PROVA 2 (GRUP MC) . ICD 2020 - 2021

1. Una succ. és acotada si $\exists M \in \mathbb{R}_+$ t.q. $|a_n| \leq M \ \forall n$.

No és cert, per exemple: $a_n = (-1)^n = \{-1, 1, -1, 1, -1, \dots, 1, \dots\}$

És acotada $|a_n| \leq 1 \leq 1$ i no té límit ja que

$$a_{2n} = 1 \longrightarrow 1 \quad \#$$
$$a_{2n+1} = -1 \longrightarrow -1$$

(el límit si $\exists$ és únic).

2. $\begin{cases} x_1 = 3 \\ x_{n+1} = \dfrac{4(x_n - 1)}{x_n}, \ n \geq 1 \end{cases}$ $\qquad$ $x_1 = 3$, $\quad x_2 = \dfrac{4 \cdot 2}{3} = \dfrac{8}{3} < 3 = x_1$

a) $x_n$ acotada? Veiem $2 \leq x_n \leq 3$ per inducció:

- $n = 1 \rightarrow 2 \leq x_1 = 3 \leq 3$ ✓
- H.I. $\rightarrow 2 \leq x_n \leq 3$.
- Cas $n+1$? $2 \underset{(i)}{\leq} x_{n+1} \underset{(ii)}{\leq} 3$ ?

(i) $\dfrac{4(x_n - 1)}{x_n} \geq 2 \underset{x_n \geq 2 > 0}{\iff} 4x_n - 4 \geq 2x_n \iff 2x_n \geq 4$
$\iff x_n \geq 2$ cert per H.I.

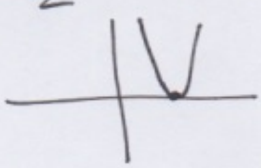
(ii) $\dfrac{4(x_n - 1)}{x_n} \leq 3 \underset{x_n \geq 2 > 0}{\iff} 4x_n - 4 \leq 3x_n \iff x_n \leq 4$
cert pq. $x_n \leq 3 \leq 4$ ✓

b) $x_n$ monòt.? Veiem decreix. i.e. $x_{n+1} \leq x_n$.

$$\iff \frac{4(x_n - 1)}{x_n} \leq x_n \underset{x_n \geq 2 > 0}{\iff} 4x_n - 4 \leq x_n^2$$

$$\iff x_n^2 - 4x_n + 4 \geq 0 \iff \text{cert per tot } x_n$$

$$x = \frac{4 \pm \sqrt{16 - 16}}{2} = 2 \qquad \Rightarrow x_n \downarrow .$$



c) $x_n$ monòt. + acotada $\Rightarrow$ convergent

$l = \lim x_n$ :

$$\underset{\downarrow\, l}{x_{n+1}} = \underset{\downarrow\, \frac{4(l-1)}{l}}{\frac{4(x_n-1)}{x_n}}$$

$$\Rightarrow l = \frac{4(l-1)}{l} \Leftrightarrow l^2 = 4l-4 \Leftrightarrow l^2-4l+4=0$$

$$\Leftrightarrow \boxed{l=2}$$

$$2 \leq \dots \leq x_n \leq \dots \leq 3$$

$x_n \downarrow$.

3. $$\lim_{n\to+\infty} \frac{1+2^{5/2}+\dots+n^{n+1/n}}{5+6^2+\dots+(n+4)^n} = \frac{a_n}{b_n}$$

$\cdot\, b_n \uparrow +\infty$. Fem Stolz:

$$\frac{a_n-a_{n-1}}{b_n-b_{n-1}} = \frac{n^{n+1/n}}{(n+4)^n} = \left(\frac{n}{n+4}\right)^n \cdot n^{1/n} \quad (*)$$

$$\cdot\ n^{1/n} = e^{\log n^{1/n}} = e^{\overset{\to 0}{\frac{1}{n}\log n}} \to e^0 = 1$$

$$\cdot\ \left(\frac{n}{n+4}\right)^n = \frac{1}{\left(\frac{n+4}{n}\right)^n} = \frac{1}{\left(1+\frac{4}{n}\right)^n} = \frac{1}{\underbrace{\left(1+\frac{1}{n/4}\right)^{n/4}}_{\to e\ (n/4\to+\infty)}{}^{4}} \to \frac{1}{e^4}$$

Així, $(*) \longrightarrow \frac{1}{e^4}\cdot 1 = \frac{1}{e^4}$ $\exists$.

$$\underset{\text{Stolz}}{\Longrightarrow} \boxed{\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{1}{e^4}}$$