LLISTA 3. ICD 2020-2021

1. a) $\lim_{x\to 1} \frac{x^2+3x-4}{\sqrt{3x+2}-\sqrt{x+4}} = \frac{1+3-4}{\sqrt{5}-\sqrt{5}} = \left[\frac{0}{0}\right] =$

$$= \lim_{x\to 1} \frac{(x^2+3x-4)(\sqrt{3x+2}+\sqrt{x+4})}{(\sqrt{3x+2}-\sqrt{x+4})\cdot(\sqrt{3x+2}+\sqrt{x+4})} =$$

$$\overset{\uparrow}{=} \lim_{x\to 1} \frac{(x^2+3x-4)(\sqrt{3x+2}+\sqrt{x+4})}{3x+2-(x+4)} \overset{(*)}{=} \cancel{\frac{(1+3-4)(\sqrt{5}+\sqrt{5})}{0}}$$

$3x+2-(x+4) = 2x-2 = 2\underbrace{(x-1)}_{\downarrow x\to 1,\ 0}$

$(a-b)(a+b) =$
$= a^2-b^2$

$$x^2+3x-4=0 \iff x = \frac{-3\pm\sqrt{9+16}}{2} = \frac{-3\pm 5}{2} \begin{cases} 1 \\ -4 \end{cases}$$

$$(*) = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+4)(\sqrt{3x+2}+\sqrt{x+4})}{2\cancel{(x-1)}} = \frac{(1+4)(\sqrt{5}+\sqrt{5})}{2}$$

$$= \frac{5\cdot 2\sqrt{5}}{2} = \boxed{5\sqrt{5}}$$

b) $\lim_{x\to -\infty} \frac{e^x+\log(1+x^2)}{\sqrt[3]{x^2}}$

$$\frac{e^x \xrightarrow[x\to-\infty]{} 0}{x^{2/3} \to +\infty} = \underbrace{\frac{1}{x^{2/3}}}_{\downarrow\, 0} \underbrace{e^{x}}_{\downarrow\, 0} \longrightarrow 0\cdot 0 = 0 \quad \left(\exists \lim_{x\to-\infty} \frac{e^x}{x^{2/3}}\right)$$

$$\frac{\log(1+x^2)}{x^{2/3}} = \frac{\log(x^2(1+1/x^2))}{x^{2/3}} = \frac{\log x^2}{x^{2/3}} + \frac{\log(1+1/x^2)}{x^{2/3}}$$

$$= 2\underbrace{\frac{\log|x|}{x^{2/3}}}_{\downarrow x\to-\infty \text{ (Teoria) } 0} + \underbrace{\frac{1}{x^{2/3}}}_{\downarrow\, 0} \underbrace{\log(\underbrace{1+\underbrace{1/x^2}_{\downarrow x\to-\infty\, 0}}_{\downarrow\, 1})}_{\downarrow\, 0} \longrightarrow 0 \quad \left(\exists \lim_{x\to-\infty} \frac{\log(1+x^2)}{x^{2/3}}\right)$$

y=x^2
log(y)/y^(1/3)—>0
y—>+infinit

$$\Rightarrow \lim_{x\to-\infty} \frac{e^x+\log(1+x^2)}{x^{2/3}} = \lim_{x\to-\infty}\frac{e^x}{x^{2/3}} + \lim_{x\to-\infty}\frac{\log(1+x^2)}{x^{2/3}}$$

$$= 0+0 = \boxed{0}.$$

2. $$\lim_{x\to 0} \frac{\sqrt{1+\sin^m x}-\sqrt{1-\sin^m x}}{x^n} \quad \forall m,n\in\mathbb{N}.$$

$$\left.\begin{array}{l} \sin x \xrightarrow[x\to 0]{} 0 \\ m\in\mathbb{N} \Rightarrow m>0 \end{array}\right\} \Rightarrow \sin^m x \xrightarrow[x\to 0]{} 0$$

$$\sqrt{1+\sin^m x}-\sqrt{1-\sin^m x} \longrightarrow \sqrt{1}-\sqrt{1}=0$$

$$x^n \xrightarrow[\substack{x\to 0\\ n>0}]{} 0.$$

$$\lim_{x\to 0}\frac{\sqrt{1+\sin^m x}-\sqrt{1+\sin^m x}}{x^n} = \lim_{x\to 0}\frac{1+\sin^m x-(1-\sin^m x)}{x^n\left(\sqrt{1+\sin^m x}+\sqrt{1-\sin^m x}\right)}$$

$$=\lim_{x\to 0}\frac{2\sin^m x}{x^n\underbrace{\left(\sqrt{1+\sin^m x}+\sqrt{1-\sin^m x}\right)}_{\downarrow \atop \sqrt{1}+\sqrt{1}=2}} = (*)$$

$$\frac{\sin^m x}{x^n} = \frac{\sin^m x}{x^m}\cdot\frac{x^m}{x^n} = \Bigg(\underbrace{\frac{\sin x}{x}}_{\substack{\downarrow x\to 0\\ 1\ \text{(TEORIA)}}}\Bigg)^m \cdot \underbrace{x^{m-n}}_{\substack{\downarrow x\to 0}} \longrightarrow \begin{cases} 0, & m>n\\ 1, & m=n\\ \infty, & m<n\end{cases}$$

$$1^m = 1 \qquad\qquad \begin{cases} 0 & m-n>0\\ 1 & m-n=0\\ \infty & m-n<0\end{cases}$$

$$(*)=\begin{cases} 2\cdot 0\cdot\frac{1}{2}=0, & m>n\\[2mm] 2\cdot 1\,\frac{1}{2}=1, & m=n\\[2mm] 2\cdot\infty\,\frac{1}{2}=\infty, & m<n\end{cases}$$

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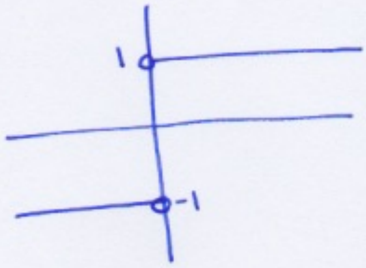
3.

(a) $f(x) = \dfrac{x}{|x|}$.

Volem trobar possible valor $f(0)$ de manera que f sigui contínua:

$$f(x) = \frac{x}{|x|} = \begin{cases} \dfrac{x}{x} = 1, & x > 0 \\ \dfrac{x}{-x} = -1, & x < 0 \end{cases}$$



$\lim_{x \to 0^-} f(x) = -1 \neq \lim_{x \to 0^+} f(x) \Rightarrow$ No és possible trobar $f(0)$ ja que els límits laterals són sempre diferents i f no podrà ser contínua al 0.

(b) $g(x) = x^2\left(1 - \dfrac{1}{x^2}\right) = x^2 - \dfrac{x^2}{x^2} = x^2 - 1$

$\lim_{x \to 0} x^2 - 1 = -1 \Rightarrow \exists \lim_{x \to 0} (x^2-1) = -1 =: g(0)$

Així, si definim $g(x) = \begin{cases} x^2(1 - 1/x^2), & x \neq 0 \\ -1, & x = 0 \end{cases}$

és una funció cont.

(c) $h(x) = x\left(1 + \dfrac{1}{\sqrt{|x|}}\right) = x + \dfrac{x}{\sqrt{|x|}} = x + \dfrac{x\sqrt{|x|}}{|x|} =$

$= x + \text{signe}(x)\sqrt{|x|} = \begin{cases} x + \sqrt{x}, & x > 0 \\ x - \sqrt{(-x)}, & x < 0 \end{cases}$

$\dfrac{x}{|x|} = \begin{cases} \dfrac{x}{x}, & x > 0 \\ \dfrac{x}{-x}, & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} = \text{signe}(x)$

$\lim_{x \to 0^-} h(x) = \lim_{\substack{x \to 0 \\ x < 0}} x - \sqrt{(-x)} = 0 - \sqrt{0} = 0$

$\lim_{x \to 0^+} h(x) = \lim_{\substack{x \to 0 \\ x > 0}} x + \sqrt{x} = 0 + \sqrt{0} = 0$

$\Rightarrow \exists \lim_{x \to 0} h(x) = 0$

Definim $\boxed{h(0) = 0}$

$\Rightarrow h$ cont.

4. $a>0$, $b\in\mathbb{R}$. $f:\mathbb{R}\to\mathbb{R}$

$$f(x)=\begin{cases} |x|, & x\le 0 \\ a x\log x, & x\in(0,1) \\ |x-b|, & x\ge 1 \end{cases}$$

f cont?

· Si $x<0$, $f(x)=|x|$ cont. $\Rightarrow f$ cont. a $(-\infty,0)$

· Si $x\in(0,1)$, $f(x)=\underbrace{ax}_{\text{cont.}}\underbrace{\log x}_{\text{cont.}}$ cont. $\Rightarrow f$ cont. a $(0,1)$

· Si $x>1$, $f(x)=|x-b|$ cont. $\Rightarrow f$ cont. a $(1,+\infty)$.

· Cal veure la continuïtat en $x=0$ i $x=1$

$\boxed{x=0}$

$$\left.\begin{array}{l} \lim\limits_{x\to 0^-} f(x)=\lim\limits_{\substack{x\to 0\\ x<0}} |x|=0 \\ \lim\limits_{x\to 0^+} f(x)=\lim\limits_{\substack{x\to 0\\ x>0}} \underbrace{\underbrace{ax}_{\to 0}\underbrace{\log x}_{\to -\infty}}_{\substack{\downarrow \text{ VISTA A TEORIA}\\ 0}}=0 \end{array}\right\}\Rightarrow \exists \lim_{x\to 0} f(x)=0$$

$\left[\, x\log x=\dfrac{\log x}{1/x}=-\dfrac{\log 1/x}{1/x} \quad \lim\limits_{x\to 0} -x\log x=\lim\limits_{y\to\infty} -\dfrac{\log y}{y}=0 \,\right]$

$f(0)\underset{\substack{\text{per}\\ \text{definició}}}{=}|0|=0.$

$\Rightarrow \exists \lim\limits_{x\to 0} f(x)=0=f(0) \Rightarrow f$ cont a $x=0$ $\forall a>0$ $\forall b\in\mathbb{R}$.

$\boxed{x=1}$

$$\left.\begin{array}{l} \lim\limits_{x\to 1^+} f(x)=\lim\limits_{\substack{x\to 1\\ x>1}} |x-b|=|1-b| \\ \lim\limits_{x\to 1^-} f(x)=\lim\limits_{\substack{x\to 1\\ x<1}} ax\log x=a\cdot 1\log 1=a\cdot 0=0 \\ f(1)=|1-b| \end{array}\right\}\Rightarrow$$

$\Rightarrow$ Necessitem: $|1-b|=0 \Leftrightarrow 1-b=0 \Leftrightarrow b=1$.

Així, si $b=1 \Rightarrow \exists \lim\limits_{x\to 1} f(x)=0=f(1) \Rightarrow f$ cont. a $x=1$.

$\boxed{f \text{ contínua (a } \mathbb{R}) \Leftrightarrow a>0,\ b=1}$