Mathematics - Numerical Analysis

Problem Sheet 0: Trapezoid Rule

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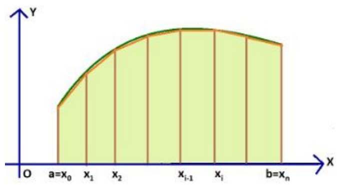
1 Trapezoid Rule

Not every integral has an exact solution. To approximate a solution we can employ numerical techniques, the simplest of which is the Trapezoidal Rule. Consider the integral of $f(x)$ over the interval[a,b] begin by dividing the interval into n segments of equal length $h = \frac{b-a}{n}$.



1. Set $a = x_0$ and $b = x_n$, then the n intervals are $[x_i, x_{i+1}]$ for $i = 0 \ldots n-1$. The area under the curve $f(x)$ now consists of integrating over each segment and summing:

$$\int_{x_i}^{x_{i+1}} f(x)\,dx$$

2. For each segment, approximate the integral with the area of a trapezoid of left height $f(x_i)$and right height $f(x_{i+1})$, giving an approximate area:

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{1}{2}h[f(x_i) + f(x_{i+1})]$$

3. To see a simple case, let $n = 3$, so $a = x_0$ and $b = x_3$:

$$\frac{1}{2}h[f(x_0) + f(x_1)] + \frac{1}{2}h[f(1) + f(x_2)] + \frac{1}{2}h[f(2) + f(x_3)]$$

which can be tidied up as:

$$\frac{1}{2}h[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)]$$

The general case is defined as:

$$I_n = \int_a^b f(x) \approx \left(\frac{b-a}{2n}\right)\left[f(x_0) + 2\left(\sum_{i=1}^{n-1} f(x_i)\right) + f(x_n)\right]$$

where $x_0 = a$, $x_1 = \frac{1}{n}$, $x_2 = 2 \times \frac{1}{n}$, $\ldots$, $x_n = b$

$$I_4 = \left(\frac{1-0}{2*4}\right)\left[\exp(0) + 2\left[\exp(\frac{1}{4})\right] + 2\left[\exp(2\times\frac{1}{4})\right] + 2\left[\exp(3\times\frac{1}{4})\right] + \exp(1)\right]$$

2 Exercise 1

- Use the Trapezoidal Rule to obtain:

  $$I = \int_0^1 e^x dx = e - 1 \approx 1.718281828$$

  where I_n is the approximate solution with n segments

  1. Begin by obtaining I_4.
  2. Increase n to improve precision.
  3. Let ϵ_n be the error made with segment length $h = \frac{(b-a)}{n}$. For each n, calculate $\epsilon_n = |I - I_n|$.

- Tabulate results and compare n and the error ϵ_n. Is there any relationship between the error and the number of segments?

- Write a simple Matlab code to investigate this:
  - Given a function $f(x)$, an interval $[a, b]$ and a number of segments n, your code should integrate the function numerically over $[a, b]$ with the interval divided into n segments. Compare with the exact solution.
  - Next, incorporate the code written into a loop to investigate the effect of increasing n on the error. Integrate for $n = 2...N$ where N is chosen to be suitably large. Plot the error against n. Try a log-log plot for ϵ_n and h, what is the conclusion?
  - Test this on several functions and intervals.

3 Exercise 2

Now consider Simpson's rule which is a refinement of the trapezoidal rule. Take n - an even number and proceed as follows:

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{3n}\right)\left[f(x_0) + 2\left(\sum_{i=1}^{\frac{n}{2}-1} f(x_{2i})\right) + 4\left(\sum_{i=1}^{\frac{n}{2}} f(x_{2i-1})\right) + f(x_n)\right]$$

To understand the rule, let $n = 6$:

$$\frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\right]$$

1. Repeat the previous exercise and error analysis with Simpson's rule.
2. What are the observations?
3.