LAB. 3 ICD 2020-2021

1. $(x_n)_n \ : \ \begin{cases} x_1 = 1/2 \\ x_{n+1} = \dfrac{2}{3-x_n}, \ n \geq 1 \end{cases}$

a) $0 < x_n \leq 1$.

INDUCCIÓ:

- Cas inicial: $0 < x_1 = 1/2 \leq 1$ ✓
- H.I.: $x_n \in (0,1]$.
- Cas $n+1$? $x_{n+1} \in (0,1]$?

$0 < x_n \leq 1 \iff -1 \leq -x_n < 0 \iff 2 \leq 3 - x_n < 3$

$\iff \frac{1}{3} < \frac{1}{3-x_n} \leq \frac{1}{2} \iff \frac{2}{3} < \underbrace{\frac{2}{3-x_n}}_{x_{n+1}} \leq \frac{2}{2} = 1$

$\iff 0 < \frac{2}{3} < x_{n+1} \leq 1 \Rightarrow 0 < x_{n+1} \leq 1$ ✓

Alternativa: · $x_{n+1} = \dfrac{2}{\underbrace{3-x_n}_{>0 \Leftarrow x_n \leq 1}} \leq 1 \iff 2 \leq 3 - x_n \iff x_n \leq 1$ cert per H.I.

· $x_{n+1} = \dfrac{2}{3-x_n} > 0$ ja que $2 > 0$ i $3 - x_n \underset{\text{H.I.}}{\geq} 2 > 0$

b) x_n monòt. i límit?

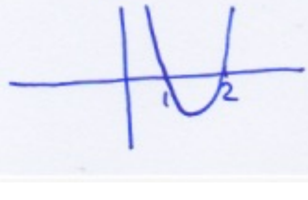
$x_1 = 1/2, \quad x_2 = \dfrac{2}{3 - 1/2} = \dfrac{2}{5/2} = \dfrac{4}{5} > 1/2$

Si és monòtona, serà creixent! Provem-ho:

$x_n \leq x_{n+1} = \dfrac{2}{\underbrace{3-x_n}_{\geq 2 > 0}} \iff (3 - x_n)x_n \leq 2$

$\iff 3x_n - x_n^2 \leq 2 \iff x_n^2 - 3x_n + 2 \geq 0 \iff x_n \in (-\infty, 1] \cup [2, +\infty)$

$x = \dfrac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot 2}}{2} = \dfrac{3 \pm 1}{2} \begin{cases} 2 \\ 1 \end{cases}$



Com $0 < x_n \leq 1 \Rightarrow x_n \in (-\infty, 1]$

$\Rightarrow x_n^2 - 3x_n + 2 \geq 0 \iff$

$\iff x_n \leq x_{n+1} \Rightarrow x_n \uparrow$

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ALTERNATIVA: Podem fer inducció també.

Calculem límit: com $x_n \uparrow$ i està acotada

$\Rightarrow x_n$ convergent. Sigui $l = \lim_n x_n = \lim_n x_{n+1}$

Aleshores:
$$\underbrace{x_{n+1}}_{\downarrow\ l} = \underbrace{\frac{2}{3-x_n}}_{\downarrow\ \frac{2}{3-l}} \Rightarrow l = \frac{2}{3-l}$$

$$\Leftrightarrow l(3-l) = 2 \Leftrightarrow l^2 - 3l + 2 = 0 \Leftrightarrow \begin{cases} l = 1 \\ \text{ó} \\ l = 2 \end{cases}$$

Com el límit és únic i $0 < x_n \leq 1$ ($x_n \uparrow$, $x_1 = 1/2$)

$$\Rightarrow \boxed{l = 1}$$

2. (a) $$\lim_{n\to+\infty} \frac{1+2+\ldots+n}{n^2} = \lim_{n\to+\infty} \frac{a_n}{b_n}$$

- $b_n = n^2 \uparrow$ i $\lim_{n\to+\infty} n^2 = +\infty$.

Aplicarem Stolz:
$$\lim_{n\to+\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to+\infty} \frac{n}{n^2-(n-1)^2} = \lim_{n\to+\infty} \frac{n}{n^2-(n^2+1-2n)}$$

$$= \lim_{n\to+\infty} \frac{n}{2n-1} = \boxed{\frac{1}{2}} \underset{\text{Stolz}}{\Longrightarrow} \boxed{\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{1}{2}}$$

(b) $$\lim_{n\to+\infty} \left(\frac{3^1+4^2+\ldots+(n+2)^n}{1^1+2^2+\ldots+n^n} \right)^2 = \left(\lim_{n\to+\infty} \frac{a_n}{b_n} \right)^2 = L$$

x^2 cont.

$b_n = (1^1+2^2+\ldots+n^n) \uparrow$ i tendeix a $+\infty$.

$$\lim_{n\to+\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to+\infty} \frac{(n+2)^n}{n^n} = \lim_{n\to+\infty} \left(\frac{n+2}{n}\right)^n = \lim_{n\to+\infty} \left(1+\frac{2}{n}\right)^n$$

$$= \lim_{n\to+\infty} \underbrace{\left(1+\frac{1}{\underset{\downarrow +\infty}{(n/2)}}\right)^{\frac{n}{2}}}_{\downarrow\ e}{}^{\cdot 2} = e^2 \Rightarrow \lim_{n\to+\infty} \frac{a_n}{b_n} = e^2$$

$$\Rightarrow L = (e^2)^2 = \boxed{e^4.}$$