LAB. 6 ICD 2020-2021

1. $n \in \mathbb{N}$, $\underset{f_n}{\overset{\shortparallel}{f}} : \mathbb{R} \longrightarrow \mathbb{R}$ def. per

$$f(x) = \begin{cases} x^n \cos\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Per quins n f és deriv. en $x=0$?

<u>OBS</u>: Si $x \neq 0$, x^n deriv. $(n>0)$

$\cos\left(\frac{1}{x^2}\right)$ deriv. (pq cos és deriv $\frac{1}{x^2}$ tb és deriv ja que $x \neq 0$).

A més, $f_n'(x) = n x^{n-1} \cos\left(\frac{1}{x^2}\right) + x^n \cdot \left(-\sin\left(\frac{1}{x^2}\right)\right) \cdot (-2) x^{-3} =$

$$= n x^{n-1} \cos\left(\frac{1}{x^2}\right) + 2 x^{n-3} \sin\left(\frac{1}{x^2}\right) =$$

$$= x^{n-3}\left[x^2 n \cos\left(\frac{1}{x^2}\right) + 2 \sin\left(\frac{1}{x^2}\right)\right] \quad \text{si } x \neq 0$$

Per veure si f és deriv. a $x=0$ hem de veure que

$$\exists \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^n \cos\left(\frac{1}{x^2}\right) - 0}{x - 0} = \lim_{x \to 0} x^{n-1} \cos\Big(\underbrace{\frac{1}{x^2}}_{\to +\infty}\Big)$$

$$= \begin{cases} \lim_{x \to 0} \cos\left(\frac{1}{x^2}\right) = \cos(\infty)\ \nexists, & n = 1 \\ \lim_{x \to 0} \underbrace{x^{n-1}}_{\to 0} \underbrace{\cos\left(\frac{1}{x^2}\right)}_{\text{acot.}} = 0, & n > 1 \\ \lim_{x \to 0} \underbrace{\boxed{\frac{1}{x^{1-n}}}}_{\to \pm\infty} \underbrace{\cos\left(\frac{1}{x^2}\right)}_{\to \cos(\infty)\ \nexists}\ \nexists, & n < 1 \end{cases}$$

Així, f és deriv. a $x=0$ si i només si $n > 1$ i $f'(0) = 0$

(b) Per quins n, f' cont. a tot $\mathbb{R}$?

$$f'(x) = \begin{cases} x^{n-3}\left(x^2 n\cos\left(\frac{1}{x^2}\right) + 2\sin\left(\frac{1}{x^2}\right)\right) & , \ x \neq 0 \\ 0 & , \ x = 0. \end{cases}$$

($\underline{n>1}$)
(pq sinó f no deriv. a $x=0$)

Si $x \neq 0$, $f'(x) = \underbrace{\underbrace{n x^{n-1}}_{\text{cont. } x\neq 0} \underbrace{\cos}_{\text{cont.}}\Big(\underbrace{\tfrac{1}{x^2}}_{\text{cont. pq } x\neq 0}\Big)}_{\text{cont. si } x\neq 0} + \underbrace{2\underbrace{x^{n-3}}_{\text{cont. } x\neq 0}\underbrace{\sin}_{\text{cont.}}\Big(\underbrace{\tfrac{1}{x^2}}_{\text{cont. } x\neq 0}\Big)}_{\text{cont. } x\neq 0}$ cont. $x \neq 0$.

Si $x=0$, veiem quan f' cont. ($\underline{n>1}$)

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \underbrace{n x^{n-1} \underbrace{\cos\left(\tfrac{1}{x^2}\right)}_{\cos(+\infty)\ \nexists \text{ però acotada}}}_{\substack{x\to 0 \\ n>1 \\ x^{n-1}\to 0 \\ 0\cdot\text{acotada} = 0.}} + \underbrace{2x^{n-3}\sin\Big(\underbrace{\tfrac{1}{x^2}}_{+\infty}\Big)}_{\sin(+\infty)\ \nexists \text{ però acotat.}}$$

$x\to 0$, $x^{n-3} \to \begin{cases} 1 & n=3 \\ 0 & n>3 \\ \pm\infty & n<3 \ (n=2) \end{cases}$

$$\begin{cases} 2\sin(\infty)\ \nexists & , n=3 \\ 0\cdot\text{acot.} = 0 & , n>3 \\ (\pm\infty)\cdot\sin(\infty)\ \nexists & , n=2 \end{cases}$$

Així, $\lim_{x\to 0} f'(x)$ existeix només quan $n>3$
i val $0 = f'(0) \Rightarrow \lim_{x\to 0} f'(x) = f(0)$ si i només si $n>3$

$\Rightarrow f'$ cont. a $x=0$ només quan $n>3$.

Així $\boxed{f' \text{ deriv. a } \mathbb{R} \iff n>3}$

2.

(a) $\lim_{x\to\pi/4} \frac{\sin x - \cos x}{\cos(2x)} = \frac{\sin(\frac{\pi}{4}) - \cos(\pi/4)}{\cos(\pi/2)} = \left[\frac{0}{0}\right]$

$\underset{\text{Hôp.}}{=} \lim_{x\to\pi/4} \frac{\cos x + \sin x}{-\sin(2x)\cdot 2} = \frac{\cos\frac{\pi}{4}+\sin\frac{\pi}{4}}{-2\underbrace{\sin(\pi/2)}_{1}} = \frac{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}}{-2} = \frac{\frac{\cancel{2}\sqrt{2}}{\cancel{2}}}{-2} =$

$= \boxed{-\frac{\sqrt{2}}{2}}$

(b) $\lim_{x\to\pi}\frac{\sin x}{1+\cos x} = \frac{\sin\pi}{1+\cos(\pi)} = \frac{0}{1-1} = \left[\frac{0}{0}\right] =$

$\underset{\text{Hôp.}}{=} \lim_{x\to\pi}\frac{\cos x}{-\sin x} = \frac{\cos(\pi)}{-\sin(\pi)} = \frac{-1}{0} = \infty = (*)$

De fet, $\sin x \to \begin{cases} 0^+ & x\to\pi^- \\ 0^- & x\to\pi^+\end{cases} \Rightarrow -\sin(x) \to \begin{cases}0^-, & x\to\pi^- \\ 0^+ & x\to\pi^+\end{cases}$

Així, $(*) = \begin{cases} \frac{-1}{0^-} = +\infty, & x\to\pi^- \\ \frac{-1}{0^+} = -\infty, & x\to\pi^+ \end{cases}$

(c) $\lim_{x\to+\infty}(\ln x)^{1/x} = \left[(+\infty)^0\right] \underset{\substack{\uparrow \\ e^y\text{ cont.}}}{=} e^{\lim_{x\to+\infty}\log(\log x)^{1/x}} =$

$= e^{\lim_{x\to+\infty}\frac{1}{x}\log(\log x)} = (*)$

$\lim_{x\to+\infty}\frac{1}{x}\log(\log(x)) = \left[\frac{+\infty}{+\infty}\right] \underset{\text{Hôp.}}{=} \lim_{x\to+\infty}\frac{\frac{1}{\log(x)}\cdot\frac{1}{x}}{1} =$

$= \lim_{x\to+\infty}\frac{1}{x\log(x)} = \frac{1}{+\infty} = \boxed{0} \Longrightarrow (*) = e^0 = \boxed{1}$

(d) $\lim_{x\to+\infty}(\ln x)^{e^{-x}} = \left[(+\infty)^0\right] \underset{e^y\text{ cont.}}{=} e^{\lim_{x\to+\infty} e^{-x}\log(\log x)} \underset{(*)}{=} e^0 = \boxed{1}$

$(*)\ \lim_{x\to+\infty} e^{-x}\log(\log x) = \lim_{x\to+\infty}\frac{\log(\log x)}{e^x} = \left[\frac{\infty}{\infty}\right] \underset{\text{Hôp.}}{=} \lim_{x\to+\infty}\frac{\frac{1}{\log x}\cdot\frac{1}{x}}{e^x} =$

$= \lim_{x\to+\infty}\frac{1}{xe^x\log x} = \frac{1}{+\infty} = 0$