EE2-08 Complex Variables and Laplace Transforms

Lecture notes

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(Based on notes developed by J.D. Gibbon)

Chapter 1

Analytic Functions

1.1 Derivation of the Cauchy-Riemann equations

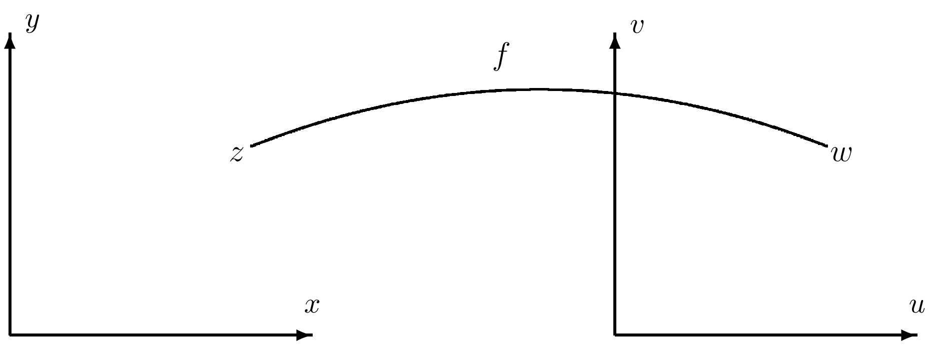
We consider a function of the complex variable $z = x + iy$

$$w = f(z)\,, \tag{1.1.1}$$

expressed in the usual manner except that the independent variable $z = x + iy$ is complex. So w will also be complex, and $w = f(z)$ has a real part $u(x, y)$ and an imaginary part $v(x, y)$: **GAP**

(1.1.2)

We think of this as a mapping from one plane to another:

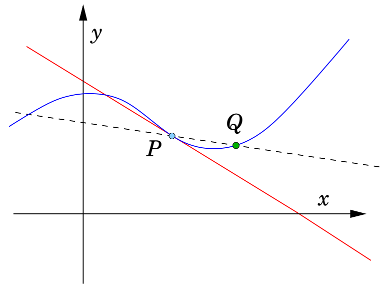


Example 1.1.1.

$$w = f(z) = z^2 \tag{1.1.3}$$

GAP

Extra difficulties appear in differentiating and integrating such functions because, for one thing, z varies in a plane and not on a line. What do we mean by a derivative? For functions of a single real variable $x \in \mathbb{R}$, the idea of an incremental change δx along the x-axis gave us an easy, intuitive grasp of differentiation:



In the complex plane, the increment δx has to be replaced by an incremental change δz. Because δz is a vector the question of the direction of this limit becomes an issue.

Firstly we look at the concept of differentiation. The definition of a derivative at a point z_0 remains the same as usual ; namely

GAP

$$(1.1.4)$$

In $\mathbb{R}$, the idea of the limit as $\delta x \to 0$ was simple: all we required is that we have the same limit approaching 0 from both sides. The subtlety here lies in the limit $\delta z \to 0$ because δz is itself a vector and therefore the limit $\delta z \to 0$ may be taken in many directions. If the limit in (1.1.4) is to be unique (to make any sense) *it is required that it be independent of the direction in which the limit $\delta z \to 0$ is taken.* If this is the case then it is said that

GAP

There is a general test on functions to determine whether (1.1.4) is independent of the direction of the limit. The simplest way is to firstly take the limit in the horizontal direction : that is $\delta z = \delta x$, in which case

$$(1.1.5)$$

GAP

Next we take the limit in the vertical direction: that is $\delta z = i\delta y$

GAP

(1.1.6)

If the limits in both directions are to be equal, then df/dz in (1.1.5) and (1.1.6) must be equal:

GAP

and equating real and imaginary parts we have: **GAP**

(1.1.7)

The boxed pair of equations above are known as **the Cauchy-Riemann equations**. If these hold at a point z then $f(z)$ is said to be differentiable at z: they are a **necessary** condition for the limits to be the same and the function to be differentiable. As it turns out, the interested reader may look in Churchill/Ward Brown, or other texts.

There is no similar requirement for the existence of a limit in single variable calculus. Thus the CR equations bring us to a further idea regarding differentiation in the complex plane:

Definition: *If $f(z)$ is differentiable at all points in a neighbourhood of a point z_0 then $f(z)$ is said to be*

GAP

A function which is analytic for all $z \in \mathbb{C}$ is said to be **holomorphic** *(entire).*

Example 1.1.2. *Show that $f(z) = z^2$ is holomorphic.*

We already have

$$u(x, y) = x^2 - y^2\,, \qquad\qquad v(x, y) = 2xy\,. \tag{1.1.8}$$

Clearly, four trivial partial derivatives show that

GAP

thus demonstrating that the CR equations hold for all values of x and y. It follows that $f(z) = z^2$ is differentiable at all points in the z-plane and every point in this plane has an (infinite) neighbourhood in which $f(z) = z^2$ is differentiable. Clearly $f(z) = z^2$ is analytic everywhere: it is holomorphic.

Some functions are analytic everywhere in the complex plane except at certain points: these points are called *singularities*. The following examples illustrate this.

Example 1.1.3. *Consider* $f(z) = \dfrac{1}{z}$. *Show that the function is analytic everywhere except the origin.*

Writing

GAP

$$\tag{1.1.9}$$

Exercise: show that the CR equations hold everywhere except at the origin $z = 0$ where the limit is indeterminate: $z = 0$ is the point where it fails to be differentiable. Hence $w = z^{-1}$ is analytic everywhere except at $z = 0$.

Example 1.1.4. *Show that* $f(z) = |z|^2$ *is analytic nowhere in the complex plane.*

We have **GAP**

$$\tag{1.1.10}$$

and so **GAP**

$$\tag{1.1.11}$$

Clearly the CR equations do **not** hold anywhere except at $z = 0$. Therefore $f(z) = |z|^2$ is not differentiable anywhere except at $z = 0$ and there is no neighbourhood around $z = 0$ in which it is differentiable. Thus the function is analytic nowhere in the z-plane.

As a final remark on this example let us look again at the limit $\delta z \to 0$, by considering $f(z) = |z|^2 = z\overline{z}$:

$$\left.\frac{df}{dz}\right|_{z=z_0} = \tag{1.1.12}$$

GAP

Now consider the last limit, with $z = z_0 + \delta z$ describing a circle of radius δz around z_0:

GAP

If we let $r = |\delta z|$ we can describe the circle in complex exponential form as

GAP

As $\delta z \to 0$, we have $r \to 0$, but there is no r in the expression for the limit, leaving

GAP

(1.1.13)

This result illustrates the problem: as θ, and with it the direction of the limit, varies, so does the limit. **This is clearly not unique except when $z_0 = 0$.** The example illustrates the idea of differentiability geometrically, but also introduces the idea of representing z on a circle as $z = re^{i\theta}$ which we will use repeatedly.

1.2 Properties of analytic functions

Let us consider the CR equations $u_x = v_y$ and $u_y = -v_x$ as a condition for the analyticity of a function $w = u(x, y) + iv(x, y)$. If we differentiate the first equation wrt x and the second equation wrt y we have

GAP

(1.2.1)

thus showing that u must always be a solution of Laplace's equation. Similarly, if we differentiate the first equation wrt y and the second equation wrt x, we get $v_{xx} + v_{yy} = 0$.

Laplace's equation is used to describe the behaviour of many physical entities: steady heat diffusion, incompressible fluid flow, elasticity, and, more to the point, electrostatics and magnetostatics. Solutions to Laplace's equation are called

GAP

It is also said that

GAP

In the following set of examples it will be shown how, given a harmonic function $u(x, y)$, its conjugate $v(x, y)$ can be constructed. The pair can then be put together as $u + iv = f(z)$ to ultimately find $f(z)$. Given a harmonic function $u(x, y$, we proceed as follows:

- Check that it is harmonic: $\nabla^2 u = 0$;
- If yes, integrate the Cauchy-Riemann equations to find the harmonic conjugate $v(x, y)$;
- Join together to form $f(z) = u + iv$.

[Note: it is equally simple to begin with a harmonic functtion $v(x, y)$ and obtain its conjugate $u(x, y)$.]

Example 1.2.1. *Given that $u = x^2 - y^2$ show (i) that it is harmonic; (ii) find $v(x, y)$ and then (iii) construct the corresponding complex function $f(z)$.*

With $u = x^2 - y^2$

GAP

so it satisfies Laplace's equation. This is a sufficient condition for v to exist and for us to write

GAP

While there are two PDEs here there can only be one solution compatible with both, recall the solution of exact first-order ODEs. Integrating them both in turn gives

GAP

It is clear that they are compatible if $A(x) = B(y) = \text{const} = c$ making the result

GAP

Example 1.2.2. *Given that $u = x^3 - 3xy^2$ find its conjugate function $v(x, y)$ and the corresponding complex function $f(z)$.*

We first check that $u = x^3 - 3xy^2$ satisfies Laplace's equation:

GAP

Thus $u_{xx} + u_{yy} = 0$ and so v exists and is found from the CR equations:

GAP

The last step is *not* inspired guesswork! Coefficients $1, 3, 3, 1$ and powers x^3, x^2y, xy^2, y^3 suggest looking for z^3 or something similar.

Exercise: For $v(x, y) = x^3 - 3xy^2$, find the conjugate function u, and hence $f(z)$.

Example 1.2.3. *Given*

$$u(x, y) = e^x \big(x \cos y - y \sin y\big)$$

show that u satisfies Laplace's equation. Also find its conjugate v and then $f(z)$. Left as exercise.

Fact (not shown, see literature): all polynomial functions $P(z)$, trigonometric functions $\cos z$ and $\sin z$ and the exponential function e^z are holomorphic, i.e. analytic everywhere. The tangent function is also analytic wherever $\cos z \neq 0 \Rightarrow z \neq (2n + 1)\pi/2$. In fact, all rational functions $\dfrac{P(z)}{Q(z)}$ where P and Q are analytic, are also analytic wherever $Q(z) \neq 0$. Similarly the composition of two analytic functions is analytic, for example $\sin(z^2 + iz)$.

Differentiation thus follows familiar patterns for analytic functions: $f(z) = \sin z \Rightarrow f'(z) = \cos z$, and so on. See Churchill/Ward Brown for more detail.

Differentiation thus follows familiar patterns for analytic functions: $f(z) = \sin z \Rightarrow f'(z) = \cos z$, and so on. Showing these facts is beyond the time available for this module.

1.3 Orthogonality

Let us finally consider the family of curves on which $u =$ const. Recall the total differential from year one: **GAP**

(1.15)

Consider the curves where u is constant. Here we have $du = 0$, giving the gradient on this family as **GAP**

(1.16)

Likewise, on the family of curves of constant v **GAP**

(1.17)

giving **GAP**

(1.18)

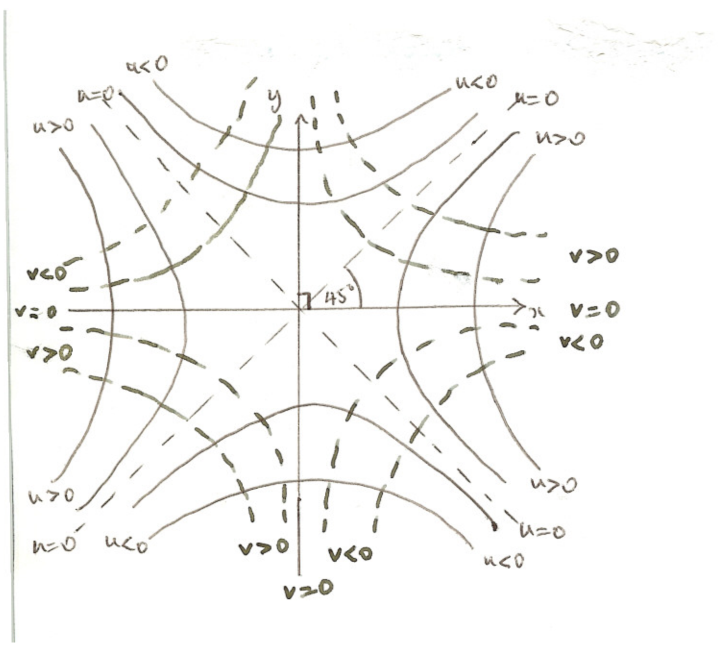
Now if $f(z)$ is analytic in a region R then the C-R equations hold there, $u_x = v_y$ and $u_y = -v_x$, and (1.18) becomes **GAP**

(1.19)

The final result is that in regions of analyticity curves of constant u and curves of constant v are always orthogonal.

Example 1.8. *We know $f(z) = z^2$ is analytic everywhere. Sketch the curves of constant u and the curves of constant v and confirm visually that they are orthogonal.*

We have $f(z) = x^2 - y^2 + 2xyi$ **GAP**



1.4 Conformal mappings

A complex mapping $w = f(z)$ maps a region R in the z-plane to a different region R^* in the w-plane.

A complex function $w = f(z)$ can be thought of as a mapping from the z-plane to the w-plane. Depending on $f(z)$ the mapping may not be unique. For instance, for $w = z^2$ for the values $\pm z_0$ there is one value w_0, so the function is not one-to-one. Complex mappings do not necessarily behave in an expected way. The concept of analyticity intrudes into these ideas in the following way.

Definition A mapping is said to be
GAP

Moreover, a mapping has a *fixed point* when $w = f(z) = z$.

Theorem 1. *The mapping defined by an analytic function $w = f(z)$ is conformal except at points where $f'(z) = 0$.*

Example 1.4.1. *$w = z^2$ is conformal everywhere except at $z = 0$ because $f'(0) = 0$. Plotting contours of $u = x^2 - y^2$ and $v = 2xy$ shows that conformality fails at the origin.*

See plot on previous page. Contours of $u, v = const$ in the z-plane: note their orthogonality except at $z = 0$ where conformality fails.

Example 1.4.2. *Consider $w = \dfrac{1}{z-1}$ which is analytic everywhere except at $z = 1$.*

We have **GAP**

(1.4.1)

in which case **GAP**

(1.4.2)

It is clear from (1.4.2) that it is always true that **GAP**

(1.4.3)

So far we have specified no shape in the z-plane on which this map operates. Some examples of specific shapes show more clearly what this map will do:

1. Consider the family of circles in the z-plane:

   GAP

   These circles are centred at (1, 0) of radius a. Clearly they map to **GAP**

   (1.4.4)

   which is a family of circles in the w-plane centred at (0, 0) of radius a^{-1}. As the value of a is increased the circles in the z-plane widen and those in the w-plane decrease. It is not difficult to show that the interior (exterior) of the circles in the z-plane map to the exterior (interior) of those in the w-plane. Thus we have

   $$\begin{array}{rl} \underline{z\text{-plane}} & \underline{w\text{-plane}} \\ \text{interior} \rightarrow & \text{exterior} \\ \text{exterior} \rightarrow & \text{interior} \end{array} \qquad (1.4.5)$$

   The circle centre (1, 0) in the z-plane maps to *the point at infinity* in the w-plane.

2. The line $x = 0$ in the z-plane maps to what? From (1.4.2) and (1.4.3) we know that

   GAP

   (1.4.6)

   (1.4.7)

   In the w-plane this is a circle of radius $\frac{1}{2}$ centred at $(-\frac{1}{2}, 0)$.

Thus we conclude that some circles can map to other circles but also straight lines can map to circles. This is investigated in the next section.

Exercise:

1) Show that the $x-$axis in the $z-$plane is mapped to a line in the $w-$plane.
2) Show that the line $y = x$ in the $z-$plane is mapped to a circle in the $w-$plane.

1.5 The map $w = \frac{1}{z}$ maps lines/circles to lines/circles

The general equation for straight lines and circles in the z-plane can be written as **GAP**

(1.5.1)

where $\alpha,\ \beta,\ \gamma$ and Δ are constants. If $\alpha = 0$ this represents a straight line but when $\alpha \neq 0$ (1.5.1) represents a circle. Writing (1.5.1) in terms of z

GAP

(1.5.2)

and then transforming to an equation in w and $\bar{w}$ through $w = \dfrac{1}{z}$ and $\bar{w} = \dfrac{1}{\bar{z}}$, (1.5.2) becomes

GAP

Since $w = u + iv$ we have **GAP**

(1.5.3)

This represents a family of circles in the $u - v$ plane when $\Delta \neq 0$ and a family of lines when $\Delta = 0$. Notice, that when $\alpha \neq 0$ and $\Delta \neq 0$ then the mapping maps circles to circles but a family of lines in the z-plane ($\alpha = 0$) also maps to a family of circles in the w-plane. However, there is also the case of a family of circles in the z-plane for which $\Delta = 0$ which map to a family of lines in the w-plane. **Thus we conclude that $w = \frac{1}{z}$ maps lines/circles to lines/circles but not necessarily lines to lines and circles to circles.**

The above analysis extends to the *fractional linear* or *Möbius* transformation

$$w = \frac{az+b}{cz+d}\,, \qquad ad \neq bc\,. \tag{1.5.4}$$

This includes cases such as:
(i) $w = \frac{1}{z}$ when $a = d = 0,\ b/c = 1$.
(ii) $w = \frac{1}{z-1}$ as in our example above where $a = 0,\ b = 1,\ c = 1,\ d = -1$.
(1.5.4) can be re-written as

$$w = c^{-1}\left\{a + \frac{bc-ad}{cz+d}\right\}. \tag{1.5.5}$$

For various special cases:

1. $w = z + b$; $(a = d = 1,\ c = 0)$ – translation.
2. $w = az$; $(b = c = 0,\ d = 1)$ – contraction/expansion + rotation
3. $w = \frac{1}{z}$; $(a = d = 0,\ b = c)$ – maps lines/circles to lines/circles.

Thus a Möbius transformation maps lines/circles to lines/circles with contraction/expansion, rotation and translation on top.

1.6 Extra: Mappings of the type $w = \frac{e^z-1}{e^z+1}$

Consider a map $w = \frac{e^z-1}{e^z+1}$ which can be re-written as

$$e^z = \frac{1+w}{1-w} = \frac{(1+u+iv)(1-u+iv)}{(1-u)^2+v^2}\,. \tag{1.6.1}$$

Real and imaginary parts give

$$e^x\cos y = \frac{1-u^2-v^2}{(1-u)^2+v^2} \qquad e^x \sin y = \frac{2v}{(1-u)^2+v^2}\,. \tag{1.6.2}$$

From these we conclude that

1. The family of lines $y = n\pi$ in the z-plane map to the line $v = 0$ for n integer. Thus an infinite number of horizontal lines in the z-plane all map to the u-axis in the w-plane.
2. The family of lines $y = \frac{1}{2}(2n+1)\pi$ in the z-plane map to the unit circle $u^2+v^2 = 1$ in the w-plane.

1.7 Extra: Conformal mappings and fluid dynamics

Why is the result of Theorem 1 on conformal mappings important? In $2D$ fluid incompressible dynamics the velocity field $\boldsymbol{u}(x,y)$ must satisfy the incompressibility condition $\operatorname{div}\boldsymbol{u} = 0$, in which case $\boldsymbol{u}$ can be written as $\boldsymbol{u} = (-\psi_y,\,\psi_x)$ where ψ is known as a *stream function*. If the fluid is also irrotational (no vorticity) then $\operatorname{curl}\boldsymbol{u} = 0$. Therefore

$$\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ -\psi_y & \psi_x & 0 \end{vmatrix} = 0\,, \tag{1.7.1}$$

in which case

$$\psi_{xx} + \psi_{yy} = 0\,. \tag{1.7.2}$$

This means that $\psi(x,y)$ satisfies Laplace's equation and is therefore a harmonic function. It also means that there must be a conjugate function, designated as $\phi(x,y)$, *the potential*, that satisfies the Cauchy-Riemann equations

$$\phi_x = \psi_y\,, \qquad\qquad \phi_y = -\psi_x\,, \tag{1.7.3}$$

and that ϕ too must be harmonic. We've now reached the situation where $2D$ incompressible ideal fluid dynamics can be cast is the language of complex mappings by writing a complex function $f(z)$ as

$$w = f(z) = \phi(x,y) + i\psi(x,y)\,. \tag{1.7.4}$$

With $\phi \equiv u$ and $\psi \equiv v$ we are now free to choose mappings $w = f(z)$, as in the previous section, to see how these may be used to solve flows around complicated shapes. To proceed in this direction requires the following result :

Theorem 2. *Harmonic functions remain harmonic under a conformal mapping.*

Proof : Let H be a harmonic function in the z-plane : that is, H satisfies

$$H_{xx} + H_{yy} = 0\,. \tag{1.7.5}$$

Now take $w = u + iv$ and with $f(z)$ analytic then the CR equations hold : $u_x = v_y$ and $v_x = -u_y$ and also $u_{xx} + u_{yy} = 0$ with $v_{xx} + v_{yy} = 0$. The chain rule says that :

$$H_x = u_x H_u + v_x H_v \qquad H_y = u_y H_u + v_y H_v \tag{1.7.6}$$

and

$$H_{xx} = u_{xx}H_u + v_{xx}H_v + u_x^2 H_{uu} + v_x^2 H_{vv} + 2u_x v_x H_{uv} \tag{1.7.7}$$

$$H_{yy} = u_{yy}H_u + v_{yy}H_v + u_y^2 H_{uu} + v_y^2 H_{vv} + 2u_y v_y H_{uv} \tag{1.7.8}$$

and so

$$H_{xx} + H_{yy} = (u_{xx} + u_{yy})H_u + (v_{xx} + v_{yy})H_v + (u_x^2 + u_y^2)H_{uu} + (v_x^2 + v_y^2)H_{vv} + 2(u_x v_x + u_y v_y)H_{uv} \tag{1.7.9}$$

From the CR equations $u_x^2 + u_y^2 = v_x^2 + v_y^2$ and $u_x v_x + u_y v_y = 0$. Hence we are left with

$$H_{xx} + H_{yy} = (u_x^2 + u_y^2)(H_{uu} + H_{vv})\,. \tag{1.7.10}$$

Thus if H satisfies Laplace's equation in the z-plane $H_{xx} + H_{yy} = 0$ then it must satisfy Laplace's equation

$$H_{uu} + H_{vv} = 0\,, \tag{1.7.11}$$

in the w-plane. The reason for the importance of this result lies in the fact that in incompressible inviscid fluid dynamics both the stream function and potential $(\psi,\,\phi)$ satisfy Laplace's equation. When this flows around a shape, such as an aerofoil, in the $z - plane$, under a conformal mapping we still have solutions of Laplace's equation in the w-plane. $\square$

The most famous of mappings in this area is **Joukowsi's** aerofoil transformation

$$w = z + \frac{1}{z}\,. \tag{1.7.12}$$

Firstly we use polar co-ordinates $x = r\cos\theta$ and $y = r\sin\theta$ so that $z = re^{i\theta}$ and

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta\,. \tag{1.7.13}$$

Therefore, with

$$a = r + \frac{1}{r} \qquad\qquad b = r - \frac{1}{r}\,, \tag{1.7.14}$$

u and v satisfy the equation for an ellipse

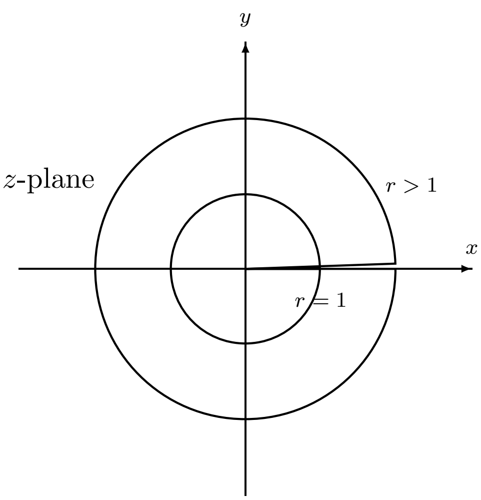
$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1\,. \tag{1.7.15}$$

Conclusions to be drawn are:

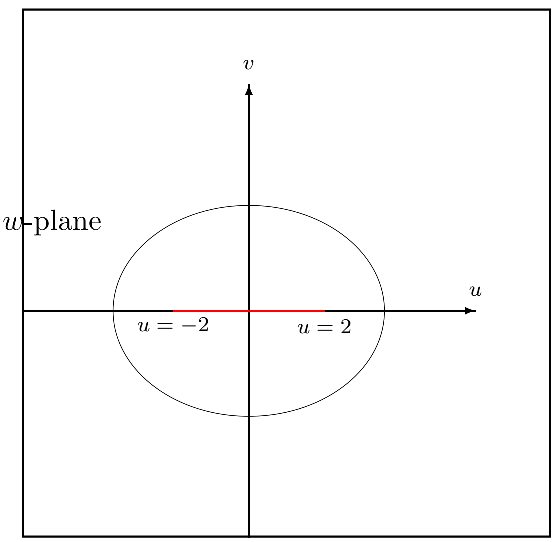
1. Circles $|z| = r = const$ centred at $z = 0$ in the z-plane map to ellipses in the w-plane.

2. The special circle $r = 1$ for which $a = 2$ and $b = 0$ maps to the red segment of the u-axis $-2 \le u \le 2$ as in the pair of figures below.

The z-plane



The w-plane



3. For circles passing through $x = \pm 1$ but not centred at $z = 0$ and for circles passing through $x = -1$ but passing outside $x = 1$ the figure below illustrated how the mapping works. Note that $dw/dz = 1 - z^{-2}$ which is zero when $z = \pm 1$. Hence conformality fails at these points.

Example Sheet 5: Functions of a complex variable

Recall that for a complex function $f(z) = u(x,y) + iv(x,y)$ the Cauchy-Riemann equations are $u_x = v_y$ and $u_y = -v_x$.

1. Verify that the following satisfy the Cauchy-Riemann equations:

   a) $u = x; \quad v = y\,,$

   b) $u = e^x \cos y; \quad v = e^x \sin y\,,$

   c) $u = x^3 - 3xy^2; \quad v = 3x^2y - y^3$

2. Show that the following functions $u(x,y)$ each satisfy Laplace's equation and then use the Cauchy-Riemann equations to determine the conjugate function v. Find also $f(z) = u + iv$.

   $$a)\ \ u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1\,, \qquad\qquad b)\ \ u = xy\,.$$

3. Show that the function
   $$u(x,y) = (x\cos y - y\sin y)\,e^x$$
   satisfies Laplace's equation. Find the conjugate function $v(x,y)$ for which u and v together satisfy the Cauchy-Riemann equations and hence find in its simplest form $w = u + iv = f(z)$ where $z = x + iy$.

4. Consider the mapping $w = \frac{1}{z-1}$ from the z-plane to the w-plane.

   a) Show that in the z-plane, the circle
   $$(x-1)^2 + y^2 = 4$$
   maps to a circle in the w-plane. What is the radius of this circle and where is its centre?

   b) To what curve does the line $x = 0$ in the z-plane map in the w-plane?

5. a) Fixed points of a map $w = f(z)$ occur when $w = z$. Show that the fixed points of $w = \frac{4z-2}{z+1}$ occur at $z = 1$ and $z = 2$.

   b) For $w = u + iv = \frac{4z-2}{z+1}$ show that the image in the w-plane of the line $x = 0$ is the circle $(u-1)^2 + v^2 = 9$. What is the image in the w-plane of the unit circle $|z| = 1$?

Answers:

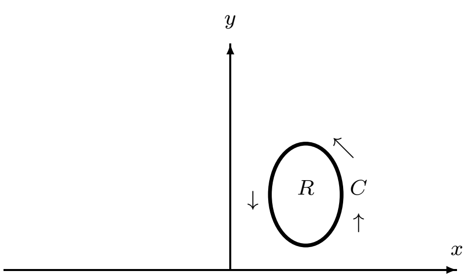
2a) $v = 3x^2y - y^3 + 6xy + c; \quad f(z) = z^3 + 3z^2 + c.$
2b) $v = \frac{1}{2}(y^2 - x^2) + c; \quad f(z) = -\frac{i}{2}z^2 + c.$
3) $v = e^x\,(x\sin y + y\cos y) + c; \quad f(z) = z\,e^z + c.$
4) a) $u^2 + v^2 = \frac{1}{4}$; b) $\left(u + \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}.$
5b) The line $u = 1$.

Chapter 2

Contour Integration

2.1 Cauchy's Theorem

How do we integrate a complex function? Let's begin with an easier question: where do we integrate it? We have seen a number of variations on the theme of integration in vector calculus, including contour integrals. This approach turns out to be particularly fruitful in complex analysis.



A closed contour C enclosing a region R in the z-plane around which the line integral is considered in the counter-clockwise direction

$$\oint_C F(z)\, dz\,. \tag{2.1.1}$$

With

$$F(z) = u + iv \qquad\qquad z = x + iy \tag{2.1.2}$$

we have **GAP**

(2.1.3)

Now Green's Theorem in a plane says that for differentiable functions $P(x, y)$ and $Q(x, y)$
GAP

(2.1.4)

For the real part of (2.1.3) we take
GAP

Similarly, for the imaginary part of (2.1.3) we take **GAP**

Therefore we have

GAP

(2.1.5)

which turns (2.1.3) into

GAP

(2.1.6)

If $F(z)$ is analytic everywhere within and on C then u and v must satisfy the CR equations: $u_x = v_y$ and $v_x = -u_y$, in which case both the real and imaginary parts on the RHS of (2.1.6) must be zero. We have established *Cauchy's Theorem:*

Theorem 3. *If $F(z)$ is analytic everywhere within and on a closed, piecewise smooth contour C then*

GAP

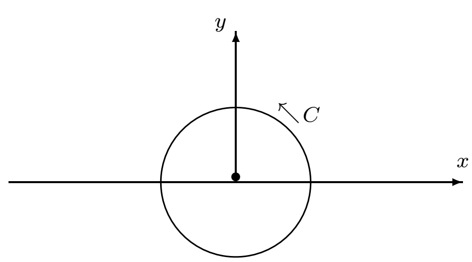
(2.1.7)

The key point is that provided $F(z)$ is analytic everywhere within and on C, any singularities of $F(z)$ *outside* of C are irrelevant.

Example 2.1.1. *The function*

$$f(z) = \frac{1}{z}$$

has its only singularity at the origin, so the integral of $f(z)$ around a contour which doesn't include the origin will be zero, by Cauchy's theorem. What about a contour that includes the origin?



For $f(z) = z^{-1}$, the circular contour C of radius a encloses a singularity • at the origin in the z-plane. The line integral is no longer zero because of this singularity.

Now write the circular contour C as

GAP

a circle around the origin of radius a. Then

GAP

so that

GAP

(2.1.8)

The singularity at $z = 0$ contributes a non-zero value of $2\pi i$ to the integral. Note that it is independent of the value of a, which is consistent with this being the only non-zero contribution to the integral: it is the same, for any circle centred at the origin.

Clearly, the only interesting behaviour happens with singularities in or on a contour. Given the powerful result of Cauchy's Theorem, our task is to see, in a more formal manner, how singularities contribute to complex integrals. Before this their nature and classification is necessary.

2.2 Poles and Residues

If a complex function fails to be analytic at a point, we call this point a *singularity*. Singularities can take many forms but the simplest class is what are called **simple poles**. A function $F(z)$ has a simple pole at $z = a$ (which could be real) if it can be written in the form

GAP

(2.2.1)

where $g(z)$ is analytic at $z = a$. Likewise, $F(z)$ has a pole of multiplicity m at $z = a$ if it can be written in the form

GAP

(2.2.2)

where, again, $g(z)$ is analytic at $z = a$, and $m = 1, 2, 3, 4, \ldots$; when $m = 2$ we have a double pole, etc.

Example 2.2.1. *The function*

$$f(z) = \frac{z^2}{(z-1)^2}$$

GAP

While all poles are singularities not all singularities are poles. For instance, $\ln z$ has a singularity at $z = 0$ but this is not a pole. Similarly, $z = a$ is not a pole when m is not an integer: $f(z) = z^2/\sqrt{z-1}$.

Definition : The **residue** of $F(z)$ at a simple pole at $z = a$ is

GAP

(2.2.3)

Definition : The **residue** of $F(z)$ at a pole of multiplicity[1] m at $z = a$ is

GAP

(2.2.4)

Note that a function may have many poles and each pole has its own residue.

Example 2.2.2. *Consider* $F(z) = \dfrac{2z}{(z-1)(z-2)}$ *which has simple poles at* $z = 1$ *and* $z = 2$.

GAP

(2.2.5)

[1] This formula will be given in an exam question , if necessary; it is found from a coefficient in what is known as a Laurent expansion – see Kreyszig's book.

Example 2.2.3. $F(z) = \dfrac{2z}{(z-1)^2(z+4)}$ *has a double pole at* $z = 1$ *and a simple pole at* $z = -4$.

GAP

It is of no significance that the residues have opposite signs.

Example 2.2.4. *The function*

$$f(z) = \frac{3z}{3z - i}$$

has a simple pole at $z = \dfrac{1}{3}i$.

GAP

2.3 The residue of $F(z) = \frac{h(z)}{g(z)}$ when $g(z)$ has a simple zero at $z = a$

Taylor series exist for analytic functions, much as for functions of a single variable. Again, this is a fact we use without proof; the interested reader may consult the literature. We expand $g(z)$ about its zero at $z = a$ in a Taylor series **GAP**

(2.3.1)

Thus, noting that $g(a) = 0$ we have
GAP

(2.3.2)

If $z = a$ were a double pole, we would also have $g'(a) = 0$ and the method would work with the obvious extension, and so on for poles of higher multiplicity.

Example 2.3.1. *Consider*

$$f(z) = \frac{1}{z^3 - 1}$$

GAP

So the residue at $z = 1$ can be calculated as

GAP

Exercise: Obtain the residues of f in the above example, at the other two poles $z = e^{\pm 2i\pi/3}$.

2.4 The Residue Theorem

Now consider a simple pole at $z = a$ as in the Figure below, showing the full contour C comprising the two edges of the cuts $C^{\pm}$ running in opposite directions, the small circle C_a and then C_1 which is the rest of C with the small piece ϵ removed.

The pole is isolated by a device which consists of taking a "cut" into the contour and inscribing a small circle of radius r around it. The full *closed* contour C consists of the two edges of the cuts $C^{\pm}$ running in opposite directions, the small circle C_a and then C_1 which is the rest of C with the small piece ϵ removed. Thus we have

GAP

(2.4.1)

This device ensures that the pole lies outside of C as it has been constructed, in which case Cauchy's Theorem says that

GAP

(2.4.2)

in which case

GAP

(2.4.3)

Two points to note are:

1. The four integrals are not closed so they don't have the $\oint$ notation. In the limit $\epsilon \to 0$ the integrals $\int_{C^+}$ and $\int_{C^-}$ cancel as they go in opposite directions;
2. The integral over C_a is clockwise, not counter-clockwise.

We are left with

GAP

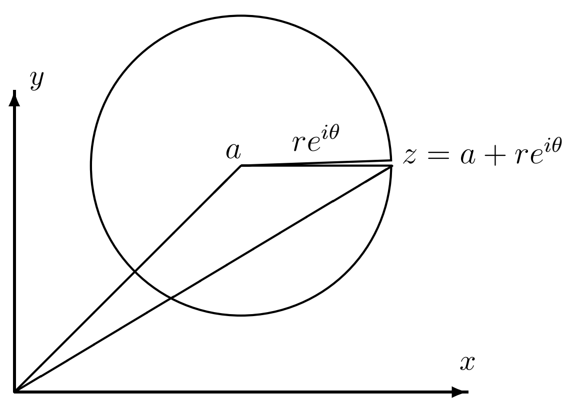
(2.4.4)

We now write

GAP

(2.4.5)

where $g(z)$ is analytic at $z = a$.



As in the figure above we write the equation of the circle of radius r, centre a in the complex plane as

GAP

giving the circle, anticlockwise.

GAP

$$(2.4.6)$$

Our next step at this stage is to take the limit $r \to 0$:

GAP

$$(2.4.7)$$

which gives **Cauchy's integral formula**

GAP

$$(2.4.8)$$

However, because $z = a$ is a simple pole, at the pole at $z = a$ the residue of $F(z)$ is

GAP

$$(2.4.9)$$

We have proved

GAP

$$(2.4.10)$$

It is clear that this procedure of making a cut and ring-fencing a pole can be performed for many simple poles and the individual residues added. The result can also be proved when poles have higher multiplicity. Altogether we have:

Theorem 4. ***(Cauchy's) Residue Theorem:*** *If the only singularities of $F(z)$ within C are poles then*

$$\oint_C F(z)\,dz = 2\pi i \times \{\textit{Sum of the residues of } F(z) \textit{ at its poles within } C\}\ . \tag{2.4.11}$$

Some examples of this immensely powerful theorem follow.

Example 2.4.1. *Obtain*

$$\oint_C \frac{2z}{(z-1)^2(z+4)}\ dz$$

where C is the circle of radius 5, centred at the origin.

In example (2.2.3) we found the residues: the simple pole at $z=-4$ and the double pole at $z=1$ had residues $\pm\frac{8}{25}$. Hence, Cauchy's residue theorem gives the integral as

GAP

Example 2.4.2. *Find*

$$\oint_{C_i} \frac{2z\,dz}{(z-1)(z-2)} \tag{2.4.12}$$

where (i) C_1 is the circle centred at $(0,0)$ of radius 3 and (ii) C_2 is the circle centred at $(0,0)$ of radius 3/2.

$F(z)$ has two simple poles: the first at $z=1$ and the second at $z=2$. Their residues have been found in (2.2.5).For C_1 both poles lie inside C_1:

GAP

(2.4.13)

whereas for C_2 only the pole at $z=1$ lie inside, thus

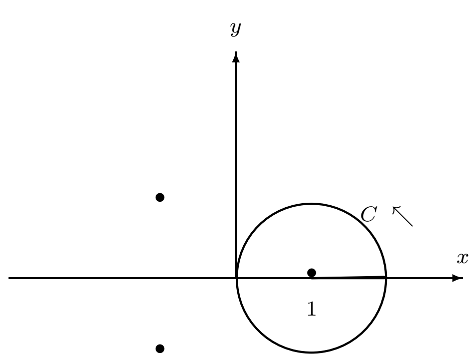
GAP

(2.4.14)

Example 2.4.3. *Find*

$$\oint_C \frac{dz}{(z^3-1)^2} \tag{2.4.15}$$

where C is the circle $|z-1|=1$.



The contour is the circle $|z-1|=1$ in the z-plane.

GAP

These are double poles for $F(z)$ but only the double pole at $z = 1$ lies inside C. Its residue there is

GAP

$$ \tag{2.4.16} $$

Therefore we deduce from the Residue Theorem that

GAP

$$\oint_C \frac{dz}{(z^3-1)^2} = \tag{2.4.17}$$

Example 2.4.4. *(Exam 2006): Find*

$$\oint_C \frac{z\,dz}{(z-1)^2(z-i)} \tag{2.4.18}$$

where C is the circle $|z| = 2$.

For the double pole at $z = 1$, the residue there is

GAP

$$ \tag{2.4.19} $$

For the simple pole at $z = i$ the residue there is

GAP

$$ \tag{2.4.20} $$

Both poles must be included within C so we conclude from the Residue Theorem that

GAP

$$\oint_C \frac{z\,dz}{(z-1)^2(z-i)} = \tag{2.4.21}$$

Example 2.4.5. *(Exam 2006): Find*

$$\oint_C \frac{z^2dz}{(z-i)^3} \tag{2.4.22}$$

where C is the circle $|z| = 2$ as above.

For the triple pole at $z = i$ the residue there is

$$\lim_{z\to 1} \frac{1}{2!}\frac{d^2}{dz^2}\left\{\frac{z^2(z-i)^3}{(z-1)^3}\right\} = 1\,. \tag{2.4.23}$$

Hence

$$\oint_C \frac{z^2dz}{(z-i)^3} = 2\pi i\,. \tag{2.4.24}$$

2.5 Improper integrals and Jordan's Lemma

We consider integrals of the type

$$\int_{-\infty}^{\infty} f(x)\ dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(x)\sin x\ dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} f(x)\cos x\ dx$$

All are instances of, *real integrals* of the type

GAP

(2.5.1)

and the Residue Theorem can be used to evaluate them, provided $f(x)$ has certain convergence properties: these are called *improper integrals* because of the infinite nature of their limits. Formally we write them as

GAP

(2.5.2)

The main idea is to consider a class of *complex integrals*

GAP

(2.5.3)

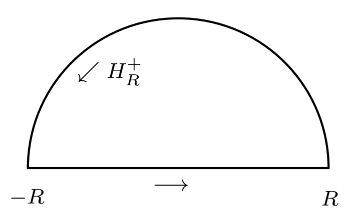
where C consists of the semi-circular arc as in the figure below. The two essential parts are the arc of the semicircle of radius R denoted by H_R and the real axis $[-R, R]$. Hence we can re-write (2.5.3) as

GAP

(2.5.4)

In principle the closed complex integral over C on the LHS can be evaluated by the Residue Theorem. Our next aim is to evaluate the real integral on the RHS in the limit $R \to \infty$. This requires a result which is called Jordan's Lemma.

Jordan's Lemma



Jordan's Lemma deals with the problem of how a contour integral behaves on the semi-circular arc H_R^+ of a closed contour C.

Lemma (**Jordan**) If the only singularities of $F(z)$ are poles, then

GAP

$$(2.5.5)$$

provided that $m > 0$ and $|F(z)| \to 0$ as $R \to \infty$. If $m = 0$ then a faster convergence to zero is required for $F(z)$.

Proof: Since H_R is the semi-circle $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$ and $dz = iRe^{i\theta}d\theta$

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{H_R} e^{imz}F(z)\,dz\right| &= \lim_{R\to\infty}\left|\int_{H_R} e^{imR\cos\theta - mR\sin\theta}F(z)R\,e^{i\theta}d\theta\right| \\ &\le \lim_{R\to\infty}\int_{H_R} e^{-mR\sin\theta}|F(z)|R\,d\theta \end{aligned} \tag{2.5.6}$$

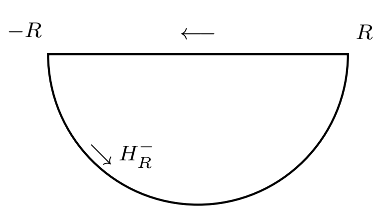
having recalled that $|e^{i\alpha}| = 1$ for any real α and $\left|\int f(z)\,dz\right| \le \int |f(z)|\,dz$. Note that in the exp-term on the RHS of (2.8.4), $\sin\theta > 0$ in the upper half plane. Hence, provided $m > 0$, the exponential ensures that the RHS is zero in the limit $R \to \infty$ (see remarks below). $\square$

Remarks:

a) When $m > 0$ forms of $F(z)$ such as $F(z) = \dfrac{1}{z}\,, F(z) = \dfrac{1}{z+a}$ or rational functions of z such as $F(z) = \dfrac{z^p \ldots}{z^q + \ldots}$ (for $0 \le p < q$ and p and q integers) will all converge fast enough as these all have simple poles and $|F(z)| \to 0$ as $R \to \infty$.

b) If, however, $m = 0$ then a modification is needed: e.g. if $F(z) = \dfrac{1}{z}$ then $|F(z)| \to 0$ but $\lim_{R\to\infty} z|F(z)| = 1$. We need to alter the restriction on the integers p and q to $0 \le p < q - 1$ which excludes cases like $F(z) = \dfrac{1}{z}\,, F(z) = \dfrac{1}{z+a}$.

c) What about $m < 0$? To ensure that the exponential is decreasing for $R \to \infty$ we need $\sin\theta < 0$. This is true in the *lower* half plane. Hence in this case we take our contour in the *lower* half plane (call this H_R^- as opposed to H_R^+ in the upper) but still in an anti-clockwise direction.



A contour in the lower $\frac{1}{2}$-plane with semi-circle H_R^- taken in the counter-clockwise direction which is used for cases when $m < 0$. See the notes on Fourier Transforms for cases when this is useful.

The conclusion is that if $F(z)$ satisfies the conditions for Jordan's Lemma then

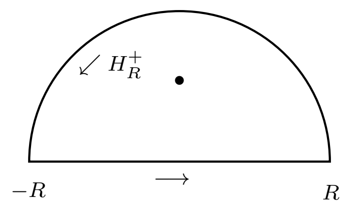
$$\int_{-\infty}^{\infty} e^{imx}F(x)\,dx = 2\pi i \times \{\text{Sum of the residues of the poles of } e^{imz}F(z) \text{ in the upper } \tfrac{1}{2}\text{-plane}\}\,. \tag{2.5.7}$$

Example 2.5.1. *Show that*

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi\,. \tag{2.5.8}$$

This integral can be solved with simple methods:

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{1+x^2} = \lim_{R\to\infty} \left[\tan^{-1} x\right]_{-R}^{R} = \frac{\pi}{2} + \frac{\pi}{2}\,.$$



C is comprised of a semi-circular arc H_R^+ and a section on the real axis from $-R$ to R. Only the simple pole at $z = i$ lies within C.

Thus we consider the complex integral over the semicircle C in the upper half-plane **GAP**

(2.5.9)

with $m = 0$. The only singularities are simple poles at **GAP**

with one pole in the upper half-plane.

For Jordan's lemma, we have $m = 0$, but
GAP

so the quadratic nature of the denominator is enough for convergence, and by Jordan's Lemma
GAP

(2.5.10)

The residue of $F(z)$ at the pole in the upper-half-plane at $z = i$ is
GAP

and so from the Residue Theorem,
GAP

(2.5.11)

Finally we have the result

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi\,. \tag{2.5.12}$$

Example 2.5.2. *Show that*

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \pi/\sqrt{2}\,. \tag{2.5.13}$$

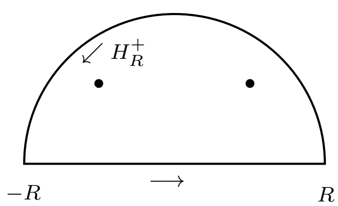
As before, we consider the complex integral over the semicircle C in the upper half-plane

$$\oint_C \frac{dz}{1+z^4} \tag{2.5.14}$$

with $m = 0$. The existence of poles as the only singularities and the quartic nature of the denominator allows us to appeal to Jordan's Lemma:

GAP

(2.5.15)



$z^4 = -1$ has four zeroes lying at $e^{i\pi/4}$, $e^{3i\pi/4}$ in the upper half-plane and $e^{-i\pi/4}$, $e^{-3i\pi/4}$ in the lower half-plane. Only the first two are relevant. Now we use the trick in (2.3.2) to find the residues of the two poles in the upper half-plane: when $f(z) = h/g$, if g has a simple zero and h is analytic at a, then the residue is $h(a)/g'(a)$. Using $h(z) = 1$ and $g(z) = 1 + z^4 \Rightarrow g'(z) = 4z^3$, the residues at $e^{i\pi/4}$ and $e^{3i\pi/4}$ are

GAP

(2.5.16)

Thus our final result is

GAP

(2.5.17)

Example 2.5.3. *Show that*

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6} = 2\pi/3\,. \tag{2.5.18}$$

Example 2.5.4. *For $m > 0$ show that*

$$\int_{-\infty}^{\infty} \frac{\cos mx\, dx}{1+x^2} = \pi e^{-m}\,. \tag{2.5.19}$$

We consider the complex integral over the semicircle C in the upper half-plane

GAP

(2.5.20)

The integrand has only one simple pole at $z = i$ in the upper half-plane whose residue is

GAP

(2.5.21)

Therefore, from the Residue Theorem

GAP

(2.5.22)

Moreover, from Jordan's Lemma, we have only simple poles, $m > 0$, and $|f(z)| \to 0$, so

GAP

(2.5.23)

Therefore

GAP

(2.5.24)

What happens to the imaginary part

$$\int_{-\infty}^{\infty} \frac{\sin mx\, dx}{1+x^2}\ ? \tag{2.5.25}$$

Notice that the integrand is an *odd function*: therefore, over $(-\infty,\, \infty)$ the part over $(-\infty,\, 0)$ will cancel with that over $(0,\, \infty)$, leaving zero as a result. Thus we have

$$\int_{-\infty}^{\infty} \frac{\cos mx\, dx}{1+x^2} = \pi e^{-m}\,. \tag{2.5.26}$$

Example 2.5.5. *For $m > 0$ show that*

$$\int_{-\infty}^{\infty} \frac{\cos mx\, dx}{(a^2 + x^2)^2} = \frac{\pi e^{-ma}}{2a^3}(1 + ma)\,. \tag{2.5.27}$$

2.6 Poles on the real axis

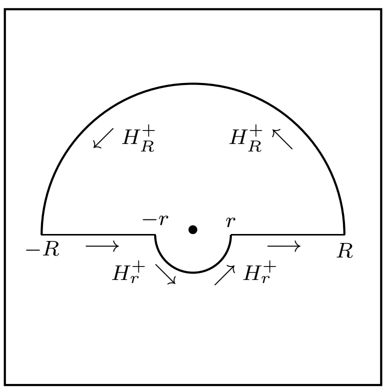
When an integrand has a pole on the real axis this causes problems by sitting on the semicircular contour, because Cauchy's Residue theorem only applies to poles *within* the contour. The trick is to change the contour to include or exclude the pole on the axis. Let us do this by example.

We would like to calculate an improper integral of the type

GAP

(2.6.1)

using the same approach as before, with Jordan's lemma. The problem is the pole at the origin, lying on the real axis.



The contour is deformed by a small semi-circle of radius r, indented into the lower half-plane, centred at the origin, thus including the pole at $z = 0$. Following the direction of the arrows, the big semicircle of radius R is designated as H_R^+ $(\theta : 0 \to \pi)$ and the little semicircle of radius r is designated as H_r^+ $(\theta : \pi \to 2\pi)$; the superscript $^+$ indicates counter-clockwise orientation.

In this version, a small semi-circular indentation into the lower half-plane has been added to the contour, thus including the pole at $z = 0$. The alternative is to have a small semi-circular indentation into the upper half-plane, thus excluding the pole from the contour.

To calculate an improper integral of the type (2.6.3) we consider:

$$\mathcal{I} = \oint_C \frac{f(z)\, dz}{z} =$$

GAP

and split the contour integral up into four distinct integrals **GAP**

(2.6.2)

For the last of the four integrals, we have zero, as $R \to \infty$, provided $f(z)$ satisfies the conditions of Jordan's lemma.

In the limit, the first and the third integral together give us:

GAP

What's new here is what happens to the second integral, around the small indented semi-circle H_r, as $r \to 0$. The approach is best outlined by an example.

Example 2.6.1. *Use Cauchy's residue theorem, with Jordan's lemma and a semi-circular indentation into the lower half-plane to calculate*

$$\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x} \tag{2.6.3}$$

As previously, we consider the complex integral

GAP

(2.6.4)

Using the small semi-circle of radius r, indented into the lower half-plane, the integrand has a simple pole inside C as $z = 0$ is included in the construction. The pole at the origin has residue

GAP

Thus, Cauchy's Theorem can be invoked to give: $2\pi i \times$(Residue at origin) $=$

(2.6.5)

GAP

In the second integral we take the limit $R \to \infty$ and, with $m = 1$, Jordan's Lemma tell us that $\int_{H_R} = 0$ because the only singularity is a pole and the function $|1/z|$ decays to zero as $R \to \infty$.

For the last integral, we note that the small semicircle H_r, as seen several times before, has the equation

GAP

Taking the two limits $R \to \infty$ and $r \to 0$ together, we have

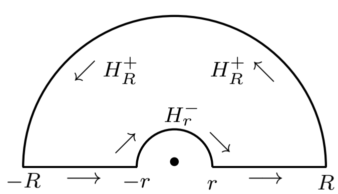
GAP

(2.6.6)

As before, we note that the integrand $\cos x/x$ is odd so the contributions on $(-\infty, 0)$ and $(0, \infty)$ cancel leaving us with

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi\,. \tag{2.6.7}$$

Exercise: Repeat the previous example, with H_r a small semi-circular indentation into the upper half-plane:



The contour is deformed by a small semi-circle of radius r, indented into the upper half-plane, centred at the origin, thus excluding the pole at $z = 0$. Following the direction of the arrows, H_R^+ is as before, but the small semicircle H_r^- has $\theta : \pi \to 0$; the superscript $^+, ^-$ indicate (counter)clockwise orientation.The same result is obtained, but care needs to be taken with the limits for θ when changing variables in the integral for H_r.

2.7 Integrals around the unit circle

We consider here integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$. As so often, the idea is best illustrated with an example.

Example 2.7.1. *Obtain*

$$I = \int_0^{2\pi} \frac{d\theta}{a + \cos\theta}\,, \qquad a > 1\,. \tag{2.7.1}$$

Take C as the unit circle:

GAP

with a similar identity available for $\sin\theta$, so we can rewrite the integral as

GAP

$$(2.7.2)$$

The next task is to determine the roots of $z^2 + 2az + 1 = 0$.

GAP

$$(2.7.3)$$

We note that for $a > 1$, while $z_1 = -a + \sqrt{a^2 - 1}$ lies *within* C, the other root, $z_2 = -a - \sqrt{a^2 - 1}$ lies *without*. Therefore we exclude the pole at z_2 and compute the residue of the integrand at z_1, which is

GAP

$$(2.7.4)$$

The Residue Theorem then gives

GAP

$$(2.7.5)$$

2.8 Extra: An application to Fourier Transforms

Now apply this to the Fourier Transform of $f(t) = \dfrac{1}{1+t^2}$ as an example

$$\overline{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{1+t^2}\, dt. \tag{2.8.1}$$

The need for Jordan's Lemma arises (with the necessity for upper and lower half-plane contours) because ω takes both positive and negative values. It obviously plays the role of $-m$ in Jordan's Lemma so we are forced to split the calculation into two halves: take a contour in the upper half-plane H_R^+ for $\omega < 0$ and take a contour in the lower half-plane H_R^- for $\omega > 0$. Hence we need to consider the complex plane where the real axis is the t-axis with the pair of complex integrals

$$\int_{H_R^+} \frac{e^{-i\omega z}}{1+z^2}\, dz \quad (\omega < 0) \qquad\qquad \int_{H_R^-} \frac{e^{-i\omega z}}{1+z^2}\, dz \quad (\omega > 0) \tag{2.8.2}$$

Use the Residue Theorem and Jordan's Lemma on H_R^+ with a simple pole at $z = i$ to obtain, in the limit $R \to \infty$,

$$2\pi i \left(\frac{e^{\omega}}{2i}\right) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{1+t^2}\, dt \quad (\omega < 0) \tag{2.8.3}$$

and, on H_R^- with a simple pole at $z = -i$, in the limit $R \to \infty$,

$$2\pi i \left(-\frac{e^{-\omega}}{2i} \right) = \int_{\infty}^{-\infty} \frac{e^{-i\omega t}}{1+t^2}\, dt \quad (\omega > 0) \tag{2.8.4}$$

Note the limits in the integral in (2.8.4) are reversed because of the contour in the lower half plane. We arrive at the result

$$\overline{f}(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{1+t^2}\, dt = \pi \begin{cases} e^{\omega}, & \omega < 0 \\ e^{-\omega}, & \omega > 0 \end{cases} = \pi e^{-|\omega|}. \tag{2.8.5}$$

Example Sheet 6: Complex Integration

1. By taking the contour C as the unit circle $|z| = 1$ (positive is anti-clockwise), evaluate the following contour integrals $\oint_C F(z)dz$:

   (a) $F(z) = (z^2 - 2z)^{-1}$,

   (b) $F(z) = (z+1)(4z^3 - z)^{-1}$,

   (c) $F(z) = z(1+9z^2)^{-1}$.

   Remember to include only those poles which lie inside C. *Answers:* a) $-\pi i$, b) 0, c) $2\pi i/9$.

2. Use the Residue Theorem to show that

$$\oint_C \frac{z\, dz}{(z-i)^2} = 2\pi i\, .$$

   where the contour C is the rectangle with vertices at $\pm\frac{1}{2} + 2i$ and $\pm\frac{1}{2} - 2i$.

3. Show that

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \tfrac{1}{2}\pi\, .$$

4. Given the real integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2} \qquad (|p| \neq 1)$$

   show that the substitution $z = e^{i\theta}$ converts it into

$$I = \frac{i}{p} \oint_C \frac{dz}{(z-p)(z-p^{-1})},$$

   where C is the unit circle $|z| = 1$. Evaluate the residues at the poles and hence show that

   (i) $I = -2\pi\, (p^2 - 1)^{-1}$ when $|p| < 1$,

   (ii) $I = +2\pi\, (p^2 - 1)^{-1}$ when $|p| > 1$.

5. By choosing a suitable contour in the upper half of the complex plane, use the Residue Theorem & Jordan's Lemma to show that for $a > b > 0$

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right) .$$

Chapter 3

Laplace Transforms

3.1 Introduction

For a function $f(t)$ uniquely defined on $0 \le t \le \infty$, its Laplace transform (LT) is defined as

GAP

$$(3.1.1)$$

where s may be complex. The LT may not exist if $f(t)$ becomes singular in $[0\,\infty]$. The LT is a **one-sided transform** in that it operates on $[0\,\infty]$ and not, like the Fourier Transform, on $[-\infty\,\infty]$. For this reason, LTs are useful for initial value problems, such as in circuit and control theory, where a function switches on at $t = 0$ and where $f(0)$ has been specified.

Because s is a complex variable the inverse transform $f(t) = \mathcal{L}^{-1}\left[\overline{f}(s)\right] = \oint_C e^{st}\,\overline{f}(s)\,ds$ is more difficult to handle because the contour C is a tricky infinite rectangle in the right-hand-half of the s-plane. Referred to as 'Bromwich integrals' the evaluation of these can be circumvented: we resort firstly to a **library of transforms** for the standard functions and secondly to ways of piecing combinations of these together for those not in the list.

3.2 Library of Laplace Transforms

1. **The exponential-function** $f(t) = e^{at}$ **:**

$$\boxed{f(t) = \exp(at)\,; \qquad \overline{f}(s) = \frac{1}{s-a}\,; \qquad \mathrm{Re}\, s > a} \tag{3.2.1}$$

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provided $\mathrm{Re}(s-a) > 0$, to guarantee convergence of the improper integral.

Hence **GAP** $f(t) = \mathcal{L}^{-1}$ $=$

and similarly, for every Laplace transform, getting around the necessity to solve the Bromwich integrals.

2. **The constant function $f(t) = 1$: GAP**

$$\boxed{f(t) = 1\,; \qquad \overline{f}(s) = \frac{1}{s} \qquad \mathrm{Re}\,(s) > 0} \tag{3.2.2}$$

This can be seen as a special case of e^{at}, with $a = 0$! Hence

GAP $f(t) = \mathcal{L}^{-1}$ $=$

3. **The sine function :**

$$\boxed{f(t) = \sin(at)\,; \qquad \overline{f}(s) = \frac{a}{s^2 + a^2}\,; \qquad \mathrm{Re}\,s > 0} \tag{3.2.3}$$

Take both the sine and cosine functions in combination : $\cos(at) + i\sin(at) = e^{iat}$

(3.2.4)

GAP

provided $\mathrm{Re}(s) > 0$. Then the imaginary (real) part gives the result for sine (cosine).

4. **The cosine function:**

$$\boxed{f(t) = \cos(at)\,; \qquad \overline{f}(s) = \frac{s}{s^2 + a^2}\,; \qquad \mathrm{Re}\,s > 0} \tag{3.2.5}$$

5. **The polynomial function $f(t) = t^n$:**

$$\boxed{f(t) = t^n; \quad \overline{f}(s) = \frac{n!}{s^{n+1}}\,; \qquad (n \geq 0)\,; \qquad \mathrm{Re}\,s > 0} \tag{3.2.6}$$

Consider $f(t) = t$:

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$$(3.2.7)$$

provided $\mathrm{Re}(s) > 0$.

This generalizes to

GAP

6. **The Heaviside function:**

$$\boxed{f(t) = H(t-a)\,; \qquad \overline{f}(s) = \frac{\exp(-sa)}{s}\,; \qquad \mathrm{Re}\,s > 0} \tag{3.2.8}$$

GAP

for $\mathrm{Re}\,(s) > 0$.

7. **The Dirac δ-function:**

$$\boxed{f(t) = \delta(t-a)\,; \qquad \overline{f}(s) = \exp(-sa)\,; \qquad a \geq 0} \tag{3.2.9}$$

GAP

and a needs to reside within the positive range of t.

8. **First Shift theorem:**

$$\boxed{\mathcal{L}\,[\exp(at)f(t)] = \overline{f}(s-a)} \tag{3.2.10}$$

GAP

provided $\mathrm{Re}(s-a) > 0$.

Example 3.2.1. *Given we know* $\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t\,,$ *what about* $\mathcal{L}^{-1}\left[\dfrac{1}{s^2-2s+5}\right]$?

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9. **Second Shift theorem :**

$$\boxed{\mathcal{L}\left[H(t-a)f(t-a)\right] = \exp(-sa)\,\overline{f}(s)} \tag{3.2.11}$$

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Let $\tau = t - a$. Then

GAP

Example 3.2.2. *Let* $\overline{f}(s) = e^{-2s}\dfrac{s}{s^2+9}$. *Then*

GAP

10. **Convolution theorem :**

$$\boxed{\mathcal{L}\left\{f \star g\right\} = \overline{f}(s)\,\overline{g}(s)} \tag{3.2.12}$$

where the convolution between two functions $f(t)$ and $g(t)$ is defined as

$$f \star g = \int_0^t f(u)g(t-u)\,du\,. \tag{3.2.13}$$

Note that the convolution is over $[0,\,t]$ and not $[-\infty,\,\infty]$ as for the Fourier Transform. The convolution integral on the RHS can also be written with f and g reversed: that is

$$\int_0^t f(t-u)g(u)\,du\,.$$

The LT of the convolution product in (3.2.13) is written down and then the order of the integrals is exchanged, as in the figure, using $\tau = t - u$.

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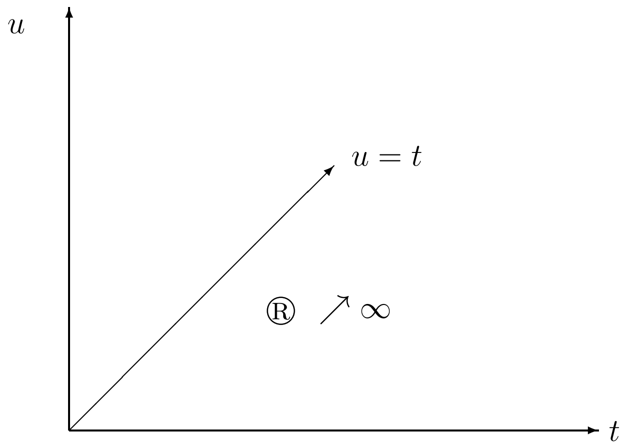

The region of integration Ⓡ can be read from the figure : the u-integration is taken in the vertical direction to cover Ⓡ but to cover this in the reverse order, the t-integration is taken in the horizontal direction.

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(3.2.14)

11. **Integral :**

$$\boxed{\mathcal{L}\left(\int_0^t f(u)\, du\right) = \frac{\overline{f}(s)}{s}} \tag{3.2.15}$$

The integral in (3.2.15) is a convolution product between $f(t)$ and $g(t) = 1$. Thus $\overline{g}(s) = 1/s$, so this is an instance of the convolution theorem.

12. **Derivative :**

$$\boxed{\mathcal{L}\left[f'(t)\right] = s\overline{f}(s) - f(0)} \tag{3.2.16}$$

Noting that $f(0)$ means $f(t=0)$

GAP

13. **Second derivative :**

$$\boxed{\mathcal{L}\left[f''(t)\right] = s^2\overline{f}(s) - sf(0) - f'(0)} \tag{3.2.17}$$

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Exercise [Exam 2013] Show that

$$\mathcal{L}\left[f'''(t)\right] = s^3\overline{f}(s) - s^2 f(0) - sf'(0) - f''(0)\,.$$

[Follow the same idea as for the second derivative.]

3.3 Using the Convolution Theorem to find inverses

If we are given an inverse LT as a function $\overline{F}(s)$ which is too complicated to appear in the Library above but can be split into composite functions $\overline{F}(s) = \overline{f}(s)\,\overline{g}(s)$ where $\overline{f}(s)$ and $\overline{g}(s)$ do belong to the Library, then the Convolution Theorem allows us to write

$$F(t) = \mathcal{L}^{-1}\left(\overline{f}(s)\,\overline{g}(s)\right) = f(t) \star g(t)\,. \tag{3.3.1}$$

Example 3.3.1. *Find* $\mathcal{L}^{-1}\left[\dfrac{1}{s(s^2+1)}\right]$.

We identify $\overline{f}(s) = \dfrac{1}{s}$ *and* $\overline{g}(s) = \dfrac{1}{s^2+1}$.

We know, see the library, that $f(t) = 1$ and $g(t) = \sin t$. Thus

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[Alternative: use partial fractions.]

Example 3.3.2. *Find* $\mathcal{L}^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$.

Identify

GAP

The Library tell us that $f(t) = \cos(at)$ and $g(t) = \dfrac{1}{a}\sin(at)$, and so

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Using $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ we find

GAP

Now integrate:

GAP

Example 3.3.3. *Find* $\mathcal{L}^{-1}\left[\dfrac{a^2}{(s^2+a^2)^2}\right]$.

As in the previous case, identify $F(s) = |\overline{f}(s)|^2$ where

$$\overline{f}(s) = \frac{a}{s^2+a^2} \qquad\qquad \overline{g}(s) = \overline{f}(s)\,. \tag{3.3.2}$$

The Library tell us that $f(t) = g(t) = \sin(at)$ so $\sin(at)$ is convolved with itself

$$\begin{aligned} F(t) &= \sin(at) \star \sin(at) \\ &= \int_0^t \sin(au)\sin[a(t-u)]\,du \\ &= \frac{1}{2a}\left[\sin(at) - at\cos(at)\right]. \end{aligned} \tag{3.3.3}$$

having used the trig-identity $\cos(A-B) - \cos(A+B) = 2\sin A \sin B$. Details are left as exercise.

3.4 Alternative for type $\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$ and $\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right]$.

Sometimes we need to find inverse Laplace transforms of the type $\dfrac{1}{(s^2+1)^2}$ or $\dfrac{s}{(s^2+1)^2}$. The square in the denominator looks ominous and it appears at first glance that this will defeat us in any attempt to get these expressions to look like any of the standard LT in the library. However, there is a trick that enables us to do just that. Consider first the sine function $\sin(at)$ where we begin by treating a as a variable, independent of t, but still take the LT over t:

$$\mathcal{L}\left[\sin(at)\right] = \frac{a}{s^2+a^2},$$

and now differentiate the last equation (partially) w.r.t. a to give

$$\mathcal{L}\left[t\cos(at)\right] = \frac{1}{s^2+a^2} - \frac{2a^2}{(s^2+a^2)^2},$$

Divide by $2a$ and use the transform of $\sin(at)$ to get

$$\mathcal{L}^{-1}\left[\frac{a}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}\sin(at) - \frac{1}{2a}t\cos(at).,$$

Let $a = 1$ for the desired result.

We repeat the process by taking

$$\mathcal{L}\left[\cos(at)\right] = \frac{s}{s^2+a^2},$$

and again differentiate w.r.t. a to obtain

$$\mathcal{L}\left[-t\sin(at)\right] = -\frac{2as}{(s^2+a^2)^2},$$

Hence

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\sin(at).$$

3.5 Examples involving partial fractions and the Shift theorem

Example 3.5.1. *Find $f(t)$ when*

$$\overline{f}(s) = \frac{6s^2+10s+2}{s(s^2+3s+2)}\,. \tag{3.5.1}$$

GAP Noting that $s^2+3s+2 =$

We can write

$$\frac{6s^2+10s+2}{s(s^2+3s+2)} = \frac{A}{\qquad} + \frac{B}{\qquad} + \frac{C}{\qquad} \quad \text{and hence}$$

GAP

Example 3.5.2. *Find $f(t)$ when*

$$\overline{f}(s) = \frac{2}{s(s-2)}\,. \tag{3.5.2}$$

Use partial fractions:

$$\overline{f}(s) = -\frac{1}{s} + \frac{1}{s-2}\,, \quad \Longrightarrow \quad f(t) = -1 + e^{2t}\,. \tag{3.5.3}$$

Example 3.5.3. *Find $f(t)$ when $\overline{f}(s) = \dfrac{1}{(s-1)^4}$.*

Recall from the library:

GAP

Example 3.5.4. *Let $f(t)$ be a square wave with amplitude 1, period 2:*

Obtain the Laplace transform.

We first write this as

GAP

and take the transform:

GAP

Recall the geometric series, for $|x| < 1$:

$$1 - x + x^2 - x^3 + \ldots = \frac{1}{1+x}$$

GAP

Example 3.5.5. *An example of both shift theorems at work. Obtain the inverse transform:*

$$\mathcal{L}^{-1}\left[\frac{se^{-2s}}{s^2+2s+5}\right]$$

We first need to identify the shift in s:

GAP

and now apply the first shift theorem with $s+1 \Rightarrow s$, so that $\overline{f}(s-a)$ inverse-transforms to $\exp(at)f(t)$ with $a = -1$:

GAP

Now take the inverse transform using the second shift theorem: $\exp(-sa)\,\overline{f}(s)$ has inverse transform $f(t-a)H(t-a)$ with $a = 2$:

GAP

3.6 Solving ODEs using Laplace Transforms

Many textbook methods are given to solve 2nd order ODEs of the type

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x = f(t) \tag{3.6.1}$$

The approach we had in year one is not always useful:

GAP

We need the LT-method to handle those cases when the forcing function is not smooth. Examples might be voltage inputs of the square wave or saw-tooth type. To approach this using LTs, the transform is taken of (3.6.1)

GAP

(3.6.2)

where the initial conditions are already present. Note that we need the intital conditions at time $t = 0$. This is rearranged into

GAP

(3.6.3)

Note that the final expression for $\overline{x}(s)$ divides conveniently into two parts corresponding to the Complementary Function and the Particular Integral:

GAP

(3.6.4)

The initial conditions appear as part of the Complementary Function. How to take the inverse depends on whether the denominator has real or complex roots, but the basic idea is to break up the RHS into simpler functions of s whose inverse transform we can get using the tools in the library.

Example 3.6.1. *Solve* $\ddot{x} + \dot{x} - 2x = e^t$ *with* $x(0) = 3$ *and* $\dot{x}(0) = 0$.

We begin by taking the transform of the ODE, term by term:

GAP

(3.6.5)

Noting that $s^2 + s - 2 = (s-1)(s+2)$ we have

GAP

(3.6.6)

We use partial fractions:

GAP

and finally can inverse transform

GAP

Example 3.6.2. *Solve* $\ddot{x} + 16x = \sin 2t$ *with* $x_0 = 0$ *and* $\dot{x}_0 = 1$.

Again, begin by applying the LT to the ODE:

GAP

and rearrange:

GAP

Partial fractions (detail omitted)

GAP

ready for inverse transform:

GAP

Example 3.6.3. *(Convolution Theorem) Solve $\ddot{x} + 3\dot{x} + 2x = f(t)$ with $x_0 = 1$ and $\dot{x}_0 = -2$. In this example $f(t)$ has not been specified although it must be assumed that its LT exists.*

GAP

Detail left as exercise. The point is we don't know what the transform $\overline{f}(s)$ is, but can write the RHS as

$$\overline{f}(s)\overline{g}_1(s) - \overline{f}(s)\overline{g}_2(s) + \overline{g}_2(s)$$

where

GAP

From these definitions it is clear that

GAP

From the Convolution Theorem we have

GAP

The power of the LT-method can be seen here in that it solves, in principle, an ODE with any forcing, provided $\overline{f}(s)$ exists. We may not be able to solve the integral, but can always apply numerical methods.

Example 3.6.4. *Solve* $\ddot{x} + 2\dot{x} + 2x = f(t)$ *with* $x_0 = 1$ *and* $\dot{x}_0 = 0$*. In this example* $f(t)$ *has not been specified although it must be assumed that its LT exists. Left as exercise.*

Example 3.6.5. *(ODE with discontinuous input) Solve*

$$\ddot{x} + \dot{x} + \frac{5}{4}x = 1 - H(t - \pi)$$

where H *is the Heaviside function, with initial conditions* $x(0) = 1$ *and* $x'(0) = -1$*.*

Transform the ODE and rearrange:

GAP

Partial fractions and rearrange:

GAP

Ready for inverse transfom. The first two terms are easy, they are a constant times the RHS of the transformed ODE. For the terms involving $s + \frac{1}{2}$ we use the first shift theorem:

GAP

The second term is ready for inverse. On the last bracket, the term $e^{-\pi s}$ requires us to use the second shift theorem with $a = \pi$:

GAP

Rearrange to finish.

GAP

Example Sheet 7: LAPLACE TRANSFORMS

1. a) For the coupled ODEs
$$2\dot{x} + \dot{y} + x + 6 = 0 \qquad \dot{x} + 2\dot{y} + y = 0$$
where $x(0) = y(0) = 1$, show that
$$\overline{y}(s) = \frac{3(s+3)}{(s+1)(3s+1)} \qquad \overline{x}(s) = \frac{3(s^2 - 3s - 2)}{s(s+1)(3s+1)}.$$
Split these expressions into partial fractions & invert to find $x(t)$ and $y(t)$.

   (b) In the same manner as part a), use Laplace transforms to solve
$$\dot{x} + 5x + 2y = e^{-t}, \quad \dot{y} + 2x + 2y = 0, \quad x(0) = 1, \;\; y(0) = 0.$$

2. A function $f(t)$ has a Laplace transform $\mathcal{L}\{f(t)\} \equiv \overline{f}(s)$. Use the 'shift property' $\mathcal{L}\left\{e^{at}f(t)\right\} = \overline{f}(s-a)$, where a is a constant, and the 'second shift property'; $\mathcal{L}\left\{H(t-a)f(t-a)\right\} = e^{-sa}\overline{f}(s)$ to show that the solution of the SHM equation with discontinous driving terms
$$\ddot{x} + x = H(t-\pi) - H(t-2\pi)$$
and with initial conditions $x(0) = \dot{x}(0) = 0$, is
$$\begin{array}{ll} x = 0 & 0 \le t \le \pi \\ x = 1 + \cos t & \pi \le t \le 2\pi \\ x = 2\cos t & 2\pi \le t \end{array}$$
where H(t) is the Heaviside step function.

3. If a function $f(t)$ is periodic in time t with fixed period T such that $f(t) = f(t-T)$ with $T > 0$ show that for $s > 0$
$$\overline{f}(s) = \frac{1}{1 - e^{-sT}} \int_0^T f(t)e^{-st}\, dt.$$
Note that this enables a Laplace Transform to be found by performing the integral only over the period $(0, T)$ for which $f(t)$ is defined.

4. Use the result of Q3 to show that the Laplace transform of the 'saw-tooth' function
$$\begin{array}{ll} f(t) = t & 0 \le t \le 1 \\ f(t) = f(t-1) & 1 \le t \end{array}$$
is given by
$$\overline{f}(s) = s^{-2} - s^{-1}\left(e^{-s} + e^{-2s} + e^{-3s} \dots\right).$$
The function $f(t)$ is often used in electronics for representing discontinuous voltages.

Answers:

1a) $x = -6 + 3e^{-t} + 4e^{-\frac{1}{3}t}$; $\;y = -3e^{-t} + 4e^{-\frac{1}{3}t}$

1b) $x = \frac{16}{25}e^{-6t} + \frac{1}{5}(t + \frac{9}{5})e^{-t}$; $\quad y = \frac{8}{25}e^{-6t} - \frac{1}{5}(\frac{8}{5} + 2t)e^{-t}$