3.4 Linearization of nonlinear state space models

The formulas for linearizing nonlinear discrete-time state space models are presented without derivation below. They can be derived in the same way as for linearizing nonlinear continuous-time models [1]. In the formulas below it assumed a second order system. I guess it is clear how the formulas can be generalized to higher orders.

Given the following discrete-time nonlinear state space model

$$\begin{aligned} x_1(k+1) &= f_1[x_1(k),\ x_2(k),\ u_1(k),\ u_2(k)] \\ x_2(k+1) &= f_2[x_1(k),\ x_2(k),\ u_1(k),\ u_2(k)] \end{aligned} \tag{3.9}$$

$$\begin{aligned} y_1(k) &= g_1[x_1(k),\ x_2(k),\ u_1(k),\ u_2(k)] \\ y_2(k) &= g_2[x_1(k),\ x_2(k),\ u_1(k),\ u_2(k)] \end{aligned} \tag{3.10}$$

where f_1 and f_2 are nonlinear functions. The corresponding linear model, which defines the system's dynamic behaviour about a specific operating point, is

$$\begin{aligned} \Delta x_1(k+1) &= \left.\frac{\partial f_1}{\partial x_1}\right|_{\text{op}} \Delta x_1(k) + \left.\frac{\partial f_1}{\partial x_2}\right|_{\text{op}} \Delta x_2(k) + \left.\frac{\partial f_1}{\partial u_1}\right|_{\text{op}} \Delta u_1(k) + \left.\frac{\partial f_1}{\partial u_2}\right|_{\text{op}} \Delta u_2(k) \\ \Delta x_2(k+1) &= \left.\frac{\partial f_2}{\partial x_1}\right|_{\text{op}} \Delta x_1(k) + \left.\frac{\partial f_2}{\partial x_2}\right|_{\text{op}} \Delta x_2(k) + \left.\frac{\partial f_2}{\partial u_1}\right|_{\text{op}} \Delta u_1(k) + \left.\frac{\partial f_2}{\partial u_2}\right|_{\text{op}} \Delta u_2(k) \end{aligned} \tag{3.11}$$

$$\begin{aligned} \Delta y_1(k) &= \left.\frac{\partial g_1}{\partial x_1}\right|_{\text{op}} \Delta x_1(k) + \left.\frac{\partial g_1}{\partial x_2}\right|_{\text{op}} \Delta x_2(k) + \left.\frac{\partial g_1}{\partial u_1}\right|_{\text{op}} \Delta u_1(k) + \left.\frac{\partial g_1}{\partial u_2}\right|_{\text{op}} \Delta u_2(k) \\ \Delta y_2(k) &= \left.\frac{\partial g_2}{\partial x_1}\right|_{\text{op}} \Delta x_1(k) + \left.\frac{\partial g_2}{\partial x_2}\right|_{\text{op}} \Delta x_2(k) + \left.\frac{\partial g_2}{\partial u_1}\right|_{\text{op}} \Delta u_1(k) + \left.\frac{\partial g_2}{\partial u_2}\right|_{\text{op}} \Delta u_2(k) \end{aligned} \tag{3.12}$$

or

$$\Delta x(k+1) = A\Delta x(k) + B\Delta u(k) \tag{3.13}$$

$$\Delta y(k) = C\Delta x(k) + D\Delta u(k) \tag{3.14}$$

where

$$\Delta x(k) = \begin{bmatrix} \Delta x_1(k) \\ \vdots \\ \Delta x_2(k) \end{bmatrix} \tag{3.15}$$

and similarly for $\Delta u(k)$ and $\Delta y(k)$. The system matrices are[1]

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\text{op}} = \left.\frac{\partial f}{\partial x^T}\right|_{\text{op}} \tag{3.16}$$

$$B = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix}\right|_{\text{op}} = \left.\frac{\partial f}{\partial u^T}\right|_{\text{op}} \tag{3.17}$$

$$C = \left.\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix}\right|_{\text{op}} = \left.\frac{\partial g}{\partial x^T}\right|_{\text{op}} \tag{3.18}$$

$$D = \left.\begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} \\ \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} \end{bmatrix}\right|_{\text{op}} = \left.\frac{\partial g}{\partial u^T}\right|_{\text{op}} \tag{3.19}$$

In the formulas above the subindex *op* is for operating point, which is a particular set of values of the variables. Often, the operating point is assumed to be an equilibrium (or static) operating point, which means that all variables have constant values there.

Example 1 ***Linearization***

Given the following non-linear state-space model:

$$x_1(k+1) = \underbrace{ax_1(k) + bx_1(k)x_2(k) + cx_1(k)u_1(k)}_{f_1} \tag{3.20}$$

$$x_2(k+1) = \underbrace{dx_2(k)}_{f_2} \tag{3.21}$$

$$y_1(k) = \underbrace{x_1(k)}_{g_1} \tag{3.22}$$

The system matrices of the corresponding linear model are

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{\text{op}} = \left.\begin{bmatrix} a + bx_2(k) + cu_1(k) & bx_1(k) \\ \\ 0 & d \end{bmatrix}\right|_{\text{op}} \tag{3.23}$$

[1] Partial derivative matrices are denoted Jacobians.

$$B = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} \\ \\ \frac{\partial f_2}{\partial u_1} \end{bmatrix}\right|_{\text{op}} = \left.\begin{bmatrix} cx_1(k) \\ \\ 0 \end{bmatrix}\right|_{\text{op}} \tag{3.24}$$

$$C = \left.\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \end{bmatrix}\right|_{\text{op}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3.25}$$

$$D = \left.\begin{bmatrix} \frac{\partial g_1}{\partial u_1} \end{bmatrix}\right|_{\text{op}} = \begin{bmatrix} 0 \end{bmatrix} \tag{3.26}$$

[End of Example 1]

3.5 Calculating responses in discrete-time state space models

3.5.1 Calculating dynamic responses

Calculating responses in discrete-time state space models is quite easy. The reason is that the model *is* the algorithm! For example, assume that Euler's forward method has been used to get the following discrete-time state space model:

$$x(k) = x(k-1) + hf(k-1) \tag{3.27}$$

This model constitutes the algorithm for calculating the response $x(k)$.

3.5.2 Calculating static responses

The static response is the response when all input variables have constant values and all output variables have converged to constant values. Assume the following possibly nonlinear state space model:

$$x(k+1) = f_1\left[x(k), u(k)\right] \tag{3.28}$$

where f_1 is a possibly nonlinear function of x and u. Let us write x_s and u_s for static values. Under static conditions (3.28) becomes

$$x_s = f_1\left[x_s, u_s\right] \tag{3.29}$$

which is an algebraic equation from which we can try to solve for unknown variables.

If the model is linear:

$$x(k+1) = Ax(k) + Bu(k) \tag{3.30}$$