PROVA 2 (GRUP MA). ICD 2020-2021

1. Una succ. és monòtona si $\forall n \begin{cases} x_n \leq x_{n+1} & \text{(creixent)} \\ \text{ó} & \\ x_n \geq x_{n+1} & \text{(decreixent)} \end{cases}$

2. No, per exemple: $x_n = n$: $x_n = n \leq n+1 = x_{n+1}$

$\Rightarrow x_n \uparrow$ i $\lim_{n\to+\infty} x_n = +\infty$.

2. $\begin{cases} x_1 = 2 \\ x_{n+1} = \dfrac{3+x_n^2}{4}, \quad n \geq 1. \end{cases}$

$x_1 = 2, \quad x_2 = \dfrac{3+4}{4} = \dfrac{7}{4} < 2 = x_1$

Veiem que $1 \leq x_n \leq 2$ per inducció:

- $n=1 \rightarrow 1 \leq \underset{x_1}{\underset{\shortparallel}{2}} \leq 2$ ✓.
- H.I. $1 \leq x_n \leq 2$
- Cas $n+1$? • $1 \leq x_{n+1} \leq 2$? $\Leftrightarrow 1 \leq \dfrac{3+x_n^2}{4} \leq 2$

$\Leftrightarrow 4 \leq 3 + x_n^2 \leq 8 \Leftrightarrow 1 \leq x_n^2 \leq 5$ cert.

Com $x_n \geq 1 \Rightarrow x_n^2 \geq 1$ ✓.

Com $x_n \leq 2 \underset{(x_n>0)}{\Rightarrow} x_n^2 \leq 4 \leq 5$ ✓.

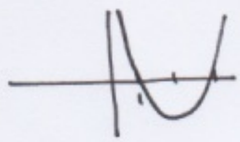
$\Rightarrow$ Així, $1 \leq x_n \leq 2 \quad \forall n$, és a dir, x_n acotada.

Veiem $x_n \downarrow$, és a dir, $x_{n+1} \leq x_n$. $\Leftrightarrow$

$\Leftrightarrow \dfrac{3+x_n^2}{4} \leq x_n \Leftrightarrow 3 + x_n^2 \leq 4x_n \Leftrightarrow 3 + x_n^2 - 4x_n \leq 0$

$\Leftrightarrow x_n \in [1,3]$

$x = \dfrac{4 \pm \sqrt{16 - 4\cdot 3\cdot 1}}{2} = \dfrac{4 \pm 2}{2} \begin{cases} 3 \\ 1 \end{cases}$



Com hem vist $x_n \in [1,2] \subset [1,3] \Rightarrow x_{n+1} \leq x_n$

i, per tant, $x_n \downarrow$

Com $x_n \downarrow$ i acotada $\Rightarrow x_n$ convergent.

Sigui $l = \lim_n x_n$, aleshores $\underset{\displaystyle l}{\underset{\downarrow}{x_{n+1}}} = \underset{\displaystyle \frac{3+l^2}{4}}{\underset{\downarrow}{\frac{3+x_n^2}{4}}}$

$$\Rightarrow l = \frac{3+l^2}{4} \Leftrightarrow l^2 - 4l + 3 = 0 \begin{cases} l = 1 \\ l = 3. \end{cases}$$

Com $\left.\begin{array}{l} 1 \leq \ldots \leq x_n \leq \ldots \leq x_1 = 2 \\ x_n \downarrow \end{array}\right\} \Rightarrow l \neq 3$

$$\Rightarrow \boxed{l = 1}$$

3. $$\lim_{n\to+\infty} \left(\frac{n!}{1+2!+\ldots+n!}\right)^{1/n} = \lim_{n\to+\infty} e^{\frac{1}{n}\log\left(\underbrace{\frac{n!}{1+2!+\ldots+n!}}_{\frac{a_n}{b_n}}\right)} = (*)$$

$\cdot\, b_n = 1+2!+\ldots+n! \leq 1!+2!+\ldots+n! + \underbrace{(n+1)!}_{\geq 0} = b_{n+1}$

$\Rightarrow b_n \uparrow$ i $b_n \to +\infty$.

Stolz: $$\frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \frac{n! - (n-1)!}{n!} = 1 - \frac{(n-1)!}{n\cdot(n-1)!} = 1 - \frac{1}{n} \xrightarrow[n\to+\infty]{} 1.$$

$\underset{\exists \text{ limit.}}{\Rightarrow} \lim_{n\to+\infty} \frac{a_n}{b_n} = 1 \underset{\text{log és cont.}}{\Rightarrow} \lim_{n\to+\infty} \log\left(\frac{a_n}{b_n}\right) = \log 1 = 0$

$$\underbrace{\frac{1}{n}}_{\to 0}\underbrace{\log\left(\frac{a_n}{b_n}\right)}_{\to 0} \longrightarrow 0 \quad \Rightarrow (*) = e^0 = \boxed{1}$$