Imperial College London

60006 Computer Vision (Term 2)

Tutorial 3: Interest Point Detection

1. Suppose we use the Harris detector for interest point detection. At each pixel, we calculate a 2x2 matrix M,

$$M = \sum_{x,y} w(x,y) \begin{bmatrix} I_x^2 & I_x I_y \\ I_x I_y & I_y^2 \end{bmatrix}$$

At one particular pixel, after calculation, we get the following matrix,

$$M = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

1.1 Please derive the eigenvalues for this matrix.

You can use the following linear algebra knowledge. For eigenvalues λ and eigenvectors $\boldsymbol{u}$ associated with matrix M, we have

$$M\boldsymbol{u} = \lambda \boldsymbol{u}$$

which can be re-written as,

$$(M - \lambda I)\boldsymbol{u} = 0$$

This means that the matrix $M - \lambda I$ is singular and its determinant $|M - \lambda I|$ is zero. For a 2x2 matrix, its determinant is defined by,

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Solution:

The determinant of $|M - \lambda I|$ can be written as,

$$|M - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 - 1 = 0$$

Solving the equation, we get the two eigenvalues,

$$\lambda_1 = 4, \lambda_2 = 2$$

1.2 Please compute the Harris detector response $R = \lambda_1\lambda_2 - k(\lambda_1 + \lambda_2)^2$ with $k = 0.05$.

Solution:

$$\begin{aligned} R &= \lambda_1\lambda_2 - k(\lambda_1 + \lambda_2)^2 \\ &= 4 \times 2 - 0.05 \times (4+2)^2 \\ &= 6.2 \end{aligned}$$

1.3 We would like to compare the Harris detector response between different scales. We decide to use the scale-adapted Harris detector, which calculates the matrix in this way,

$$M = \sum_{x,y} w(x,y)\sigma^2 \begin{bmatrix} I_x^2(\sigma) & I_x(\sigma)I_y(\sigma) \\ I_x(\sigma)I_y(\sigma) & I_y^2(\sigma) \end{bmatrix}$$

Suppose we get the following matrix at a pixel,

$$M = \sigma^2 \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

where $\sigma = 2$. Please derive the eigenvalues for this matrix and the corresponding Harris detector response.

Solution:

When $\sigma = 2$, the matrix $M = \begin{bmatrix} 12 & 4 \\ 4 & 12 \end{bmatrix}$.

Similarly, we write the determinant of $|M - \lambda I|$ as,

$$|M - \lambda I| = \begin{vmatrix} 12-\lambda & 4 \\ 4 & 12-\lambda \end{vmatrix} = (12-\lambda)^2 - 16 = 0$$

Solving the equation, we get the two eigenvalues,

$$\lambda_1 = 16, \lambda_2 = 8$$

The Harris detector response is,

$$\begin{aligned} R &= \lambda_1\lambda_2 - k(\lambda_1+\lambda_2)^2 \\ &= 16\times 8 - 0.05\times(16+8)^2 \\ &= 99.2 \end{aligned}$$

1.4 For the above matrix

$$M = \sigma^2\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Directly calculate the scale-adapted Harris detector using equation $R = \det(M) - k(\mathrm{tr}(M))^2$, with $k = 0.05$ without performing eigen-decomposition. Check whether the result is the same.

Solution:

When $\sigma = 2$, the matrix $M = \begin{bmatrix} 12 & 4 \\ 4 & 12 \end{bmatrix}$.

The Harris detector response is,

$$\begin{aligned} R &= \det(M) - k\big(\mathrm{tr}(M)\big)^2 \\ &= (12\times 12 - 4\times 4) - 0.05\times(12+12)^2 \\ &= 99.2 \end{aligned}$$

2. SIFT uses the difference of Gaussian (DoG) filter to calculate the detector response. The DoG approximates the scale-normalised Laplacian of Gaussian (LoG) filter. Let us denote the Gaussian filter as,

$$G(x,y,\sigma) = \frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$$

2.1 Please prove the following equation holds (also known as the heat diffusion equation),

$$\frac{\delta G}{\delta\sigma} = \sigma\nabla^2 G$$

where ∇^2 is the Laplacian operator and $\nabla^2 G = \frac{\delta^2 G}{\delta x^2} + \frac{\delta^2 G}{\delta y^2}$.

Solution:

Take the derivative of G regarding σ, we have

$$\frac{\delta G}{\delta\sigma} = \frac{1}{2\pi\sigma^3}\left(\frac{x^2+y^2}{\sigma^2} - 2\right)e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Take the derivative of G regarding x, we have

$$\frac{\delta G}{\delta x} = -\frac{1}{2\pi\sigma^4}xe^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$\frac{\delta^2 G}{\delta x^2} = \frac{1}{2\pi\sigma^4}\left(\frac{x^2}{\sigma^2} - 1\right)e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Similarly, we have

$$\frac{\delta^2 G}{\delta y^2} = \frac{1}{2\pi\sigma^4}\left(\frac{y^2}{\sigma^2} - 1\right)e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Therefore,

$$\frac{\delta^2 G}{\delta x^2} + \frac{\delta^2 G}{\delta y^2} = \frac{1}{2\pi\sigma^4}\left(\frac{x^2+y^2}{\sigma^2} - 2\right)e^{-\frac{x^2+y^2}{2\sigma^2}}$$

It follows that

$$\frac{\delta G}{\delta\sigma} = \sigma\left(\frac{\delta^2 G}{\delta x^2} + \frac{\delta^2 G}{\delta y^2}\right) = \sigma\nabla^2 G$$

2.2 The difference of Gaussian is defined as $DoG(x, y, \sigma) = G(k\sigma) - G(\sigma)$. Given that we can approximate the derivative $\frac{\delta G}{\delta \sigma}$ using the finite difference,

$$\frac{\delta G}{\delta \sigma} \approx \frac{G(k\sigma) - G(\sigma)}{k\sigma - \sigma}$$

please derive the relationship between $DoG(x, y, \sigma)$ and the scale-normalised Laplacian of Gaussian $\sigma^2 \nabla^2 G$.

Solution:

Since

$$\frac{\delta G}{\delta \sigma} \approx \frac{G(k\sigma) - G(\sigma)}{k\sigma - \sigma}$$

and

$$\frac{\delta G}{\delta \sigma} = \sigma \nabla^2 G$$

it follows that,

$$DoG(x, y, \sigma) = G(k\sigma) - G(\sigma) \approx (k - 1) \cdot \sigma^2 \nabla^2 G$$

$DoG(x, y, \sigma)$ is proportional to $\sigma^2 \nabla^2 G$ with a constant $k - 1$.