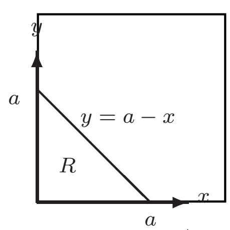
EE2-08C Mathematics
Solutions to Example Sheet 3: Multiple Integrals

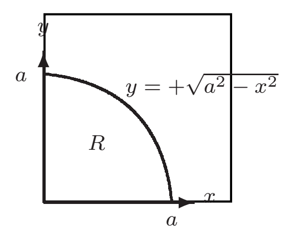
1. Evaluate the double integral $\int\int_R dx\,dy$ where the region R is the finite triangular region bounded by the positive x-axis and y-axis and the line $y = a - x$ for $a > 0$. Make sure you sketch the region of integration.

   Solution: The area under $y = a - x$ and between the x-axis and the y-axis is

$$\begin{aligned} A &= \int\int_R dxdy = \int_0^a \left(\int_0^{a-x} dy \right) dx \\ &= \int_0^a (a - x)\, dx = \tfrac{1}{2}a^2 \end{aligned}$$

   The pictures of the two regions in (1) and (2) are:





2. The co-ordinates (x_0, y_0) of the centre of gravity of a thin uniform lamina R are given by

$$Ax_0 = \int\int_R x\,dx\,dy \qquad\qquad Ay_0 = \int\int_R y\,dx\,dy$$

   where A is the area of R. If R is a quarter circle of radius a in the first quadrant, show that

$$(x_0,\, y_0) = \left(\frac{4a}{3\pi}, \frac{4a}{3\pi} \right) .$$

   Solution: The integral to find Ax_0 for the quarter circle is $(A = \frac{1}{4}\pi a^2)$

$$Ax_0 = \int\int_R x\,dxdy = \int_0^a x \left(\int_0^{\sqrt{a^2-x^2}} dy \right) dx = \int_0^a x\sqrt{a^2 - x^2}\,dx$$

   We evaluate the final integral using the substitution $x = a\cos\theta$ which gives

$$Ax_0 = \int_0^a x\sqrt{a^2 - x^2}\,dx = a^3 \int_0^{\pi/2} \sin^2\theta\cos\theta\,d\theta = \tfrac{1}{3}a^3$$

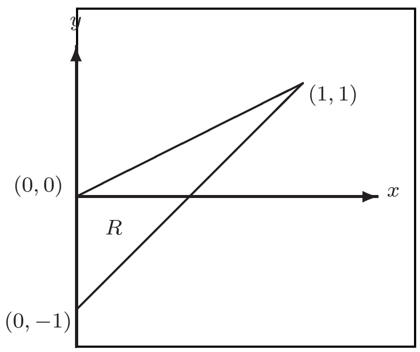
   Also $y_0 = x_0$ by symmetry so we have $(x_0,\, y_0) = \left(\frac{4a}{3\pi},\, \frac{4a}{3\pi}\right)$.

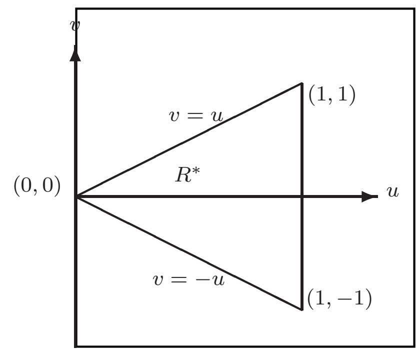
3. A region R in the right-hand diagram below consists of a triangle with vertices $(0,0)$, $(0,-1)$ and $(1,1)$. Show that the transformation $u = 2x - y, \;\; v = y$, maps R to the triangular region R^* which has vertices $(0,0)$, $(1,-1)$ and $(1,1)$.

Using the Jacobian transformation, show that $dxdy = \frac{1}{2}dudv$. Hence show that

$$I = \int\int_R \cos\left[(2x-y)^2\right]\, dxdy = \tfrac{1}{2}\sin 1$$

Solution: The region R is the left hand diagram below. To show that the transformation $u = 2x-y, \;\; v = y$, maps R to the region R^* in the right hand diagram it is only necessary to show that the line $y = x$ (from $(0,0)$ to $(1,1)$) becomes $v = u$, the line $y = 2x-1$ (from $(0,-1)$ to $(1,1)$) becomes $u = 1$ and the line $x = 0$ (from $(0,0)$ to $(0,-1)$) becomes $v = -u$. The co-ordinates map accordingly.





To show that $dx\,dy = \frac{1}{2}du\,dv$ we need the Jacobian transformation $dxdy = J_{u,v}(x,y)\,dudv$. For these elements solve $u = 2x-y$ and $v = y$ to get $x = \frac{1}{2}(u+v)$ and $y = v$. Thus

$$J_{u,v}(x,y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{vmatrix} = \tfrac{1}{2}$$

and so $dx\,dy = \frac{1}{2}du\,dv$; from this

$$I = \int\int_R \cos\left[(2x-y)^2\right]\, dxdy = \tfrac{1}{2}\int\int_{R^*} \cos(u^2)\,dudv$$

Now R^* is the diagram on the right with $v = -u$ as the lower boundary and $v = u$ as the upper boundary. Integrating over v first

$$I = \tfrac{1}{2}\int\int_{R^*} \cos(u^2)\,dudv = \tfrac{1}{2}\int_0^1 \left(\int_{v=-u}^{v=u} dv\right)\cos(u^2)du = \int_0^1 u\cos(u^2)du$$

The last integral can be evaluated using the substitution $\theta = u^2$ to give $\frac{1}{2}\int_0^1 \cos\theta\, d\theta = \frac{1}{2}\sin 1$.

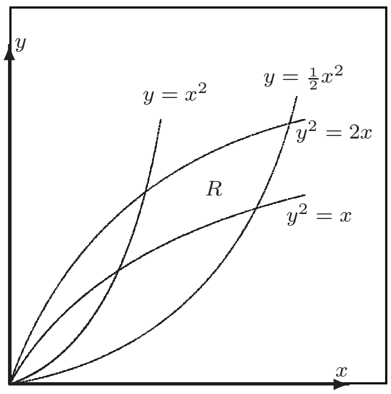
4. A region R is such that its upper and lower boundaries are the pair of parabolae $y^2 = 2x$, $y^2 = x$ and its left and right boundaries are the pair of parabolae $y = x^2$, $y = \frac{1}{2}x^2$. Show that the transformation

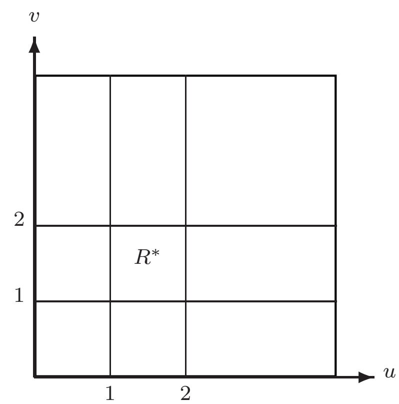
$$u = x^2/y, \qquad v = y^2/x$$

maps R into a square R^* in the (u, v) plane. Show also that $du\,dv = 3\,dx\,dy$ and

$$I = \int\int_R xy\,dxdy = 3/4.$$

Solution: The two regions regions R and R^* are sketched below; it is clear that the region R between the 4 parabolae maps to the region R^* between the 4 lines $u = 1$, 2 and $v = 1$, 2.





The Jacobian for this transformation is more easily found in the form $J_{x,y}(u, v)$ because we more easily have u and v as functions of x and y

$$u = x^2/y, \qquad v = y^2/x$$

$$J_{x,y}(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x/y & -x^2/y^2 \\ -y^2/x^2 & 2y/x \end{vmatrix} = 3$$

Thus $dudv = J_{x,y}(u, v)dxdy = 3dxdy$ or $dxdy = \frac{1}{3}dudv$. Noting that $xy = uv$ we have

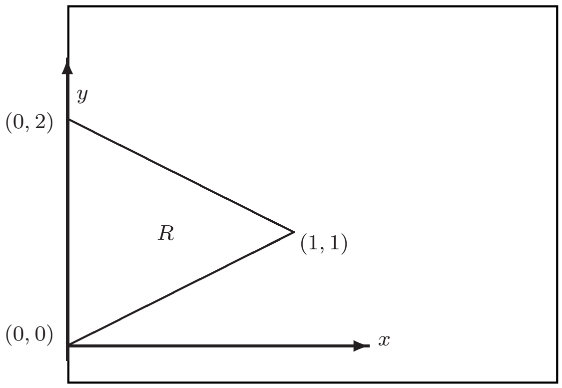
$$I = \int\int_R xy\,dxdy = \tfrac{1}{3}\int\int_{R^*} uv\,dudv = \tfrac{1}{3}\int_1^2 \left(\int_1^2 vdv\right) udu = 3/4$$

5. It is possible to evaluate the double integral

$$I = \int_0^1 x \left(\int_x^{2-x} \frac{dy}{y} \right) dx$$

as it stands but it can be done differently by changing the order of integration.

Solution: The diagram below, deduced from the limits $y = x$ and $y = 2 - x$, shows the region of integration R:



Because of the geometry, we split R into two right-angled triangles, both sharing the line segment parallel to the $x-$axis from $(0, 1)$ to $(1, 1)$, with the top triangle having its third vertex at $(0, 2)$ and the bottom one at the origin.

So for the bottom triangle, integration over y is now from 0 to 1 in the vertical, and horizontally, over x from 0 to the diagonal $x = y$, and the integral is

$$\int_0^1 \frac{1}{y} \left(\int_0^y x\,dx \right) dy = \int_0^1 \frac{1}{y} \left[\frac{x^2}{2} \right]_0^y dy = \frac{1}{2} \int_0^1 \frac{1}{y} (y^2)\ dy = \frac{1}{4}$$

Similarly, for the top triangle, integration over y is now from 1 to 2 in the vertical, and horizontally over x from 0 to the diagonal $x = 2 - y$ and the integral is

$$\int_1^2 \frac{1}{y} \left(\int_0^{2-y} x\,dx \right) dy = \int_1^2 \frac{1}{y} \left[\frac{x^2}{2} \right]_0^{2-y} dy = \frac{1}{2} \int_1^2 \frac{1}{y} (2-y)^2\ dy = \frac{1}{2} \left[4 \ln y - 4y + \frac{y^2}{2} \right]_1^2 = 2 \ln 2 - \frac{5}{4}\,,$$

so the total result is $I = 2 \ln 2 - 1\,$.

6. The following integral cannot be evaluated in its present form

$$I = \int_0^{a/2} \left(\int_{2x}^{a} \exp\left(y^2\right)\, dy \right)\, dx.$$

Reverse the order of integration and solve. In this problem there is no splitting of the region of integration. Assume $a > 0$.

Solution: The region of integration, deduced from the limits, is a triangle in the first quadrant with one side on the $y-$axis, corners at $(0, a)\,, (a/2, a)$ and the origin. Reversing the order of integration we have integrate first with respect to x, using a horizontal thin strip from $x = 0$ to $x = \dfrac{1}{2}y$, then wrt y, limits $0 \dots a$. Then

$$I = \int_{y=0}^{a} \exp\left(y^2\right) \left(\int_{x=0}^{x=y/2} dx \right)\, dy\,,$$

and the first integration is easy:

$$I = \int_{y=0}^{a} \exp\left(y^2\right) \frac{1}{2} y\, dy$$

and so

$$I = \frac{1}{4}\left[\exp\left(y^2\right)\right]_{y=0}^{a} = \frac{1}{4}\left[\exp\left(a^2\right) - 1\right]\,.$$