

EE2-08 Mathematics:

Solutions to Example Sheet 6: Laplace Transforms

1. a) Recalling¹ that $\mathcal{L}(\dot{x}) = s\bar{x}(s) - x(0)$, Laplace Transform the pair of ODEs using the initial conditions $x(0) = y(0) = 1$ to get

$$2(s\bar{x} - 1) + (s\bar{y} - 1) + \bar{x} = -6/s \quad (s\bar{x} - 1) + 2(s\bar{y} - 1) + \bar{y} = 0$$

Solve these simultaneous equations in \bar{x} and \bar{y} to get

$$\bar{x}(s) = \frac{3(s^2 - 3s - 2)}{s(s+1)(3s+1)} \quad \bar{y}(s) = \frac{3(s+3)}{(s+1)(3s+1)}$$

Split these expressions into partial fractions

$$\bar{x}(s) = -\frac{6}{s} + \frac{3}{s+1} + \frac{4}{s+\frac{1}{3}} \quad \bar{y}(s) = -\frac{3}{s+1} + \frac{4}{s+\frac{1}{3}}$$

and then invert to find the solutions from the tables

$$x(t) = -6 + 3e^{-t} + 4e^{-\frac{1}{3}t} \quad y(t) = -3e^{-t} + 4e^{-\frac{1}{3}t}$$

- (b) In the same manner as part a), use Laplace transforms on the ODEs to get

$$(s\bar{x} - 1) + 5\bar{x} + 2\bar{y} = \frac{1}{s+1} \quad s\bar{y} + 2\bar{x} + 2\bar{y} = 0$$

Solving these simultaneous equations we obtain

$$\bar{x}(s) = \frac{(s+2)^2}{(s+1)^2(s+6)} \quad \bar{y}(s) = -\frac{2(s+2)}{(s+1)^2(s+6)}$$

which split into partial fractions thus

$$\bar{x}(s) = \frac{9}{25(s+1)} + \frac{1}{5(s+1)^2} + \frac{16}{25(s+6)} \quad \bar{y}(s) = -\frac{8}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{8}{25(s+6)}$$

which invert to

$$x(t) = \frac{1}{5} \left(\frac{9}{5} + t \right) e^{-t} + \frac{16}{25} e^{-6t} \quad y(t) = -\frac{2}{5} \left(\frac{4}{5} + t \right) e^{-t} + \frac{8}{25} e^{-6t}$$

2. Laplace transforming the ODE and using the shift theorem, we get

$$(s^2 + 1)\bar{x}(s) = \frac{e^{-s\pi}}{s} - \frac{e^{-2s\pi}}{s} \quad \Rightarrow \quad \bar{x}(s) = \left(\frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-s\pi} - \left(\frac{1}{s} - \frac{s}{s^2 + 1} \right) e^{-2s\pi}$$

Noting that the $\mathcal{L}^{-1} \left(\frac{s}{s^2+1} \right) = \cos t$, and using the second shift theorem (formula sheet) which says that $\mathcal{L} [H(t-a)f(t-a)] = e^{-sa} \bar{f}(s)$, we find that the inversion becomes

$$x(t) = H(t-\pi) [1 - \cos(t-\pi)] - H(t-2\pi) [1 - \cos(t-2\pi)]$$

Noting that $H(t-\pi) = 1$ for $t > \pi$ but is zero for $t < \pi$ (with equivalent results for $H(t-2\pi)$), we obtain

$$\begin{aligned} x &= 0 & 0 \leq t \leq \pi \\ x &= 1 + \cos t & \pi \leq t \leq 2\pi \\ x &= 2 \cos t & 2\pi \leq t \end{aligned}$$

Note that when $t \geq 2\pi$ the two cosines add.

¹In the Formula Sheet.

3. The function $f(t)$ is periodic in time t with fixed period T such that $f(t) = f(t - T)$ with $T > 0$. Laplace transform (for $s > 0$) and split the domain up into an infinite set of successive intervals

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st} dt = \int_0^T f(t)e^{-st} dt + \int_T^{2T} f(t)e^{-st} dt + \int_{2T}^{3T} f(t)e^{-st} dt + \dots$$

Now consider the integrals on the RHS: typically they all have the form $\int_{nT}^{(n+1)T} f(t)e^{-st} dt$ on the time interval $[nT, (n+1)T]$. Use a substitution $\tau_n = t - nT$ and appeal to the fact that $f(t)$ is periodic $f(\tau_n + nT) = f(\tau_n)$ to obtain

$$\int_{nT}^{(n+1)T} f(t)e^{-st} dt = e^{-snT} \int_0^T f(\tau_n)e^{-s\tau_n} d\tau_n$$

The labels on the dummy variables τ_n within the integrals don't matter if the limits are the same; that is $\int_0^T f(\tau_n)e^{-s\tau_n} d\tau_n = \int_0^T f(t)e^{-st} dt$. Hence we have

$$\bar{f}(s) = (1 + e^{-sT} + e^{-2sT} + \dots) \int_0^T f(t)e^{-st} dt$$

For $s > 0$ the series sums to $(1 - e^{-sT})^{-1}$ to give the answer.

4. This example uses $T = 1$ and the results of Q3 on the sawtooth function, which is a piece-wise linear periodic function

$$\bar{f}(s) = (1 - e^{-s})^{-1} \int_0^1 te^{-st} dt$$

Now it is easily shown that

$$\int_0^1 te^{-st} dt = \frac{1 - e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

and so

$$\bar{f}(s) = (1 - e^{-s})^{-1} \left[\frac{1 - e^{-s}}{s^2} - \frac{e^{-s}}{s} \right] = \left[\frac{1}{s^2} - \frac{1}{s} \left(\frac{e^{-s}}{1 - e^{-s}} \right) \right]$$

On expanding $(1 - e^{-s})^{-1}$ as a series ($s > 0$) we have

$$\bar{f}(s) = \frac{1}{s^2} - \frac{1}{s} \left(\frac{e^{-s}}{1 - e^{-s}} \right) = \frac{1}{s^2} - \frac{1}{s} (e^{-s} + e^{-2s} + e^{-3s} + \dots)$$