# EE2 Mathematics – Probability & Statistics

## Solution 5

1. (a) $X \sim \mathrm{N}(\mu = 20, \sigma^2 = 0.04^2)$, so

$$\begin{aligned} P(X > 20.06) &= 1 - P(\sigma Z + \mu \leq 20.06) \\ &= 1 - P(Z \leq \frac{20.06 - 20}{0.04}) \\ &= 1 - \Phi\left(\frac{20.06 - 20}{0.04}\right) = 1 - \Phi(1.5) = 1 - 0.933 = 0.067 \end{aligned}$$

   (b) We need $P(X < 20 - k) + P(X > 20 + k) = 0.02$. By symmetry, this comes to $2P(X > 20 + k) = 0.02$, i.e. $P(X > 20 + k) = 0.01$. Hence $P(X \leq 20 + k) = P(Z \leq \frac{20+k-20}{0.04}) = 0.99 = \Phi^{-1}(2.326)$, so

$$\frac{20 + k - 20}{0.04} = 2.326 \Leftrightarrow k = 0.093$$

   (c) The limits are $(20 - 0.093, 20 + 0.093) = (19.907, 20.093)$. If $X \sim \mathrm{N}(20.027, 0.04^2)$, we have

$$\begin{aligned} P(19.907 \leq X \leq 20.093) &= P(X \leq 20.093) - P(X \leq 19.907) \\ &= P(Z < \frac{20.093 - 20.027}{0.04}) - P(Z < \frac{19.907 - 20.027}{0.04}) \\ &= \Phi(1.65) - \Phi(-3) \\ &= \Phi(1.65) - (1 - \Phi(3)) \\ &= 0.950 - 0.001 = 0.949, \end{aligned}$$

   so the proportion that is scrapped is $1 - 0.949 = 0.051$.

2. (a) Poisson($\theta$)

$$\begin{aligned} M_X(t) &= \sum_{\forall x} e^{tx} f_X(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\theta}\theta^x}{x!} \\ &= e^{-\theta} \sum_{x=0}^{\infty} \frac{(\theta e^t)^x}{x!} = e^{-\theta} e^{\theta e^t} = e^{\theta(e^t - 1)} \end{aligned}$$

(Recall $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$)

(b) Geometric($\theta$)

$$\begin{aligned} M_X(t) &= \sum_{\forall x} e^{tx} f_X(x) = \sum_{x=1}^{\infty} e^{tx}(1-\theta)^{x-1}\theta \\ &= \theta e^t \sum_{x=1}^{\infty}\{(1-\theta)e^t\}^{x-1} = \frac{\theta e^t}{1-(1-\theta)e^t} \end{aligned}$$

(Recall $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$)

(c) N(0,$\theta^2$)

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi\theta^2}}e^{-\frac{x^2}{2\theta^2}}dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\theta^2}}e^{-\frac{x^2-2\theta^2 tx}{2\theta^2}}dx \\ &= e^{\theta^2 t^2/2}\underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\theta^2}}e^{-\frac{(x-\theta^2 t)^2}{2\theta^2}}dx}_{=1\,(\text{valid pdf N}(\theta^2 t,\theta^2))} = e^{\theta^2 t^2/2} \end{aligned}$$

(d) Exp($\theta$)

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_{0}^{\infty} e^{tx}\theta e^{-\theta x}dx \\ &= \frac{\theta}{\theta-t}\underbrace{\int_0^{\infty}(\theta-t)e^{-(\theta-t)x}dx}_{=1\,(\text{valid pdf Exp}(\theta-t))} = (1-t/\theta)^{-1}, \end{aligned}$$

but we also need $t < \theta$ for the interval to converge.

3. (a) This one is straightforward; the probability (3,4) functions is $1-p^2$, so the probability the top path functions is $(1-p)(1-p^2)$. The probability the bottom path functions is $(1-p)^2$. Putting the two together, the probability the system functions is

$$1-[1-(1-p)^2][1-(1-p)(1-p^2)].$$

(b) The easiest way to do this one is to consider separately the case where component 4 functions, and the case where it fails. If it fails, the top and bottom path each consist of two components connected in series. The probability the system functions is then $1-(1-(1-p)^2)^2$.

If component 4 functions, we can ignore components 3 and 5. The system functions if (1,2) functions, so the probability is $1-p^2$.

Let $F$ be the event 'the system functions', and $C_4$ be the event 'component 4 functions'. The law of total probability gives

$$\begin{aligned} P(F) &= P(F|C_4)P(C_4) + P(F|\overline{C_4})P(\overline{C_4}) \\ &= (1-p^2)(1-p) + (1-(1-(1-p)^2)^2)p. \end{aligned}$$

(c) Same approach again; this time we condition on component 2. If it fails, the top and bottom path each consist of two components connected in series. If it functions, we can ignore components 1 and 3. We have

$$\begin{aligned} P(F) &= P(F|C_2)P(C_2) + P(F|\overline{C_2})P(\overline{C_2}) \\ &= (1-p^2)(1-p) + (1-(1-(1-p)^2)^2)p. \end{aligned}$$

4. The definition states that $Y = |X - a|$, so

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(|X-a| \le y) = P(-y \le X - a \le y) \\ &= P(a - y \le X \le a + y) = \begin{cases} F_X(a+y) - F_X(a-y) & \text{if } y \ge 0 \\ 0 & \text{if } y < 0\,. \end{cases} \end{aligned}$$

We now differentiate wrt $y$ to find

$$f_Y(y) = \begin{cases} f_X(a+y) + f_X(a-y) & \text{if } y \ge 0 \\ 0 & \text{if } y < 0\,. \end{cases}$$

Please explain this bit visually if possible.

If $X \sim \mathrm{N}(\mu, \sigma^2)$ and $a = \mu$, we have

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ f_Y(y) &= \begin{cases} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\mu+y-\mu)^2}{2\sigma^2}} + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\mu-y-\mu)^2}{2\sigma^2}} & \text{if } y \ge 0 \\ 0 & \text{if } y < 0\,, \end{cases} \\ &= \begin{cases} \sqrt{\frac{2}{\pi\sigma^2}} e^{-y^2/2\sigma^2} & \text{if } y \ge 0 \\ 0 & \text{if } y < 0\,, \end{cases} \end{aligned}$$

i.e. the half-normal distribution.