LAB. 2 ICD 2020-2021

1. $\lim\limits_{n\to+\infty} \frac{2n}{3n-1} = \frac{2}{3}$ a partir de la def.

V.v. $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ t.q. $\forall n \geq n_0 \ \left|\frac{2n}{3n-1} - \frac{2}{3}\right| < \varepsilon$

$$\left|\frac{2n}{3n-1} - \frac{2}{3}\right| = \left|\frac{6n-6n+2}{9n-3}\right| = \frac{2}{|9n-3|} \underset{\substack{\uparrow \\ n\geq 1 \to 9n-3>0}}{=} \frac{2}{9n-3} < \varepsilon \Leftrightarrow$$

$$\underset{\substack{\uparrow \\ 9n-3>0}}{\Leftrightarrow} 2 < (9n-3)\cdot\varepsilon = 9n\varepsilon - 3\varepsilon$$

$$\Leftrightarrow 2+3\varepsilon < 9n\varepsilon \underset{\substack{\uparrow \\ 9\varepsilon>0}}{\Leftrightarrow} n > \frac{2+3\varepsilon}{9\varepsilon}$$ No és natural en general.

Així, si prenem $n_0 := \left[\frac{2+3\varepsilon}{9\varepsilon}\right]+1$ ja tindrem la def. de límit:

$\forall \varepsilon > 0 \quad \exists n_0 := \left[\frac{2+3\varepsilon}{9\varepsilon}\right]+1 \in \mathbb{N}$ t.q. $\forall n \geq n_0$

$$\left|\frac{2n}{3n-1} - \frac{2}{3}\right| < \varepsilon.$$

2. $(x_n)_n$ definida com: $\begin{cases} x_1 = 1 \\ x_{n+1} = \frac{9x_n - 4}{2x_n+1}, \ n \geq 1 \end{cases}$

Dem: i) x_n monòt.

ii) x_n acotada

iii) límit si $\exists$.

Obs 1: $x_1 = 1, \quad x_2 = \frac{9-4}{2+1} = \frac{5}{3} \geq x_1 = 1 \Rightarrow$ Si és monòtona serà creixent.

Obs 2: $2x_n + 1 \geq 1 > 0$. De fet veurem $x_n \geq 1$

· Cas inicial: $x_1 = 1 \geq 1$

$x_2 = \frac{5}{3} \geq 1$

· H.I. x_n t.q. $x_n \geq 1$

· Cas n+1? $x_{n+1} \geq 1$?

$$x_{n+1} = \frac{9x_n-4}{\underbrace{2x_n+1}_{>0}} \geq 1 \Leftrightarrow 9x_n - 4 \geq 2x_n + 1 \Leftrightarrow 7x_n \geq 5 \Leftrightarrow x_n \geq \frac{5}{7} \ (*)$$

(*) OK! De fet, $x_n \geq 1 \geq 5/7$ ✓

i) Vegem x_n monòtona creixent, i.e. $x_n \leq x_{n+1}$.

Fem inducció:

· Cas inicial: $\underset{\substack{\| \\ 1}}{x_1} \leq \underset{\substack{\| \\ \frac{5}{3}}}{x_2}$ OK!

· H.I. $x_n \leq x_{n+1}$.

· Cas n+1: $x_{n+1} \leq x_{n+2}$?

$$x_{n+1} = \frac{9x_n - 4}{2x_n + 1} \overset{?}{\leq} \frac{9x_{n+1} - 4}{2x_{n+1} + 1} \underset{\substack{\uparrow \\ 2x_m + 1 > 0 \\ \forall m \in \mathbb{N} \\ \text{(Obs. 2.)}}}{\iff}$$

$$\iff (9x_n - 4)(2x_{n+1} + 1) \leq (2x_n + 1)(9x_{n+1} - 4)$$

$$\iff \cancel{18x_n x_{n+1}} + 9x_n - 8x_{n+1} \cancel{-4} \leq \cancel{18x_n x_{n+1}} - 8x_n + 9x_{n+1} \cancel{-4}$$

$$\iff 17x_n \leq 17x_{n+1} \iff x_n \leq x_{n+1}$$ CERT PER H.I. ✓.

Així, x_n monòtona creixent.

ii) x_n acotada:

Per l'Obs 2, ja sabem que $x_n \geq 1$ (de fet, per ser creixent i $x_1 = 1 \Rightarrow x_n \geq 1$). Veiem ara una cota superior. Demostrarem $x_n \leq 2 + \sqrt{2}$:

~~fent inducció~~ Farem inducció:

· Cas inicial: $x_1 = 1 \leq 2 + \sqrt{2}$.

$x_2 = \frac{5}{3} \leq 2 + \sqrt{2}$

· H.I.: $x_n \leq 2 + \sqrt{2}$

· Cas n+1: $x_{n+1} \leq 2 + \sqrt{2}$?

$$x_{n+1} = \frac{9x_n - 4}{2x_n + 1} \leq 2 + \sqrt{2} \underset{\substack{\uparrow \\ 2x_n + 1 > 0}}{\iff} 9x_n - 4 \leq 2(2 + \sqrt{2})x_n + 2 + \sqrt{2}$$

$$\iff x_n \underbrace{(5 - 2\sqrt{2})}_{> 0} \leq 6 + \sqrt{2} \iff x_n \leq \frac{6 + \sqrt{2}}{5 - 2\sqrt{2}} \underset{(*)}{=} 2 + \sqrt{2}$$

CERT PER H.I. ✓.

$\Rightarrow$ Hem vist: $1 \leq x_n \leq 2 + \sqrt{2} \quad \forall n$

$\Rightarrow x_n$ acotada.

(*) $\dfrac{6+\sqrt{2}}{5-2\sqrt{2}} = \dfrac{(6+\sqrt{2})(5+2\sqrt{2})}{(5-2\sqrt{2})(5+2\sqrt{2})} = \dfrac{34+17\sqrt{2}}{17} = 2+\sqrt{2}$.

iii) Com x_n és creixent i acotada (superiorment)

$\Rightarrow$ és convergent i $\exists\, l = \lim x_n = \lim x_{n+1}$

Així,
$$\underset{\displaystyle l}{\underset{\downarrow}{x_{n+1}}} = \underset{\displaystyle \frac{9l-4}{2l+1}}{\underset{\downarrow}{\frac{9x_n-4}{2x_n+1}}} \Rightarrow l = \frac{9l-4}{2l+1} \iff$$

$\iff 2l^2 + l = 9l - 4 \iff 2l^2 - 8l + 4 = 0$

$\iff l^2 - 4l + 2 = 0 \iff \begin{cases} l = 2-\sqrt{2} \\ \text{ó} \\ l = 2+\sqrt{2} \quad (**) \end{cases}$

El límit si $\exists$ és únic:

$2-\sqrt{2} \leq x_1 = 1 \leq x_2 \leq x_3 \leq \dots \leq x_n \leq \dots \leq 2+\sqrt{2}$

Com $x_n \uparrow$ i $x_1 > 2-\sqrt{2}$ i $x_n \leq 2+\sqrt{2}$

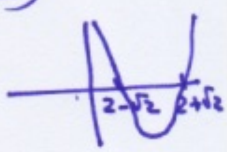
$\Rightarrow \boxed{l = 2+\sqrt{2}}$

(**) Quan fem aquest càlcul, això ens dóna la intuició de quina cota superior hem de posar per demostrar que x_n está acotada superiorment.

ALTERNATIVA PER VEURE x_n creixent: Si primer demostrem que x_n està acotada: $1 \leq x_n \leq 2+\sqrt{2}$, aleshores:

$x_n \leq x_{n+1} = \dfrac{9x_n-4}{2x_n+1} \underset{2x_n+1>0}{\iff} 2x_n^2 + x_n \leq 9x_n - 4 \iff$

$\iff 2x_n^2 - 8x_n + 4 \leq 0 \iff x_n^2 - 4x_n + 2 \leq 0 \iff$



$\iff x_n \in [2-\sqrt{2}, 2+\sqrt{2}]$ CERT PERQUÈ:

$x_n \in [\underset{2-\sqrt{2}}{\underset{\vee\!\mid}{1}}, 2+\sqrt{2}] \subset [2-\sqrt{2}, 2+\sqrt{2}]$ #.