Problema 12

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ \frac{x^2}{2} & 0 \le x < 1 \\ 1 - \frac{(2-x)^2}{2} & 1 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

a) X és absolutamente continua

$$0 = \boxed{F(0^-) = F(0)} = \frac{0^2}{2} = 0$$

$$\frac{1^2}{2} = \boxed{F(1^-) = F(1)} = 1 - \frac{(2-1)^2}{2} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$1 - \frac{(2-2)^2}{2} = \boxed{F(2^-) = F(2)} = 1$$

b) $f(x) = F'(x)$

$$f(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ -\frac{2(2-x)}{2}(-1) = (2-x) & 1 \le x < 2 \\ 0 & x \ge 2 \end{cases}$$

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$$f(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ -\dfrac{2(2-x)}{2}(-1) = (2-x) & 1 \leq x < 2 \\ 0 & x \geq 2 \end{cases}$$

$$P(X=1) = P(X \leq 1) - P(X<1) = F(1) - F(1^-) = \boxed{0}$$

$$P(X \in (1,2]) = P(1 < X \leq 2) = F(2) - F(1) = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

Altra forma:

$$P(1 < X \leq 2) = \int_1^2 (2-x)\,dx =$$

$$= 2x\Big]_1^2 - \frac{x^2}{2}\Big]_1^2$$

$$= 2(2-1) - \left(\frac{2^2}{2} - \frac{1}{2}\right)$$

$$= 2 - \left(2 - \frac{1}{2}\right) = \frac{1}{2}$$

$$P\left(X \in \left[\frac{1}{2}, \frac{3}{2}\right]\right) = P\left(\frac{1}{2} \leq X \leq \frac{3}{2}\right)$$

$$= F\left(\frac{3}{2}\right) - F\left(\frac{1}{2}\right) =$$

$$= 1 - \frac{\left(2-\frac{3}{2}\right)^2}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} = 1 - \frac{1}{8} - \frac{1}{8} = \boxed{\frac{3}{4}}$$

Altra forma:

$$P\left(\frac{1}{2} \leq X \leq \frac{3}{2}\right) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(x)\,dx$$

$$= \int_{\frac{1}{2}}^{1} x\,dx + \int_{1}^{3/2} (2-x)\,dx = \frac{3}{4}$$

b) $P(X \in [1,\infty) \mid X \in (0, \frac{3}{2}])$

$= P(X \geq 1 \mid 0 < X \leq \frac{3}{2})$

$= \dfrac{P(X \geq 1,\ 0 < X \leq \frac{3}{2})}{P(0 < X \leq \frac{3}{2})} = \dfrac{P(1 \leq X \leq \frac{3}{2})}{P(0 < X \leq \frac{3}{2})}$

$= \dfrac{F(\frac{3}{2}) - F(1)}{F(\frac{3}{2}) - F(0)} = \dfrac{\frac{7}{8} - \frac{1}{2}}{\frac{7}{8} - 0} = \boxed{\frac{3}{7}}$

$F(\frac{3}{2}) = 1 - \dfrac{(2 - \frac{3}{2})^2}{2} = 1 - \frac{1}{8} = \frac{7}{8}$

$F(1) = 1 - \dfrac{(2-1)^2}{2} = \frac{1}{2}$

$F(0) = \dfrac{0^2}{2} = 0$

c) $E(X) = \displaystyle\int_{-\infty}^{\infty} x f(x)\, dx$

$= \displaystyle\int_0^1 x \cdot x\, dx + \int_1^2 x \cdot (2-x)\, dx = \int_0^1 x^2 dx + \int_1^2 (2x - x^2) dx$

$= \dfrac{x^3}{3}\Big]_0^1 + 2 \cdot \dfrac{x^2}{2}\Big]_1^2 - \dfrac{x^3}{3}\Big]_1^2$

$= \frac{1}{3} + (2^2 - 1^2) - \left(\frac{2^3}{3} - \frac{1^3}{3}\right) = \frac{1}{3} + 3 - \frac{7}{3} = \boxed{1}$