LAB. 9 ICD 2020-2021

1.

(a) $\underbrace{\frac{1}{e^x}}_{= e^{-x}} \geq 1-x, \quad x \in \mathbb{R}$

V.v. $\forall x \in \mathbb{R} \quad e^{-x} \geq 1-x \iff e^{-x}-1+x \geq 0 \quad \forall x \in \mathbb{R}$

$f(x) = e^{-x} \to f(0) = 1$

$f'(x) = -e^{-x} \to f'(0) = -1$

$f''(x) = e^{-x}$

$\underset{\substack{\text{Taylor} \\ a=0}}{\Longrightarrow} e^{-x} = f(x) = 1 - x + \frac{e^{-c}}{2!}x^2$

$c \in \langle 0, x \rangle$

$$\Longrightarrow e^{-x} - 1 + x = \frac{\overset{>0}{e^{-c}} \overset{\geq 0}{x^2}}{\underset{>0}{2}} \geq 0 \quad \forall x \in \mathbb{R} \qquad \#.$$

(b) $x^2 - \frac{x^4}{2} \underset{(1)}{\leq} \underbrace{\ln(1+x^2)}_{f(x)} \underset{(2)}{\leq} x^2$

$g(t) = \ln(1+t) \to g(0) = \ln(1) = 0$

$g'(t) = \frac{1}{1+t} \to g'(0) = 1$

$g''(t) = \frac{-1}{(1+t)^2} \to g''(0) = -1$

$g'''(t) = \frac{2}{(1+t)^3} \to g'''(0) = 2$

$\Longrightarrow g(t) = \ln(1+t) = 0 + t + \frac{(-1)}{2!}t^2 + \frac{2}{3!(1+c)^3}t^3, \quad c \in \langle 0, t \rangle$

(grau 2) $a=0$

$= t - \frac{t^2}{2} + \frac{1}{3(1+c)^3}t^3, \quad c \in \langle 0, t \rangle$

També: $g(t) = \ln(1+t) = 0 + t - \frac{1}{2(1+c)^2}t^2, \quad c \in \langle 0, t \rangle$

(grau 1)

$= t - \frac{1}{2(1+c)^2}t^2, \quad c \in \langle 0, t \rangle$

Fent servir polin. de grau 2 de $g(t)$ veurem (1):

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{1}{3(1+c)^3} x^6 \geq x^2 - \frac{x^4}{2}$$

$c \in \langle 0, x^2 \rangle = (0, x^2)$

$\Rightarrow c > 0$

Fent servir polin. de grau 1 de $g(t)$ veurem (2):

$$\ln(1+x^2) = x^2 - \frac{1}{2(1+c)^2} x^4 \leq x^2$$

$c \in (0, x^2) \Rightarrow c > 0$

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(c) $\underbrace{\sqrt[5]{1+x}}_{f(x)} \leq 1 + \frac{x}{5} - \frac{2x^2}{25} + \frac{6x^3}{125}$, $x > -1$.

$f(x) = (1+x)^{1/5} \rightarrow f(0) = 1$

$f'(x) = \frac{1}{5}(1+x)^{-4/5} \rightarrow f'(0) = 1/5$

$f''(x) = \frac{-4}{25}(1+x)^{-9/5} \rightarrow f''(0) = \frac{-4}{25}$

$f'''(x) = \frac{36}{125}(1+x)^{-14/5} \rightarrow f'''(0) = \frac{36}{125}$

$f^{(4)}(x) = \frac{-36 \cdot 14}{5 \cdot 125}(1+x)^{-19/5}$, $f^{(4)}(0) = \frac{-36 \cdot 14}{5 \cdot 125}$

$$f(x) = \sqrt[5]{1+x} = 1 + \frac{1}{5}x - \frac{4}{25 \cdot 2!}x^2 + \frac{36}{3! \cdot 125}x^3 - \underbrace{\frac{36 \cdot 14 (1+c)^{-\frac{19}{5}}}{4! \cdot 5 \cdot 125} x^4}_{\geq 0}$$

$c \in \langle 0, x \rangle$

$x > -1$

$\Rightarrow c > -1 \rightarrow 1 + c > 0$

$$\leq 1 + \frac{x}{5} - \frac{2}{25}x^2 + \frac{6}{125}x^3 \quad \forall x > -1$$

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2. (a) $\lim_{x\to 0} \dfrac{\left(\sqrt[3]{1+x} - 1 - \frac{x}{3} + \frac{x^2}{9}\right)^2}{(x-\sin x)^{2n}} = L$

$f(x) = \sqrt[3]{1+x} = (1+x)^{1/3} \longrightarrow f(0) = 1$

$f'(x) = \frac{1}{3}(1+x)^{-2/3} \longrightarrow f'(0) = \frac{1}{3}$

$f''(x) = \frac{-2}{9}(1+x)^{-5/3} \longrightarrow f''(0) = -\frac{2}{9}$

$f'''(x) = \frac{10}{27}(1+x)^{-8/3} \longrightarrow f'''(0) = \frac{10}{27}$

$\sqrt[3]{1+x} = 1 + \frac{x}{3} - \frac{2}{9\cdot 2!}x^2 + \frac{10}{27 \underbrace{3!}_{=3\cdot 2}}x^3 + o(x^3)$

$= 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5}{81}x^3 + o(x^3)$

$g(x) = \sin x \implies x - \frac{x^3}{6} + o(x^3)$

$\Rightarrow x - \sin x = \frac{x^3}{6} + o(x^3)$

Així,

$L = \lim_{x\to 0} \dfrac{\left(\frac{5}{81}x^3 + o(x^3)\right)^2}{\left(\frac{x^3}{6} + o(x^3)\right)^{2n}} = \lim_{x\to 0} \dfrac{x^6\left(\frac{5}{81} + \frac{o(x^3)}{x^3}\right)^2}{x^{6n}\left(\frac{1}{6} + \frac{o(x^3)}{x^3}\right)^{2n}}$

$= \underbrace{x^{6-6n}} \cdot \dfrac{\left(\frac{5}{81} + \underbrace{\frac{o(x^3)}{x^3}}_{x\to 0}\to 0\right)^2}{\left(\frac{1}{6} + \underbrace{\frac{o(x^3)}{x^3}}_{\downarrow x\to 0 \; 0}\right)^{2n}} = \begin{cases} \dfrac{\frac{5}{81}}{1/6} = \dfrac{30}{81}, & n = 1 \\ +\infty, & n > 1 \end{cases}$

$x^{6-6n} \to \begin{cases} 1 & 6-6n = 0 \; (n=1) \\ 0 & 6-6n > 0 \; (n<1) \to \text{No pot ser pq } n\in\mathbb{N}. \\ +\infty & 6-6n < 0 \; (n>1) \end{cases}$

$x^{6-6n} = \underbrace{(x^6)}_{\downarrow 0^+}{}^{\overbrace{(1-n)}^{<0}} \longrightarrow +\infty$

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(b) $\lim\limits_{x\to 0} \dfrac{e^{x^2}-1-x^2}{x^{2n}} = L$

$$e^t = 1 + t + \frac{t^2}{2} + o(t^2)$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} + o(x^4)$$

$$L = \lim_{x\to 0} \frac{\frac{x^4}{2} + o(x^4)}{x^{2n}} = \lim_{x\to 0} \frac{x^4\left(\frac{1}{2} + \frac{o(x^4)}{x^4}\right)}{x^{2n}} =$$

$$= \lim_{x\to 0} x^{4-2n}\left(\frac{1}{2} + \frac{o(x^4)}{x^4}\right) = \lim_{x\to 0} \underbrace{(x^2)^{2-n}}_{\begin{cases} 1, & n=2 \\ 0, & n<2 \ (n=1) \\ +\infty, & n>2 \end{cases}} \underbrace{\left(\underbrace{\frac{1}{2}}_{\frac{1}{2}} + \underbrace{\frac{o(x^4)}{x^4}}_{\xrightarrow{x\to 0}\ 0}\right)}_{\xrightarrow{x\to 0}\ \frac{1}{2}}$$

$$= \begin{cases} \frac{1}{2}, & n=2 \\ 0, & n=1 \\ +\infty & n\geq 3 \end{cases}$$

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