Written assignment

1.

For a Lambertian Model, diffuse reflection is $L = \frac{\rho_d}{\pi} I(\vec{n} \cdot \vec{s})$

Let's suppose distance of the two light sources s_1, s_2 are r_1, r_2 respectively. And θ is the angle between $\vec{s}_1$, $\vec{s}_2$.

For two light sources with equal intensity I

$$L_1 = \frac{\rho_d}{\pi} I(\vec{n} \cdot \vec{s}_1)$$

$$L_2 = \frac{\rho_d}{\pi} I(\vec{n} \cdot \vec{s}_2)$$

$$L = L_1 + L_2 = \frac{\rho_d}{\pi} I(\vec{s}_1 + \vec{s}_2) \cdot \vec{n}$$

Now normalize vector $(\vec{s}_1 + \vec{s}_2)$ so that it has unit length. According to law of cosines, length of $(\vec{s}_1 + \vec{s}_2)$ is $\sqrt{1 + 1 + 2cos\theta}$. Thus rewrite L

$$L = \frac{\rho_d}{\pi} I\sqrt{2 + 2cos\theta}\left(\vec{n} \cdot \frac{\vec{s}_1 + \vec{s}_2}{\sqrt{2 + 2cos\theta}}\right)$$

Compare this equation with the origin one $L = \frac{\rho_d}{\pi} I(\vec{n} \cdot \vec{s})$, we can regard this L as the diffuse reflection resulted from one single light source s_3. And the intensity $I_3 = I\sqrt{2 + 2cos\theta}$, direction $\vec{s}_3 = \frac{\vec{s}_1 + \vec{s}_2}{\sqrt{2+2cos\theta}}$.

For two light sources with different intensities I_1 and I_2, similarly we have

$$L_1 = \frac{\rho_d}{\pi} I_1(\vec{n} \cdot \vec{s}_1)$$

$$L_2 = \frac{\rho_d}{\pi} I_2(\vec{n} \cdot \vec{s}_2)$$

$$L = L_1 + L_2 = \frac{\rho_d}{\pi}(I_1\vec{s}_1 + I_2\vec{s}_2) \cdot \vec{n}$$

And the length of $(I_1\vec{s}_1 + I_2\vec{s}_2)$ is $\sqrt{(I_1)^2 + (I_2)^2 + 2I_1I_2cos\theta}$. Then, rewrite L

$$L = \frac{\rho_d}{\pi}\sqrt{(I_1)^2 + (I_2)^2 + 2I_1I_2cos\theta}\left(\vec{n} \cdot \frac{I_1\vec{s}_1 + I_2\vec{s}_2}{\sqrt{(I_1)^2 + (I_2)^2 + 2I_1I_2cos\theta}}\right)$$

This time, the intensity of 'equal effect' light source is $I_3 = \sqrt{(I_1)^2 + (I_2)^2 + 2I_1I_2cos\theta}$, direction $\vec{s}_3 = \frac{I_1\vec{s}_1 + I_2\vec{s}_2}{\sqrt{(I_1)^2+(I_2)^2+2I_1I_2cos\theta}}$.

2.

a. The contour is a circle.

The equation of a Gaussian sphere is $x^2 + y^2 + z^2 = 1$, thus for a point lies on the surface of sphere (x_n, y_n, z_n), the normal of that point is $\frac{(x_n, y_n, z_n)}{\|(x_n, y_n, z_n)\|} = (x_n, y_n, z_n)$

According to Lambertian law $L = \frac{\rho_d}{\pi} I(\vec{n} \cdot \vec{s})$. As there is only one light source, thus the only factor which affects the diffuse reflection intensity is $(\vec{n} \cdot \vec{s})$.

Suppose the light source direction is $\vec{s} = (x_s, y_s, z_s)$, then $(\vec{n} \cdot \vec{s}) = x_n x_s + y_n y_s + z_n z_s$. The contour is a set of points with the same intensity which means $(\vec{n} \cdot \vec{s})$ should be a constant value. So the contour must comply $x_n x_s + y_n y_s + z_n z_s = constant.$

If we regard it as a function of (x_n, y_n, z_n), it is a function of a plane. As the point (x_n, y_n, z_n) also lies on the surface of the sphere, it is on the intersection of the plane and the Gaussian sphere. And thus the contour is a circle.

b.

Now that there are two different light sources, suppose they have the direction $\vec{s}_1$ and $\vec{s}_2$. The possible normal $\vec{n}$ lies on the two intersection circles of $\vec{s}_1$ and $\vec{s}_2$ as proven in part a.

Suppose these two circles lies on two planes P_1 and P_2 respectively. It is obvious that the intersection points of two circles must also lie on the intersection of two planes. Suppose the intersection of these two planes is line l, also note that the intersection of these two circle is on the Gaussian sphere, thus the intersection points must be on the intersection of the line l and the Gaussian sphere. And as there are at most two intersection points of a line and a sphere, the intersection points of two circles could not exceed two as well. There are at most two surface orientations that give rise to a given pair of brightness.