PROVA 3 GRUP MA ICD 2020-2021

1. Dem si $\lim_{x\to a} f(x) = l \in \mathbb{R} \Rightarrow \exists$ un entorn d'a on f és acot.

Sabem que $\forall \varepsilon > 0 \ \exists \delta > 0$ t.q. $|x-a| < \delta \Rightarrow$

$$|f(x) - l| < \varepsilon$$

Fixem $\varepsilon = 1$, aleshores $\exists \delta > 0$ t.q. $|x-a| < \delta$

$$\Rightarrow \underbrace{|f(x) - l| < 1}_{-1 < f(x) - l < 1 \iff l-1 < f(x) < l+1}$$

$\Rightarrow \forall x \in I(a, \delta)\ l-1 < f(x) < l+1 \Rightarrow f$ acotada.

2. Calculem els límits següents si $\exists$.

a) $$\lim_{x\to 2} \frac{x-2}{\sqrt{\underbrace{x^2-2}_{\downarrow 2}} - \sqrt{\underbrace{x^2-2x+2}_{\downarrow 2}}} = \frac{0}{\sqrt{2}-\sqrt{2}} = \left[\frac{0}{0}\right] =$$

$$= \lim_{x\to 2} \frac{(x-2)\left(\sqrt{x^2-2} + \sqrt{x^2-2x+2}\right)}{\cancel{x^2}-2-(\cancel{x^2}-2x+2)} = \lim_{x\to 2} \frac{\cancel{(x-2)}(\sqrt{x^2-2} + \sqrt{x^2-2x+2})}{\underset{2(x-\cancel{2})}{\overset{\|}{2x-4}}}$$

$$= \lim_{x\to 2} \frac{\sqrt{x^2-2} + \sqrt{x^2-2x+2}}{2} = \frac{\sqrt{2}+\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \boxed{\sqrt{2}}$$

b) $$\lim_{x\to -\infty} \frac{e^x + \log(x^2-2) + x^6 - 2}{x^6}$$

$$\frac{e^x}{x^6} = \frac{1}{\underset{\downarrow +\infty}{x^6}} e^{x} \longrightarrow 0 \cdot e^{-\infty} = 0 \cdot 0 = 0 .$$

A teoria hem vist que $\lim_{x\to +\infty} \frac{e^x}{x^m} = 0$. ~~En el nostre cas,~~

i també $\lim_{x\to +\infty} \frac{\log x}{x^m} = 0$

$$\frac{\log(x^2-2)}{x^6} = \underbrace{\frac{\log(x^2-2)}{x^2-2}}_{\substack{\| \; y=x^2-2 \xrightarrow[x\to-\infty]{} +\infty \\ \frac{\log(y)}{y} \xrightarrow[y\to+\infty]{} 0}} \cdot \underbrace{\frac{x^2-2}{x^6}}_{\substack{x\to-\infty \downarrow \text{ grau num} < \text{grau den.} \\ 0}} \xrightarrow[x\to+\infty]{} 0$$

$$\frac{x^6-2}{x^6} \xrightarrow[x\to-\infty]{} 1 \quad (\text{grau num} \leq \text{grau den.})$$

Així, hem vist que $\exists \lim_{x\to-\infty} \frac{e^x}{x^6}$, $\exists \lim_{x\to-\infty} \frac{\log(x^2-2)}{x^2-2}$

i $\exists \lim_{x\to-\infty} \frac{x^2-2}{x^6}$. Pertant,

$$\lim_{x\to-\infty} \frac{e^x + \log(x^2-2) + x^2-2}{x^6} = \lim_{x\to-\infty} \frac{e^x}{x^6} + \lim_{x\to-\infty} \frac{\log(x^2-2)}{x^6} + \lim_{x\to-\infty} \frac{x^6-2}{x^6}$$

$$= 0 + 0 + 1 = \boxed{1}$$

3. Estudiem per a quins valors d'a i b $\in \mathbb{R}$, la funció

$$f(x) = \begin{cases} a\cos(x/2), & x \le 0 \\ \log(1+x^2)e^{-1/x}, & 0 < x \le 1 \\ b + |x|, & x > 1 \end{cases}$$

és contínua.

Si $x<0$, $f(x) = a\cos\left(\frac{x}{2}\right)$ contínua

Si $0<x<1$, $f(x) = \log(\underbrace{1+x^2}_{>0})\, e^{-1/x}$, $x \neq 0$, contínua.

Si $x>1$, $f(x) = b+|x|$ cont.

Així, f és cont. a $\mathbb{R}\setminus\{0,1\}$. Estudiem $x=0$ i $x=1$.

$\boxed{x=0}$ $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{\substack{x\to 0\\ x<0}} a\cos\left(\frac{x}{2}\right) = a\cdot\cos 0 = a\cdot 1 = a$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{\substack{x\to 0\\ x>0}} \log(\underbrace{1+x^2}_{1})\, e^{-\overbrace{1/x}^{\to+\infty}} = \log(1)e^{-\infty} = 0\cdot 0 = 0$

$f(0) = a\cos(0) = a$

f cont. a $x=0 \iff \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)$

$\iff \boxed{a=0}$

$\boxed{x=1}$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{\substack{x\to 1\\ x<1}} \log(1+x^2)\, e^{-1/x} = \log(2)\, e^{-1} = \frac{\log 2}{e}$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{\substack{x\to 1\\ x>1}} b+|x| = b+|1| = b+1$

$f(1) = \log 2\, e^{-1}$

f cont. a $x=1 \iff \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1) \iff$

$\iff \frac{\log 2}{e} = b+1 \iff \boxed{b = \frac{\log 2}{e} - 1}$

Així, $\boxed{f \text{ cont. a } \mathbb{R} \iff \begin{cases} a=0 \\ b = \frac{\log 2}{e} - 1 \end{cases}}$

4. Calculeu el següent límit si $\exists$:

$$\underbrace{\lim_{n\to+\infty} \frac{1+6^2+\dots+(n+4)^n}{1+2^{3/2}+\dots+n^{n-1/n}}}_{\displaystyle \frac{a_n}{b_n}}$$

- $a_n \in \mathbb{R}$.
- $b_n = 1+2^{3/2}+\dots+n^{n-1/n} = \sum_{i=1}^{n} i^{i-1/i} < \sum_{i=1}^{n+1} i^{i-1/i} = b_{n+1}$

$\quad \uparrow (n+1)^{n+1-\frac{1}{n+1}} > 0$

$\Rightarrow b_n$ estrict. creix.

- $b_n \longrightarrow +\infty$ (cada terme és positiu i $\lim_{n\to+\infty} n^{n-1/n} =$

$$= \lim_{n\to+\infty} e^{(n-1/n)\log n} = e^{\lim_{n\to+\infty} \underbrace{n\log n}_{+\infty\cdot+\infty=+\infty} - \underbrace{\frac{1}{n}\log n}_{0}} = e^{+\infty-0} = e^{+\infty} = +\infty.$$

<u>Stolz:</u> $$\lim_{n\to+\infty} \frac{a_n-a_{n-1}}{b_n-b_{n-1}} = \lim_{n\to+\infty} \frac{(n+4)^n}{n^{n-1/n}} =$$

$$= \lim_{n\to+\infty} \frac{(n+4)^n}{n^n n^{-1/n}} = \lim_{n\to+\infty} \frac{(n+4)^n}{n^n}\cdot n^{1/n} = \lim_{n\to+\infty} \left(\frac{n+4}{n}\right)^n \cdot n^{1/n}$$

$= (*)$

- $$\left(\frac{n+4}{n}\right)^n = \left(1+\frac{4}{n}\right)^n = \left(1+\frac{1}{n/4}\right)^n = \Big(\underbrace{\big(1+\frac{1}{\underset{\downarrow +\infty}{n/4}}\big)^{\frac{n}{4}}}_{\downarrow e}\Big)^4 \longrightarrow e^4$$

- $$n^{1/n} = e^{\frac{1}{n}\log n} \xrightarrow[n\to+\infty]{} e^0 = 1 \qquad \left(\frac{1}{n}\log n \xrightarrow[n\to+\infty]{} 0\right)$$

Així, $(*) = e^4 \cdot 1 = e^4 \quad \exists$

$$\underset{\text{Stolz}}{\Longrightarrow} \lim_{n\to+\infty} \frac{a_n}{b_n} = e^4.$$