PROVA 3 GRUP MC ICD 2020-2021

1. Dem. que si f i g són funcions t.q.

$$\lim_{x\to+\infty} f(x) = l \quad , \quad \lim_{x\to+\infty} g(x) = +\infty$$

$$\Rightarrow \lim_{x\to+\infty} (f(x)+g(x)) = +\infty$$

Fixem $M>0 \Rightarrow \exists\, K_f, K_g > 0$ t.q. $|f(x)-l|<1 \quad \forall x > K_f$

$g(x) > M-l+1 \quad \forall x > K_g$

$K := \max(K_f, K_g) \Rightarrow \forall x > K,$

$f(x)+g(x) > l-1+M-l+1 = M$

$\neq$

2. Calculeu els següents límits si $\exists$.

(a) $$\lim_{x\to-1} \frac{x+1}{\sqrt{\underbrace{x^4+1}_{2}} - \sqrt{\underbrace{x^4+3x+4}_{-2+4=2}}} = \frac{0}{\sqrt{2}-\sqrt{2}} = \left[\frac{0}{0}\right]$$

$$= \lim_{x\to-1} \frac{(x+1)\left(\sqrt{x^4+1}+\sqrt{x^4+3x+4}\right)}{\cancel{x^4}+1-(\cancel{x^4}+3x+4)} = \lim_{x\to1} \frac{\cancel{(x+1)}\left(\sqrt{x^4+1}+\sqrt{x^4+3x+4}\right)}{\underbrace{-3x-3}_{-3\cancel{(x+1)}}}$$

$$= \lim_{x\to-1} \frac{\sqrt{x^4+1}+\sqrt{x^4+3x+4}}{-3} = \frac{\sqrt{2}+\sqrt{2}}{-3} = \boxed{\frac{-2}{3}\sqrt{2}}$$

(b) $$\lim_{x\to+\infty} \frac{e^{-x}+\log(x^4+3)+x^3}{x^5}$$

$$\frac{e^{-x}}{x^5} = \frac{1}{x^5}e^{-x} \longrightarrow 0\cdot e^{-\infty} = 0\cdot 0 = 0$$

A teoria, hem vist que $\lim_{y\to+\infty} \frac{\log(y)}{y^n} = 0$

$$\frac{\log(x^4+3)}{x^5} = \underbrace{\frac{\log(x^4+3)}{x^4+3}}_{\substack{\| \; y=x^4+3\to+\infty \\ \frac{\log y}{y} \xrightarrow[y\to+\infty]{} 0}} \cdot \underbrace{\frac{x^4+3}{x^5}}_{0} \longrightarrow 0\cdot 0 = 0$$

gran num < gran den.

$$\frac{x^3}{x^5} = \frac{1}{x^2} \xrightarrow[x \to +\infty]{} 0$$

Així, hem vist que $\exists \lim_{x \to +\infty} \frac{e^{-x}}{x^5}$, $\exists \lim_{x \to +\infty} \frac{\log(x^4+3)}{x^5}$ i $\exists \lim_{x \to +\infty} \frac{x^3}{x^5}$. Per tant,

$$\lim_{x \to +\infty} \frac{e^{-x} + \log(x^4+3) + x^3}{x^5} = \lim_{x \to +\infty} \frac{e^{-x}}{x^5} + \lim_{x \to +\infty} \frac{\log(x^4+3)}{x^5}$$

$$+ \lim_{x \to +\infty} \frac{x^3}{x^5} = 0 + 0 + 0 = \boxed{0}$$

3. Estudieu per quins valors d'a i b, la funció

$$f(x) = \begin{cases} b(|x|-1), & x \le 0 \\ \log\left(\frac{x^3+e}{x^2+1}\right), & 0 < x \le 1 \quad \text{és cont.} \\ a\,\frac{\sin(x-1)}{x-1}, & x > 1 \end{cases}$$

Si $x<0$, $f(x) = b(|x|-1)$ cont.

Si $0<x<1$, $f(x) = \log\Big(\underbrace{\frac{x^3+e}{x^2+1}}_{>0}\Big)$ cont. (den. no s'anul·la!!)

$x^3+e > 0$, $x^2+1>0$

Si $x>1$, $f(x) = a\,\frac{\sin(x-1)}{x-1}$ cont. $\quad x-1 \neq 0$

f és cont. a $\mathbb{R}\setminus\{0,1\}$. Anem a veure què passa a $x=0,1$.

$\boxed{x=0}$ $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{\substack{x\to 0\\ x<0}} b(|x|-1) = b(0-1) = -b$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{\substack{x\to 0\\ x>0}} \log\left(\frac{x^3+e}{x^2+1}\right) = \log\left(\frac{e}{1}\right) = 1$

$f(0) = b(0-1) = -b$

f cont. a $x=0 \iff \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)$

$\iff -b = 1 \iff \boxed{b=-1}$

$\boxed{x=1}$ $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{\substack{x\to 1\\ x<1}} \log\left(\frac{x^3+e}{x^2+1}\right) = \log\left(\frac{1+e}{2}\right)$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{\substack{x\to 1\\ x>1}} a\,\frac{\sin(\overset{\shortparallel\, y}{x-1})}{\underbrace{x-1}_{=y}} = \lim\limits_{\substack{y\to 0\\ y>0}} a\,\underbrace{\frac{\sin y}{y}}_{\to 1 \;(y\to 0)} = a$

$f(1) = \log\left(\frac{1+e}{2}\right)$

f cont. a $x=1 \iff \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = f(1)$

$\iff \boxed{\log\left(\frac{1+e}{2}\right) = a}$

Així, $\boxed{f \text{ cont. a } \mathbb{R} \iff \begin{cases} a = \log\left(\frac{1+e}{2}\right) \\ b = -1 \end{cases}}$

4. Calcular el següent límit si $\exists$.

$$\lim_{n\to+\infty} \underbrace{\frac{3^3+4^6+\dots+(n+2)^{2n}}{2+3^{7/2}+\dots+(n+1)^{2n-1/n}}}_{\displaystyle = \frac{a_n}{b_n}}$$

- $a_n \in \mathbb{R}$.
- $b_n = 2+3^{7/2}+\dots+(n+1)^{2n-1/n} = \sum_{i=1}^{n}(i+1)^{2i-1/i} \underset{\uparrow}{<} \sum_{i=1}^{n+1}(i+1)^{2i-1/i} = b_{n+1}$, $\quad (n+1+1)^{2(n+1)-1/(n+1)} > 0$

$\Rightarrow b_n \uparrow$

- $b_n \uparrow +\infty$ pq tots els termes són positius

i $\lim_{n\to+\infty}(n+1)^{2n-1/n} = e^{\lim_{n\to+\infty}(2n-1/n)\log(n+1)} = e^{\lim_{n\to+\infty} \underbrace{2n\log(n+1)}_{\to+\infty} - \underbrace{\frac{\log(n+1)}{n}}_{\to 0}}$

$$= e^{+\infty-0} = e^{+\infty} = +\infty.$$

<u>Stolz</u>: $\lim_{n\to+\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}} = \lim_{n\to+\infty}\frac{(n+2)^{2n}}{(n+1)^{2n-1/n}} =$

$$= \lim_{n\to+\infty}\left(\frac{n+2}{n+1}\right)^{2n}(n+1)^{1/n} = (*)$$

- $\left(\frac{n+2}{n+1}\right)^{2n} \underset{\substack{\uparrow \\ n+2=(n+1)+1}}{=} \left(1+\frac{1}{n+1}\right)^{2n} = \left(\underbrace{\left(1+\frac{1}{\underset{\downarrow +\infty}{n+1}}\right)^{(n+1)}}_{\downarrow e}\right)^{\frac{2n}{n+1}\to 2} \to e^2$

- $(n+1)^{1/n} = e^{\frac{1}{n}\log(n+1)\to 0} \to e^0 = 1$

$(*) = e^2 \cdot 1 = e^2 \quad \exists$

$$\underset{\text{Stolz}}{\Longrightarrow} \boxed{\lim_{n\to+\infty}\frac{a_n}{b_n} = e^2}$$