ELEC40001

MATHEMATICS: YEAR 1

Independence, Basis and Dimension

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Abstract

Matrix calculations can be seen as only numbers. To engineers, it involves vectors i.e. columns of $A\mathbf{x}$ and AB are linear combinations of n vectors - the columns of A. Now matrix calculations are abstracted from numbers and vectors to a third highest level of understanding. Instead of individual columns, vector "spaces" are considered. Without seeing vector spaces and their subspaces, everything cannot be understood about $A\mathbf{x} = b$.

This section is about the true size of a subspace. There are n columns in an $[m \times n]$ matrix but the true **dimension** of the column space is not necessarily n. The dimension is measured by counting **independent** columns - the true dimension of the column space is the rank r. The idea of independence applies to any vectors in any vector space. Most of this section concentrates on the common subspaces such as the column space and the nullspace of A.

The goal is to understand a **basis**: independent vectors that "span the space".

Contents

1 Introduction

1.1 Column Rank and Row Rank

Given a matrix A of dimension $[m \times n]$, there can be at most n pivots because a column cannot have more than one pivot.

Full Row Rank: Rank of A is equal to the number of rows

$$r = m$$

Column Space $C(A)$ is R^m which is always solvable.

Full Row Rank properties:

- All rows have pivots.
- $Ax = b$ has a solution for every right-sided b.
- Column space spans the whole space of R^m.
- There are $n - r = n - m$ special solutions in nullspace of A.

Full Column Rank: Rank of A is equal to the number of columns

$$r = n$$

n pivots with no free variables indicate that only $x = 0$ is in the nullspace meaning that the columns are independent.

Full Column Rank properties:

- All columns of A are pivot columns.
- $A\mathbf{x} = b$ can only have one solution or no solution.
- There are no free variables so no special solutions.
- The nullspace $N(A)$ contains only the zero vector.

Rank can be easily identified by finding the **Reduced Row Echelon Form** R. There are four possible cases with r, m and n:

- $r = m = n$: Square matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R = I$$

  - R is the Identity Matrix.
  - A is invertible.

- No free variables and nullspace is **only** the zero vector.
- $A\mathbf{x} = b$ always has a solution - an unique solution: $\mathbf{x} = A^{-1}b$.
- Rows are independent.

- $r = m$ and $m < n$: Flat matrix (Full Row Rank)

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 0 & 1 & \frac{17}{5} & \frac{14}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{4}{5} & -\frac{3}{5} \\ 0 & 1 & \frac{17}{5} & \frac{14}{5} \end{bmatrix} = R$$

  - There are $n - m = 2$ free columns thus two free variables.
  - Nullspace is the linear combination of $n - m$ special solution vectors.
  - There are infinitely many solutions: $m_p + x_n$

- $r = m$ and $m > n$: Tall matrix (Full Column Rank)

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = R$$

  - No free variables thus no special solutions.
  - Columns are independent.
  - One solution or, if b is not in the column space, there is no solution.

- $r < m$ and $r < n$: Not full rank
  - ∞ solutions or 0 if b is not in the column space.

$r = m = n$	$r = n < m$	$r = m < n$	$r < m$ and $r < n$
full rank	full column rank	full row rank	
square	tall	flat	any of these
$R = I$	$R = \begin{pmatrix} I \\ 0 \end{pmatrix}$	$R = (I F)$	$R = \begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix}$
1 sol	0 or 1 sol	∞ many sols	0 or ∞ many sols

1.2 Echelon form

Echelon form of the matrix can be presented in two states:

- Row echelon form.
- Reduced row echelon form.

This means that the matrix meets the following three requirements:

1. The first number (leading coefficient) in the row is 1.
2. Every leading 1 is to the right of the one above it.
3. Any non-zero rows are always above rows with all zeros.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & \cdot 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 1: Common examples of echelon form matrices

1.2.1 Row echelon form

A matrix is in **row echelon form** if it meets the following requirements:

1. The first non-zero number from the left is always to the right of the first non-zero number in the row above.
2. Rows consisting of all zeros are at the bottom of the matrix.

$$\begin{bmatrix} 1 & a_0 & a_1 & a_2 & a_3 \\ 0 & 0 & 2 & a_4 & a_5 \\ 0 & 0 & 0 & 1 & a_6 \end{bmatrix}$$

Figure 2: Row echelon form

The row echelon form is used to determine basic matrix information such as **rank**, **free columns** and **identity columns**.

1.2.2 Reduced row echelon form

Reduced row echelon form has four requirements:

1. The first non-zero number in the first row is the number 1.
2. The second row also starts with the number 1, which is further to the right than the leading entry in the first row. For every subsequent row, the number 1 must be further to the right.
3. The leading entry in each row must be the only non-zero number in its column.
4. Any non-zero rows are at the bottom of the matrix.

$$\begin{bmatrix} 1 & 0 & a_1 & 0 & b_1 \\ 0 & 1 & a_2 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \end{bmatrix}$$

Figure 3: Reduced row echelon form

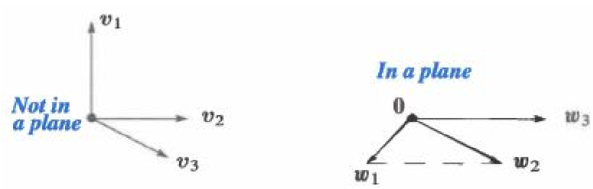
Reduced row echelon form is a type of matrix used to solve systems of linear equations. It is used to find the **basis** and **dimension** of matrices.

2 Linear Independence

Linear combination (of $v_1, v_2, \dots, v_n$): There exists scalars defined as $x_1, x_2, \dots, x_n$ such that the vector $v = (x_1 \times v_1) + (x_2 \times v_2) + \cdots + (x_n \times v_n)$.

Linearly independence: The only solution to $Ax = 0$ is $x = 0$ meaning the only combination that gives the zero vector is $0\,v_1 + 0\,v_2 + \cdots + 0\,v_n$.

In order for there to be linear independence between vectors, the vectors must not be in the same plane. If the vectors are in the same plane then the vectors are **linearly dependent** so the nullspace does not only contain the zero vector $N(A) \neq \underline{\mathbf{0}}$.



Example 1: Columns of A are dependent thus $Ax = 0$ has a non-zero solution.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

- The rank is 2 so this means that there is a free column.
- Through Gaussian elimination, the special solution is a combination of the pivot columns.

A easy way to check for linear independence is if the number of vector columns 'n' is greater than number of vector rows 'm'

$$n < m$$

If $n > m$ then the vector columns R^m must be linearly dependent.

3 Vectors that 'Span' a Subspace

Span: A set of vectors *span* a space if its linear combinations fill the space i.e. two vectors span a line, three could at best span all of R^3.

There are two kinds of subspaces:

- **Column subspace** $C(A)$: Spans the Column Space i.e. contains all combinations of the columns of A and is a subspace of R^m.
- **Row subspace** $C(A^T)$: Spans the Row Space i.e. contains all combinations of the rows of A and is a subspace of R^m.

Example 1: Describe the Column Space and Row Space of A.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix} \text{; m = 3 and n = 2}$$

1. Find the m and n.

$$m = 3 \text{ and } n = 2$$

2. Describe the Column Space with $m = 3$.

   The Column Space spans two columns of A thus is a plane in R^3.

3. Describe the Row Space.

   The Row Space of A spans three rows and spans all of R^2.

3.1 Basis for a Vector Space

A minimal set of vectors in V that spans V is called a **basis** for V.

Basis (of a vector space): A sequence of vectors with the following properties:
- All vectors are linearly independent i.e. a maximal linearly independent set.
- Span the entire space V.

The properties of basis are important since every vector v in vector space is **a unique combination of basis vectors**. This means that **basis vectors are independent**.

1. Columns of all invertible $n \times n$ matrix give a basis for R^n.

Invertible matrix Independent columns Column space is $\mathbf{R}^3$	$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	**Singular matrix** Dependent columns Column space $\neq \mathbf{R}^3$	$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

2. The **pivot columns** of A are a **basis** for its column space and the **pivot rows** of A are a basis for its row space.

   Example 1: Given $c_1 * \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 * \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 * \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ does it form a basis?
   The system form a Basis, the system spans the space R^3 and are linearly independent.

Example 2: Given vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ does it form a basis? If not, find a basis that includes the vectors.

1. Note the linear combinations form a plane:

$$p * \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + q * \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

2. Analyse the linear combination:

   - The vectors are linearly independent since the vectors are not multiples of each other.

   - Plane clearly does not span the entire R^3 so another independent vector is required.

3. Add a suitable vector to satisfy the conditions of a basis:

$$c_1 * \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 * \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + c_3 * \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

4. Verify the conditions for a basis are met:

   - A square matrix is produced.

   - Linearly independent, there is a unique solution for $x = A^{-1} * b$.

   - The system spans the entire R^3 hence forms a basis.

A simple test to ensure Linear Independence, simply find the Reduced Row Echelon form of the matrix.

4 Dimension

Dimension (of a space): The number of vectors in a basis. For example, R^3 has 3 as a dimension and R^2 has 2 as a dimension.

Theorems:

- Dimension of column space is equal to the rank of the matrix

$$rank(A) = dimC(A)$$

- Dimension of nullspace of $m \times n$ matrix is equal to number of columns minus the rank

$$n - rank(A) = dimN(A)$$

- Existence of solution of $Ax = b$:
  - $Ax = b$ has a solution for all b
  - Column space of A is the entire R^m
  - rank(A)=m
  - Full Row Rank - Rows are linearly independent
  - If $m = n$, rows of A form the basis
- Uniqueness of solution of $Ax = b$:
  - If $Ax = b$ has a solution, it is unique
  - Nullspace contains only the Zero Vector
  - $rank(A) = n$
  - Columns are linearly independent (Full Column Rank)
  - If $m = n$, rows of A form the basis

5 The Four Fundamental Subspaces

Rank i.e. number of pivots and **dimension** i.e. number of vectors in a basis are important terms used to describe the four fundamental subspaces. Those subspaces are the column space and the nullspace of A and A^T. Together, they lift the understanding of $A\mathbf{x} = b$ to a higher level - the subspace level.

They are all connected by the Fundamental Theorem of Linear Algebra. The four fundamental subspaces are:

- Column space $C(A)$ of A - The subspace of $\Re^m$ spanned by the columns of A.
- Row space $C(A^T)$ of A - The subspace of $\Re^n$ spanned by the rows of A.
- Nullspace $N(A)$ of A - The subspace of $\Re^n$ of solutions of $Ax = 0$.
- Left Nullspace $N(A^T)$ of A - The subspace of $\Re^m$ of solutions of $A^T x = 0$

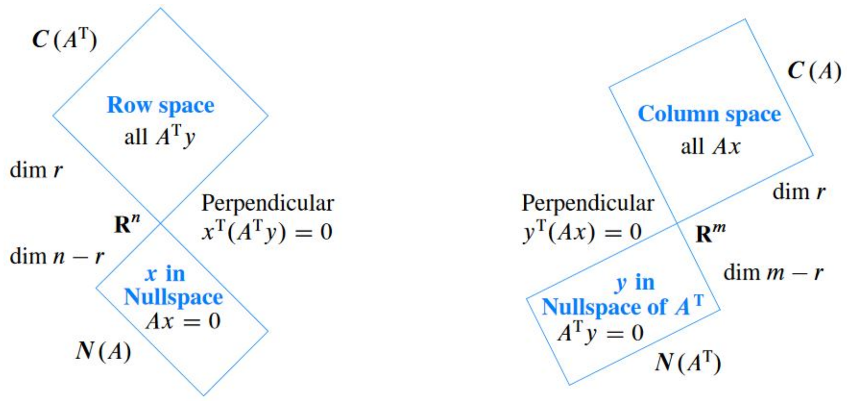


Figure 4: Dimensions and orthogonality for any m by n matrix A of rank r.

5.1 The basis and dimension of the Four Fundamental Subspaces

> **Tip**: Always obtain the Reduce Row Echelon Form of a given matrix before finding the four bases.

Finding the basis of these vector spaces to determine the dimensions are important to understanding the four fundamental subspaces.

5.1.1 Column space $C(A)$:

Note that elementary row operations do change the column space and it is not obvious that $\dim\,C(A) = \dim\,C(A^T)$ [See: Proof of column space dimension]. The columns of U that contain the pivots are linearly independent and the corresponding columns of A are also linearly independent. The other dependent columns of A are dependent on the independent columns thus they will also span $C(A)$. This means that the independent columns i.e. pivot columns form a basis for this space.

As the dimension is the number of number of independent variables:

$$\dim\,C(A) = r$$

For the column space $C(A)$ by expressing it in reduced row echelon form.:
1. Basis: Number of independent variables i.e. pivot columns.
2. Dimension: The rank of A defined as $\dim\,C(A) = \text{rank}(A) = r$

5.1.2 Row space $C(A^T)$

If B is obtained from A by elementary row operations, then the rows of B are **linear combinations** of the rows of A. Since the elementary row operations can all be reversed, the rows of A are also linear combinations of the rows of B

$$C(A^T) = C(B^T)$$

Building on, taking A to its row echelon form U by elementary row operations, the non-zero rows of U form a basis for the row space of U and hence for the row space of A. By finding the basis and letting $r = \text{rank}(A)$, the dimension is defined as:

$$\dim C(A^T) = r$$

For the row space $C(A^T)$ by expressing it in reduced row echelon form.:
1. Basis: Number of independent variables i.e. non-zero rows.
2. Dimension: The rank of A defined as $\dim\,C(A^T) = \text{rank}(A)$

5.1.3 Nullspace $N(A)$:

Notice that elementary row operations do not change this space at all i.e. if A is expressed in $A = LU$ through Gaussian elimination to find the nullspace, the lower triangular L will have all its diagonal coefficients equal to 1. This means that with the definition of the nullspace $Ax = 0$:

$$Ax = 0 \Leftrightarrow (LU)x = 0 \Leftrightarrow L^{-1}LUx = L^{-1}0 = 0$$

Allowing $Ax = 0$ to be simplified to:

$$Ax = 0 \Leftrightarrow Ux = 0$$

Then, a basis for $N(A)$ is actually a basis for $N(U)$ which is **the number of free variables present in the system** $Ux = 0$.

If $\text{rank}(A) = r$ and the number of variables is n, then the number of pivots is r and the number of free variables is $n - r$ thus the dimension of the null space is defined as:

$$\dim N(A) = n - r$$

Example 1: Given a matrix A already expressed in echelon form to find U, find the special solutions thus a basis and dimension.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

Process:

1. Identify the **free columns** (2 and 4) and **pivot columns** (1 and 3).
2. Assign any value to the 2 free variables to find the 2 special solutions:
   - $x_2 = 1$ and $x_4 = 0$

     $$\begin{aligned} 2x_3 + 4x_4 = 0 &\Rightarrow x_3 = 0 \\ x_1 + 2x_2 + 2x_3 + 2x_4 = 0 &\Rightarrow x_1 = -2 \end{aligned}$$

     Thus solution 1 is:

     $$x = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

   - $x_2 = 0$ and $x_4 = 1$

     $$\begin{aligned} 2x_3 + 4x_4 = 0 &\Rightarrow x_3 = -2 \\ x_1 + 2x_2 + 2x_3 + 2x_4 = 0 &\Rightarrow x_1 = 2 \end{aligned}$$

     Thus solution 2 is:

     $$x = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

3. Find the basis:

   $$\text{A basis of null space: } \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

4. Find the dimension:

$$\dim N(A) = n - r \Rightarrow \dim N(A) = 4 - 1 = 3$$

For the null space $N(A)$ by expressing it in reduced row echelon form.:
1. Basis: The number of free variables present in the system $Ux = 0$.
2. Dimension: The difference between the number of columns n and the rank r of A is the same as the number of pivots:

$$\dim N(A) = n - r$$

5.1.4 Left null space $N(A^T)$:

If A is $[m \times n]$ then A^T is $[n \times m]$ and left null space is a subspace of $\Re^n$. For the left null space $N(A^T)$, it is clear that since rank$(A) =$ rank(A^T):

$$\dim N(A^T) = m - r$$

A basis for $N(A^T)$ is found by using A^T in its row echelon form and using the free variables to find the basis vectors like the with the null space $N(A)$ i.e. giving each free variable in turn the value 1 while all other independent variables take the value 0.

If it is observed that R has no zero rows then there is no combination of the rows of A that gives a row of zeroes, i.e. there is no y such that

$$\underline{\mathbf{y}}^T A = \underline{\mathbf{0}}^T$$

This means that there is only the zero vector as solution, so the only vector in the left null-space is 0, and it is the basis.

Example 1: Find a basis and the dimensions of the left null space of A:

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 6 & 7 & 8 \\ 0 & 10 & 11 & 12 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$$

Process:

1. Find the row echelon form of A^T:

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{2} & 6 & 10 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \mathbf{2} & 6 & 10 \\ 0 & \mathbf{-2} & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. Find the dimension:

$$\dim N(A^T) = 3 - 2 = 1$$

3. Note that only one non-zero vector x such that $A^T x = 0$ is needed since the dimension is 1

4. Find the special solution of A^T:

   - One free variable x_3

   - Set $x_3 = 1$

   $$-2(x_2) - 4(x_3) = 0 \Rightarrow x_2 = -2$$

   - Find all the unknown variables:

   $$2x_1 + 6x_2 + 10x_3 = 0 \Rightarrow x_1 = 1$$

5. Form the basis from the special solutions:

$$\text{Basis for } N(A^T) : \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

For the left null space $N(A^T)$ by expressing it in reduced row echelon form.:

1. Basis: The number of free variables present in the system $Ux = 0$.
2. Dimension: The difference between the number of columns n and the rank r of A is the same as the number of pivots:

$$\dim N(A) = m - r$$

5.2 Examples

Example 1: Given the matrix A, find a basis for each of the four fundamental subspaces $C(A)$, $N(A)$, $C(A^T)$ and $N(A^T)$. Verify the relations linking the dimensions of the subspaces to the size of the matrix $m \times n$ and rank r.

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Process:

1. Column space $C(A)$:
   - Basis:
     - Note: A basis is formed by pivot columns
     - Find the echelon form:

$$\text{Echelon form} = \begin{bmatrix} \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{1} & 2 & 2 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

     - Select the pivot columns.

$$\text{A basis for } C(A) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

   - Dimension:
     - dim $C(A) = r =$ number of pivot columns

$$\dim C(A) = r = 3$$

2. Null space $N(A)$:
   - Basis:
     - Note: A basis is formed by special solutions
     - Find the echelon form:

$$\text{Echelon form} = \begin{bmatrix} \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{1} & 2 & 2 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

     - Find the number of special solutions:

$$\text{Number of special solutions} = \text{Number of free columns} = 1$$

- Set $x_3 = 1$ and find the values of the other variables:

$$\begin{aligned} x_3 &= 1 \\ x_4 &= 0 \\ x_1 &= x_3 = 1 \\ x_2 &= -2x_3 = -2 \end{aligned}$$

- Find the special solution thus a basis for $N(A)$:

$$N(A) = p \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

- Dimension:
  - $\dim C(A) = n - r$

$$\dim C(A) = n - r = 4 - 3 = 1$$

3. Row space $C(A^T)$:
   - Basis:
     - Note: A basis is formed by special solutions
     - Find the reduced echelon form of A:

$$\text{Reduced echelon form of } A = \begin{bmatrix} \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{1} & 2 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

     - Find the reduced echelon form of A^T:

$$\text{Reduced echelon form of } A^T = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}$$

     - Find the number of special solutions:

$$\text{Number of special solutions} = \text{Number of pivot columns} = 3$$

     - Find the basis by listing the pivot columns:

$$\text{A basis for } C(A^T) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

- Dimension:
    - dim $C(A) = n - r$

$$\dim C(A) = r = 3$$

4. Left null space $N(A^T)$:
    - Basis:
        - Note: A basis is the number of zero rows.
        - Find the reduced echelon form of A:

$$\text{Reduced echelon form of } A = \begin{bmatrix} \mathbf{1} & 0 & -1 & 0 \\ 0 & \mathbf{1} & 2 & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$$

        - Find the reduced echelon form of A^T:

$$\text{Reduced echelon form of } A^T = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ -1 & 2 & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}$$

        - Identify number of zero rows to determine special solutions:

$$\text{Number of zero rows} = \text{Number of special solutions} = 0$$

    - Dimension:
        - dim $C(A) = m - r$

$$\dim C(A) = m - r = 3 - 3 = 0$$