EE2-08C Mathematics
Solutions to Example Sheet 4: Green's/Stokes/Gauss Theorems

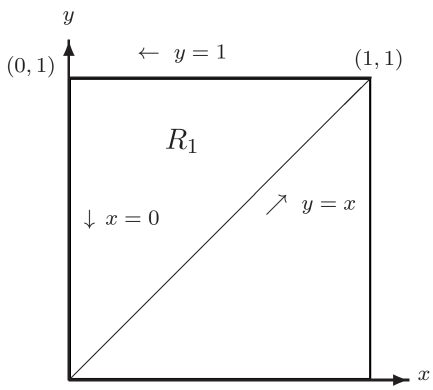
1. Use Green's Theorem to convert the following line integrals to double integrals and hence evaluate them:

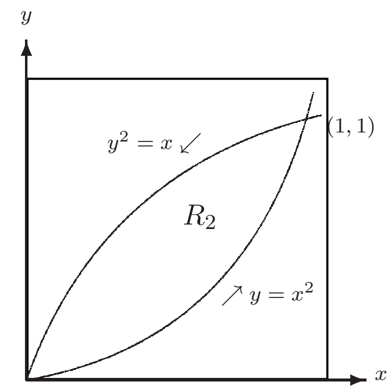
   (a) $\oint_C \left[6xy\,dx + (2x^3y + 3x^2)\,dy\right]$ where C is the triangle with vertices $(0,0)$, $(1,1)$ and $(0,1)$.

   (b) $\oint_C \left[(2xy - x^2)\,dx + (x + y^2)\,dy\right]$. C is the boundary of the area enclosed by the parabolae $y = x^2$ and $y^2 = x$.

   Solution:

   The pictures for the two paths in a) and b) are given below:





a) C is the triangle with vertices $(0,0)$, $(1,1)$ and $(0,1)$. $P = 6xy$, $\quad Q = 2x^3y + 3x^2$ so $Q_x - P_y = 6x^2y$. By Green's Theorem

$$\begin{aligned}\oint_C \left[6xy\,dx + (2x^3y + 3x^2)\,dy\right] &= 6\int\int_{R_1} x^2 y dx dy \\ &= 6\int_0^1 x^2 \left(\int_{y=x}^{y=1} y dy\right) dx \\ &= 3\int_0^1 x^2(1 - x^2) dx = 2/5\end{aligned}$$

b) C is the boundary of the area enclosed by the parabolae $y = x^2$ and $y^2 = x$. $P =$

$2xy - x^2$, $\;Q = x + y^2$. Thus $Q_x - P_y = 1 - 2x$ and Green's Theorem gives

$$\begin{aligned}
\oint_C \left[(2xy - x^2)\,dx + (x + y^2)\,dy\right] &= \int\int_{R_2} (1-2x)dxdy \\
&= \int_0^1 (1-2x)\left(\int_{x^2}^{x^{1/2}} dy\right)dx \\
&= \int_0^1 (1-2x)\left(x^{1/2} - x^2\right)dx = 1/30
\end{aligned}$$

2. By choosing $P = \frac{x^2}{x+y}$ and $Q = -\frac{y^2}{x+y}$ in Green's Theorem, show that

$$\int\int_R \frac{x^2+y^2}{(x+y)^2}\,dx\,dy = \tfrac{1}{2}\,,$$

where R is the first quadrant of the circle $x^2 + y^2 = 1$. *Hint: In the line integral you will have 3 sections. On the curved part of C, look for a factorization which gives a term which will cancel with the denominator.*

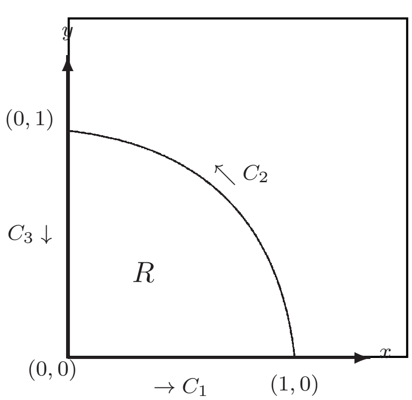
Solution:
Take $P = \frac{x^2}{x+y}$ and $Q = -\frac{y^2}{x+y}$ so

$$Q_x - P_y = \frac{x^2+y^2}{(x+y)^2}$$

which we use in Green's Theorem. Thus

$$\int\int_R \frac{x^2+y^2}{(x+y)^2}\,dx\,dy = \oint \frac{x^2dx - y^2dy}{x+y}$$

where R is the first quadrant of the circle $x^2 + y^2 = 1$.



C_1: $y = 0$ with $x:\ 0 \to 1$ and $\int_0^1 x\,dx = \frac{1}{2}$.

C_3: $x = 0$ with $y:\ 1 \to 0$ and $-\int_1^0 y\,dy = \frac{1}{2}$.

C_2: On the quarter circle $x^2 + y^2 = 1$ with $x:\ 1 \to 0$ we have $xdx = -ydy$, so

$$\int_{C_2} \frac{x^2dx - y^2dy}{x+y} = \int_1^0 x\,dx = -\tfrac{1}{2}$$

Thus $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} = \frac{1}{2}$.

3. Given $\vec{F} = F\vec{k}$ and S the hemisphere of radius R, shown in the figure (see exercise sheet).

obtain $\displaystyle\int\int_S \vec{F}\cdot\mathbf{d}\vec{s}$.

Solution: The integral measures the flux passing through the hemisphere S. As $\vec{F}$ is only in the $\vec{k}-$direction, the flux leaving the hemisphere through S is the same as the flux entering the hemisphere through its base, a disc of radius R: we note that both have the same boundary, a circle of radius R in the $xy-$plane. Call this disc S', then

$$\int\int_S \vec{F}\cdot\mathbf{d}\vec{s} = \int\int_{S'} \vec{F}\cdot\mathbf{d}\vec{s}.$$

For S', a vector element of area $d\vec{s}$ is perpendicular to the $xy-$plane, so $d\vec{s} = \vec{k}\ dx\ dy$. Hence

$$F\cdot\mathbf{d}\vec{s} = F\vec{k}\cdot\vec{k}\ dx\ dy = F\ dx\ dy$$

and

$$\int\int_{S'} \vec{F}\cdot\mathbf{d}\vec{s} = \int\int_{S'} F\ dx\ dy\ =\ F(\text{ Area of } S') = F\pi R^2\,.$$

4. Consider the vector field $\boldsymbol{F} = (y^2 - \sin x)\underline{\boldsymbol{i}} + xy^2\underline{\boldsymbol{j}} + (5 - z^3)\underline{\boldsymbol{k}}$, and $I = \displaystyle\oint_C \boldsymbol{F}\cdot d\boldsymbol{r}$, where C is the unit circle: $x^2 + y^2 = 1$. Obtain the value of I.

Solution: This is good for an application of Stokes' theorem in 2D:

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int\int_R (\text{curl}\boldsymbol{F})\cdot\underline{\boldsymbol{k}}\ dx\ dy$$

because the curve C is very simple and in 2D encloses the unit disc. We begin by calculating $\text{curl}(\boldsymbol{F}) = \underline{\boldsymbol{k}}(\partial_x(xy^2) - \partial_y(y^2 - \sin x) = \underline{\boldsymbol{k}}(y^2 - 2y)$ so that $(\text{curl}\boldsymbol{F})\cdot\underline{\boldsymbol{k}} = y^2 - 2y$, a scalar field, which we now integrate over the unit disc. Stokes' Theorem now gives that

$$\oint_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int\int_R y^2 - 2y\ dx\ dy$$

and we change to polar coordinates $y = r\sin\theta$ and $dx\ dy = r\ dr\ d\theta$ giving

$$\int\int_R y^2 - 2y\ dx\ dy = \int_{\theta=0}^{2\pi}\left\{\int_0^1 \left[(r\sin\theta)^2 - 2r\sin\theta\right] r\ dr\right\}\ d\theta = \int_0^{2\pi} \sin^2\theta\left[\frac{r^4}{4}\right]_0^1 - \sin\theta\left[\frac{2r^3}{3}\right]_0^1\ d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}\sin^2\theta - \frac{2}{3}\sin\theta\ d\theta = \frac{\pi}{4}\,,$$

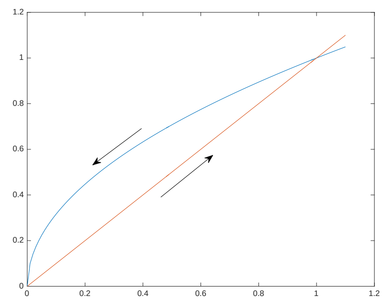
substituting $\sin^2 x = (1 - \cos 2x)/2$ in the last step, before integrating.

5. The volume $\mathcal{V}$ is the cube with one corner at the origin and another at the point $(1, 1, 1)$. The (closed) surface of $\mathcal{V}$ is denoted by $\mathrm{d}S$. Define $\vec{G} = 2x^2yz\underline{\boldsymbol{i}} - xy^2z\underline{\boldsymbol{j}} - xyz^2\underline{\boldsymbol{k}}$. Find the value of

$$\oiint_S \vec{G} \cdot \mathrm{d}\vec{S}\,.$$

Solution: Use divergence/Gauss' theorem.
$\nabla \cdot \vec{G} = 4xyz - 2xyz - 2xyz = 0 \Rightarrow \oiint_S \vec{G} \cdot d\vec{S} = \int\int\int_{\mathcal{V}} \nabla \cdot \vec{G}\ dV = 0$, trivially.

6. The figure below shows a closed contour C, given by the line $y = x$ from the origin to the point $(1, 1)$, and the curve $x = y^2$ from $(1, 1)$ back to the origin.



Find $\displaystyle\oint_C (y^3 + 3x^2y)\ dx + (x^3 - 2y^2)\ dy\,.$

Solution: This is good to apply Green's theorem. Identify $P(x, y) = y^3 + 3x^2y$ and $Q(x, y) = x^3 - 2y^2$, then $Q_x - P_y = 3x^2 - (3y^2 + 3x^2) = -3y^2$. Hence by GT, the line integral is equal to the double integral over the region R enclosed by the contour C:

$$\oint_C (y^3 + 3x^2y)\ dx + (x^3 - 2y^2)\ dy = \int\int_R -3y^2\ dx\ dy$$

and we use horizontal strips to integrate first w.r.t x, so that

$$I = -3\int_0^1 y^2 \left(\int_{x=y^2}^{x=y} 1\ dx \right)\ dy = -3\int_0^1 y^2(y - y^2)\ dy = -\frac{3}{20}$$

after a bit of calculation.