ELEC40001

Mathematics: Year 1

Vector spaces and subspaces

Xin Wang

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Abstract

Dealing with 'Square Matrices' is easy but it is not always square and some have 0's in the pivot positions.

This section leads with 'Vector Spaces and Subspaces' and following by solving $Ax = b$ for all cases.

Contents

1 Vector Revision

1.1 Linear Combinations

Linear combinations that form the heart of linear algebra is composed of two vector operations: Addition and Multiplication.

$$\text{Linear Combination of } v \text{ and } w \text{ is: } cv + dw$$

Four special linear combinations are:

- Sum of vectors: $1v + 1w$
- Difference of vectors: $1v - 1w$
- Zero vector: $0v + 0w$
- Vector cv in direction of v: $cv + 0w$

Possible linear combinations (Vectors with three elements: x, y and z) are defined by the space they take in the 3D-dimension(R^3):

- cu: Line in
- $cu + dv$: A plane
- $cu + dv + ew$: Entire 3D space

1.2 Dot Product

Dot product (Inner product) of $v = (v_1, v_2)$ and $w = (w_1, w_2)$ can be defined Algebraically and Geometrically.

Algebraic definition:

$$v.w = w.v = v_1 * w_1 + v_2 * w_2$$

Geometric definition:

$$v.w = w.v = \|v\| * \|w\| * \cos(\theta)$$

A dot product of 0 means that the vectors are **perpendicular**.

Dot product of a vector with itself is the square product of the vector's magnitude:

$$v.v = \|v\|^2$$

1.3 Magnitude

Length $\|v\|$ of a vector is square root of the its dot product:

$$\|v\| = \sqrt{v.v}$$

The unit vector u of a specified vector has a magnitude of 1. It is defined as:

$$u = \frac{v}{\|v\|}$$

1.4 Angle between two vectors

The dot product of the two **unit** vectors are always between 1 and -1 (Schwarz Inequality) and is defined as:

$$u.U = \cos(\theta)$$

$$\frac{v.w}{\|v\| * \|w\|} = \cos(\theta)$$

1.5 Matrices

A system of matrices is defined as:

$$Ax = b$$

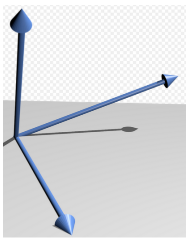
Matrix b is usually known and matrix x unknown. Linear equations solve for unknown matrix x defined as:

$$x = A^{-1} * b$$

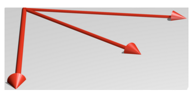
1.6 Independence and Dependence

Determines the question whether a vector v is in the plane defined by w and u.

- Independence: A vector v is not in the plane defined by w and u.
- Dependent: A vector v' is in the plane defined by w and u.



(a) Independent vectors



(b) Dependent vectors

Figure 1: Three vectors showing dependency

When solving a linear equation, the dependency of the matrix columns defines the number of solutions. The three vectors either lie in a plane or they do not lie in a plane.

- Independent vectors: $Ax = 0$ has **one** solution. Matrix A is invertible.
- Dependent vectors: $Ax = 0$ has **many** solutions. Matrix A is non-invertible.

1.7 Linear Equations

Linear algebra solves a system of equations which are linear (Only multiplied by numbers).

When a system of equations are expressed as $Ax = b$, matrix A is called **Coefficient Matrix**.

Linear algebra can be viewed as Rows or Columns.

- Row Picture: Shows two lines meeting at a single point (Solution)

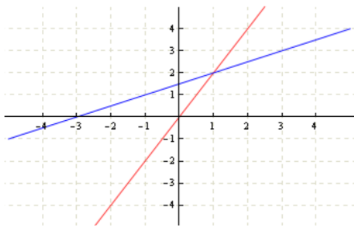


Figure 2: Row picture indicating the point where two lines meet

- Column Picture: Combines the column vectors to produce a vector b

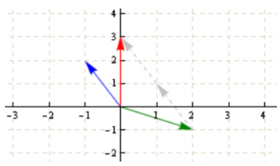


Figure 3: Column picture showing the resulting vector

For example:

$$x + 2y + 3z = 6$$
$$2x + 5y + 2z = 4$$
$$6x - 3y + z = 2$$

The system of equations can be viewed as follows:

- Row: Three planes meeting at a single point
- Column: Three columns to produce a vector b

1.8 Elimination

The elimination method is a way to solve linear equations where two unknowns are reduced to one unknown in one of the equations.

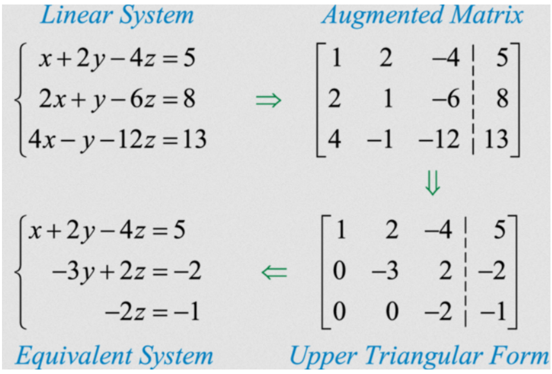


Figure 4: Process of Elimination obtaining one unknown to solve

The Elimination method produces a **Upper Triangular System** from which then system is solved from bottom up (**Back Substitution**).

Graphically, the point of intersection of lines do not change.

Elimination method fails sometimes and could lead to three cases:

- No solution
- Infinite solution
- Fixable by exchanging rows

1.8.1 No solutions

$$\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 11 \end{aligned} \qquad\qquad \begin{aligned} x - 2y &= 1 \\ 0y &= 8 \end{aligned}$$

Graphically speaking:

- Row Picture: This is two **parallel lines** that never cross hence there is never a solution.
- Column Picture: Columns do not combine to produce a third vector b [1]

1.8.2 Infinite solutions

$$\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 3 \end{aligned} \qquad\qquad \begin{aligned} x - 2y &= 1 \\ 0y &= 0 \end{aligned}$$

Graphically speaking:

[1] Page 47 of Strang: Introduction to linear algebra

- Row Picture: This is two **parallel lines** that have become the same line. Every point on that line satisfies both line equations.
- Column Picture: Two columns become the same column and the vector b is the same as the two initial vectors. [2]

1.9 Matrix

There are types of matrices:

- Elimination matrix E: Subtracts/adds a multiple of a row to another row
- Identity matrix: Contains only 1 along the major diagonal
- Permutation matrix: Matrix containing information about row exchanges
- Augmented matrix: Matrix containing the Coefficient Matrix A and right side b matrix
- Zero matrix: A matrix containing 0 only

1.9.1 Matrix operation rules

Matrix multiplication:

$$\begin{bmatrix} m \\ n \end{bmatrix} * \begin{bmatrix} n \\ p \end{bmatrix} = \begin{bmatrix} m \\ p \end{bmatrix}$$

To multiply matrices A and B, A has n columns and B must have n rows.

1.9.2 Matrix operation laws

Addition laws:

- Commutative law: $A + B = B + A$
- Distributive law: $c(A + B) = cA + cB$
- Associative law: $A + (B + C) = (A + B) + C$

Multiplication laws:

- Commutative law: $AB \neg BA$
- Right Distributive law: (A+B)C=AC+BC
- Left Distributive law: C(A+B)=CA+CB
- Associative law: A(BC)=(AB)C

[2] Page 47 of Strang: Introduction to linear algebra

1.10 Inverse matrix

Not all matrices have inverses. Matrix A is invertible if there exists a matrix A^{-1} which

$$A * A^{-1} = A^{-1} * A = I$$

Notes:

- An invertible matrix cannot have a 0 determinant
- Inverse exists if elimination produces n pivots (Same number of rows)
- There exists only one inverse for each matrix
- If there is a non-zero x which $Ax = 0$ then A has no inverse - $x = A^{-1} * 0$ is impossible
- 2×2 matrix is invertible if $ad - bc$ is not 0
- The product $A \times B$ has an inverse if A and B each have their own inverses

Inverse matrices can be found by using the Gauss-Jordan method.

1.11 Transposes and Permutations

The rows and columns of a matrix are swapped.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$$

Converting to transposes are handled differently during operations:

- Sum: $A + B$ is $A^T + B^T$
- Product: AB is $(AB)^T = B^T A^T$
- Inverse: A^{-1} is $(A^{-1})^T = (A^T)^{-1}$

2 Vector Spaces and Subspaces

There are three levels of understanding in Linear Algebra: Numbers, Vectors and Spaces.

It is important to consider the **spaces** of vectors and their **subspaces**. The most important vector spaces are $\mathbf{R}^n$ where n represents the number of components within in a vector. This space consists of all column vectors with n components.

The two essential vector operations (Addition and Multiplication) occur inside the vector space to produce linear combinations.

> *Can add any vectors in $\boldsymbol{R}^n$ and multiple any vector by any scalar. The results must **stay within the vector space** to satisfy the laws of the problem set.*[a]
>
> [a]Refer to page 121 of Strang Intro to Linear Algebra for the eight laws

There are other vector spaces besides $\mathbf{R}^n$:

- M: Vector space of all real 2×2 matrices
- F: Vector space of all real functions $f(x)$
- Z: Vector space consisting of only a zero vector. No space cannot exist without a zero vector, it is the smallest possible vector space. Hence, every vector space has a Z vector space.

2.1 Subspaces

There are subspaces of $\mathbf{R}^n$ that can be investigated. For example, in $\mathbf{R}^3$ space, a plane through the origin or can be considered a vector space as well as Z. They exists within the vector space of $\mathbf{R}^3$.

> **Subspace**: A subset of a vector space (Including Zero Vector) that satisfies two requirements, where v and w are vectors in the subspace:
> - $v + w$ is in the subspace
> - $c * v$ is in the subspace (where c is a scalar)

For example the possible subspaces for $\mathbf{R}^3$ are:

- L any lines through origin
- P any plane through origin
- Z single vector at origin
- $\mathbf{R}^3$ whole space

2.2 Column space of A

Linear algebra solves for $Ax = b$ and important subspaces are tied directly to matrix A. If A is not invertible then the system is solvable for *some values of* b. The values that form the possible matrix b form the **column space** of A.

In the system, values of Ax and, as a result b is defined by the possible combinations of the columns of A. hence, to get all the values of b, all possible values of x are used (All linear combinations).

This section has a relation to 'Independence'. Some columns/rows could be combinations of others meaning that it would not span the entire R^n.

> **Column Space**: Consists of all linear combinations of the columns of A to form all possible vectors Ax. Forms the column space $C(A)$.

This means that to solve $Ax = b$, b has to be in the column space produced by A or there is no solution.

Example: A is $[m \times n]$ so its columns have m components. The column space of A is a subspace of R^m. Ax is defined as:

$$\begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

b is defined as the linear combination of columns of A:

$$b = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

The b vector lies in the plane created by the two column vectors which is a subspace within the R^3 space.

2.3 Nullspace of A

The nullspace of A is found by solving for $Ax = 0$. The solutions form the nullspace of A and are denoted by $N(A)$. $x = 0$ is always in the nullspace so every matrix A has a nullspace.

The solution vectors form the nullspace subspace when solving for ($Ax = 0$ and $Ay = 0$) - this is tested with two conditions:

- $A(x + y) = 0 + 0$
- $A(cx) = c0$

Any combinations made will remain in the nullspace hence it is a subspace.

As previous, the matrix A is defined by $[m \times n]$. When solving $Ax = 0$, the vector x are in R^n, meaning the vector has n components.

Example: $x + 2y + 3z = 0$ where $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

The equation $Ax = 0$ produces a plane through the origin. The plane is a subspace of R^3 - the nullspace of A.

To describe the nullspace of A, one point on the line (a **special solution**) is chosen. All the points on the line are multiples of this special solution. It is important to sometimes realise that rows in a matrix can be multiples of other rows - row picture: line of that row is the same line as the other row, does not contribute anything to the picture.

Example:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 3x_1 + 6x_2 &= 0 \end{aligned}$$

This can be simplified to:

$$\begin{aligned} x_1 + 2x_2 &= 0 \\ 0 &= 0 \end{aligned}$$

The nullspace of this system of equations can be found by choosing a special solution: let $x_2 = 1$. The nullspace consists of all combinations of special solutions:

$$Nullspace = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

There could be more than one special solution. For example:

$$x + 2y + 3z = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \text{ has special solutions } s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

The *free components* are usually set to 1 or 0. *Free Components* are columns without a pivot which is column 2 and 3 in the above equation. The resulting matrix can be further simplified by finding *Reduced Form: R.*

2.3.1 Process to finding Nullspace

Elimination is required to simplify A by row operations before solutions can be obtained. The two stages:

- Forward Elimination: Convert A to Upper Triangular U (Echelon Form)
- Back Substitution: Produces x from Ux

In Forward Elimination, the difference is that some pivots encountered will be 0. The process simply continues to the next column until the process is completed.

Example:

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$$

Column 2 is a 0 pivot:

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & \mathbf{0} & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$

Moving to next 'pivot' point and continue:

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & \mathbf{0} & 0 \end{bmatrix}$$

At the end of Forward Elimination, row 2 and row 3 are the same hence row 3 does not contribute to the picture. The 'Echelon Form' is found and the following can be identified:

- Pivot Columns: x_1 and x_3
- Free Columns (No pivots): x_2 and x_4
- Rank (Number of Pivot Columns - equations that contributed): 2

> If a matrix is invertible, there will be **no free pivots**. All variables are pivot variables.

In Back Substitution, the Free Columns are substituted with 1s and 0s to obtain Special Solutions. The number of Special Solutions are determined by the number of Free Columns.

After finding the values of Free Columns, these are combined to form the complete solution denoted as:

$$x = x_2 * \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 * \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}$$

> Summary of steps to find the nullspace of A:
> 1. Obtain Echelon Form through Elimination
> 2. Identify: Pivot Columns and Free Columns
> 3. Set first free column variable equal to 1 and the others to 0
> 4. Find corresponding values from the equation created
> 5. Repeat for each Free Column identified
> 6. The resulting linear combinations are the nullspace

2.4 Echelon Matrices

An echelon matrix is a matrix form as the result of Gaussian Elimination.

$$U = \begin{bmatrix} p & x & x & x & x & x & x \\ 0 & p & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & p & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Three Pivot Variables: x_1, x_2, x_6
- Four Free Variables: x_3, x_4, x_5, x_7
- Four Special Solutions

To repeat, the Column Space is a subspace of R^4 and Nullspace is a subspace of R^7. The dimension of the nullspace is the number of free variables, 4 in this case.

2.5 Reduced Row Echelon Matrix: R

Provided the Echelon Matrix:

$$U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The Reduced Row Echelon Matrix has 0's above and below the pivots, and 1's as pivots.

Divide row 2 by 4 and subtract rows:

$$R = rref(A) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form makes it easy to find special solutions.

- Set $x_2 = 1$ and $x_4 = 0$. Solve.
- Set $x_2 = 0$ and $x_4 = 1$. Solve.

If the matrix is invertible, the Reduced Row Echelon matrix is the identity matrix.

2.6 The Rank and Row Reduced Form

The size of the matrix is given by m and n but it is not the true size of the linear system. The true size of the Matrix A is determined by its rank - the number of pivots r. This is due to eliminations like columns or rows that are multiples or a combination of rows - these will all reduce to 0s in the Reduced Row Echelon form.

Every free columns **are** combinations of earlier pivot columns.
Every pivot columns **are not** combinations of earlier pivot columns.

Example: Given the matrix $A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix}$

- Column 1 and 2 are going in different directions.
- Column 2 is the multiple of Column 3.
- Column 4 is the addition of Column 1, 2 and 3.

The matrix will only have 2 pivots and 2 free column.

- Find the Upper Triangular form:

$$\begin{aligned} U &= \begin{pmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

- Find the Reduced Row Echelon R form:

$$R = \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2.6.1 Rank One

2.6.2 Pivot Columns

Pivot columns consists of 1's in pivots and 0's in the rest.

2.6.3 Special Solutions

Each special solutions has one variable set to 1 and the other variables to 0. There is a special solution for every free variable.

$Ax = 0$ has r pivots and $n - r$ free variables. The nullspace matrix N contains $n - r$ special solutions.

Special solutions are easily identified with R. The R format is simplified into:

$$N = \begin{bmatrix} -F \\ I \end{bmatrix}$$

where I is defined by $n - r$ and F is defined by r.

Example: Given the special solutions: $Rx = x_1 + 2x_2 + 3x_3 = 0$

$$R = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad N = \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The rank is one. The special solutions are $(-2, 1, 0)$ and $(-3, 0, 1)$.

2.7 The Complete Solution to $Ax = b$

2.7.1 Solvability of $Ax = b$

The existence of solutions to $Ax = b$ needs to be established first. There is always a solution to $Ax = 0$ but there is not always a solution to $Ax = b$ - b needs to be in the column space ($C(A)$).

The process for checking for solvability is:
1. Form Augmented Matrix
2. Elimination process
3. Last row indicates condition required for solvability (Solvability Condition)
    - Consistent: Last row is all 0's and has a solution
    - Inconsistent: Last row is not all 0's and has no solution

Example: Given the matrix:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

Going through the process listed above:

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 & | & b_1 \\ 2 & 4 & 6 & 8 & | & b_2 \\ 3 & 6 & 8 & 10 & | & b_3 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 2 & 2 & | & b_1 \\ 0 & 0 & 2 & 4 & | & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & | & b_3 - 3b_1 \end{bmatrix} \xrightarrow[R_3 - R_2]{} \begin{bmatrix} 1 & 2 & 2 & 2 & | & b_1 \\ 0 & 0 & 2 & 4 & | & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & | & b_3 - b_2 - b_1 \end{bmatrix}$$

The last row equation indicates solutions exists only if: $b_3 + b_2 + b_1 = 0$

2.7.2 Finding the Complete Solution

The complete solution $Ax = b$ is composed of:

$$x = x_p \text{ (Particular solution) } + x_n \text{ (Other solutions in nullspace.)}$$

As mentioned previously, complete solution is defined as: $x = x_p + x_n$ where:

- $x_{particular}$: Solves for $Ax_p = b$
- $x_{nullspace}$: Solves for $Ax_n = 0$

Summary of steps to find the complete solution of A:
1. Find 'Echelon Form' of **Augmented Coefficient Matrix**
2. Establish if system is solvable
3. Solve Homogenous Equation $Ax = 0$ for nullspace
4. Find Particular Solution $Ax = b$: x_p
5. Solution is the combination of Particular Solution and Nullspace Solution

Example: Given $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} * \underline{\mathbf{x}} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ solve.

Augmented matrix in Echelon form:

$$[A : \underline{\mathrm{b}}] \sim \begin{bmatrix} \mathbf{1} & 2 & 2 & 2 & | & 1 \\ 0 & 0 & \mathbf{2} & 4 & | & 3 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Finding the Nullspace Solution by setting Free Column Variables to 1's and 0's

$$x_n = p * \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + q * \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

The Particular Solution is found with Free Column Variables all set to 0

$$x_p = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

The complete solution is:

$$\underline{\mathrm{x}} = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix} + p * \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + q * \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Summary:

- Elimination on $\begin{bmatrix} A & b \end{bmatrix}$ leads to $\begin{bmatrix} R & d \end{bmatrix}$. This means $Ax = b$ is equal to $Rx = d$.
- $Ax = b$ and $Rx = d$ are solvable only if rows in R with 0s also have 0s in d.
- When $Rx = d$ is solvable, one *particular solution:* x_p *has all free variable equal to zero.*