11.1 $f: \mathbb{R}^3 \to \mathbb{R}^2$ i $g: \mathbb{R}^2 \to \mathbb{R}^3$, definides per $\begin{cases} \\ g:(x,y) \mapsto (x, x+y, x-y) \end{cases}$

i) Troba les matrius de f, g, $f \circ g$, $g \circ f$.

$M(f) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $f(0,0,1) = (1,0)$

$f(1,0,0) = (1,0)$ $f(0,1,0) = (1,0)$

$M(g) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$ $f(0,1) = (0,1,-1)$

$f(1,0) = (1,1,1)$

$M(g \circ f) = Mg \cdot Mf$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$M(f \circ g) = Mf \cdot Mg$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

Efectivament, les columnes de la matriu d'una aplicació lineal són la imatge de $e_1, e_2, \ldots e_n$.

El profe diu que també podem trobar $f \circ g$ i $g \circ f$ de la següent manera:

$$g \circ f : (x,y,z) \longrightarrow (x+y+z, 0) \overset{?}{\longrightarrow} (x+y+z, x+y+z, x+y+z)$$

$$f \circ g : (x,y) \longrightarrow (x, x+y, x-y) \longrightarrow (3x, 0)$$

11.2 Sigui E un espai de $\mathbb{R}^3$, $f: E \to E$ de matriu $\begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & -5 \\ 1 & 2 & -1 \end{pmatrix}$

Calcula $\ker f$, $\operatorname{Im} f$, $\ker f \cap \operatorname{Im} f$, $\ker f + \operatorname{Im} f$.

<u>Kerf</u>: Sabem que són: $\{v \in E : f(v) = 0\}$. Per tant mirarem els vectors que compleixen aquesta condició.

$$\begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & -5 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \left.\begin{aligned} x + 4y + z &= 0 \\ 2x + y - 5z &= 0 \\ x + 2y - z &= 0 \end{aligned}\right\} \sim \begin{aligned} y &= y \\ z &= -y \\ x &= -3y \end{aligned}$$

$\ker f = \langle (3, -1, 1) \rangle$ $\quad \ker f = \{(3\lambda, -\lambda, \lambda), \forall \lambda \in \mathbb{R}\}$

$\dim \ker f = 1$

$\mathrm{Im}f = \{w \in F \mid w = f(v),\ v \in E\}$ * conjunt de vectors que pertanyen a la imatge de l'aplicació.

$$M_f = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & -5 \\ 1 & 2 & -1 \end{pmatrix}$$

$\downarrow\ \downarrow\ \downarrow$

$f(e_1)\ f(e_2)\ f(e_3)$

$$\underbrace{\begin{vmatrix} 1 & 4 & 1 \\ 2 & 1 & -5 \\ 1 & 2 & -1 \end{vmatrix}}_{\text{volem saber el rang}} = 0 \qquad \begin{vmatrix} 1 & 4 \\ 2 & 1 \end{vmatrix} \neq 0$$

1a i 2a col són LI

*

Quantes columnes són LI?

* $\mathrm{Im}f = \langle (1,2,1), (4,1,2) \rangle$ $\dim \mathrm{Im}f = 2$

$\underline{\mathrm{Ker}f + \mathrm{Im}f}$ $\begin{pmatrix} 3 & 1 & 4 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ Kerf Imf

Mirem el rang x saber-ne la dimensió.

$$\begin{vmatrix} 3 & 1 & 4 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 0 \Rightarrow \dim \mathrm{Ker}f + \mathrm{Im}f = 2$$

Tenim $\underbrace{\dim E}_{3} = \underbrace{\dim \mathrm{Ker}f}_{1} + \underbrace{\dim \mathrm{Im}f}_{2}$

Tenim $\underbrace{\dim \mathrm{Ker}f + \mathrm{Im}f}_{2} = \underbrace{\dim \mathrm{Im}f}_{2} \Rightarrow \mathrm{Im}f \supset \mathrm{Ker}f$.

Per tant, $\mathrm{Ker}f \cap \mathrm{Im}f = \mathrm{Ker}f$.

11.9 Demostra que si $f: E \to E$, llavors $\mathrm{Ker}f \cap \mathrm{Im}f = \{0\}$ sii $\mathrm{Ker}f = \mathrm{Ker}f^2$

$\boxed{\Rightarrow}$ $\mathrm{Ker}f \subset \mathrm{Ker}f^2$

$x \in \mathrm{Ker}f \Leftrightarrow x \in f(x) = 0 \Rightarrow f(f(x)) = 0 \Rightarrow f^2(x) = 0 \Rightarrow x \in \mathrm{Ker}f^2$

$\mathrm{Ker}f^2 \subset \mathrm{Ker}f$

$x \in \mathrm{Ker}f^2 \Rightarrow f(f(x)) = 0 \Rightarrow \begin{cases} f(x) \in \mathrm{Im}f \\ f(x) \in \mathrm{Ker}f \end{cases} \xrightarrow{\text{hipòtesi}} f(x) = 0 \Rightarrow x \in \mathrm{Ker}f$

$\boxed{\Leftarrow}$

$\mathrm{Ker}f = \mathrm{Ker}f^2 \Rightarrow f(f(x)) = 0 \Rightarrow x \in f(x) \Rightarrow \mathrm{Ker}f \cap \mathrm{Im}f = \{0\}$.

Tan sols tenen en comú el vector $\{0\}$.

$\ker f \supset \operatorname{Im} f \iff f^2 = 0$.

$\boxed{\Longrightarrow}$

$f^2 = 0 \Rightarrow f^2(x) = 0 \quad \forall x$. $\underbrace{f^2(x) = f(f(x)) = 0}_{\operatorname{Im} f \subset \ker f, \text{ per tant és } 0.}$

$\boxed{\Longleftarrow}$

$f^2 = 0 = f^2(x) = 0$. $f(f(x)) = 0$. $\Rightarrow f(x) \in \ker f, \ \forall f(x) \in \operatorname{Im} f$.

11.5 $f: E \to F \quad g: F \to G$

$\ker f \subset \ker(g \circ f), \ \operatorname{Im} g \supset \operatorname{Im}(g \circ f) \rightarrow$ demostrar-ho.

$f: E \longrightarrow F \quad g: F \longrightarrow G$

$$\left.\begin{array}{l} \ker f = \{v \in E \mid f(v) = 0\} \\ \ker(g \circ f) = \{u \in G \mid g(u) = 0\} \end{array}\right\} \begin{array}{l} g(u) = 0 \iff g(f(v)) = 0 \\ \Longrightarrow \ker f \subset \ker(g \circ f) \end{array}$$

$\operatorname{Im} g \supset \operatorname{Im} g \circ f$

$$\left.\begin{array}{l} \operatorname{Im} g = \{w \in F \mid w = f(v), v \in E\}. \\ \operatorname{Im} g \circ f = \{u \in G \mid u = g(s), s \in F\} \end{array}\right\} g(s) = g(f(v))$$

Si $v \in \ker f \Rightarrow f(v) = 0 \Rightarrow g(f(v)) = 0 \iff (g \circ f)(v) = 0 \Rightarrow v \in \ker f (g \circ f)$.

Si $v \in \operatorname{Im} g \circ f \Rightarrow \exists w \ (g \circ f)(w) = v \iff \exists w \ g(f(w)) = v \Rightarrow v \in \operatorname{Im} f$.

11.4 $E \to E$ $\qquad$ f és injectiva $\iff \ker f = \{0\}$

Imposem rang $M_a < 3$

$$M = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix} \qquad 0 = \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = a^3 + 1 + 1 - a - a - a = a^3 - 3a + 2 \begin{cases} \boxed{a = 1} \\ \boxed{a = -2} \end{cases}$$

$\boxed{a = 1}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x + y + z = 0$ $\qquad$ $\mathbb{R}^3 - 1 \text{ eq} = 2 \dim \ker f$

$\ker f = \langle (1, 0, -1), (0, 1, -1) \rangle$

$\boxed{a=-2}$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left.\begin{array}{l} -2x+y+z=0 \\ x-2y+z=0 \end{array}\right\} \quad (\lambda,\lambda,\lambda) \ \lambda\in\mathbb{R}$$

$$\ker f = \langle (1,1,1) \rangle$$

$\boxed{a \neq -2 \quad a \neq -1}$ $\Rightarrow$ f isomorfisme.

$\begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{pmatrix}$ Per a quins valors de a no és inj... Calcula'n el nucli en aquests casos. Per als valors restants de a, f és un isomorfisme, calcula la matriu del seu invers.

11.10 Sigui $f\ \mathbb{R}^3 \to \mathbb{R}^3$ l'aplicació lineal de la matriu

$\begin{pmatrix} 1 & 2 & -a \\ 0 & a & 0 \\ -1 & b & 1 \end{pmatrix}$. Determina kerf i imf segons a i b.

Kerf $\{e \in E \mid f(e) = 0\}$

$$\begin{pmatrix} 1 & 2 & -a \\ 0 & a & 0 \\ -1 & b & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \left.\begin{array}{l} x + 2y - az = 0 \\ ay = 0 \\ -x + by + z = 0 \end{array}\right] \begin{array}{l} x + 2y - az = 0 \\ ay = 0 \\ (2-b)y + (-a-1)z = 0 \end{array}$$

$$(2-b)y = (a+1)z \ ; \quad y = \frac{a+1}{2-b} z$$

$$x + 2\left(\frac{a+1}{2-b}\right)z - az = 0 \ ; \quad x + \left(\frac{2a+2}{2-b}\right)z - az = 0 \ ;$$

$$x + \left(\frac{2a+2}{2-b} - \frac{2a+ab}{2-b}\right)z = 0 \quad x + \left(\frac{2-ab}{2-b}\right)z = 0 \ ;$$

$$x = -\left(\frac{2-ab}{2-b}\right)z = 0$$

$$\begin{array}{l} x + 2y - z = 0 \\ y = 0 \\ -x + by + z = 0 \end{array} \left\{ \begin{array}{l} x = z \\ y = 0 \end{array} \right. \quad \begin{array}{l} (x,y,z) = \\ = (z,0,z) = \\ = z(1,0,1) \end{array}$$

$\boxed{\text{Si } a = 1}$ $\text{Kerf} = \{(+\lambda, 0, \lambda)\}$ $\quad \text{Kerf} = \langle(-1,0,1)\rangle$

$\boxed{\text{Si } a \neq 1}$ $\text{Kerf} = \left\{-\left(\frac{2-ab}{2-b}\right)\lambda, 0, \lambda\right\}$, $\text{Kerf} = \left\langle -\left(\frac{2a-ab}{2-b}\right), 0, 1\right\}$

$\boxed{b \neq 2}$

Imf

$$\begin{vmatrix} 1 & 2 & -a \\ 0 & a & 0 \\ -1 & b & 1 \end{vmatrix} = a - a^2 \qquad \begin{array}{l} a - a^2 = 0 \ ; \quad a = a^2 \ ; \\ a(1-a) = 0 \quad \boxed{a = 1} \end{array}$$

Si $\boxed{a=1}$ $\text{Imf} = \langle (1,0,-1), (2,1,b) \rangle$ ✓

Si $a \neq 1$ $\text{Imf} = \langle (1,0,-1), (2,a,b), (-a,0,1) \rangle$.

2) És f exhaustiva? f exhaustiva $\Leftrightarrow$ dim $Im f = 3$

f exhaustiva $\Leftrightarrow f$ inj $\Leftrightarrow f$ bij

Aplicació inversa $\Leftrightarrow$ isomorfisme.

11.10

Si $a \neq 0$ $a \neq 1$ $\Rightarrow \det A \neq 0 \Rightarrow f$ inj $\Rightarrow f$ bij, $\ker f = \{0\}$

$Im f = \mathbb{R}^3$

Si $a = 0$ $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ -1 & b & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{matrix} -x+by+z = 0 \\ x+2y = 0 \end{matrix}$

$\ker f = \langle (2,-1,b+2) \rangle$

$Im f = \langle (1,0,-1), (0,0,1) \rangle$

Si $a = 1$

$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & b & 1 \end{pmatrix}$ Fet a l'altre full

Si $a = 0$

$\ker f = \langle (2,-1,b+2) \rangle$

$Im f = \langle (1,0,-1)(0,0,1) \rangle$

$\mathbb{R}^3 = \ker f + Im f$

$\begin{vmatrix} 2 & -1 & b+2 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$

dim $\ker f \cap$ dim $Im f = 3-2-1 = 0$

$(2\lambda, -\lambda, (b+2)\lambda) = x(1,0,-1) + y(0,0,1)$

$\left\{ \begin{matrix} x = 2\lambda \\ -\lambda = 0 \\ (b+2)\lambda = -x+y \end{matrix} \right\} \Rightarrow x = y = 0 \Rightarrow \ker f \cap Im f = \{0\}$

Si $a = 1$

$\begin{vmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & b & 1 \end{vmatrix} \neq 0$