5. $(x_n)_n : \begin{cases} x_1 = 2 \\ x_{n+1} = \dfrac{x_n^2+2}{2x_n}, \ n \geq 1 \end{cases}$.

(a) $x_n > 0$ i $x_n^2 \geq 2$ $\forall n$.

<u>Obs</u>: $x_{n+1} = \dfrac{x_n^2+2}{2x_n} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$

<u>Inducció</u>: . $n=1$: $x_1 = 2 > 0$ $\quad x_1^2 = 4 \geq 2$

$x_2 = \frac{2}{2} + \frac{1}{2} > 0$. $\quad x_2^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4} \geq 2$ ✓

· H.I. $x_n > 0$, $x_n^2 \geq 2$.

· x_{n+1}?

$x_{n+1} = \underbrace{\frac{x_n}{2}}_{>0} + \underbrace{\frac{1}{x_n}}_{>0} > 0$ ✓

$x_{n+1}^2 \geq 2 \iff |x_{n+1}| \geq \sqrt{2} \underset{x_n>0}{\iff} x_{n+1} \geq \sqrt{2}$.

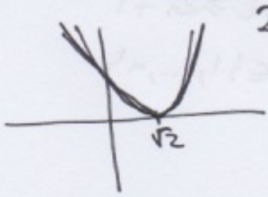
$\iff \dfrac{x_n}{2} + \dfrac{1}{x_n} \geq \sqrt{2} \iff$

$\iff x_n^2 + 2 \geq 2\sqrt{2}x_n$

$\iff x_n^2 - 2\sqrt{2}x_n + 2 \geq 0$.

$x^2 - 2\sqrt{2}x + 2 \geq 0$

$x = \dfrac{2\sqrt{2} \pm \sqrt{8-4\cdot 2 \cdot 1}}{2 \cdot 1} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$



$\Rightarrow x \in \mathbb{R}$ <u>OK</u>.

$\Rightarrow x_n^2 - 2\sqrt{2}x_n + 2 \geq 0$ sempre (de fet $x_n^2 - 2\sqrt{2}x_n + 2 = (x_n - \sqrt{2})^2 \geq 0$)

$\Rightarrow$ $x_n > 0$ i $x_n^2 \geq 2$ $(x_n \geq \sqrt{2})$.

(b) x_n monòt.

$x_1 = 2, \quad x_2 = \frac{3}{2} < x_1$

Si és monòt. $\Rightarrow$ serà decreix.

$x_{n+1} \leq x_n$? $\Leftrightarrow \frac{x_n^2+2}{2x_n} \leq x_n$? $\Leftrightarrow$

$$\underset{\substack{\uparrow \\ x_n > 0}}{\Leftrightarrow} \frac{x_n^2+2}{2x_n^2} \leq 1 \Leftrightarrow \frac{1}{2} + \frac{1}{x_n^2} \leq 1 \Leftrightarrow \frac{1}{x_n^2} \leq \underbrace{1 - \frac{1}{2}}_{1/2}$$

$\Leftrightarrow \frac{1}{x_n^2} \leq \frac{1}{2} \Leftrightarrow x_n^2 \geq 2$ VIST ABANS

$\Rightarrow x_n \downarrow$.

(c) És converg.? Límit?

$x_n \downarrow$ acot. inf. $\Rightarrow$ és converg.

$0 < \ldots < \sqrt{2} \leq \ldots \leq x_{n+1} \leq x_n \leq \ldots \leq x_1$

(De fet x_n és acotada $x_n \in [\sqrt{2}, x_1]$.)

Sigui l límit, $l = \frac{l^2+2}{2l}$

$$\Leftrightarrow 2l^2 = l^2 + 2 \Leftrightarrow l^2 = 2 \Leftrightarrow \begin{cases} l = \sqrt{2} \\ \text{ó} \\ l = -\sqrt{2} \end{cases}$$

Com $x_n \geq \sqrt{2} \Rightarrow l \neq -\sqrt{2}$

$$\Longrightarrow \boxed{l = \lim_n x_n = \sqrt{2}}$$

6.

(a) $\lim\limits_{n\to+\infty} \dfrac{1+\sqrt{2}+\sqrt[3]{3}+\ldots+\sqrt[n]{n}}{n}$

Aplicarem Criteri de Stolz:

$a_n = 1+\sqrt{2}+\sqrt[3]{3}+\ldots+\sqrt[n]{n}$

$b_n = n$

HIPÒTESIS: . $b_n = n$ estrict. creixent i $\lim\limits_{n\to+\infty} b_n = +\infty$

$(b_n = n < n+1 = b_{n+1}\ ,\ \lim\limits_{n\to+\infty} n = +\infty)$.

Calculem $\lim\limits_{n\to+\infty} \dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim\limits_{n\to+\infty} \dfrac{(1+\sqrt{2}+\ldots+\sqrt[n]{n}) - (1+\sqrt{2}+\ldots+\sqrt[n-1]{n-1})}{n-(n-1)} =$

$= \lim\limits_{n\to+\infty} \dfrac{\sqrt[n]{n}}{1} = \lim\limits_{n\to+\infty} \sqrt[n]{n} = \lim\limits_{n\to+\infty} e^{\log(\sqrt[n]{n})} =$

$\underset{\substack{\uparrow \\ e^x \text{ cont.}}}{=} e^{\lim\limits_{n\to+\infty} \log(n^{1/n})} = e^{\lim\limits_{n\to+\infty} \frac{1}{n}\log(n)} \underset{\substack{\uparrow \\ \log(x) \text{ creix} \\ \text{més lent que } x}}{=} e^0 = \boxed{1}$

Fet com a exemple de Stolz a apunts de teoria.

$\underset{\text{Stolz}}{\Longrightarrow} \boxed{\lim\limits_{n\to+\infty} \dfrac{a_n}{b_n} = 1}$

(b) $\lim\limits_{n\to+\infty} \dfrac{\log(n^n)}{\log(n!)} =$

$= \lim\limits_{n\to+\infty} \dfrac{n\log(n)}{\log(n(n-1)(n-2)\cdots 1)} = \lim\limits_{n\to+\infty} \dfrac{n\log n}{\sum_{k=1}^{n}\log k} = \lim\limits_{n\to+\infty} \dfrac{a_n}{b_n}$

Obs: . $b_n = \sum_{k=1}^{n}\log k \underset{\substack{\uparrow \\ \log x > 0,\ x>1}}{<} \sum_{k=1}^{n+1}\log k = b_{n+1} \Rightarrow b_n$ estrict. creixent

. $b_n \nearrow +\infty$ $(\log n \to +\infty)$

Podem aplicar Stolz:

$\lim\limits_{n\to+\infty} \dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim\limits_{n\to+\infty} \dfrac{n\log(n) - (n-1)\log(n-1)}{\sum_{k=1}^{n}\log k - \sum_{k=1}^{n-1}\log k} = \lim\limits_{n\to+\infty} \dfrac{n(\log n) - (n-1)\log(n-1)}{\log n}$

$= \underbrace{\lim\limits_{n\to+\infty} \dfrac{n\log\left(\frac{n}{n-1}\right)}{\log n}}_{i)} + \underbrace{\lim\limits_{n\to+\infty} \dfrac{\log(n-1)}{\log(n)}}_{ii)} = 0+1 = 1$

i) $\dfrac{n\log\left(\frac{n}{n-1}\right)}{\log n} = \dfrac{n}{n-1}\cdot\dfrac{n-1}{\log n}\log\left(\dfrac{n-1+1}{n-1}\right)$

$= \underbrace{\dfrac{n}{n-1}}_{\downarrow\, 1}\ \underbrace{\dfrac{1}{\log n}}_{\downarrow\, 0}\ \log\underbrace{\left(1+\dfrac{1}{n-1}\right)^{n-1}}_{\downarrow\, n\to+\infty\; e} \xrightarrow[n\to+\infty]{} 1\cdot 0\cdot\log(e) = 0.$

$\log n \to +\infty$

ii) $\dfrac{\log(n-1)}{\log(n)} \longrightarrow 1 \quad (n-1 \simeq n,\ n\to+\infty).$

$\underset{\text{Stolz}}{\Longrightarrow}\ \boxed{\lim_{n\to+\infty} \dfrac{a_n}{b_n} = 1}$

7. (a) $\displaystyle\lim_{n\to+\infty} \frac{\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}}{\sqrt{n}} = \lim_{n\to+\infty}\frac{a_n}{b_n}$

<u>Obs</u>: $b_n=\sqrt{n} \longrightarrow +\infty$ i és estrict. creixent ja que $b_n=\sqrt{n} < \sqrt{n+1} = b_{n+1}$.

Aplicarem Stolz:

$\displaystyle\lim_{n\to+\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}} = \lim_{n\to+\infty}\frac{\frac{1}{\sqrt{n}}}{\sqrt{n}-\sqrt{n-1}} = \lim_{n\to+\infty}\frac{1}{\sqrt{n}(\sqrt{n}-\sqrt{n-1})}$

$\displaystyle = \lim_{n\to+\infty}\frac{1}{\sqrt{n}}\cdot\frac{(\sqrt{n}+\sqrt{n-1})}{(\sqrt{n}-\sqrt{n-1})(\sqrt{n}+\sqrt{n-1})} = \lim_{n\to+\infty}\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}(n-(n-1))}$

$\displaystyle = \lim_{n\to+\infty}\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}} = \lim_{n\to+\infty} 1+\underbrace{\sqrt{\underbrace{\frac{n+1}{n}}_{\downarrow\,1}}}_{\downarrow\,\sqrt{1}=1} = 2$

$\underset{\text{Stolz}}{\Longrightarrow}\ \boxed{\lim_{n\to+\infty}\dfrac{a_n}{b_n} = 2}$

(b) $\lim_{n\to+\infty} \dfrac{2^2+3^4+\ldots+(n+1)^{2n}}{2^1+5^2+\ldots+(n^2+1)^n} = \lim_{n\to+\infty} \dfrac{a_n}{b_n}$

- $b_n = \sum_{k=1}^{n} (k^2+1)^k \underset{\uparrow}{<} \sum_{k=1}^{n+1} (k^2+1)^k = b_{n+1} \Rightarrow b_n$ estrict. creix.

$k=n+1 \to ((n+1)^2+1)^{n+1} > 0$

i $\lim_{n\to+\infty} b_n = +\infty$.

Aplicarem Stolz:

$$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \frac{(n+1)^{2n}}{(n^2+1)^n} = \left[\frac{(n+1)^2}{n^2+1}\right]^n =$$

$$= \left[\frac{(n^2+1)+2n}{n^2+1}\right]^n = \left[1+\frac{2n}{n^2+1}\right]^n =$$

$$= \left[1+\frac{1}{\frac{n^2+1}{2n}}\right]^n = \left[1+\frac{1}{\left(\frac{n^2+1}{2n}\right)}\right]^{\left(\frac{n^2+1}{2n}\right)\cdot\frac{2n}{n^2+1}\cdot n}$$

$$= \underbrace{\left[\left[1+\frac{1}{\frac{n^2+1}{2n}}\right]^{\frac{n^2+1}{2n}}\right]}_{\downarrow (*)\ e}^{\underbrace{\frac{2n^2}{n^2+1}}_{\downarrow\ 2}} \xrightarrow[n\to+\infty]{} e^2$$

$$\underset{\text{Stolz}}{\Longrightarrow} \boxed{\lim_{n\to+\infty} \frac{a_n}{b_n} = e^2}$$

(*) $\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e$

$\lim_{n\to+\infty} \left(1+\frac{1}{a_n}\right)^{a_n} = e$ si $\lim_{n\to+\infty} a_n = +\infty$

$a_n = \dfrac{n^2+1}{2n} \longrightarrow +\infty$.

(c) $\lim\limits_{n\to+\infty} \dfrac{\sin(n^2)\cdot\cos(n^2)}{\sqrt{n}}$

<u>Obs</u>: $\underbrace{0}_{\substack{\downarrow n\to+\infty \\ 0}} \le \left| \dfrac{\sin(n^2)\cdot\cos(n^2)}{\sqrt{n}} \right| = \dfrac{|\sin(n^2)\cos(n^2)|}{\sqrt{n}} \underset{\uparrow}{\le} \underbrace{\dfrac{1}{\sqrt{n}}}_{\substack{\downarrow n\to+\infty \\ 0}}$

$-1 \le \sin(x) \le 1$
$-1 \le \cos(x) \le 1$

$$\Longrightarrow \lim_{n\to+\infty} \left| \frac{\sin(n^2)\cdot\cos(n^2)}{\sqrt{n}} \right| = 0$$

$$\underset{\uparrow}{\Longrightarrow} \boxed{\lim_{n\to+\infty} \frac{\sin(n^2)\cdot\cos(n^2)}{\sqrt{n}} = 0}$$

$|a_n| \to 0 \iff a_n \to 0$.

<u>Dem</u>: $\forall \varepsilon > 0 \ \exists n_0 : \forall n \ge n_0 \ \underbrace{\big||a_n| - 0\big|}_{\substack{\shortparallel \\ ||a_n|| = |a_n| = |a_n - 0|}} < \varepsilon$

(d) $\lim\limits_{n\to+\infty} \sqrt[n+4]{(n+1)(n+2)\cdot\ldots\cdot(n+n)} = e^{\lim\limits_{n\to+\infty} \log \sqrt[n+4]{(n+1)(n+2)\cdot\ldots\cdot(n+n)}}$

$$\log \sqrt[n+4]{(n+1)(n+2)\cdot\ldots\cdot(n+n)} = \frac{1}{n+4} \log\big((n+1)(n+2)\cdot\ldots\cdot(n+n)\big)$$

$$= \frac{1}{n+4} \sum_{k=1}^{n} \underbrace{\log(n+k)}_{\substack{\vee\!/ \\ \log(n+1)}} \geqslant \frac{1}{n+4} \sum_{k=1}^{n} \underbrace{\log(n+1)}_{\substack{\text{no depèn} \\ \text{de } k \text{ !!}}} =$$

$$= \underbrace{\left(\frac{1}{n+4}\cdot n\right)}_{\downarrow\, 1} \underbrace{\log(n+1)}_{\downarrow\, +\infty} \longrightarrow +\infty$$

$$\Longrightarrow \lim_{n\to+\infty} \log \sqrt[n+4]{(n+1)\cdot\ldots\cdot(n+n)} = +\infty$$

$$\Longrightarrow \lim_{n\to+\infty} \sqrt[n+4]{(n+1)(n+2)\cdot\ldots\cdot(n+n)} = e^{+\infty} = \boxed{+\infty}$$