D_n v & \text{for } x \leq g_n \end{cases} \quad (4)$$

Possible parameters:

$K_p$	$10^6$	$[kg/s^2]$
$K_n$	$10^6$	$[kg/s^2]$
$D_p$	350	$[kg/s]$
$D_n$	350	$[kg/s]$
$g_n$	0	$[m]$
$g_p$	0.2	$[m]$

### 1.4 General physical principles

#### 1.4.1 Thermodynamics

$p$	$Pa$	pressure
$V$	$m^3$	volume
$m$	$kg$	mass
$n$	$mol$	amount of substance
$R$	$Jkg^{-1}K^{-1}$	ideal gas constant
$r$	$Jkg^{-1}K^{-1}$	mass-specific gas constant
$T$	$K$	temperature
$S$	$m$	area
$z$	$m$	hight
$w$	$ms^{-1}$	flow speed
$H$	$J$	enthalpy
$\nu$	$m^3kg^{-1}$	specific volume
$Q$	$J$	heat shared with environment
$W_T$	$J$	work
$c_p$	$Jkg^{-1}K^{-1}$	is the specific heat at constant pressure
$c_v$	$Jkg^{-1}K^{-1}$	is the specific heat at constant volume
$g = 9.81$	$ms^{-2}$	gravity acceleration
$\gamma = 1.4(\text{air})$	—	heat capacity ratio (isentropic expansion factor)

#### 1.4.2 Equation of state

Generally  $pV = nRT$  but for air purpose were  $r = \frac{pv}{T} = R = 287.1[Jkg^{-1}K^{-1}]$  following equation can be used 6:

$$pV = mrT \quad (5)$$

### 1.4.3 Isothermal process

Used in some papers 7:

$$p_1 V_1 = p_2 V_2 = \text{const} \quad (6)$$

### 1.4.4 Adiabatic process

Adiabatic process 8:

$$p_1 V_1^\gamma = p_2 V_2^\gamma = \text{const} \quad (7)$$

Heat capacity ratio:

$$\gamma = \frac{c_p}{c_v} \quad (8)$$

Mayer's relation:

$$c_p = c_v + R \quad (9)$$

### 1.4.5 Bernoulli's principle

Bernoulli's principle 11:

$$H_1 + \frac{mw_1^2}{2} + mgz_1 + Q = H_2 + \frac{mw_2^2}{2} + mgz_w + W_T \quad (10)$$

$$H_1 - H_2 = - \int_1^2 V dp = c_p(T_1 - T_2) = c_p T_1 \left(1 - \frac{T_2}{T_1}\right) \quad (11)$$

Differential form:

$$\nu dp + w dw + g dz + dw_T = 0 \quad (12)$$

### 1.4.6 Fluid mechanics

$\dot{m}$	$kg s^{-1}$	mass flow
$c$	$ms^{-2}$	speed of sound
$w_k$	$ms^{-2}$	critical flow velocity
$\psi$	—	flow coefficient
$\psi_{max}$	—	critical flow coefficient
$\beta$	—	ration of pressure differential
$\beta_k$	—	critical ratio of pressure differential

Continuity equation 14:

$$\dot{m} = S_1 w_1 \rho_1 = S_2 w_2 \rho_2 = \text{const} \quad (13)$$

### 1.4.7 Air expansion from tank

Assuming  $W_T = 0$ ,  $z_1 = z_2$ ,  $Q = 0$  conditions and combine with 11 we will get 15 equation:

$$w_2 = \sqrt{2(H_1 - H_2)} \quad (14)$$

$$w_2 = \sqrt{2RT_1 \left(\frac{\gamma}{\gamma-1}\right) \left(1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right)} \quad (15)$$

$$\rho_2 = \frac{p_1}{RT_1} \left( \frac{p_2}{p_1} \right)^{\frac{1}{\gamma}} \quad (16)$$

Together 14 16 17:

$$\dot{m} = Sp_1 \sqrt{\frac{2}{RT_1}} \cdot \sqrt{\frac{\gamma}{\gamma-1} \left( \left( \frac{p_2}{p_1} \right)^{\frac{2}{\gamma}} - \left( \frac{p_2}{p_1} \right)^{\frac{\gamma+1}{\gamma}} \right)} \quad (17)$$

where:

$$\psi \left( \frac{p_2}{p_1} \right) = \sqrt{\frac{\gamma}{\gamma-1} \left( \left( \frac{p_2}{p_1} \right)^{\frac{2}{\gamma}} - \left( \frac{p_2}{p_1} \right)^{\frac{\gamma+1}{\gamma}} \right)} \quad (18)$$

Finally 20:

$$\dot{m} = Sp_1 \sqrt{\frac{2}{RT_1}} \psi \left( \frac{p_2}{p_1} \right) \quad (19)$$

#### 1.4.8 Critical flow velocity

Speed of sound:

$$c = \sqrt{\frac{dp}{d\rho}} = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma RT} \quad (20)$$

Assume  $c = w_2$  (16, 21) we will get the critical flow velocity:

$$c_2 = w_k = \sqrt{\gamma RT} = \sqrt{2RT_1 \frac{\gamma}{\gamma-1} - 2w_k^2 \frac{1}{\gamma-1}} \quad (21)$$

$$w_k^2 = 2RT_1 \frac{\gamma}{\gamma-1} - 2w_k^2 \frac{1}{\gamma-1} \quad (22)$$

$$w_k = \sqrt{2RT_1 \frac{\gamma}{\gamma-1}} = \sqrt{2p_1 \nu_1 \frac{\gamma}{\gamma+1}} \quad (23)$$

For calculating critical pressure ratio assume  $w_k = w_2$  24 16:

$$\sqrt{2RT_1 \frac{\gamma}{\gamma-1}} = \sqrt{2RT_1 \frac{\gamma}{\gamma-1} \left( 1 - \left( \frac{p_2}{p_1} \right)^{\frac{\gamma+1}{\gamma}} \right)} \quad (24)$$

$$\left( \frac{p_2}{p_1} \right)^{\frac{\gamma-1}{\gamma}} = \frac{2}{\gamma+1} \quad (25)$$

$$(26)$$

$$\left( \frac{p_2}{p_1} \right)_k = \left( \frac{p_k}{p_1} \right) = \left( \frac{2}{\gamma+1} \right)^{\frac{\gamma}{\gamma-1}} = \beta_k \quad (27)$$

Critical pressure condition is  $p_k = p_1 \beta_k$ .

Applying 28 to 19:

$$\psi_{max}(\beta_k) = \left( \frac{2}{\gamma+1} \right)^{\frac{\gamma}{\gamma-1}} \sqrt{\frac{\gamma}{\gamma+1}} \quad (28)$$

For air  $\beta_k = 0.528$ ,  $\psi_{max} = 0.484$   
 Final equation for  $\psi$ :

$$\psi\left(\frac{p_2}{p_1}\right) = \begin{cases} \sqrt{\frac{\gamma}{\gamma-1} \left( \left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}} - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma+1}{\gamma}} \right)} & 0.528 < \frac{p_2}{p_1} \leq 1 \\ \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma+1}} \sqrt{\frac{\gamma}{\gamma+1}} & 0 \geq \frac{p_2}{p_1} \geq 0.528 \end{cases} \quad (29)$$

## 1.5 Pneumatic actuator model

$p_A, p_B$	$Pa$	pressure in chamber A, B
$\dot{m}_A, \dot{m}_B$	$kg \cdot s^{-1}$	mass flow on way to chamber A, B
$S_A, S_B$	$m^2$	piston area
$V_A, V_B$	$m^3$	volume of chamber A,B
$V_{0A}, V_{0B}$	$m^3$	"dead" volume of chamber A,B
$m$	$kg$	piston mass
$F_{load}$	$N$	load
$x$	$m$	piston position
$l$	$m$	maximum piston position

Mechanical equation, where  $F_f$  represent all friction and viscous forces.

$$m\ddot{x} = (p_A S_A - p_B S_B) - F_f \quad (30)$$

Volumes of chambers:

$$V_A = S_A x + V_{0A} \quad (31)$$

$$V_B = S_B(l - x) + V_{0B} \quad (32)$$

$$\dot{V}_A = S_A \dot{x} \quad (33)$$

$$\dot{V}_B = -S_B \dot{x} \quad (34)$$

### 1.5.1 Pressure models

#### Isothermal model

$$m = \rho V \quad (35)$$

$$\dot{m} = \dot{\rho} V + \rho \dot{V} \quad (36)$$

Applying 6:

$$\rho = \frac{p}{RT} \quad (37)$$

$$\dot{\rho} = \frac{\dot{p}}{RT} \quad (38)$$

Finally get 40:

$$\dot{p} = -\frac{p}{V} \dot{V} + \frac{RT}{V} \dot{m} \quad (39)$$

**Adiabatic model** For simple adiabatic model following equation can be used 41:

$$\dot{p} = -\frac{\gamma p}{V} \dot{V} + \frac{\gamma RT}{V} \dot{m} \quad (40)$$

$$\dot{p}_A = \frac{\gamma}{S_A x + V_{0A}} (-p_A S_A \dot{x} + RT_A \dot{m}_A) \quad (41)$$

$$\dot{p}_B = \frac{\gamma}{S_B(l-x) + V_{0B}} (p_B S_B \dot{x} + R T_B \dot{m}_B) \quad (42)$$

$T_A, T_B$  calculated from 6, or in adiabatic model this parameters can remain constant same as atmospheric temperature.

### 1.5.2 Input/Output mass flows

$$\dot{m}T = \dot{m}_{in}T_s - \dot{m}_{out}T_{A/B} \quad (43)$$

### 1.5.3 Differential equation for Temperature change

$$T = \int (\gamma T_s - T_{A/B}) \frac{R \dot{m}_{A/Bin}}{p_{A/B} V_A} T_{A/B} - (\gamma - 1) \frac{R \dot{m}_{A/Bout}}{p_{A/B} V_{A/B}} T_{A/B}^2 - (\gamma - 1) \frac{\dot{V}_{A/B}}{V_{A/B}} T_{A/B} \quad (44)$$

### 1.5.4 Valve model

$S_{eq}$	$m^2$	Equivalent cross section
$S_{max}$	$m^2$	Maximum cross section
$C_d$	—	Coefficient of contraction
$u$	—	Regulation variable

**Valve flow model with simply input control signal** For regulation flow this model used input control signal directly without spool mechanics.

Coefficient of contraction 46:

$$C_d = \frac{S_{eq}}{S_{max}} \quad (45)$$

For flow control regulation  $u \in \langle -1, 1 \rangle$  can be used.

$$u = \begin{cases} u \in \langle -1, 0 \rangle & \text{discharge the chamber} \\ u = 0 & \text{valve closed} \\ u \in (0, 1) & \text{filling the chamber} \end{cases} \quad (46)$$

$$\dot{m} = u S_{max} C_d p_1 \sqrt{\frac{2}{RT_1}} \cdot \psi \left( \frac{p_2}{p_1} \right) \quad (47)$$

**For filling the chamber:**

- $p_1 = p_s$
- $p_2 = p_A$  or  $p_B$
- $T_1 = T_s$

**For discharge the chamber:**

- $p_1 = p_A$  or  $p_B$
- $p_2 = p_0$
- $T_1 = T_A, T_B$

where  $p_s$  is supply pressure.  $p_0$  atmospheric pressure. As  $T_i$  - atmospheric temperature using according to isothermal process.

$$\dot{m}_A = \begin{cases} u S_v C_d p_s \sqrt{\frac{2}{RT_s}} \cdot \psi\left(\frac{p_A}{p_s}\right) & , u \in (0, 1) \\ 0 & , u = 0 \\ u S_v C_d p_A \sqrt{\frac{2}{RT_A}} \cdot \psi\left(\frac{p_0}{p_A}\right) & , u \in \langle -1, 0 \rangle \end{cases} \quad (48)$$

$$\dot{m}_B = \begin{cases} u S_v C_d p_s \sqrt{\frac{2}{RT_s}} \cdot \psi\left(\frac{p_B}{p_s}\right) & , u \in (0, 1) \\ 0 & , u = 0 \\ u S_v C_d p_A \sqrt{\frac{2}{RT_B}} \cdot \psi\left(\frac{p_0}{p_B}\right) & , u \in \langle -1, 0 \rangle \end{cases} \quad (49)$$

**Valve flow with spool mechanic included** With respect to valve spool modeled as 1DOF system 1 and mechanical and geometrical properties following equation were used.

**Valve flow with spool** In this model we accept a spool displacement  $x_s$ , controlled by input voltage  $u$ .

$$\dot{m}(P_u, P_d) = \begin{cases} C_f A_v \left( \frac{\gamma}{R} \left( \frac{2}{\gamma-1} \right) \right)^{\frac{1}{2}} \cdot \frac{P_u}{\sqrt{T}} \left( \frac{P_d}{P_u} \right)^{\frac{1}{\gamma}} \cdot \sqrt{1 - \left( \frac{P_d}{P_u} \right)^{\frac{\gamma-1}{\gamma}}} & , \text{ if } \frac{P_d}{P_u} > P_{cr} \text{ (subsonic)} \\ C_f A_v \frac{P_u}{\sqrt{T}} \cdot \sqrt{\frac{\gamma}{R} \left( \frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{\gamma-1}}} & , \text{ if } \frac{P_d}{P_u} \leq P_{cr} \text{ (sonic)} \end{cases} \quad (50)$$

where  $C_f$  is discharge coefficient,  $A_v$  is the effective are of valve orifice.

$$A_v = \frac{\pi x_s^2}{4} \quad (51)$$

$$x_s = C_v u \quad (52)$$

where  $C_v$  is the valve constant.

**Valve model by Endler** Require fitting constants and generally system identification. Mass flow rates are given by following equations:

$$\begin{aligned} \dot{m}_A(u, p_A) &= g_1(p_A, \text{sign}(u)) \arctg(2u) \\ \dot{m}_B(u, p_B) &= g_2(p_B, \text{sign}(u)) \arctg(2u) \end{aligned} \quad (53)$$

where  $g_1, g_2$  are signal functions given:

$$\begin{aligned} g_1(p_A, \text{sign}(u)) &= \beta \Delta p_A = \begin{cases} (p_s - p_A) \beta^{ench} & , \text{ if } u \geq 0 \\ (p_A - p_0) \beta^{evs} & , \text{ if } u < 0 \end{cases} \\ g_2(p_B, \text{sign}(u)) &= \beta \Delta p_B = \begin{cases} (p_s - p_B) \beta^{ench} & , \text{ if } u < 0 \\ (p_B - p_0) \beta^{evs} & , \text{ if } u \geq 0 \end{cases} \end{aligned} \quad (54)$$

where  $\beta^{ench}, \beta^{evs}$  are constant coefficients. For fitting model stop piston (speed of piston is null). This mean that volume is constant. We can measure flow rate  $\dot{m}$  versus input voltage  $u$  with given pressure difference.

**Valve dead-zone** For more precision control and modeling of the valve system, valve dead-zone can be used 56.

$$u_z = \begin{cases} g_z(u) < 0 & , \text{ if } u \leq u_n \\ 0 & , \text{ if } u_n < u < u_p \\ h_z(u) > 0 & , \text{ if } u \geq u_p \end{cases} \quad (55)$$

## 1.6 Mechanical assembly

Mechanical assembly basically represented by following equation 57.

$$\ddot{x} = \frac{1}{m} (S_A p_A - S_B p_B - S_0 p_0 - F_f) \quad (56)$$

where  $F_f$  is a friction force. Friction force can be modeled in the different ways.

As an example of possible model is following equation. That consist from complex friction forces including viscous friction and Coulomb friction 58.

$$F_f = \begin{cases} C\dot{x} + \left( f_c + (f_s - f_c)e^{-\left(\frac{\dot{x}}{v_s}\right)^\delta} \right) \text{sign}(\dot{x}) & , \text{ if } \dot{x} \leq v_e \\ \mu\dot{x} & , \text{ if } \dot{x} > v_e \end{cases} \quad (57)$$

where  $C$  - viscous friction coefficient,  $f_c$  - Coulomb friction,  $f_s$  - maximum static friction,  $\mu$  - dynamic friction factor,  $v_s$  - Stribeck velocity,  $\delta$  - arbitrary index,  $v_e$  critical velocity.

## 2 Models based on approximation

Generally with dataset of input-output signals approximation model can be fit. Using System Identification Toolbox and modeled as Black-Box or Gray-Box models. This section attempted to fit some models using data from SimScape and Equation model presented before.

Fit approximation model make sense only if we know what to fit. Using signal process techniques and identify dominant signals that providing best classification features we will train models with respect to this signals.

Demonstration scripts are done and waiting for signals :)

### 2.1 State-space model

### 2.2 ARX model



### 3 Models comparison

#### 3.1 Model based on equations

This model 2 was developed with respect to equations represented in previous section.

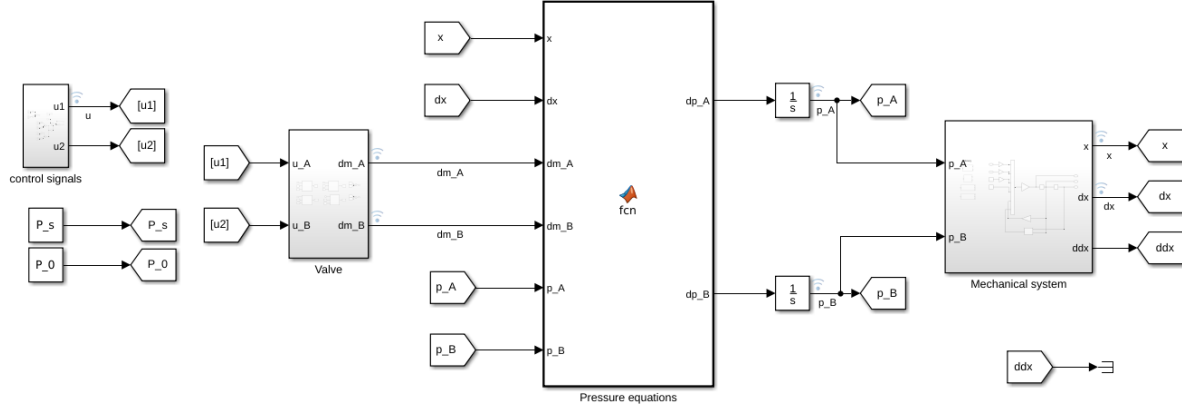


Figure 2: Simulink model based on equations

#### 3.2 Model Simscape

Model 3 was developed using SimScape toolbox.

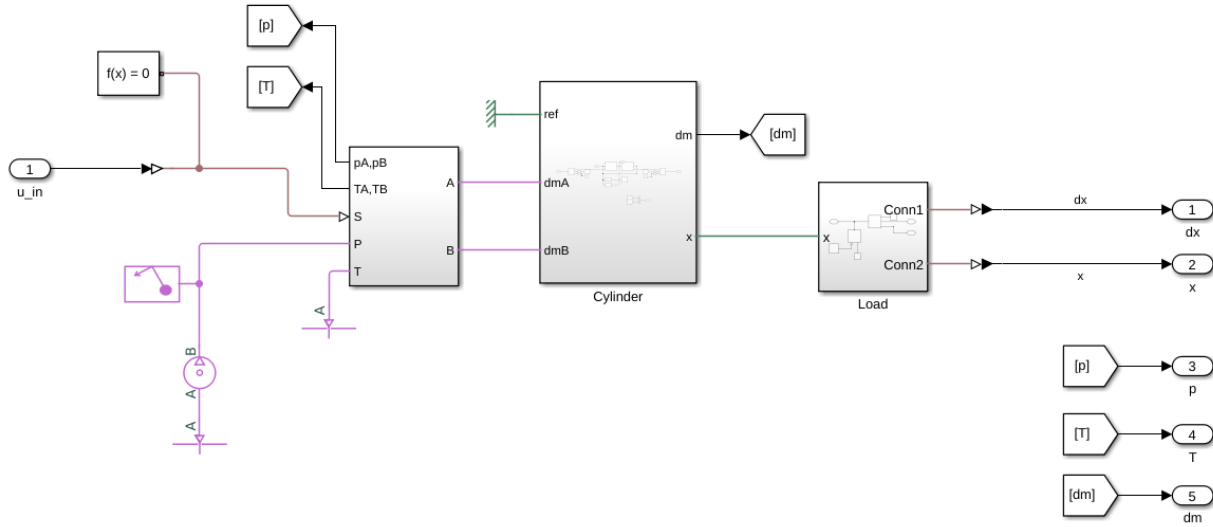


Figure 3: Simulink model using SimScape Toolbox

#### 3.3 Comparison

Following figure 4 represent comparison of 2 models (Simscape and based on equations) using same parameters for simulation: There is slight difference between models causing Valve dynamics simplifications in model based on equations.

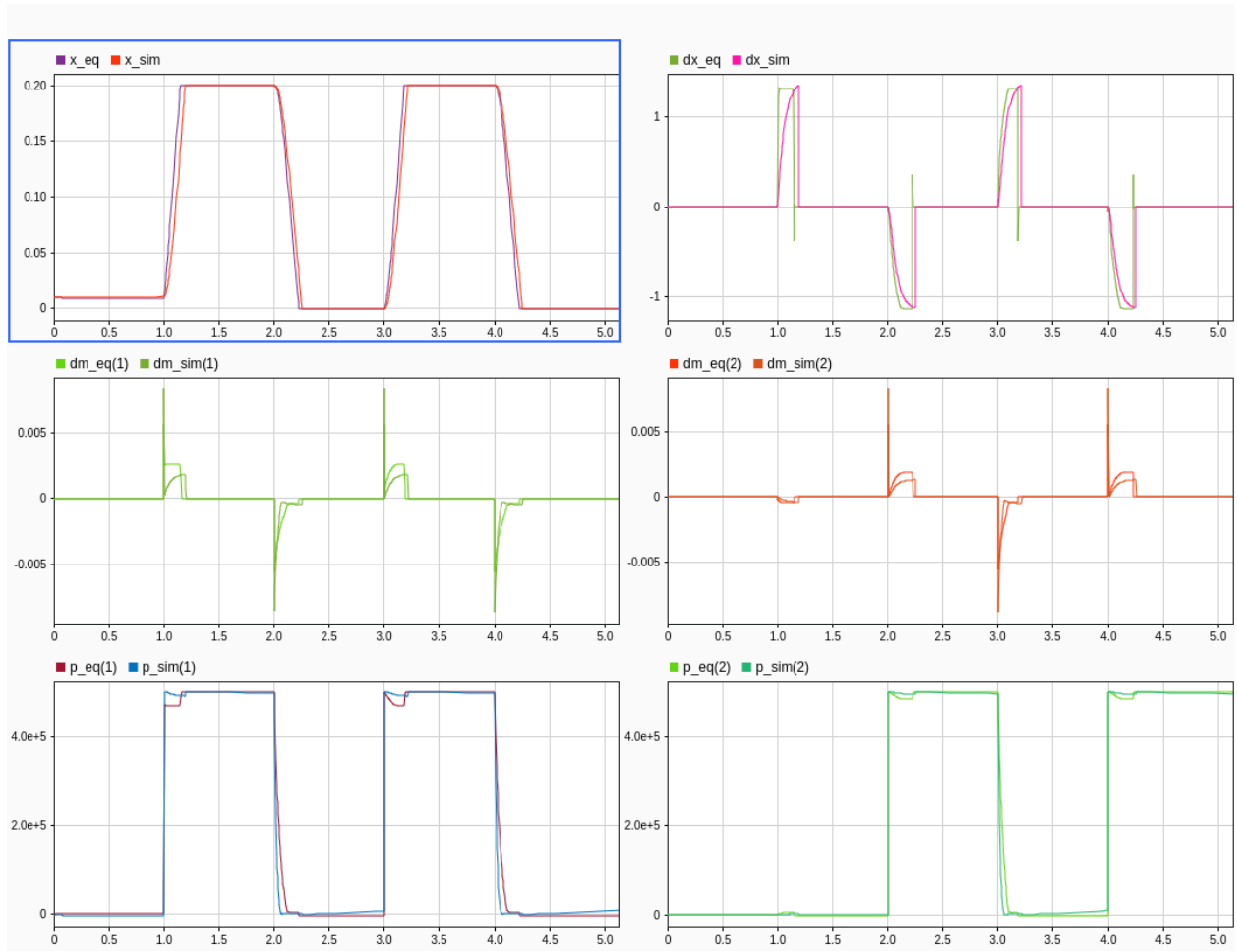


Figure 4: Comparison of simscape and model based on equations

## 4 Parameter identification

### 4.1 Mechanical assembly

In mechanical system there is  $F_f$  force represented by frictions accruing in the system. This force can be modeled by different friction models with respect to 1.6. Friction force parameters can be estimated using "gray-box" method. Using  $\dot{m}$  mass flow data versus  $x$  position measured on real assembly and use these data as an input and output, we can fit  $F_f$ . Simplify model can contain TODO:

- $F_C$  static friction
- $C_v$  viscous
- $C_p$  Pressure difference

### 4.2 Cylinder

Dead volume:  $p_1 V_1^n = p_2 V_2^n$  or datasheet.

### 4.3 Valve

For valve system there are two parameters that need to be estimated. According to equation 59 with constant  $p_1$  (pressure supply) and  $p_2$  (atmospheric pressure), we can estimate  $C$  if we neglect Valve Spool dynamic. If in experiment we determine that spool dynamic necessary to include. We provide same experiment with spool model including to "Gray-box" fitting model.

$$\dot{m} = \mathbf{u}(x_s) C p_1 \sqrt{\frac{2}{RT_1}} \cdot \psi\left(\frac{p_2}{p_1}\right) \quad (58)$$