

EE2-08 Mathematics

Solutions to Example Sheet 5: Functions of a complex variable

1. Verify that the following satisfy the Cauchy-Riemann equations:

- a) $u = x; \quad v = y,$
- b) $u = e^x \cos y; \quad v = e^x \sin y,$
- c) $u = x^3 - 3xy^2; \quad v = 3x^2y - y^3$

Solution:

To verify that the following satisfy the Cauchy-Riemann equations $u_x = v_y$ $v_x = -u_y$:

- a) $u_x = 1 \quad v_y = 1; u_y = v_x = 0. \quad \therefore$ CR equations satisfied.
 - b) $u = e^x \cos y \Rightarrow u_x = e^x \cos y, \quad u_y = -e^x \sin y.$
 $v = e^x \sin y \Rightarrow v_y = e^x \cos y, \quad v_x = e^x \sin y. \quad \therefore$ CR equations satisfied.
 - c) $u = x^3 - 3xy^2 \Rightarrow u_x = 3x^2 - 3y^2, \quad u_y = -6xy.$
 $v = 3x^2y - y^3 \Rightarrow v_x = 6xy, \quad v_y = 3x^2 - 3y^2. \quad \therefore$ CR equations satisfied.
2. Show that the following functions $u(x, y)$ each satisfy Laplace's equation and then use the Cauchy-Riemann equations to determine the conjugate function v . Find also $f(z) = u + iv$.

- a) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1,$
- b) $u = xy.$

Solution:

(a) With $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ we have $u_x = 3x^2 - 3y^2 + 6x$ and $u_y = -6xy - 6y$. Therefore $u_{xx} = 6x + 6$ and $u_{yy} = -6x - 6$ and so $u_{xx} + u_{yy} = 0$. Because u satisfies Laplace's equation, there exists a conjugate function $v(x, y)$ that satisfies the CR equations: $u_x = v_y, \quad v_x = -u_y$. To find v we integrate these

$$v_y = u_x = 3x^2 - 3y^2 + 6x \Rightarrow v = \int (3x^2 - 3y^2 + 6x) dy + A(x)$$

$$v_x = -u_y = 6xy + 6y \Rightarrow v = \int (6xy + 6y) dx + B(y)$$

where $A(x)$ and $B(y)$ are arbitrary functions of x and y respectively. The solution(s) for v must be the same from each equation; together we find that $v = 3x^2y - y^3 + 6xy + c$ where $A = c$ and $B = c - y^3$ with c as an arbitrary constant. In combination $f(z) = u + iv = z^3 + 3z^2 + ci$.

(b) $u = xy$ we have $u_x = y$ and $u_y = x$. Therefore $u_{xx} = 0$ and $u_{yy} = 0$ and so $u_{xx} + u_{yy} = 0$. Because u satisfies Laplace's equation, there exists a conjugate function $v(x, y)$ that satisfies the CR equations: $u_x = v_y, \quad v_x = -u_y$. To find v we integrate these

$$v_y = u_x = y \Rightarrow v = \int y dy + A(x)$$

$$v_x = -u_y = -x \Rightarrow v = -\int x dx + B(y)$$

Together we find that $v = \frac{1}{2}(y^2 - x^2) + c$ where $A(x) = -\frac{1}{2}x^2 + c$ and $B(y) = \frac{1}{2}y^2 + c$. In combination we find that $f(z) = u + iv = -\frac{1}{2}iz^2 + \text{const.}$

3. Show that the function

$$u(x, y) = (x \cos y - y \sin y) e^x$$

satisfies Laplace's equation. Find the conjugate function $v(x, y)$ for which u and v together satisfy the Cauchy-Riemann equations and hence find in its simplest form $w = u + iv = f(z)$ where $z = x + iy$.

Solution:

To show that the function $u(x, y) = e^x (x \cos y - y \sin y)$ satisfies Laplace's equation:

$$\begin{aligned} u_x &= e^x (\cos y + x \cos y - y \sin y) \Rightarrow u_{xx} = e^x (2 \cos y + x \cos y - y \sin y) \\ u_y &= -e^x (x \sin y + \sin y + y \cos y) \Rightarrow u_{yy} = -e^x (x \cos y + 2 \cos y - y \sin y) \end{aligned}$$

Thus Laplace's equation $u_{xx} + u_{yy} = 0$ is satisfied and we can find a conjugate function v :

$$\begin{aligned} v_y = u_x &\Rightarrow v = \int [e^x (\cos y + x \cos y - y \sin y)] dy + A(x) \\ v_x = -u_y &\Rightarrow v = \int [e^x (x \sin y + \sin y + y \cos y)] dx + B(y) \end{aligned}$$

The (partial) integrations are messy but give

$$v = e^x (x \sin y + y \cos y) + C$$

where $A = B = C = \text{const.}$ For $f(z) = u + iv$ together we have

$$\begin{aligned} f(z) &= e^x (x \cos y - y \sin y + ix \sin y + iy \cos y) + c \\ &= e^x z (\cos y + i \sin y) + c \\ &= e^z z + c \end{aligned}$$

having used $e^{iy} = \cos y + i \sin y$.

4. Consider the mapping $w = \frac{1}{z-1}$ from the z -plane to the w -plane.
a) Show that in the z -plane, the circle

$$(x-1)^2 + y^2 = 4$$

maps to a circle in the w -plane. What is the radius of this circle and where is its centre?

Solution:

The mapping $w = \frac{1}{z-1}$ from the z -plane to the w -plane can be written as

$$w = u + iv = \frac{1}{x-1+iy} = \frac{(x-1)-iy}{(x-1)^2+y^2}$$

$$u = \frac{x-1}{(x-1)^2 + y^2} \quad v = -\frac{y}{(x-1)^2 + y^2} \quad \Rightarrow \quad u^2 + v^2 = \frac{1}{(x-1)^2 + y^2}$$

Then the circle $(x-1)^2 + y^2 = 4$ maps to $u^2 + v^2 = \frac{1}{4}$, which is a circle in the w -plane, of radius $\frac{1}{2}$ centred at $(0,0)$.

b) To what curve does the line $x = 0$ in the z -plane map in the w -plane?

Solution:

The line $x = 0$ in the z -plane gives values of u, v

$$u = -\frac{1}{1+y^2} \quad v = -\frac{y}{1+y^2} \quad \Rightarrow \quad u^2 + v^2 = \frac{1}{1+y^2}$$

Hence $u^2 + v^2 = -u$ which, on completing the square, becomes $(u + \frac{1}{2})^2 + v^2 = \frac{1}{4}$. This is a circle in the w -plane, of radius $\frac{1}{2}$ centred at $(-\frac{1}{2}, 0)$.

5. a) Fixed points of a map $w = f(z)$ occur when $w = z$. Show that the fixed points of $w = \frac{4z-2}{z+1}$ occur at $z = 1$ and $z = 2$.

Solution:

For fixed points of $w = \frac{4z-2}{z+1} = z$ solve $z(z+1) = 4z-2$. Roots occur at $z = 1$ and $z = 2$.

- b) For $w = u + iv = \frac{4z-2}{z+1}$ show that the image in the w -plane of the line $x = 0$ is the circle $(u-1)^2 + v^2 = 9$. What is the image in the w -plane of the unit circle $|z| = 1$?

Solution:

For $w = u + iv = \frac{4z-2}{z+1}$ we have

$$u + iv = \frac{4z-2}{z+1} = \frac{4x-2+4iy}{x+1+iy}$$

Thus solving for u and v through rationalisation of the denominator

$$u = \frac{4(x^2 + y^2) + 2(x-1)}{(x+1)^2 + y^2} \quad v = \frac{6y}{(x+1)^2 + y^2} \quad \Rightarrow \quad (u-1)^2 + v^2 = \frac{9[x^2 + y^2 - 1]^2 + 36y^2}{[(x+1)^2 + y^2]^2}$$

- (i) When $x = 0$ in the z -plane then this reduces to $(u-1)^2 + v^2 = 9$. This is a circle in the w -plane of radius 3 centred at $(1,0)$.

- (ii) For the circle $|z| = 1$ in the z -plane we have $x^2 + y^2 = 1$ which means that

$$u = \frac{2x+2}{2x+2} = 1 \quad v = \frac{6y}{2x+2}$$

Hence in the w -plane we have the vertical line $u = 1$ for all values of v .