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Discrete Mathematics: Year 2

Set Theory and Proof

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Abstract

Set Theory is an important language and tool for reasoning. It is the basis of Mathematics and Computer Science since Logic is basically Set Theory. It is a useful tool for formalizing and reasoning about computations and its objects.

Proof is the activity of finding and discovering and confirming the truth. This chapter discusses how to write down proofs that are able to be communicated effectively. There is a balance to proofs between too much detail and too little (being too strict) - just like a sating in Italy: "It requires two people to make a good salad dress; a generous person to add oil and a mean person to add the vinegar". A tolerant openness and awareness is critical in discovering/understanding proofs while a strictness and discipline is required to write it down.

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1 Mathematical argument

Introduces basic mathematical notation and the various method of "argument".

1.1 Logical notation

For statements A and B, there are common abbreviations:

1. **Conjunction** of A and B: A and B must both be TRUE or otherwise it is FALSE.
$$A \ \& \ B$$
2. A **implies** B: B TRUE only if A is TRUE.
$$A \Rightarrow B$$
3. A **if** B: A TRUE if and only if B TRUE.
$$A \iff B$$
4. not A: A TRUE if A is FALSE
$$\neg A$$

Common logic quantifiers:

1. "There exists": $\exists$
2. "For all": $\forall$
3. "There exists x such that $P(x)$":
$$\exists x \ . \ P(x)$$
4. "There exists some x satisfying a property $P(x)$ but also that x is the **unique** (only) object satisfying $P(x)$":
$$(\exists x.P(x)) \ \& \ (\forall y, z.P(y) \ \& \ P(z)) \Rightarrow y = z$$
Simplifies to:
$$\exists! x.P(x)$$

1.2 Patterns of proof

1.2.1 Chains of implications

To prove $A \Rightarrow B$, possible to show that assuming A_1 can be proved then possible to prove B.

$$\begin{aligned} A &\Rightarrow A_1 \\ &\Rightarrow \dots \\ &\Rightarrow A_n \\ &\Rightarrow B \end{aligned}$$

Take care to differentiate between $\Rightarrow$ and $\iff$ since sometimes the reverse direction is untrue.

1.2.2 Proof by contradiction

To show that A is TRUE, sometimes it is only possible to show that $\neg A$ leads to a conclusion which is FALSE - show $\neg A$ is not the case so A.

Example 1: Proof that $\sqrt{2}$ is irrational through proof by contradiction.

1. Suppose $\sqrt{2}$ **is** rational - it can be written as ratio of two integers p and q:
$$\sqrt{2} = \frac{p}{q}$$
where assumed that p and q have no common factors - if there are then they are cancelled out

   Since p and q are simplified to the lowest terms, both p and q cannot be even - one must be odd.

2. Square both sides:
$$2 = \frac{p^2}{q^2}$$

3. Implies that:
$$p^2 = 2q^2$$
p^2 is an even number since it is $2 \times q^2$.

4. Note q is **also even** since if q is odd then q^2 will be odd - odd number times odd number is always odd.

1.2.3 Argument by cases

Proof of $A_k \Rightarrow C$ requires the proof of $A_1 \Rightarrow C$ to/and $A_k \Rightarrow C$. Most often a proof breaks down into k cases showing $A_1 \Rightarrow C$,...,$A_k \Rightarrow C$.

Example 1: For all non-negative integers $a > b$, the difference of squares $a^2 - b^2$ does not give a remainder of 2 when divided by 4.

1. Observe that:
$$a^2 - b^2 = (a+b)(a-b)$$

2. Establish cases:
   - a and b are both even
   - a and b are both odd
   - One is odd and the other one even.

3. Test the cases:

- Case 1: Both a and b are even: The product of two even numbers are divisible by 4, giving remainder 0.
- Case 2: Both a and b are odd: The product of two odd numbers are divisible by 4, giving remainder 0.
- Case 3: One is odd and the other even: Product is odd and remainder of 2 when divided by 4 - **a contradiction**.

1.2.4 Existential properties

Proving $\exists x \, . \, A(x)$, it suffices by showing an object a such that $A(a)$.

For example, one can show that $\exists x.A(x)$ by obtaining a contradiction - $\exists x.\neg A(x)$

1.2.5 Universal properties

Simplest way to prove $\forall x \, . \, A(x)$ is to let x be an arbitrary element and show $A(x)$. There are famous examples such as *mathematical induction*.

1.3 Mathematical induction

2 Sets