EE2 Mathematics : Vector Calculus

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(Based on notes developed by J.D. Gibbon and R. Syms)

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These notes are not identical word-for-word with my lectures which will be given on a BB/WB. Some of these notes may contain more examples than the corresponding lecture while in other cases the lecture may contain more detailed working. I will **NOT** be handing out copies of these notes – **you are therefore advised to attend lectures and take your own**.

1. The material in them is dependent upon the Vector Algebra you were taught at A-level and in year one. A summary of what you need to revise lies in **Handout 1 : "Things you need to recall about Vector Algebra"** which is also §1 of this document. Refer to your first-year notes for detail.

2. Further handouts are :

   (a) **Handout 2 :** "The role of grad, div and curl in vector calculus" summarizes most of the material in §3.

   (b) **Handout 3 :** "Changing the order in double integration" is incorporated in §5.5.

   (c) **Handout 4 :** "Green's, Divergence & Stokes' Theorems plus Maxwell's Equations" summarizes the material in §6, §7 and §8.

   Technically, while Maxwell's Equations themselves are not in the syllabus, three of the four of them arise naturally out of the Divergence & Stokes' Theorems and they connect all the subsequent material with that given from lectures on e/m theory given in your own Department.

1 Revision : Things you need to recall about Vector Algebra

Notation: $\boldsymbol{a} = a_1\underline{\boldsymbol{i}} + a_2\underline{\boldsymbol{j}} + a_3\underline{\boldsymbol{k}} \equiv (a_1, a_2, a_3)$.

1. The *magnitude or length of a vector* $\boldsymbol{a}$ is

$$|\boldsymbol{a}| = a = \left(a_1^2 + a_2^2 + a_3^2\right)^{1/2}. \tag{1.1}$$

2. The *scalar (dot) product* of two vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ & $\boldsymbol{b} = (b_1, b_2, b_3)$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3\,. \tag{1.2}$$

Since $\boldsymbol{a} \cdot \boldsymbol{b} = ab\cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$, assuming neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ are null.

3. The *vector (cross) product*[1] between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \tag{1.3}$$

Recall that $\boldsymbol{a} \times \boldsymbol{b}$ can also be expressed as

$$\boldsymbol{a} \times \boldsymbol{b} = (ab\sin\theta)\,\hat{\boldsymbol{n}} \tag{1.4}$$

where $\hat{\boldsymbol{n}}$ is a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ in a direction determined by the right hand rule. If $\boldsymbol{a} \times \boldsymbol{b} = 0$ then $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel if neither vector is null.

4. The *scalar triple product* between three vectors $\boldsymbol{a},\ \boldsymbol{b}$ and $\boldsymbol{c}$ is

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \qquad \begin{array}{ccccc} & & \boldsymbol{a} & & \\ & \nearrow & & \searrow & \\ \boldsymbol{c} & & \longleftarrow & & \boldsymbol{b} \end{array} \qquad \begin{array}{c}\text{Cyclic Rule:}\\ \text{clockwise +ve}\end{array} \tag{1.5}$$

According to the cyclic rule

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \tag{1.6}$$

One consequence is that if **any two of the three vectors are equal (or parallel)** then their scalar product is zero: e.g. $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{a}) = \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{a}) = 0$. If $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = 0$ and no pair of $\boldsymbol{a},\ \boldsymbol{b}$ and $\boldsymbol{c}$ are parallel then the three vectors must be coplanar.

5. The *vector triple product* between three vectors is

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})\,. \tag{1.7}$$

The placement of the brackets on the LHS is important: the RHS is a vector that lies in the same plane as $\boldsymbol{b}$ and $\boldsymbol{c}$ whereas $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{b}(\boldsymbol{c} \cdot \boldsymbol{a}) - \boldsymbol{a}(\boldsymbol{c} \cdot \boldsymbol{b})$ lies in the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$. Thus, $\boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{c}$ without brackets is a meaningless statement!

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[1] It is acceptable to use the notation $\boldsymbol{a} \wedge \boldsymbol{b}$ as an alternative to $\boldsymbol{a} \times \boldsymbol{b}$.

2 Scalar Fields

Our first aim is to step up from single variable calculus – that is, dealing with functions of one variable – to functions of two, three or even four variables. The physics of electro-magnetic (e/m) fields requires us to deal with the three co-ordinates of space $(x,\ y,\ z)$ and also time t. There are two different types of functions of the four variables, scalar and vector fields. The latter will be discussed in §3.

1. A **scalar field** [2] is written as

$$\psi = \psi(x, y, z, t)\,., \tag{2.1}$$

   a function of four variables, so the terms **scalar field** and **scalar function** are interchangeable. Note that one cannot plot ψ as a graph in the conventional sense as ψ takes values at *every point in space and time*. A good example of a scalar field is the temperature of the air in a room. If the box-shape of a room is thought of as a co-ordinate system with the origin in one corner, then every point in that room can be labelled by a co-ordinate (x, y, z). If the room is poorly air-conditioned the temperature in different parts may vary widely : moving a thermometer around will measure the variation in temperature from point to point (spatially) and also in time (temporally). Another example of a scalar field is the concentration of salt or a dye dissolved in a fluid: Take a container of water and drop some dye in at one end, then observe and measure the concentration as the dye spreads. An electric potential is a scalar fucntion: $V_E = \phi(x, y, z)$.

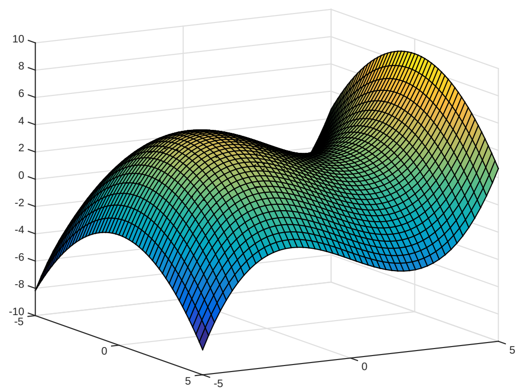
2.1 Visualization

Visualizing a scalar field/function is more difficult for functions of 3 or 4 variables, but we have several reasonably easy ways of looking at functions of 2 variables.

Example 1

$$\phi(x, y) = \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2}$$

To visualise this function we can plot the graph of the surface given by $z = \phi(x, y)$, using Matlab, for example.

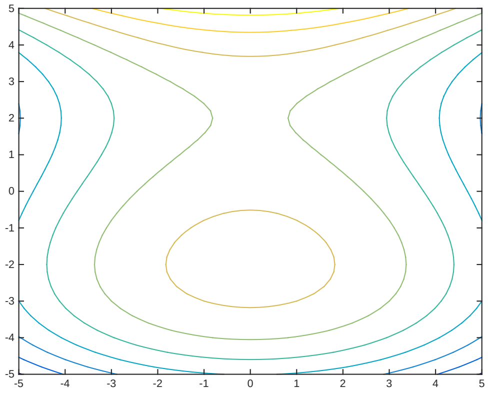


An alternative is to plot the **contour lines** or**level surfaces** given by $z = c$ =constant. Thus we plot the curves

$$\frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2} = c$$

[2]The convention is to use Greek letters for scalar fields and bold Roman for vector fields.

for a number of equally spaced contours. For the above function this gives the picture:



The matlab code for these two figures is simple:

```
v=-5:0.2:5;
[x,y]=meshgrid(v);
z= y.^3/12-y-x.^2/4+3.5;
figure;
surf(x,y,z);
figure;
contour(x,y,z);
```

Higher dimensions is, of course, more difficult to visualize, but there are ways around this.

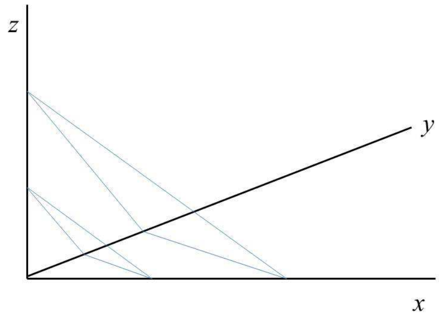
Example 2

$$\phi(x, y, z) = x + y + z$$

Each point in 3-D space is assigned one value ϕ. For each c, we can draw $\phi(x, y, z) = c$. For a function of two variables, in the previous example, we saw that these were contour lines, one-dimensional. For a function of three variables, the contours are two-dimensional surfaces, hence we refer to them as **level surfaces**, which are again easier to plot. Here, each value of c gives us the level surface

$$x + y + z = c$$

which we recognize as a series of parallel planes with common normal vector $(1, 1, 1)$.



A scalar field ϕ can represent an electric potential. Taking $\phi = c$ we obtain so-called **equipotentials**. For a function of two variables, $\phi(x, y) = c$ gives equipotential lines; for a function of three variables, $\phi(x, y, z) = c$ gives equipotential surfaces. In both cases, they represent constant potential energy. If the contour lines are those on a map, they represent altitude above sea level and the potential energy is gravitational: the higher above some zero marker, e.g. sea level, the higher the potential energy, which will be familiar to those who have done mechanics.

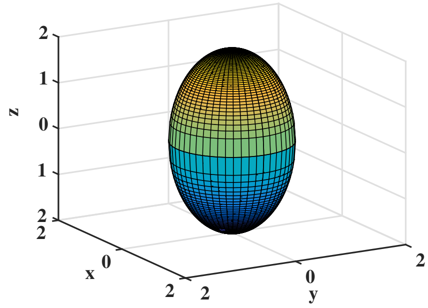
Example 3 The equipotentials of

$$\phi(x, y, z) = x^2 + y^2 + \frac{z^2}{4}$$

are given by

$$\phi(x, y, z) = x^2 + y^2 + \frac{z^2}{4} = c$$

which are ellipsoids:



Note: the surface of a conductor is an equipotential surface.

2.2 Scalar fields varying with time

Timevarying scalar fields present another problem: how to display the variation in time.

Example 4 A travelling wave in 1-D with temporal frequency F and wavelength λ has the equation

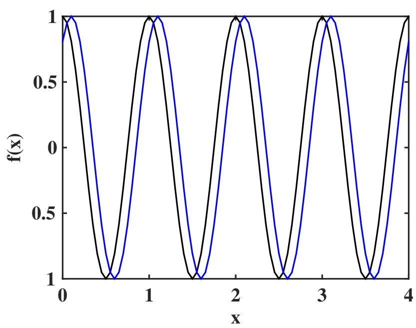
$$f(x, t) = A\cos\left[2\pi\left(Ft - \frac{1}{\lambda}x\right)\right]$$

or

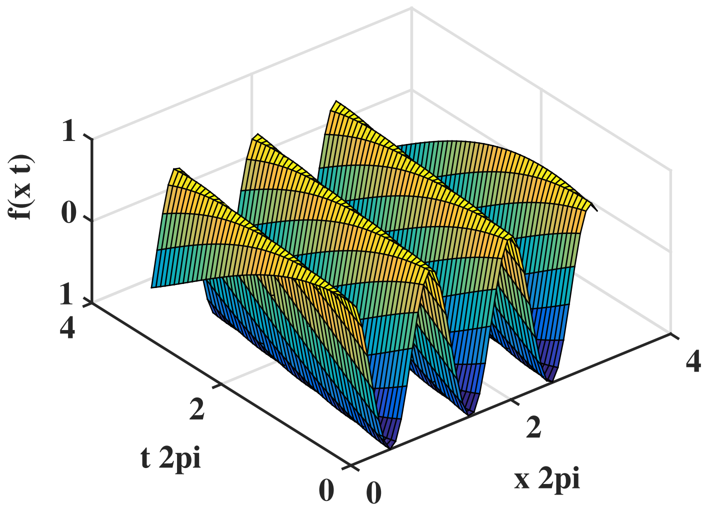
$$f(x, t) = A\cos(\omega t - kx)$$

with angular frequency $\omega = 2\pi F$ and propagation constant $k = 2\pi/\lambda$. This can also be seen as the real part of the complex function $\phi(x, t) = A\exp[i(\omega t - kx)]$.

Taking two different values of t we obtain a clear picture of the movement of the wave



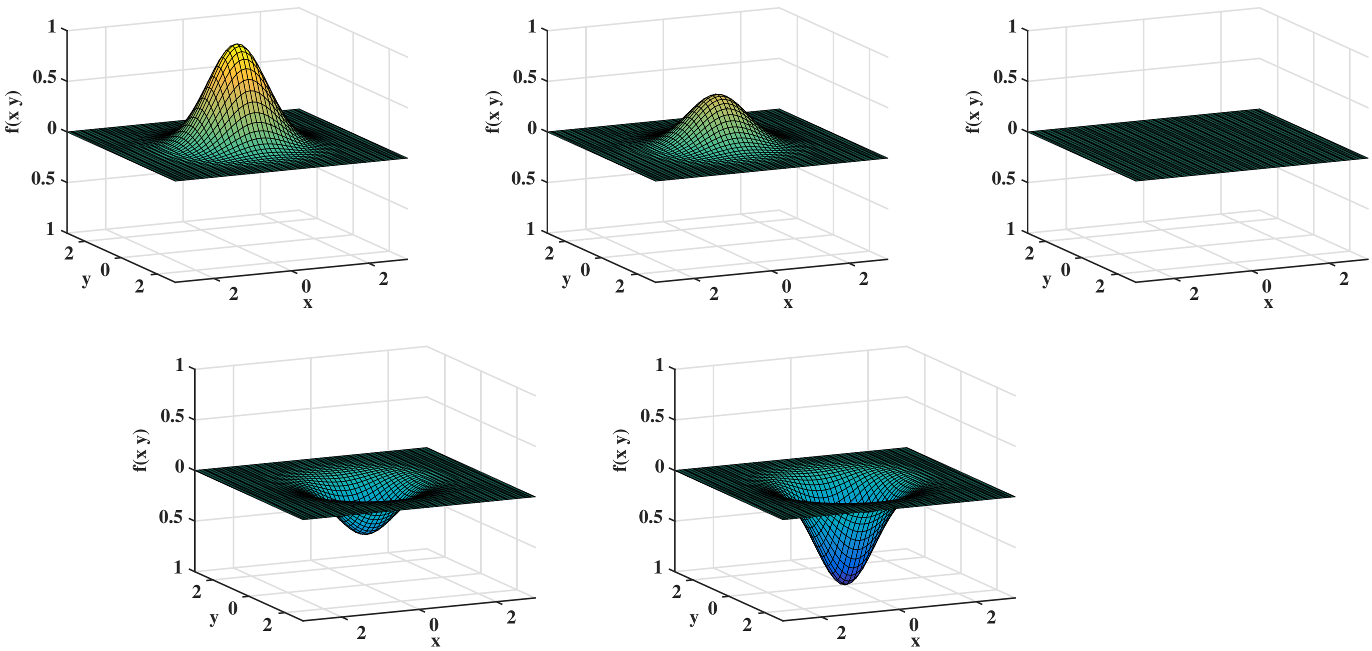
As a surface plot we can see the change over continuous time:



For a time-varying field in 2-D, we can take snapshots of the surface at various times:

Example 5

$$\phi(x,y,t) = e^{-(x^2+y^2)}\cos(\omega t)\,.$$



3 Vector Fields

A **vector field** $\boldsymbol{B}(x,y,z,t)$ has components $(B_1,\ B_2,\ B_3)$ in terms of the three unit vectors $(\underline{\boldsymbol{i}},\ \underline{\boldsymbol{j}},\ \underline{\boldsymbol{k}})$

$$\underline{\boldsymbol{B}} = \underline{\boldsymbol{i}}B_1(x,y,z,t) + \underline{\boldsymbol{j}}B_2(x,y,z,t) + \underline{\boldsymbol{k}}B_3(x,y,z,t)\,. \tag{3.1}$$

These components can each be functions of any one or all of (x,y,z,t), so B_1, B_2, B_3 are all scalar fields. Three physical examples of vector fields are :

1. An **electric field :**

$$\underline{\boldsymbol{E}}(x,y,z,t) = \underline{\boldsymbol{i}}E_1(x,y,z,t) + \underline{\boldsymbol{j}}E_2(x,y,z,t) + \underline{\boldsymbol{k}}E_3(x,y,z,t)\,, \tag{3.2}$$

2. A **magnetic field :**

$$\underline{\boldsymbol{H}}(x,y,z,t) = \underline{\boldsymbol{i}}H_1(x,y,z,t) + \underline{\boldsymbol{j}}H_2(x,y,z,t) + \underline{\boldsymbol{k}}H_3(x,y,z,t)\,, \tag{3.3}$$

3. The **velocity field** $\boldsymbol{u}(x,y,z,t)$ in a fluid.

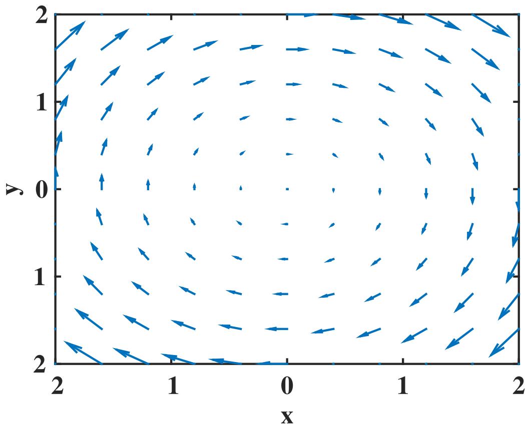
A classic illustration of a three-dimensional vector field in action is the e/m signal received by a mobile phone which can be received anywhere in space. And, of course, the gravitational field.

3.1 Visualization

Again, visualization is easiest for 2-D.

Example 1. $\underline{\boldsymbol{F}}(x,y) = y\underline{\boldsymbol{i}} - x\underline{\boldsymbol{j}}$.

At each point in the $x,y-$plane we draw an arrow representing the direction/magnitude of the field, the vector at the point (x,y) is $(y,-x)$:



To generate the plot, matlab code:

```
g=-2:0.4:2;
[x,y]=meshgrid(g);
figure;
u=y;
v=-x;
quiver(x,y,u,v);
```

We can draw **field lines** which are everywhere parallel to the vector field. If the field is a force field, motion under the force follows the field lines. In a gravitational field, bodies move along the (simple) field lines, everywhere pointing to the origin. Electric fields exert forces on charges, which will then move along the trajectories given by the field lines.

There is a simple relation allowing the construction of field lines. For a given 2-D field $\underline{\boldsymbol{F}} = F_x\underline{\boldsymbol{i}} + F_y\underline{\boldsymbol{j}}$, the field lines $y = y(x)$ satisfy

$$\frac{dy}{dx} = \frac{F_y}{F_x}\,.$$

To see this, we note that the line giving a small displacement $\delta\underline{\mathbf{x}} = \delta x\underline{\boldsymbol{i}} + \delta y\underline{\boldsymbol{j}}$ is in the same direction as the field, and therefore

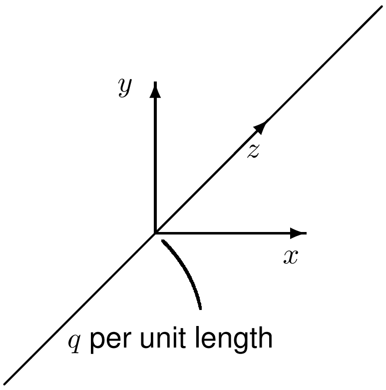
$$\frac{\delta x}{F_x} = \frac{\delta y}{F_y} \Rightarrow F_x\delta y - F_y\delta x$$

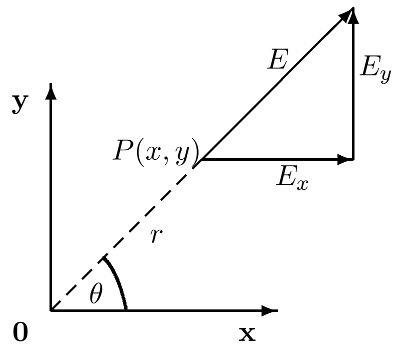
and the result follows by dividing and taking the limit $d\underline{\mathbf{x}} \to 0$.

In the case of the above example, we have $\dfrac{dy}{dx} = -\dfrac{x}{y} \Rightarrow y\ dy = -x\ dx$ and we can integrate to obtain the field lines as circles $x^2 + y^2 = c$, confirming what was evident in the above figure. In general, for $F_x(x, y)$ and $F_y(x, y)$ this will not be so easy; in 3-D this can be much harder. Note that the equation does not give the direction of flow.

Example 2. Given a line of electric charge through the origin and parallel to the $z-$axis, the electric field $\underline{\boldsymbol{E}}$ is independent of z, and we need only consider the case $z = 0$, so that at any point $P(x, y)$ the field is entirely radial and can be written as

$$\underline{\boldsymbol{E}} = \frac{q}{2\pi\epsilon_0 r}\underline{\hat{\mathbf{r}}}$$

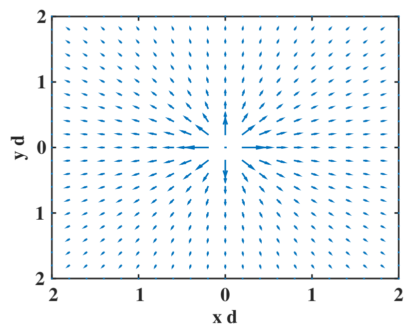




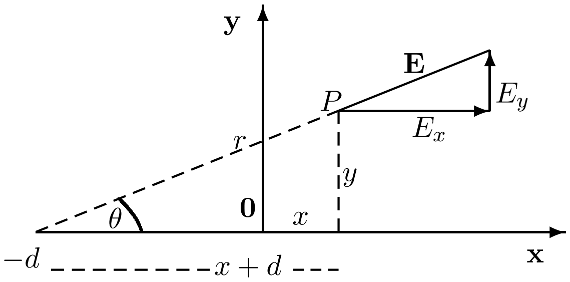
Here $r = \sqrt{x^2 + y^2}$, the distance from the origin, q is the charge per unit length, ϵ_0 is the permittivity of free space and $\underline{\hat{\mathbf{r}}}$ is a unit vector in the radial direction. At P, we write $\underline{\boldsymbol{E}}$ in terms of cartesian components:

$$\underline{\boldsymbol{E}} = E_x\underline{\boldsymbol{i}} + E_y\underline{\boldsymbol{j}} = \frac{k}{r}\cos\theta\underline{\boldsymbol{i}} + \frac{k}{r}\sin\theta\underline{\boldsymbol{j}}\text{, where } k = q/(2\pi\epsilon_0).$$

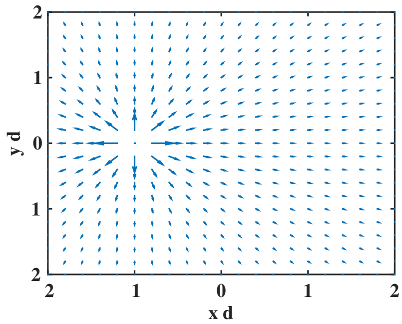
Thus, given $\cos\theta = x/r$ and $\sin\theta = y/r$ we obtain $\underline{\boldsymbol{E}} = \dfrac{kx}{x^2 + y^2}\underline{\boldsymbol{i}} + \dfrac{ky}{x^2 + y^2}\underline{\boldsymbol{j}}$, which is easy to plot:



If we shift the line of charge, still parallel to $z-$axis, from the origin to $(-d, 0)$ we have the same details as before except that now $r^2 = (x+d)^2 + y^2$.

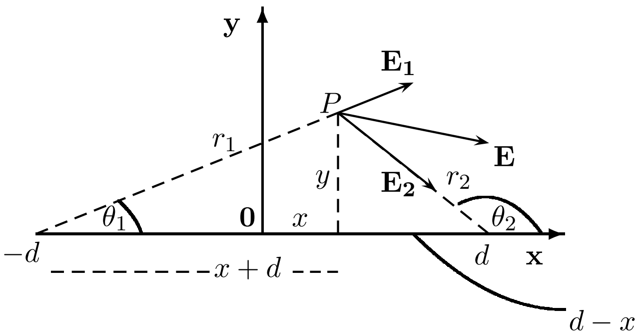


We now have the field as $\underline{\boldsymbol{E}} = \dfrac{k(x+d)}{(x+d)^2 + y^2}\underline{\boldsymbol{i}} + \dfrac{ky}{(x+d)^2 + y^2}\underline{\boldsymbol{j}}$.



Finally, if we take two line charges at $(\pm d, 0)$ with strength $\mp q$ per unit length, we can use superposition to draw the field, as the total electric field is the sum of the individual fields, $\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_1 + \underline{\boldsymbol{E}}_2$ where

$$\underline{\boldsymbol{E}}_1 = \frac{q}{2\pi\epsilon_0 r_1}\hat{\underline{\mathbf{r}}}_1 = \frac{k}{r_1}\hat{\underline{\mathbf{r}}}_1 \quad \text{and} \quad \underline{\boldsymbol{E}}_2 = -\frac{q}{2\pi\epsilon_0 r_2}\hat{\underline{\mathbf{r}}}_2 = -\frac{k}{r_2}\hat{\underline{\mathbf{r}}}_2\,,$$

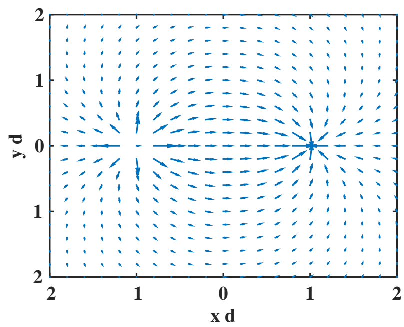


with $r_1^2 = (x+d)^2 + y^2$ and $r_2^2 = (x-d)^2 + y^2$. As before
$\underline{\boldsymbol{E}}_1 = E_{1,x}\underline{\boldsymbol{i}} + E_{1,y}\underline{\boldsymbol{j}} = \dfrac{k}{r_1}\cos\theta_1\underline{\boldsymbol{i}} + \dfrac{k}{r_1}\sin\theta_1\underline{\boldsymbol{j}}$ and $\underline{\boldsymbol{E}}_2 = E_{2,x}\underline{\boldsymbol{i}} + E_{2,y}\underline{\boldsymbol{j}} = -\dfrac{k}{r_2}\cos\theta_2\underline{\boldsymbol{i}} - \dfrac{k}{r_2}\sin\theta_2\underline{\boldsymbol{j}}$
and using $\cos\theta_1 = \dfrac{x+d}{r_1}$ and $\cos\theta_2 = \dfrac{x-d}{r_2}$ we get $\underline{\boldsymbol{E}} = E_x\underline{\boldsymbol{i}} + E_y\underline{\boldsymbol{j}}$, where

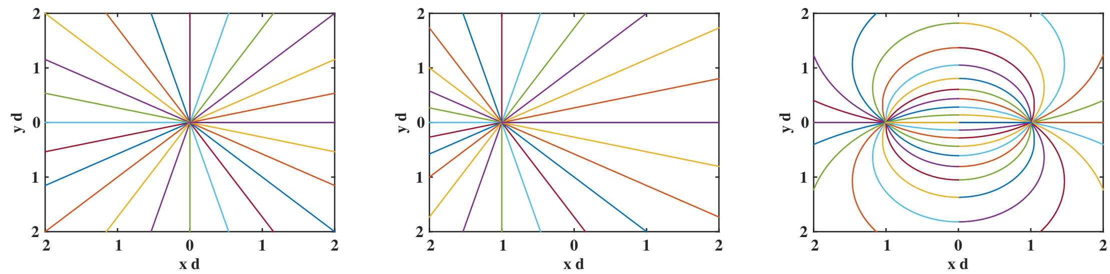
$$E_x = E_{1,x} + E_{2,x} = \frac{k}{r_1}\frac{x+d}{r_1} - \frac{k}{r_2}\frac{x-d}{r_2} = k\left(\frac{x+d}{(x+d)^2 + y^2} - \frac{x-d}{(x-d)^2 + y^2}\right)$$

and, similarly,

$$E_y = E_{1,y} + E_{2,y} = \frac{k}{r_1}\frac{y}{r_1} - \frac{k}{r_2}\frac{y}{r_2} = k\left(\frac{y}{(x+d)^2+y^2} - \frac{y}{(x-d)^2+y^2}\right).$$



When bodies move under gravity, the gravitational field lines show direction of force exerted and the trajectory of the bodies in motion. Similarly, electric fields exert forces on charges, which will then move along trajectories given by the field lines. For the previous example, we get

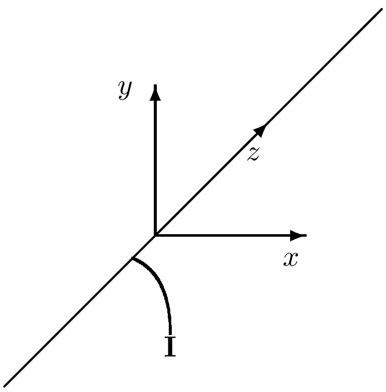


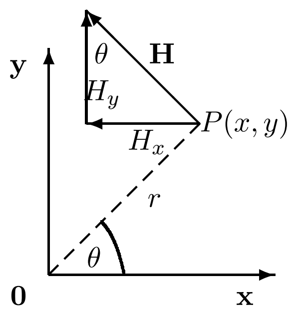
[Note: the directions of motion along the field lines are not shown.]

Example 3. Consider a current-carrying wire, passing through the origin and parallel to the $z-$axis: this is the same symmetry as in the previous example, and we can consider $z = 0$ as representative. Here, the magnetic field at a point P(x, y) is entirely tangential and can be written as

$$\underline{\boldsymbol{H}} = \frac{I}{2\pi r}\underline{\hat{\theta}},$$

where I is the current and $\underline{\hat{\theta}}$ is a unit vector in the direction of increasing θ, orthogonal to $\underline{\hat{r}}$.



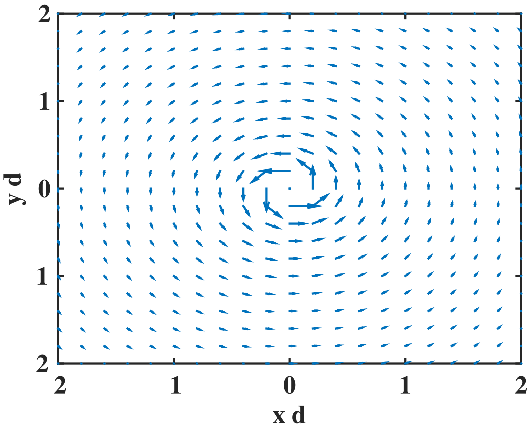


Again, resolve into cartesian components:

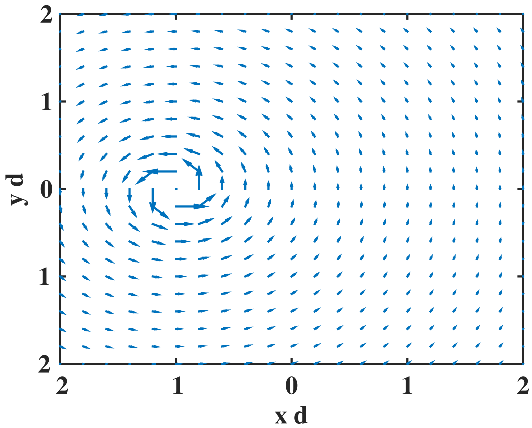
$$\underline{\boldsymbol{H}} = H_x\underline{\boldsymbol{i}} + H_y\underline{\boldsymbol{j}} = -\frac{I}{2\pi r}\sin\theta\underline{\boldsymbol{i}} + \frac{I}{2\pi r}\cos\theta\underline{\boldsymbol{j}}$$

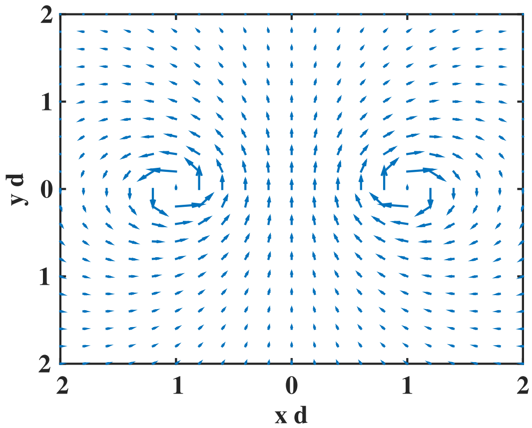
and again, with $\cos\theta = x/r$ and $\sin\theta = y/r$ we have

$$\underline{\boldsymbol{H}} = \frac{I}{2\pi}\left(-\frac{y}{x^2+y^2}\underline{\boldsymbol{i}} + \frac{x}{x^2+y^2}\underline{\boldsymbol{j}}\right) .$$



A similar calculation to the previous example gives the results for one wire, coordinate shifted to $(-d, 0)$ (below, left), and two wires at $(\pm d, 0)$ carrying currents of $\mp I$ (below, right).





In the second of these, the fields circle round the wires in opposite directions, reinforcing each other in between.

4 The vector operators : grad, div and curl

4.1 Definition of the gradient operator ∇

Consider a scalar field ϕ which depends on space,

$$\phi = \phi(x, y, z) .$$

If we consider how quickly the scalar field (or portential) varies in the x direction then we examine the quantity

$$\frac{\partial \phi}{\partial x}\,.$$

Similarly if we consider how quickly the scalar field varies in the y or z directions we should examine (respectively)

$$\frac{\partial \phi}{\partial y} \quad \text{or} \quad \frac{\partial \phi}{\partial z}\,.$$

We can summarise how quickly the potential varies at any point by looking at the **gradient**,

$$\nabla \phi = \underline{\boldsymbol{i}}\frac{\partial \phi}{\partial x} + \underline{\boldsymbol{j}}\frac{\partial \phi}{\partial y} + \underline{\boldsymbol{k}}\frac{\partial \phi}{\partial z}\,. \tag{4.1}$$

The gradient is written as $\boldsymbol{\nabla}\phi$ or gradϕ. **Note that while ϕ is a scalar field, $\nabla\phi$ itself is a vector field.**

We shall use the shorthand of the symbol ∇ to indicate this vector derivative. This is therefore the *definition* of the vector differential operator $\boldsymbol{\nabla}$ (called *nabla*, an Assyrian word). Typically we write this vector derivative in the form

$$\nabla = \underline{\boldsymbol{i}}\frac{\partial}{\partial x} + \underline{\boldsymbol{j}}\frac{\partial}{\partial y} + \underline{\boldsymbol{k}}\frac{\partial}{\partial z}\,. \tag{4.2}$$

We now show what this vector derivative means physically and how it may be used. Note that the vector derivative of a scalar function is a vector and therefore has direction and magnitude.

4.2 Directional derivatives

We want to consider the rate of change of the scalar ϕ in a particular direction. We shall denote this direction by the unit vector $\hat{\boldsymbol{m}}$. (always ensure the direction is given by a *unit* vector). Now if the direction is $\hat{\boldsymbol{m}} = \underline{\boldsymbol{i}}$ the answer is $\frac{\partial \phi}{\partial x}$ while if $\hat{\boldsymbol{m}} = \underline{\boldsymbol{j}}$ the answer is $\frac{\partial \phi}{\partial y}$. Recall that, in general, the component of a vector in the direction of a unit vector is given by their scalar product. Hence we find that rate of change of ϕ in the direction $\hat{\boldsymbol{m}}$ is given by

$$\frac{\partial \phi}{\partial \hat{\boldsymbol{m}}} = \boldsymbol{\nabla}\phi \cdot \hat{\boldsymbol{m}}. \tag{4.3}$$

This is called the directional derivative of ϕ in the $\hat{\boldsymbol{m}}$ direction and note, in contrast to the vector derivative, it is a scalar.

Figure 1 gives a geometric interpretation of the directional derivative of a 2-dimensional scalar field.

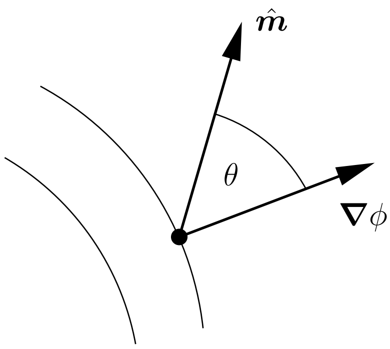


Figure 1: The directional derivative is given by the dot product between the direction vector and the gradient of the field.

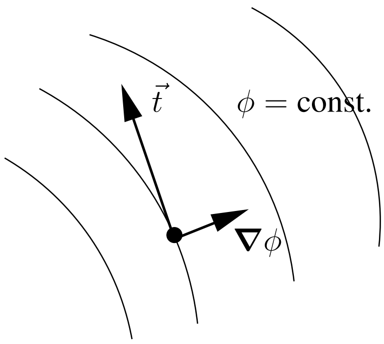


Figure 2: The gradient of a scalar field is perpendicular to the tangent to the level surfaces of that field.

4.3 The gradient of a scalar field

To understand what the gradient of a scalar field represents physically consider a scalar field in just two dimensions $\phi(x, y)$ (for example the electric voltage in a 2-D plate). Now draw (in the 2-D region) the curves along which the voltage is constant. These are the contour lines or level surfaces and can be thought of as the potential lines, shown below in figure 2.

Figure 2 shows $\boldsymbol{\nabla}\phi$. In practice if the scalar field is, for example, the electric potential ϕ then the vector field $\nabla\phi$ is related to the electric field by $\mathbf{E} = -\nabla\phi$.

We now wish to understand what the quantity $\nabla\phi$ is. Taking a point in the region consider how quickly the potential varies in different directions. If we were to consider a direction that goes along a level surface then we see that the potential does not change (by definition – the field is constant along the level surface). Hence we have that if $\hat{\boldsymbol{m}}$ points along the level surface then $\boldsymbol{\nabla}\phi \cdot \hat{\boldsymbol{m}} = 0$. Written another way, if we have the vector $\vec{t}$ which is tangent to a level surface then

$$\boldsymbol{\nabla}\phi \cdot \vec{t} = 0. \tag{4.4}$$

This gives us a geometric interpretation of $\boldsymbol{\nabla}\phi$. Equation (4.4) implies that the vector field $\boldsymbol{\nabla}\phi$ is everywhere perpendicular to the (tangent vector $\vec{t}$) and hence perpendicular to the level surfaces. Alternative simple statements are that $\boldsymbol{\nabla}\phi$ must point in the direction of steepest ascent or descent of ϕ, or that $\boldsymbol{\nabla}\phi$ points in the direction in which ϕ varies most rapidly. The magnitude of $\boldsymbol{\nabla}\phi$ tells us how quickly the potential ϕ varies in its steepest direction. Therefore we have that

$$\boldsymbol{\nabla}\phi = \left(\hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z} \right) \phi \tag{4.5}$$

$$= \hat{\boldsymbol{n}}\,\frac{\partial \phi}{\partial \vec{n}}, \tag{4.6}$$

where the vector

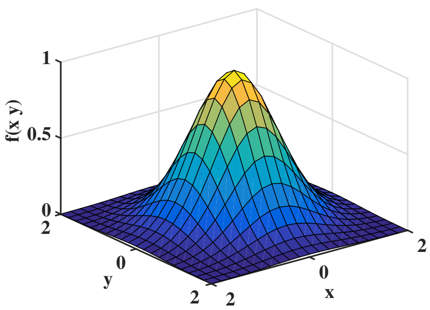
$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{\nabla}\phi}{|\boldsymbol{\nabla}\phi|} \tag{4.7}$$

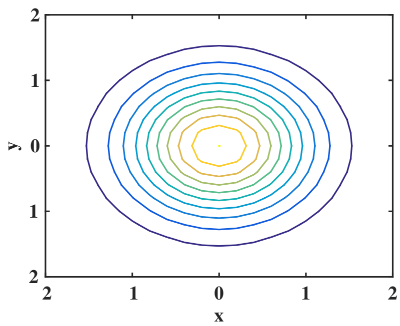
$$= \text{normal vector in direction of greatest increase of } \phi(\vec{r}). \tag{4.8}$$

If you've ever walked using a map which shows contour lines, you're used to this: walking along the contour is horizontal while walking perpendicular to the contour is the steepest ascent or descent

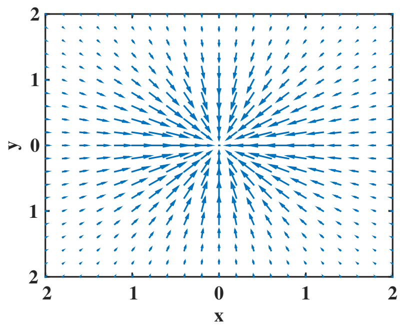
Example 1: Consider the 2-D scalar field $\phi(x, y) = \exp(-x^2 - y^2)$.

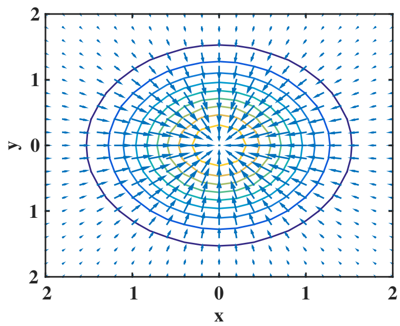
Clearly, $\phi = C$ implies that $x^2 + y^2$ =constant, so the equipotentials are circles:





Now obtain $\nabla\phi = -2x\exp(-x^2 - y^2)\underline{\boldsymbol{i}} - 2y\exp(-x^2 - y^2)\underline{\boldsymbol{j}}$ and plot:





The equipotential lines and gradient field are clearly perpendicular.

- **Example 2) :** With $\psi = \frac{1}{3}\left(x^3 + y^3 + z^3\right)$

$$\nabla\psi = \underline{\boldsymbol{i}}x^2 + \underline{\boldsymbol{j}}y^2 + \underline{\boldsymbol{k}}z^2. \tag{4.9}$$

  As explained above, the RHS is a vector field whereas ψ is a scalar field. The gradient can be thought of as a function from scalar to vector fields.

- **Example 3) :** With $\psi = xyz$ the vector $\nabla\psi$ is

$$\nabla\psi = \underline{\boldsymbol{i}}yz + \underline{\boldsymbol{j}}xz + \underline{\boldsymbol{k}}xy\,. \tag{4.10}$$

4.4 Definition of the divergence of a vector field div $\boldsymbol{B}$

Because vector algebra allows two forms of product - the scalar and vector products - analogously, there are two ways of operating ∇ on a vector field

$$\boldsymbol{B} = \underline{\boldsymbol{i}}B_1(x, y, z, t) + \underline{\boldsymbol{j}}B_2(x, y, z, t) + \underline{\boldsymbol{k}}B_3(x, y, z, t)\,. \tag{4.11}$$

The first is through the scalar or dot product

$$\text{div}\,\boldsymbol{B} \;=\; \nabla\cdot\boldsymbol{B} = \left(\underline{\boldsymbol{i}}\frac{\partial}{\partial x} + \underline{\boldsymbol{j}}\frac{\partial}{\partial y} + \underline{\boldsymbol{k}}\frac{\partial}{\partial z}\right)\cdot\left(\underline{\boldsymbol{i}}B_1 + \underline{\boldsymbol{j}}B_2 + \underline{\boldsymbol{k}}B_3\right). \tag{4.12}$$

Recalling that $\underline{\boldsymbol{i}}\cdot\underline{\boldsymbol{i}} = \underline{\boldsymbol{j}}\cdot\underline{\boldsymbol{j}} = \underline{\boldsymbol{k}}\cdot\underline{\boldsymbol{k}} = 1$ but $\underline{\boldsymbol{i}}\cdot\underline{\boldsymbol{j}} = \underline{\boldsymbol{i}}\cdot\underline{\boldsymbol{k}} = \underline{\boldsymbol{k}}\cdot\underline{\boldsymbol{j}} = 0$, the result is

$$\text{div}\,\boldsymbol{B} = \nabla\cdot\boldsymbol{B} = \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\,. \tag{4.13}$$

Note that div B is a scalar field because div is formed through the dot product. The best physical explanation that can be given is that div $\boldsymbol{B}$ is a measure of the compression or expansion of a vector field through the 3 faces of a cube.

- If div $\boldsymbol{B} = 0$ the the vector field $\boldsymbol{B}$ is incompressible ;
- If div $\boldsymbol{B} > 0$ the the vector field $\boldsymbol{B}$ is expanding ;
- If div $\boldsymbol{B} < 0$ the the vector field $\boldsymbol{B}$ is compressing .

If $\boldsymbol{B}$ represents a physical flow, say a liquid, and we consider a small sphere moving with the flow (field) lines then div$\boldsymbol{B} = 0$ implies that the volume flowing into the sphere is equal to the volume flowing out, so the volume of the sphere stays the same, but the actual shape may change with the flow. Note: for all e-m fields $\underline{\boldsymbol{F}}$, div$\underline{\boldsymbol{F}} = 0$.

Note : the usual rule in vector algebra that $a\cdot b = b\cdot a$ (that is, a and b commute) doesn't hold when one of them is an operator. Thus

$$\boldsymbol{B}\cdot\nabla = B_1\frac{\partial}{\partial x} + B_2\frac{\partial}{\partial y} + B_3\frac{\partial}{\partial z} \neq \nabla\cdot\boldsymbol{B} \tag{4.14}$$

4.5 Examples

Example 1 : Let $\boldsymbol{B} = \underline{\boldsymbol{i}}x^2 + \underline{\boldsymbol{j}}y^2 + \underline{\boldsymbol{k}}z^2$ then

$$\text{div}\, \boldsymbol{B} = 2x + 2y + 2z \tag{4.15}$$

Example 2
Now consider the vector field that is the radial vector in 3 dimensions,

$$\vec{F} = x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}. \tag{4.16}$$

Notice that this vector field points *radially* outwards at every point, like a hedgehog, with the magnitude increasing at larger distances. Figure 3 shows this vector field in 3D and in a slice through the centre. In this case the divergence is

$$\begin{aligned} \boldsymbol{\nabla} \cdot \vec{F} &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) && (4.17)\\ &= 1 + 1 + 1 && (4.18)\\ &= 3\;. && (4.19) \end{aligned}$$

Notice that the divergence here is constant everywhere and that it is positive $(3 > 0)$.

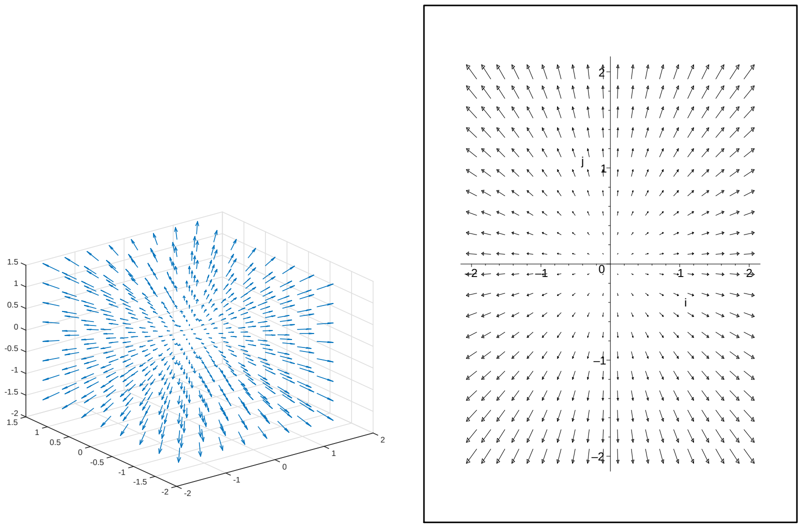


Figure 3: The vector field $\vec{F} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ from example 2. There is a “net outflow” of arrows. The second panel is a cut in the $z = 0$ plane.

Matlab for 3d:

```
[x,y,z]=meshgrid(-1.5:0.4:1.5);
u=x;v=y;w=z;
quiver3(x,y,z,u,v,w)
```

Example 3

Now consider the same sort of radially symmetric vector field as in Example 2 but with a negative sign.

$$\vec{F} = -\left(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}}\right), \tag{4.20}$$

This vector field points radially *inflow* at each point with the magnitude increasing at larger distances and is shown in Figure 4. In this case the divergence is

$$\boldsymbol{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(-z) \tag{4.21}$$

$$= -1 - 1 - 1 = -3\,. \tag{4.22}$$

Note that again the divergence is constant but here it is negative ($-3 < 0$). In fact at any point the divergence of a vector field indicates how rapidly the vector field is spreading out. Positive values indicate spreading vectors while negative values indicate the vectors are converging.

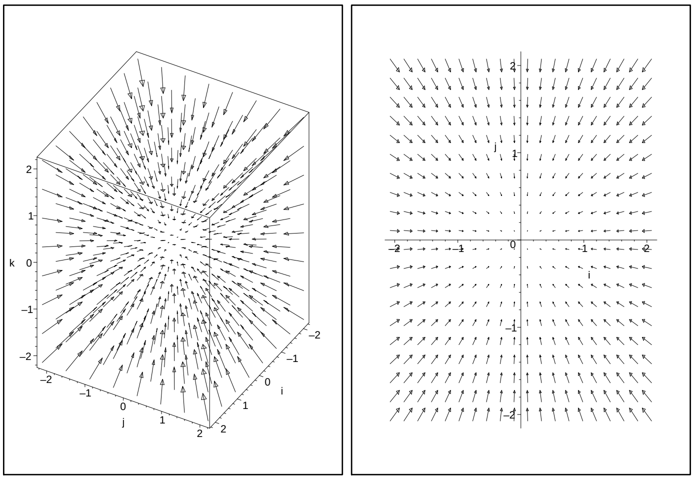


Figure 4: The vector field $\vec{F} = -x\hat{\boldsymbol{\imath}} - y\hat{\boldsymbol{\jmath}} - z\hat{\boldsymbol{k}}$ from example 3. There is a "net inflow" of arrows. The second panel is a cut in the $z = 0$ plane.

Example 4

Another example of a flow field where some flows in and some out is given by

$$\vec{F} = x\,\hat{\boldsymbol{\imath}} - y\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}. \tag{4.23}$$

and a 3D picture and a 2D slice of this vector field is shown in Figure 5 In this case the divergence is

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(1) \tag{4.24}$$

$$= 1 - 1 + 0 \tag{4.25}$$

$$= 0. \tag{4.26}$$

and hence divergence is constant and the vector field is neither spreading or converging at any point.

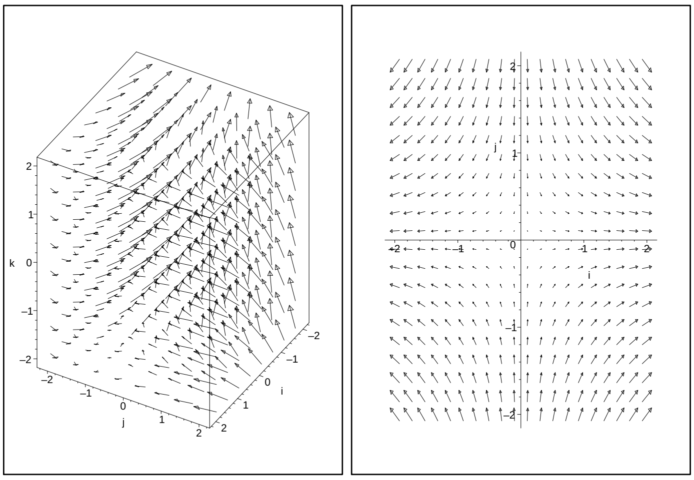


Figure 5: The vector field $\vec{F} = x\hat{\boldsymbol{\imath}} - y\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$ from example 4. Some of the flow is into the domain and some out. The second panel is a cut in the $z = 0$ plane.

Example 5

Now we consider a different vector field with a similar property to above

$$\vec{F} = y\,\hat{\boldsymbol{\imath}} - x\,\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}. \tag{4.27}$$

In this case the divergence is

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) + \frac{\partial}{\partial z}(1) \tag{4.28}$$

$$= 0 - 0 + 0 \tag{4.29}$$

$$= 0. \tag{4.30}$$

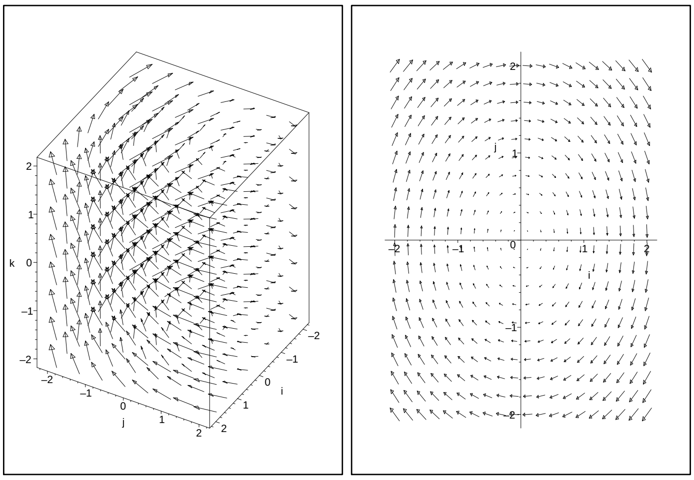


Figure 6: The vector field $\vec{F} = y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}}$ from example 5. There is no flow in to the centre of the domain. The second panel is a cut in the $z = 0$ plane.

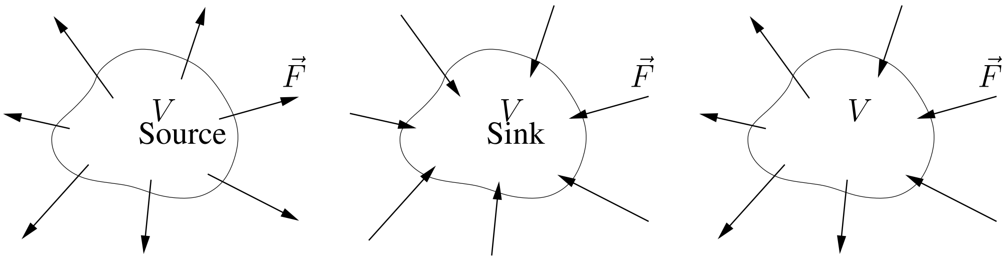


Figure 7: The three possible cases for the divergence of a vector field $\vec{F}$. In the first case the divergence $\boldsymbol{\nabla} \cdot \vec{F} > 0$ and there is a net outflow (flux) of fluid (or other material) from the region: there is a *source* of material within the region V. In the second case the divergence $\boldsymbol{\nabla} \cdot \vec{F} < 0$ and there is a net inflow (flux) of material into the region: there is a *sink* of material within V. The final cases is where the divergence $\boldsymbol{\nabla} \cdot \vec{F} = 0$. There is no *net* outflow or inflow into the region V.

4.6 Interpretation of examples

These examples suggest the following physical interpretation. Figure 7 gives a picture of what may be occurring in a small region V.

In summary:

- If the divergence of a field is *positive* in a region, that region contains a *source* (or many sources) of the field (the vectors spread out due to these sources).
- If the divergence of a field is *negative* in a region, that region contains a *sink* (or many sinks) of the field (the vectors converge).

Associated with this is the idea of a *flux* through a surface of the small region V due to the vector field.

4.7 Divergence and Electric Fields

To help with understanding the previous ideas and the next section it is worth considering $\vec{F}$ to be the electric flux density $\vec{D}$ where

$$\vec{D} = \epsilon_0 \vec{E} \tag{4.31}$$

and here $\vec{E}$ is the electric field and ϵ_0 the permittivity of free space. We shall now derive the equation that governs the behaviour of $\vec{D}$.

The flux density $\vec{D}$ changes due to the electrostatic charge. We shall assume the electric charge is distributed with density $\rho(x, y, z)$.
(Note the total charge within a volume Q is given by $Q = \int_V \rho(x, y, z)\, \mathrm{d}V$).

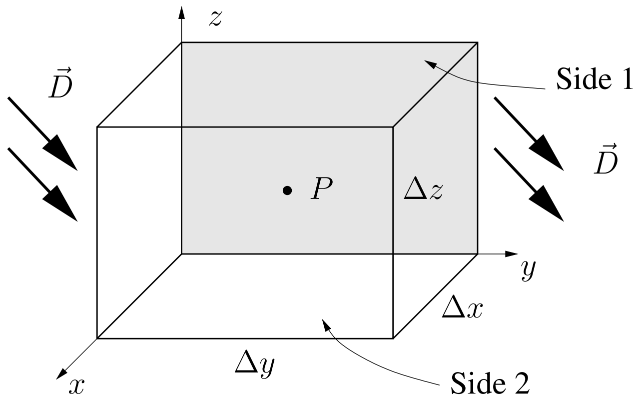


Figure 8: An electric field acting on a small volume. The divergence can be interpreted in terms of the change of the electric flux density.

To derive an equation governing $\vec{D}$ we consider a small cube surrounding the central point $P = (x, y, z)$. We will take the sides of the box to have length $\Delta x, \Delta y, \Delta z$ and will take these to be very small distances (we formally will take the limit of them going to zero). The volume of this cube is clearly $\Delta x\, \Delta y\, \Delta z$. and is illustrated in Figure 8 We write the electric field density as $\vec{D} = D_x \hat{\boldsymbol{\imath}} + D_y \hat{\boldsymbol{\jmath}} + D_z \hat{\boldsymbol{k}}$. We then consider the change (gain) of charge per unit volume per unit time in the small box with centre at P.

Noting that side 1 is at $x - \Delta x/2$ while side 2 is at $x + \Delta x/2$ and using a linear approximation to the flux density (the simplest Taylor series) we can find that

$$x \text{ component of flux density at } P = D_x \tag{4.32}$$

$$x \text{ component of flux density at side 1} \approx D_x - \frac{1}{2}\frac{\partial D_x}{\partial x}\Delta x \tag{4.33}$$

$$x \text{ component of flux density at side 2} \approx D_x + \frac{1}{2}\frac{\partial D_x}{\partial x}\Delta x. \tag{4.34}$$

The flux of charge density across the surface is the density $\vec{D}$ through the surface times the area of the surface So in this example the flux of charge density crossing side 1 per unit time is

$$\left(D_x - \frac{1}{2}\frac{\partial D_x}{\partial x}\Delta x \right) \Delta y \Delta z. \tag{4.35}$$

Similarly the flux of charge density crossing side 2 per unit time is

$$\left(D_x + \frac{1}{2}\frac{\partial D_x}{\partial x}\Delta x\right)\Delta y \Delta z. \tag{4.36}$$

Therefore the net flux in the x direction, giving the change in the charge density per unit time in the x direction, is given by equation (4.36) minus equation (4.35), which is

$$\frac{\partial D_x}{\partial x}\Delta x \Delta y \Delta z. \tag{4.37}$$

Clearly there are similar results for the other sides and directions. By adding together the results for each direction we get that the total gain is given by

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z. \tag{4.38}$$

The only way electric fields can be generated is from *charge* distributions. If the total charge inside the volume is $\rho(x, y, z)\Delta x \Delta y \Delta z$ then we can use Gauss law:

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

and write

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z = \rho(x, y, z)\Delta x \Delta y \Delta z \tag{4.39}$$

Taking the limit $\Delta x, \Delta y, \Delta z \to 0$, we have one of Maxwell's equations, expressing the change of the electric flux density in terms of the net charge in a volume and can be written in the form

$$\boldsymbol{\nabla} \cdot \vec{D} = \rho(x, y, z), \tag{4.40}$$

or, using (4.31)

$$\boldsymbol{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}. \tag{4.41}$$

Another of Maxwell's equations follows from considering a bar magnet and shrinking it to a point, in a similar way as taking the limit of the size of the cube in the previous example. As there is no net flow of magnetic flux in or out of the region enclosing the magnet, then we can write down that

$$\boldsymbol{\nabla} \cdot \vec{B} = 0, \tag{4.42}$$

where the magnetic flux density is denoted by $\vec{B}$. This can also be interpreted as enforcing the non-existence of magnetic monopoles.

Similar reasoning can also be used to derive equations for the motion of fluids. For example, in an *incompressible* fluid with velocity $\vec{v}$ we find that $\boldsymbol{\nabla} \cdot \vec{v} = 0$.

4.8 Definition of the curl of a vector field curl $\boldsymbol{B}$

The alternative in vector multiplication is to use ∇ in a cross product with a vector $\boldsymbol{B}$:

$$\text{curl}\,\boldsymbol{B} = \nabla\times\boldsymbol{B} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ B_1 & B_2 & B_3 \end{vmatrix}. \tag{4.43}$$

The best physical explanation that can be given is to visualize in colour the intensity of $\boldsymbol{B}$ then curl $\boldsymbol{B}$ is a measure of the curvature/swirl/rotation in the field lines of $\boldsymbol{B}$. Imagine again, the small sphere moving along the field lines of a field $\boldsymbol{B}$. Then curl$\boldsymbol{B}$ measures how much the sphere rotates as it moves. The magnitude of curl tells how rapidly the vectors rotate and the direction indicates the axis around which the rotation occurs.

Example 1) : Take the vector denoting a straight line from the origin to a point (x, y, z) denoted by $\boldsymbol{r} = \underline{\boldsymbol{i}}x + \underline{\boldsymbol{j}}y + \underline{\boldsymbol{k}}z$. Then

$$\text{curl}\,\boldsymbol{r} = \nabla\times\boldsymbol{r} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = 0\,. \tag{4.44}$$

Example 2) : Choose $\boldsymbol{B} = \frac{1}{2}\left(\underline{\boldsymbol{i}}x^2 + \underline{\boldsymbol{j}}y^2 + \underline{\boldsymbol{k}}z^2\right)$ then

$$\text{curl}\,\boldsymbol{B} = \nabla\times\boldsymbol{B} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ \frac{1}{2}x^2 & \frac{1}{2}y^2 & \frac{1}{2}z^2 \end{vmatrix} = 0\,. \tag{4.45}$$

Example 3) : Take the vector denoted by $\boldsymbol{B} = \underline{\boldsymbol{i}}y^2z^2 + \underline{\boldsymbol{j}}x^2z^2 + \underline{\boldsymbol{k}}x^2y^2$. Then

$$\text{curl}\,\boldsymbol{B} = \nabla\times\boldsymbol{B} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ y^2z^2 & x^2z^2 & x^2y^2 \end{vmatrix} = 2x^2\underline{\boldsymbol{i}}(y-z) - 2y^2\underline{\boldsymbol{j}}(x-z) + 2z^2\underline{\boldsymbol{k}}(x-y)\,. \tag{4.46}$$

Example 4) : For the curl of a two-dimensional vector $\boldsymbol{B} = \underline{\boldsymbol{i}}B_1(x, y) + \underline{\boldsymbol{j}}B_2(x, y)$ we have

$$\text{curl}\,\boldsymbol{B} = \nabla\times\boldsymbol{B} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ B_1(x,y) & B_2(x,y) & 0 \end{vmatrix} = \underline{\boldsymbol{k}}\left(\frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y}\right), \tag{4.47}$$

which points in the vertical direction only because there are no $\underline{\boldsymbol{i}}$ and $\underline{\boldsymbol{j}}$ components.

4.9 Repeated use of the ∇ operator

If we have a scalar field $\phi = \phi(x, y, z)$ then the divergence of the gradient in Cartesian coordinates has a simple form. Note that the gradient of ϕ is a vector field and the divergence of this gradient is a scalar field. Hence we write

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi) = \left(\hat{\boldsymbol{\imath}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial x}\,\hat{\boldsymbol{\imath}} + \frac{\partial\phi}{\partial y}\,\hat{\boldsymbol{\jmath}} + \frac{\partial\phi}{\partial z}\,\hat{\boldsymbol{k}}\right) \tag{4.48}$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{4.49}$$

This is often given the symbol

$$\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi)\equiv\nabla^2\phi. \tag{4.50}$$

This notation can be very useful for example if ϕ is an electrostatic potential we know that

$$\vec{E}=-\boldsymbol{\nabla}\phi$$

.

Then if we apply Maxwell's equations we have that

$$\boldsymbol{\nabla}\cdot\vec{E}=\frac{\rho(x,y,z)}{\epsilon_0} \tag{4.51}$$

giving *Poisson's equation*

$$\nabla^2\phi=-\frac{\rho(x,y,z)}{\epsilon_0}. \tag{4.52}$$

If the region is charge free (for example it is a conductor) then $\rho=0$, and Maxwell's equation reduces to *Laplace's equation*

$$\nabla^2\phi=0. \tag{4.53}$$

If instead of taking the divergence of the gradient of a scalar field we take the curl of the gradient of a scalar field, $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi)$ then we also find a simplification:

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\phi)=\begin{vmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix} \tag{4.54}$$

$$=\hat{\boldsymbol{\imath}}\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z}\end{vmatrix}-\hat{\boldsymbol{\jmath}}\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial z}\end{vmatrix}+\hat{\boldsymbol{k}}\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y}\end{vmatrix} \tag{4.55}$$

$$=\hat{\boldsymbol{\imath}}\left(\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}\right)-\hat{\boldsymbol{\jmath}}\left(\frac{\partial^2\phi}{\partial x\partial z}-\frac{\partial^2\phi}{\partial z\partial x}\right)+\hat{\boldsymbol{k}}\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{\partial^2\phi}{\partial y\partial x}\right) \tag{4.56}$$

$$=\vec{0} \tag{4.57}$$

as all the second partial derivatives commute, i.e.

$$\frac{\partial^2\phi}{\partial y\partial z}=\frac{\partial^2\phi}{\partial z\partial y}. \tag{4.58}$$

Hence the curl of the gradient of a scalar field is *always* zero. A field with zero curl is called "curl-free". For example if the electric field is given by a potential

$$\vec{E} = -\boldsymbol{\nabla}\phi$$

then we immediately know that this electric field is curl-free since we find that

$$\boldsymbol{\nabla} \times \vec{E} = 0$$

whatever the potential field actually is.

The curl is used in two of the other Maxwell's equations:

$$\boldsymbol{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{4.59}$$

$$\boldsymbol{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \tag{4.60}$$

The first is *Faraday's* law which relates the electric field $\vec{E}$ to the magnetic flux density $\vec{B}$. The second relates the magnetic field itself, $\vec{H}$, to the current density $\vec{J}$ and the electric field density $\vec{D}$. We also have that

$$\vec{D} = \epsilon\vec{E} \tag{4.61}$$

$$\vec{B} = \mu\vec{H} \tag{4.62}$$

where ϵ is the *permittivity* and μ the *permeability*.

4.10 Five vector identities

There are 5 useful vector identities (see hand out No 2).

1. The gradient of the product of two scalars ψ and ϕ

$$\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi\,. \tag{4.63}$$

2. The divergence of the product of a scalar ψ with a vector $\boldsymbol{b}$

$$\text{div}\,(\psi\boldsymbol{B}) = \psi\,\text{div}\,\boldsymbol{B} + (\nabla\psi)\cdot\boldsymbol{B}\,. \tag{4.64}$$

Note: the brackets in the last term, $(\nabla\psi)\cdot\boldsymbol{B}$ are not actually necessary. If we write $\nabla\psi\cdot\boldsymbol{B}$, this could not be interpreted as $\nabla(\psi\cdot\boldsymbol{B})$, as there is no such thing as $\psi\cdot\boldsymbol{B}$.

3. The curl of the product of a scalar ψ with a vector $\boldsymbol{B}$

$$\text{curl}\,(\psi \boldsymbol{B}) = \psi\,\text{curl}\,\boldsymbol{B} + (\nabla\psi) \times \boldsymbol{B}\,. \tag{4.65}$$

   or equivalently,

$$\nabla \times (\psi \boldsymbol{B}) = \psi\,\nabla \times \boldsymbol{B} + (\nabla\psi) \times \boldsymbol{B}\,.$$

   All three of these follow directly from the product rule.

4. The curl of the gradient of any scalar ψ

$$\text{curl}\,(\nabla\psi) = \nabla \times \nabla\psi = 0\,, \tag{4.66}$$

   as seen in the previous section.

5. The divergence of the curl of any vector $\boldsymbol{B}$

$$\text{div}\,(\text{curl}\,\boldsymbol{B}) = \nabla \cdot (\nabla \times \boldsymbol{B}) = 0\,. \tag{4.67}$$

   The cyclic rule for the scalar triple product in (4.67) shows that this is zero for all vectors $\boldsymbol{B}$ because two vectors (∇) in the triple are the same.

4.11 Irrotational and solenoidal vector fields

Consider identities 4) & 5) above (see also Handout 2 “The role of grad, div & curl ...”)

$$\text{curl}\,(\nabla\phi) \;=\; \nabla \times \nabla\phi = 0\,, \tag{4.68}$$

$$\text{div}\,(\text{curl}\,\boldsymbol{B}) \;=\; 0\,. \tag{4.69}$$

(4.68) says that if any vector $\boldsymbol{B}(x,y,z)$ can be written as the gradient of a scalar $\phi(x,y,z)$ (which can’t always be done)

$$\boldsymbol{B} = \nabla\phi \tag{4.70}$$

then automatically curl $\boldsymbol{B} = 0$. Such vector fields are called curl-free or irrotational vector fields. It is equally true that for a given field $\boldsymbol{B}$, if it is found that curl $\boldsymbol{B} = 0$, then we can write [3]

$$\boldsymbol{B} = \pm\nabla\phi \tag{4.71}$$

ϕ is called the “scalar potential”. Note that not every vector field has a corresponding scalar potential, but only those that are curl-free.

Example 1: Consider the vector field $\underline{\boldsymbol{F}} = 2xyz^3\underline{\boldsymbol{i}} + x^2z^3\underline{\boldsymbol{j}} + 3x^2yz^2\underline{\boldsymbol{k}}$. Can this be the gradient of a scalar field ϕ? If so, we can resonstruct ϕ from $\underline{\boldsymbol{F}}$. For ϕ to exist, we require $\nabla \times \underline{\boldsymbol{F}} = 0$. Check:

$$\text{curl}\,\boldsymbol{B} = \nabla\times\boldsymbol{B} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix} = (3x^2z^2 - 3x^2z^2)\underline{\boldsymbol{i}} - (6xyz^2 - 6xyz^2)\underline{\boldsymbol{j}} + (2xz^3 - 2xz^3)\underline{\boldsymbol{k}} = 0\,. \tag{4.72}$$

[3] Whether we use $+$ or $-$ in (4.71) depends on convention : in e/m theory normally uses a minus sign whereas fluid dynamics uses a plus sign.

Therefore ϕ exists and

$$\frac{\partial\phi}{\partial x} = F_x = 2xyz^3 \qquad \frac{\partial\phi}{\partial y} = F_y = x^2z^3 \quad \text{and} \quad \frac{\partial\psi}{\partial z} = F_z = 3x^2yz^2\,.$$

Integrating F_x w.r.t x we have

$$\phi(x,y,z) = \int 2xyz^3\;dx = x^2yz^3 + f(y,z)$$

where $f(y,z)$ is an arbitrary function. Similarly

$$\phi = \int x^2z^3\;dy = x^2yz^3 + g(x,z) \quad \text{and} \quad \phi = \int 3x^2yz^2\;dz = x^2yz^3 + h(x,y)\,.$$

Equating the three expressions, we see that $f(y,z) = g(x,z) = h(x,y)$ =constant, and $\phi = x^2yz^3 + c$.
Likewise we now turn to (4.69) : vector fields $\boldsymbol{A}$ for which div $\boldsymbol{A} = 0$ are called **solenoidal of divergence-free**, in which case $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \text{curl}\,\boldsymbol{B} \tag{4.73}$$

where the vector B is called a "vector potential". Note that only vectors that are div-free have a corresponding vector potential.The last lecture on Maxwell's Equations stresses that all magnetic fields in the universe are div-free as no magnetic monopoles have yet been found : thus all magnetic fields are solendoidal vector fields.

Example 2 : The Newtonian gravitational force between masses m and M (with gravitational constant G) is

$$\underline{\boldsymbol{F}} = -GmM\frac{\boldsymbol{r}}{r^3}\,, \tag{4.74}$$

where $\boldsymbol{r} = \underline{\boldsymbol{i}}x + \underline{\boldsymbol{j}}y + \underline{\boldsymbol{k}}z$ and $r^2 = x^2 + y^2 + z^2$.

1. Let us first calculate curl $\underline{\boldsymbol{F}}$

$$\text{curl}\,\underline{\boldsymbol{F}} = -GmM\,\text{curl}\,(\psi\boldsymbol{r})\,, \qquad \text{where} \qquad \psi = r^{-3}\,. \tag{4.75}$$

The 3rd in the list of vector identities gives

$$\text{curl}\,(\psi\boldsymbol{r}) = \psi\,\text{curl}\,\boldsymbol{r} + (\nabla\psi)\times\boldsymbol{r} \tag{4.76}$$

and we already know that curl $\boldsymbol{r} = 0$. It remains to calculate $\nabla\psi$:

$$\nabla\psi = \nabla\left\{\left(x^2+y^2+z^2\right)^{-3/2}\right\} = -\frac{3(\underline{\boldsymbol{i}}x + \underline{\boldsymbol{j}}y + \underline{\boldsymbol{k}}z)}{(x^2+y^2+z^2)^{5/2}} = -\frac{3\boldsymbol{r}}{r^5}\,. \tag{4.77}$$

Thus, from (4.74),

$$\text{curl}\,\underline{\boldsymbol{F}} = -GmM\left(0 - \frac{3\boldsymbol{r}}{2r^5}\times\boldsymbol{r}\right) = 0\,. \tag{4.78}$$

Thus the Newton gravitational force field is curl-free, which is why a gravitational potential exists. We can write [4]

$$\underline{\boldsymbol{F}} = -\nabla\phi\,. \tag{4.79}$$

One can find ϕ by inspection : it turns out that $\phi = -GmM\,1/r$.

[4] In this case a negative sign is adopted on the scalar potential.

2. Now let us calculate div $\underline{\boldsymbol{F}}$

$$\text{div}\,\underline{\boldsymbol{F}} = -GmM\,\text{div}\,(\psi\boldsymbol{r}) \qquad \text{where} \qquad \psi = r^{-3}\,. \tag{4.80}$$

The 2nd in the list of vector identities gives

$$\text{div}\,(\psi\boldsymbol{r}) = \psi\,\text{div}\,\boldsymbol{r} + (\nabla\psi)\cdot\boldsymbol{r} \tag{4.81}$$

and we already know that div $\boldsymbol{r} = 3$ and we have already calculated $\nabla\psi$ in (4.77).

$$\text{div}\,\underline{\boldsymbol{F}} = -GmM\left(\frac{3}{r^3} - \frac{3\boldsymbol{r}}{r^5}\cdot\boldsymbol{r}\right) = 0\,. \tag{4.82}$$

Thus the Newton gravitational force field is also div-free, which is why a gravitational vector potential A also exists.

As a final remark it is noted that with $\underline{\boldsymbol{F}}$ satisfying both $\underline{\boldsymbol{F}} = -\nabla\phi$ and div $\underline{\boldsymbol{F}} = 0$, then

$$\text{div}\,(\nabla\phi) = \nabla^2\phi = 0\,, \tag{4.83}$$

so we see that the gravitational field also satisfies **Laplace's equation.**

5 Line (path) integration

In single variable calculus the idea of the integral

$$\int_a^b f(x)\,dx = \sum_{i=1}^{N} f(x_i)\delta x_i \tag{5.1}$$

is a way of expressing the sum of values of the function $f(x_i)$ at points x_i multiplied by the area of small strips δx_i : correctly it is often expressed as the area under the curve $f(x)$. Pictorially the concept of an area sits very well in the plane with $y = f(x)$ plotted against x. **However, the idea of area under a curve has to be dropped when line integration is considered because we now wish to place our curve C in 3-space where a scalar field $\psi(x, y, z)$ or a vector field $\underline{\boldsymbol{F}}(x, y, z)$ take values *at every point* in this space.** Instead, we consider a specified continuous curve C in 3-space – known as the path[5] of integration – and then work out methods for summing the values that either ψ or $\underline{\boldsymbol{F}}$ take on that curve. **It is essential to realize that the curve C sitting in 3-space and the scalar/vector fields ψ or $\underline{\boldsymbol{F}}$ that take values at every point in this space are wholly independent quantities and must not be conflated.**

[5] The same idea arises in complex integration where, by convention, a closed C is known as a 'contour'.

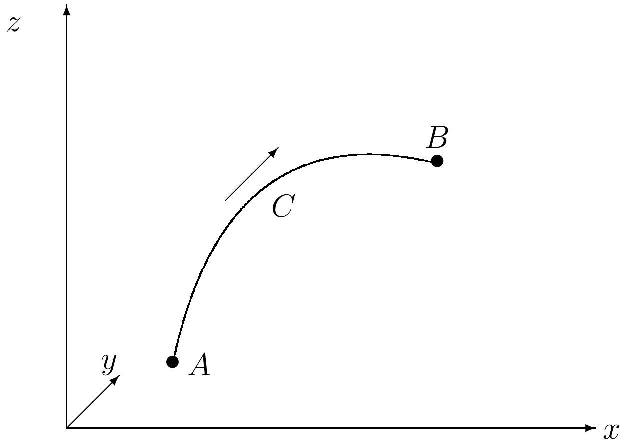


Figure 3.1 : *The curve C in 3-space starts at the point A and ends at B.*

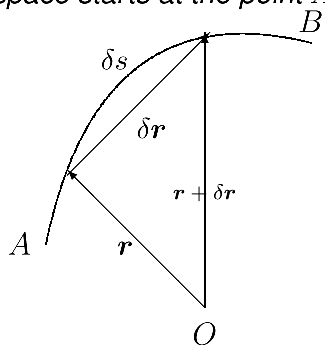


Figure 3.2 : *On a curve C, small elements of arc length δs and the chord $\delta \boldsymbol{r}$, where O is the origin.*

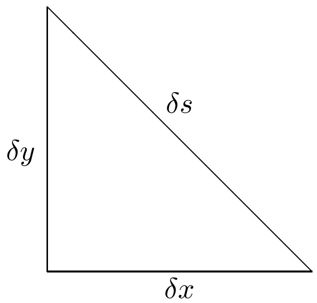


Figure 3.3 : *In 2-space we can use Pythagoras' Theorem to express δs in terms of δx and δy.*

Pythagoras' Theorem in 3-space (see Fig 3.3 for a 2-space version) express δs in terms of δx, δy and δz : $(\delta s)^2 = (\delta x)^2 + (\delta y)^2 + (\delta z)^2$. There are two types of line integral :

Type 1 : The first concerns the integration of a scalar field ψ along a path C

$$\int_C \psi(x, y, z)\, ds \tag{5.2}$$

Type 2 : The second concerns the integration of a vector field $\underline{\boldsymbol{F}}$ along a path C

$$\int_C \underline{\boldsymbol{F}}(x, y, z) \cdot d\boldsymbol{r} \tag{5.3}$$

Remark : In either case, if the curve C is closed then we use the designations

$$\oint_C \psi(x, y, z)\, ds \qquad \text{and} \qquad \oint_C \underline{\boldsymbol{F}}(x, y, z) \cdot d\boldsymbol{r} \tag{5.4}$$

5.1 Line integrals of Type 1 : $\int_C \psi(x,y,z)\,ds$

The general case follows from the arc length formula seen in year one. If we consider that $ds = \sqrt{dx^2+dy^2}$ is an infinitesimal element of length, then $\int_C 1\,ds$ gives the length of the curve C. With the parametrization $x(t), y(t)$ of the curve, where $a \le t \le b$ marks the endpoints, we have

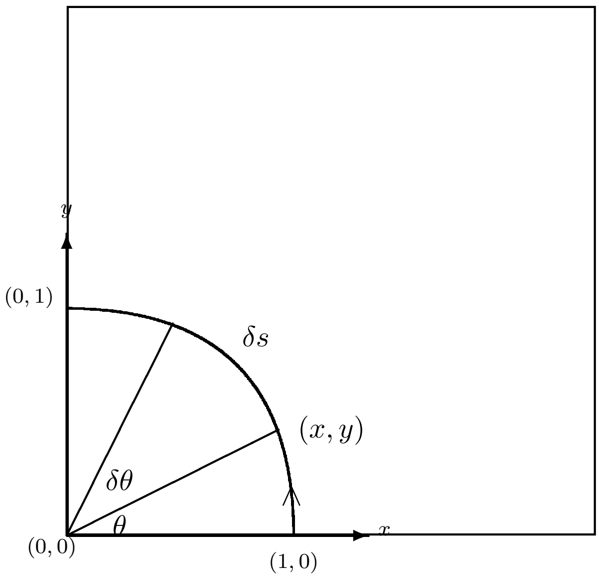
$$\frac{ds}{dt} = \sqrt{\dot{x}^2+\dot{y}^2} \Rightarrow ds = \sqrt{\dot{x}^2+\dot{y}^2}dt\,,$$

and arc length is given by

$$\int_C 1 ds = \int_a^b \sqrt{\dot{x}^2+\dot{y}^2}dt\,.$$

How to evaluate these integrals is best shown by a series of examples keeping in mind that, where possible, one should always draw a picture of the curve C :

Example 1) : Show that $\int_C x^2y\,ds = 1/3$ where C is the circular arc in the first quadrant of the unit circle, oriented anti-clockwise

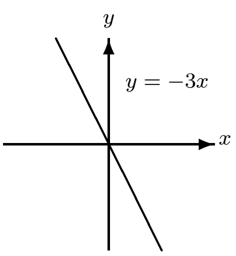


C is the arc of the unit circle $x^2+y^2=1$ with the parametrization in polar coordinates:
$x = \cos t \Rightarrow \dot{x} = -\sin t\,, y = \sin t \Rightarrow \dot{y} = \cos t$
for $0 \le t \le \pi/2$.
Thus the small element of arc length is

$$\delta s = \sqrt{\sin^2 t + \cos^2 t}\;\delta t = \delta t\,.$$

$$\begin{aligned} \int_C x^2y\,ds &= \int_0^{\pi/2} \cos^2 t \sin t (dt) \\ &= 1/3\,. \end{aligned} \qquad (5.5)$$

Example 2) : Show that $\int_C xy^3\,ds = -54\sqrt{10}/5$ where C is the line $y=-3x$ from $x=-1 \to 1$.



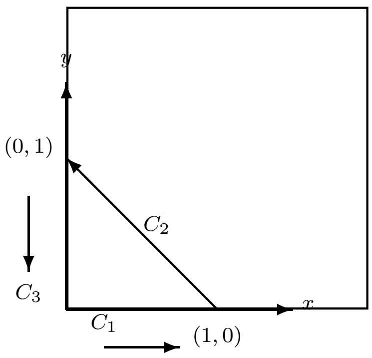
ds is an element on the line $y = -3x$. Thus we can parametrize the segment with $x = t\,, y = -3t$ with $-1 \le t \le 1$. so

$$ds = \sqrt{\dot{x}^2+\dot{y}^2}dt = \sqrt{10}dt$$

$$\int_C xy^3\,ds \;=\; \sqrt{10}\int_{-1}^{1} t(-3t)^3\,dt = -54\sqrt{10}/5\,.$$

Note that we could altenatively let $x = t/3$ and $y = -t$ with $-3 \le t \le 3$. The key is for t to increase over the interval to guarantee uniqueness.

Example 3) : Show that $\oint_C x^2 y\, ds = \sqrt{2}/12$ where C is the closed triangle in the figure.



On C_1 : $y = 0$ so $ds = dt$ but $\int_{C_1} = 0$ (because $y = 0$).
On C_3 : $x = 0$ so $ds = dt$ but $\int_{C_3} = 0$ (because $x = 0$).
In both of these, ds depends on the parametrization, but as the integral is zero, we can skip the detail.
On C_2 : $y = 1 - x$ so $y = t$ and $x = 1 - t$ with $0 \le t \le 1$ is a valid parametization. Hence $ds = \sqrt{2}dt$.

Therefore

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + \int_0^1 (1-t)^2 t\sqrt{2}\, dt + 0 = \sqrt{2}/12\,. \tag{5.6}$$

Note that on C_2, we could altenatively let $x = -t$ and $y = 1 + t$ with $-1 \le t \le 0$. The key is for t to increase over the interval.

Example 4) : (see Sheet 2). Find $\int_C (x^2 + y^2 + z^2)\, ds$ where C is the helix

$$\boldsymbol{r} = \underline{\boldsymbol{i}} \cos\theta + \underline{\boldsymbol{j}} \sin\theta + \underline{\boldsymbol{k}}\theta\,, \tag{5.7}$$

with one turn : that is θ running from $0 \to 2\pi$.

From (5.7) we note that $x = \cos\theta$, $y = \sin\theta$ and $z = \theta$, already in parametrized form. Therefore $dx/d\theta = -\sin\theta$, $dy/d\theta = \cos\theta$ and $dz/d\theta = 1$. Thus

$$ds = \sqrt{\sin^2\theta + \cos^2\theta + 1}\, d\theta = \sqrt{2}\, d\theta\,. \tag{5.8}$$

Therefore we can write the integral as

$$\int_C (x^2 + y^2 + z^2)\, ds = \sqrt{2} \int_0^{2\pi} \left(1 + \theta^2\right)\, d\theta = 2\pi\sqrt{2}\left(1 + 4\pi^2/3\right)\,. \tag{5.9}$$

5.2 Line integrals of Type 2 : $\int_C \underline{\boldsymbol{F}}(x, y, z) \cdot d\boldsymbol{r}$

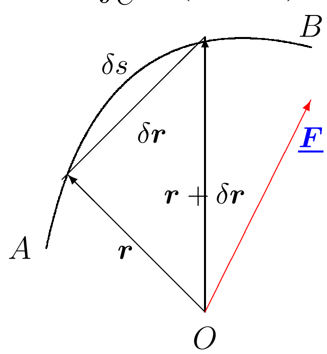
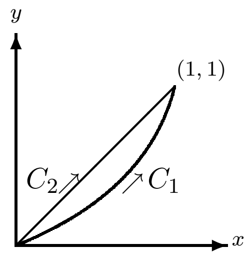


Figure 3.4 : *A vector $\underline{\boldsymbol{F}}$ and a curve C with the chord $\delta\boldsymbol{r}$: O is the origin.*

1. Think of $\underline{\boldsymbol{F}}$ as a force on a particle being drawn through the path of the curve C. Then the work done δW in pulling the particle along the curve with arc length δs and chord $\delta \boldsymbol{r}$ is $\delta W = \underline{\boldsymbol{F}} \cdot \delta \boldsymbol{r}$. Thus the full work W is
$$W = \int_C \underline{\boldsymbol{F}} \cdot d\boldsymbol{r}\,. \tag{5.10}$$
2. The second example revolves around taking $\underline{\boldsymbol{F}}$ as an electric field $\underline{\boldsymbol{E}}(x,y,z)$. Then the mathematical expression of Faraday's Law says that the electro-motive force on a particle of charge e travelling along C is precisely $e\int_C \underline{\boldsymbol{E}}(x,y,z)\cdot d\boldsymbol{r}$.

Example 1 : Evaluate $\int_C \underline{\boldsymbol{F}} \cdot d\boldsymbol{r}$ given that $\underline{\boldsymbol{F}} = \underline{\boldsymbol{i}}x^2y + \underline{\boldsymbol{j}}(x-z) + \underline{\boldsymbol{k}}xyz$ and the path C is the parabola $y = x^2$ in the plane $z = 2$ from $(0,0,2) \to (1,1,2)$.



C_1 is along the curve $y = x^2$ in the plane $z = 2$ in which case $dz = 0$ and $dy = 2x\,dx$. C_2 is along the straight line $y = x$ in which case $dy = dx$.

$$\begin{aligned}\int_{C_1} \underline{\boldsymbol{F}} \cdot d\boldsymbol{r} &= \int_{C_1} (F_1dx + F_2dy + F_3dz)\\ &= \int_{C_1} \left(x^2y\,dx + (x-z)\,dy + xyz\,dz\right)\\ &= \int_{C_1} \left(x^2y\,dx + (x-2)\,dy\right)\end{aligned} \tag{5.11}$$

Using the fact that $dy = 2x\,dx$ we have

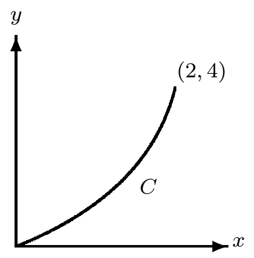
$$\begin{aligned} I &= \int_0^1 \left(x^4\,dx + (x-2)2x\,dx\right)\\ &= \int_0^1 \left(x^4 + 2x^2 - 4x\right)dx = -17/15\,. \end{aligned} \tag{5.12}$$

Finally, with the same integrand but along C_2 we have

$$\begin{aligned}\int_{C_2} \underline{\boldsymbol{F}} \cdot d\boldsymbol{r} &= \int_0^1 \left(x^3\,dx + (x-2)\,dx\right)\\ &= 1/4 + 1/2 - 2 = -5/4\,. \end{aligned} \tag{5.13}$$

This example illustrates the point that with the same integrand and start/end points the value of an integral can differ when the route between these points is varied.

Example 2 : Evaluate $I = \int_C (y^2dx - 2x^2dy)$ given that $\underline{\boldsymbol{F}} = (y^2, -2x^2)$ with the path C taken as $y = x^2$ in the $z = 0$ plane from $(0,0,0) \to (2,4,0)$.



C is the curve $y = x^2$ in the plane $z = 0$, in which case $dz = 0$ and $dy = 2x\,dx$. The starting point has co-ordinates $(0,0,0)$ and the end point $(2,4,0)$.

$$
\begin{aligned}
I &= \int_C (y^2 dx - 2x^2 dy) \\
&= \int_0^2 \left(x^4 - 4x^3\right) dx = -48/5\,. \qquad (5.14)
\end{aligned}
$$

Example 3 : Given that $\underline{\boldsymbol{F}} = \underline{\boldsymbol{i}}x - \underline{\boldsymbol{j}}z + 2\underline{\boldsymbol{k}}y$, where C is the curve $z = y^4$ in the $x = 1$ plane, show that from $(1,0,0) \to (1,1,1)$ the value of the line integral is $7/5$.

On C we have $dz = 4y^3 dy$ and in the plane $x = 1$ we also have $dx = 0$. Therefore

$$
\begin{aligned}
\int_C \underline{\boldsymbol{F}} \cdot d\boldsymbol{r} &= \int_C (x\,dx - z\,dy + 2y\,dz) \\
&= \int_0^1 \left(-y^4 + 8y^4\right) dy = 7/5\,. \qquad (5.15)
\end{aligned}
$$

5.3 Independence of path in line integrals of Type 2

Are there circumstances in which a line integral $\int_C \underline{\boldsymbol{F}} \cdot d\boldsymbol{r}$, for a given $\underline{\boldsymbol{F}}$, takes values which are independent of the path C?

To explore this question let us consider the case where $\underline{\boldsymbol{F}} \cdot d\boldsymbol{r}$ is an *exact differential* : that is $\underline{\boldsymbol{F}} \cdot d\boldsymbol{r} = -d\phi$ for some scalar[6] ϕ. For starting and end co-ordinates A and B of teh curve C we have

$$
\int_C \underline{\boldsymbol{F}} \cdot d\boldsymbol{r} = -\int_C d\phi = \phi[A] - \phi[B]\,. \qquad (5.16)
$$

This result is independent of the route or path taken between A and B. Thus we need to know what $\underline{\boldsymbol{F}} \cdot d\boldsymbol{r} = -d\phi$ means. Firstly, from the chain rule

$$
d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \nabla\phi \cdot d\boldsymbol{r}\,. \qquad (5.17)
$$

Hence, if $\underline{\boldsymbol{F}} \cdot d\boldsymbol{r} = -d\phi$ we have $\underline{\boldsymbol{F}} = -\nabla\phi$, **which means that $\underline{F}$ must be a curl-free vector field**

$$
\text{curl}\,\underline{\boldsymbol{F}} = 0\,, \qquad (5.18)
$$

[6]As stated earlier, the choice of sign is by convention.

where ϕ is the scalar potential. A curl-free $\underline{\boldsymbol{F}}$ is also known as a **conservative** vector field.

Result : The integral $\int_C \underline{\boldsymbol{F}}\cdot d\boldsymbol{r}$ is independent of path only if curl $\underline{\boldsymbol{F}} = 0$. Moreover, when C is closed then

$$\oint_C \underline{\boldsymbol{F}} \cdot d\boldsymbol{r} = 0\,. \tag{5.19}$$

Example 1 : Is the line integral $\int_C (2xy^2\,dx + 2x^2y\,dy)$ independent of path?

We can see that $\underline{\boldsymbol{F}}$ is expressed as $\underline{\boldsymbol{F}} = 2xy^2\underline{\boldsymbol{i}} + 2x^2y\underline{\boldsymbol{j}} + 0\underline{\boldsymbol{k}}$. Then

$$\text{curl}\,\underline{\boldsymbol{F}} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ 2xy^2 & 2x^2y & 0 \end{vmatrix} = (4xy - 4xy)\underline{\boldsymbol{k}} = 0\,. \tag{5.20}$$

Thus the integral is independent of path and we should be able to calculate ϕ from $\underline{\boldsymbol{F}} = -\nabla\phi$.

$$-\frac{\partial\phi}{\partial x} = 2xy^2\,, \qquad -\frac{\partial\phi}{\partial y} = 2x^2y\,, \qquad -\frac{\partial\phi}{\partial z} = 0\,. \tag{5.21}$$

Partial integration of the 1st equation gives $\phi = -x^2y^2 + A(y)$ where $A(y)$ is an arbitrary function of y only, whereas from the second $\phi = -x^2y^2 + B(x)$. Thus $A(y) = B(x) = \text{const} = C$. Hence

$$\phi = -x^2y^2 + C\,. \tag{5.22}$$

Example 2 : Find the work done $\int_C \underline{\boldsymbol{F}} \cdot d\boldsymbol{r}$ by the force $\underline{\boldsymbol{F}} = (yz,\, xz,\, xy)$ moving from $(1,1,1) \to (3,3,2)$.

Is this line integral independent of path? We check to see if curl $\underline{\boldsymbol{F}} = 0$.

$$\text{curl}\,\underline{\boldsymbol{F}} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ yz & xz & xy \end{vmatrix} = 0\,. \tag{5.23}$$

Thus the integral is independent of path and ϕ must exist. In fact $-\phi_x = yz,\; -\phi_y = xz$ and $-\phi_z = xy$. Integrating the first gives $\phi = -xyz + A(y,z)$, the second gives $\phi = -xyz + B(x,z)$ and the third $\phi = -xyz + C(x,y)$. Thus $A = B = C = \text{const}$ and

$$\phi = -xyz + \text{const}\,. \tag{5.24}$$

$$W = -\int_{(1,1,1)}^{(3,3,2)} d\phi = \big[xyz\big]_{(1,1,1)}^{(3,3,2)} = 18 - 1 = 17\,. \tag{5.25}$$

Example 3 : Find $\int_C \underline{\boldsymbol{F}} \cdot d\boldsymbol{r}$ on every path between $(0,0,1)$ and $(1, \pi/4,\, 2)$ where

$$\underline{\boldsymbol{F}} = \big(2xyz^2,\; (x^2z^2 + z\cos yz),\; (2x^2yz + y\cos yz)\big)\,. \tag{5.26}$$

We check to see if curl $\underline{\boldsymbol{F}} = 0$.

$$\text{curl}\,\underline{\boldsymbol{F}} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ 2xyz^2 & (x^2z^2 + z\cos yz) & (2x^2yz + y\cos yz) \end{vmatrix} = 0\,. \tag{5.27}$$

Thus the integral is independent of path and ϕ must exist. Then $\phi_x = -2xyz^2$, $\phi_y = -(x^2z^2 + z\cos yz)$ and $\phi_z = -(2x^2yz + y\cos yz)$. Integration of the first gives

$$\phi = -x^2yz^2 + A(y,z)\,, \tag{5.28}$$

of the second

$$\phi = -(x^2yz^2 + \sin yz) + B(x,z)\,, \tag{5.29}$$

and the third

$$\phi = -(x^2yz^2 + \sin yz) + C(x,y)\,. \tag{5.30}$$

The only way these are compatible is if $A(y,z) = \sin yz + c$ and $B = C = c =$ const. and so $\phi = -(x^2yz^2 + \sin yz) + c$. In consequence

$$\int_C \underline{\boldsymbol{F}} \cdot d\boldsymbol{r} = -\left[\phi\right]_{(0,0,1)}^{(1,\pi/4,\,2)} = \pi + 1\,. \tag{5.31}$$

6 Double and multiple integration

Now we move on to yet another different concept of integration : that is, summation over an area instead of along a curve. Consider a region R in the $x - y$ plane with boundary curve C which is always taken in the anti-clockwise direction.

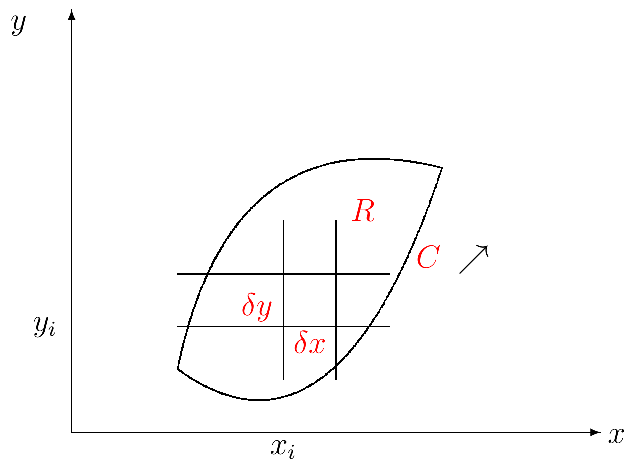


Figure 4.1 : *In the $x - y$ plane the curve C is the boundary of the region and R denotes the area inside. The small element of area is $\delta A = \delta x \delta y$.*

Consider the value of a scalar function $\psi(x_i, y_i)$ at the co-ordinate point (x_i, y_i) at the lower left hand corner of the square of area $\delta A_i = \delta x_i \delta y_i$. Then

$$\sum_{i=1}^{N}\sum_{j=1}^{M} \psi(x_i, y_i)\,\delta A_i \to \underbrace{\int\!\!\int_R \psi(x,y)\,dxdy}_{double\ integral} \qquad \text{as} \qquad \delta x \to 0,\ \delta y \to 0\,. \tag{6.1}$$

We say that the RHS is the “double integral of ψ over the region R”. **Note : do not confuse this with the area of R itself, which is**

$$\text{Area of R} = \int\!\!\int_R dxdy\,. \tag{6.2}$$

6.1 How to evaluate a double integral

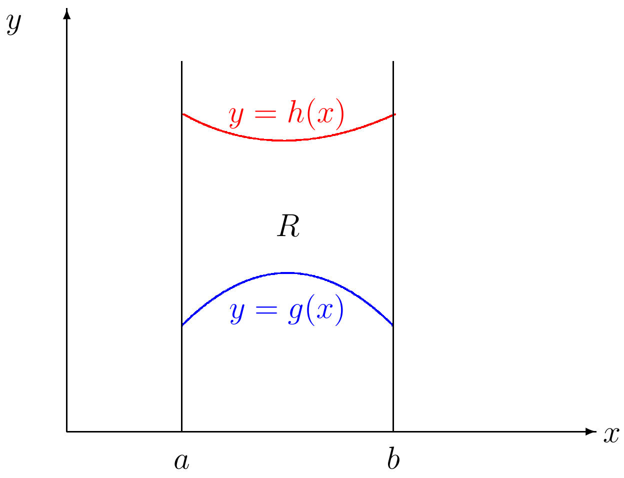


Figure 4.2 : *The region R is bounded between the upper curve $y = h(x)$, the lower curve $y = g(x)$ and the vertical lines $x = a$ and $x = b$.*

$$\int\int_R \psi(x,y)\,dxdy = \int_a^b \left\{\int_{y=g(x)}^{y=h(x)} \psi(x,y)\,dy\right\}\,dx \tag{6.3}$$

The inner integral is a partial integral over y holding x constant. Thus the inner integral is a function of x

$$\int_{y=g(x)}^{y=h(x)} \psi(x,y)\,dy = P(x) \tag{6.4}$$

and so

$$\int\int_R \psi(x,y)\,dxdy = \int_a^b P(x)\,dx\,. \tag{6.5}$$

Moreover the area of R itself is

$$\text{Area of R} = \int_a^b \left\{\int_{y=g(x)}^{y=h(x)} dy\right\}\,dx = \int_a^b \{h(x) - g(x)\}\,dx \tag{6.6}$$

6.2 Applications

1. **Area under a curve:** For a function of a single variable $y = f(x)$ between limits $x = a$ and $x = b$

$$\text{Area} = \int_a^b \left\{\int_0^{f(x)} dy\right\} dx = \int_a^b f(x)\,dx\,. \tag{6.7}$$

2. **Volume under a surface :** A surface is 3-space may be expressed as $z = f(x,y)$

$$\begin{aligned} \text{Volume} &= \int\int\int_V dxdydz = \int_R \left\{\int_0^{f(x,y)} dz\right\} dxdy \\ &= \int\int_R f(x,y)\,dxdy \end{aligned} \tag{6.8}$$

By this process, a 3-integral has been reduced to a double integral. A specific example would be the volume of an upper unit hemisphere $z = +\sqrt{1-(x^2+y^2)} \equiv f(x,y)$ centred at the origin.

3. **Mass of a solid body:** Let $\rho(x,y,z)$ be the variable density of the material in a solid body. Then the mass δM of a small volume $\delta V = \delta x \delta y \delta z$ is $\delta M = \rho\, \delta V$,

$$\text{Mass of body} = \int\int\int_V \rho(x,y,z)\, dV\,. \tag{6.9}$$

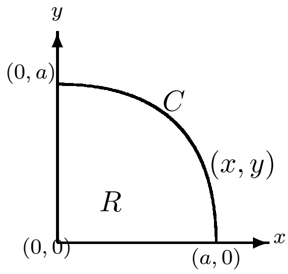
6.3 Examples of multiple integration

Example 1: Consider the first quadrant of a circle of radius a. Show that:

$$(i) \qquad \text{Area of R} = \pi a^2/4$$

$$(ii) \qquad \int\int_R xy\, dxdy = a^4/8$$

$$(iii) \qquad \int\int_R x^2y^2\, dxdy = \pi a^6/96$$



i): The area of R is given by

$$A = \int_0^a \left\{ \int_0^{\sqrt{a^2-x^2}} dy \right\} dx = \int_0^a \sqrt{a^2-x^2}\, dx\,. \tag{6.10}$$

Let $x = a\cos\theta$ and $dx = -a\sin\theta d\,\theta$ then $A = \frac{1}{2}a^2 \int_0^{\pi/2}(1-\cos 2\theta)\, d\theta = \pi a^2/4$.

ii):

$$\begin{aligned} \int\int_R xy\, dxdy &= \int_0^a x \left(\int_0^{\sqrt{a^2-x^2}} y\, dy \right) dx \\ &= \tfrac{1}{2} \int_0^a x(a^2-x^2)\, dx \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}x^2a^2 - \tfrac{1}{4}x^4\right]_0^a = a^4/8\,. \end{aligned} \tag{6.11}$$

iii):

$$\begin{aligned} \int\int_R x^2y^2\, dxdy &= \int_0^a x^2 \left(\int_0^{\sqrt{a^2-x^2}} y^2\, dy \right) dx \\ &= \tfrac{1}{3} \int_0^a x^2(a^2-x^2)^{3/2}\, dx \\ &= \tfrac{1}{3}a^6 \int_0^{\pi/2} \cos^2\theta \sin^4\theta\, d\theta \\ &= \tfrac{1}{3}a^6 \big(I_4 - I_6\big)\,. \end{aligned} \tag{6.12}$$

where $I_n = \int_0^{\pi/2} \sin^n\theta\, d\theta$. This is an integral recurrence relation which we met in year one:

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}\,, \qquad \text{with} \qquad I_0 = \pi/2\,. \tag{6.13}$$

Thus $I_2 = \pi/4\,, I_4 = 3\pi/16$ and $I_6 = 5\pi/32$ and so from (6.12) the answer is $\pi a^6/96$.

6.4 Changing the order in double integration

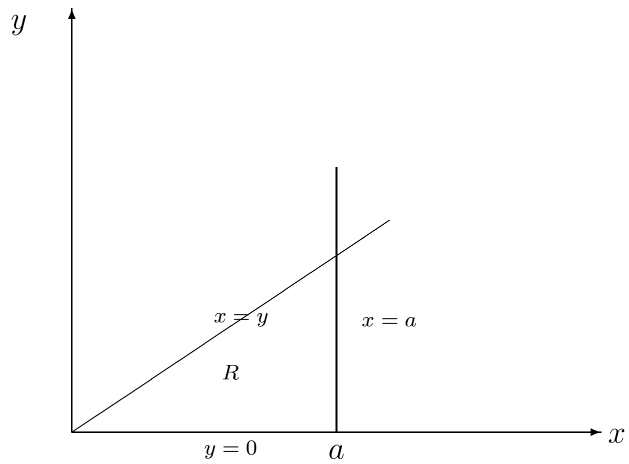
An example is used to illustrate this : consider

$$I = \int_0^a \left(\int_y^a \frac{x^2\,dx}{x^2+y^2}\right) dy\,. \tag{6.14}$$

Performing the integration in this order is hard as it involves a term in $\tan^{-1}\left(\frac{a}{y}\right)$. To make evaluation easier we change the order of the x-integration and the y-integration; first, however, we must deduce what the area of integration is from the internal limits in (6.14). The internal integral is of the type

$$\int_{x=y}^{x=a} f(x,y)\,dx. \tag{6.15}$$

Being an integration over x means that we are summing horizontally so we have *left and right hand limits*. The *left limit* is $x = y$ and the *right limit* is $x = a$. This is shown in the drawing of the graph where the region of integration is labelled as R:



Now we want to perform the integration over R but in reverse order to that above: we integrate vertically first by reading off the *lower limit* as $y = 0$ and the *upper limit* as $y = x$. After integrating horizontally with limits $x = 0$ to $x = a$, this reads as

$$I = \int_0^a \left(\int_{y=0}^{y=x} \frac{x^2\,dy}{x^2+y^2}\right) dx. \tag{6.16}$$

The inside integral can be done with ease. x is treated as a constant because the integration is over y. Therefore define $y = x\theta$ where θ is the new variable. We find that

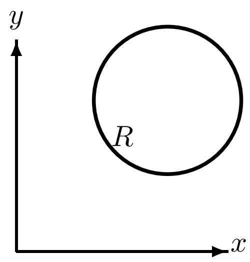
$$\int_0^x \frac{x^2\,dy}{x^2+y^2} = x\int_0^1 \frac{d\theta}{1+\theta^2} = \frac{x\pi}{4}. \tag{6.17}$$

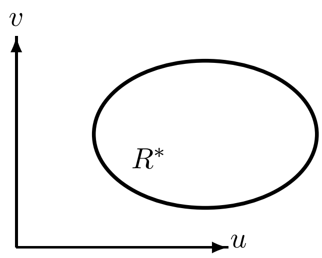
Hence

$$I = \frac{\pi}{4}\int_0^a x\,dx = \frac{\pi a^2}{8}. \tag{6.18}$$

6.5 Change of variable and the Jacobian

In the last example it might have been easier to have invoked the natural circular symmetry in the problem. Hence we must ask how $\delta A = \delta x \delta y$ would be expressed in polar co-ordinates. This suggests considering a more general co-ordinate change. In the two figures below we see a region R in the $x - y$ plane that is distorted into R^* in the plane of the new co-ordinates $u = u(x, y)$ and $v = v(x, y)$





The transformation relating the two small areas $\delta x \delta y$ and $\delta u \delta v$ is given here by (the modulus sign is a necessity) :

Result 1 :

$$dxdy = |J_{u,v}(x, y)|\; dudv\,, \tag{6.19}$$

where the Jacobian $J_{u,v}(x, y)$ is defined by

$$J_{u,v}(x, y) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{6.20}$$

Note that

$$\frac{\partial x}{\partial u} \neq \left(\frac{\partial u}{\partial x}\right)^{-1} \tag{6.21}$$

because u_x is computed at $y = const$ whereas x_u is computed at $v = const$. However, luckily there is an inverse relationship between the two Jacobians

$$J_{x,y}(u, v) = [J_{u,v}(x, y)]^{-1} \tag{6.22}$$

$$dudv = |J_{x,y}(u, v)|\; dxdy \tag{6.23}$$

Either (6.19) and (6.23) can therefore be used at one's convenience.

Proof : (not examinable). Consider two sets of orthogonal unit vectors ; $(\underline{\boldsymbol{i}}\,, \underline{\boldsymbol{j}})$ in the $x-y$-plane, and $(\hat{\boldsymbol{u}}\,, \hat{\boldsymbol{v}})$ in the $u-v$-plane. Keeping in mind that the component of $\delta \boldsymbol{x}$ along $\hat{\boldsymbol{u}}$ is $(\delta x/\delta u)\delta u$ (v is constant along $\hat{\boldsymbol{u}}$), in vectorial notation we can write

$$\delta \boldsymbol{x} = \hat{\boldsymbol{u}}\,\frac{\delta x}{\delta u}\delta u + \hat{\boldsymbol{v}}\,\frac{\delta x}{\delta v}\delta v \tag{6.24}$$

$$\delta \boldsymbol{y} = \hat{\boldsymbol{u}}\,\frac{\delta y}{\delta u}\delta u + \hat{\boldsymbol{v}}\,\frac{\delta y}{\delta v}\delta v \tag{6.25}$$

and so the cross-product is

$$\delta \boldsymbol{x} \times \delta \boldsymbol{y} = \left(\hat{\boldsymbol{u}}\,\frac{\delta x}{\delta u}\delta u + \hat{\boldsymbol{v}}\,\frac{\delta x}{\delta v}\delta v\right) \times \left(\hat{\boldsymbol{u}}\,\frac{\delta y}{\delta u}\delta u + \hat{\boldsymbol{v}}\,\frac{\delta y}{\delta v}\delta v\right)\,, \tag{6.26}$$

Therefore

$$\begin{aligned} dx\,dy &= |\delta \boldsymbol{x} \times \delta \boldsymbol{y}| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}\right| \, |\hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}}|\, dudv \\ &= |J_{u,v}(x,y)|\; dudv\,. \end{aligned} \tag{6.27}$$

The inverse relationship (6.23) can be proved in a similar manner □

Result 2 :

$$\int\!\!\int_R f(x,y)\,dxdy = \int\!\!\int_{R^*} f(x(u,v),\,y(u,v))\,|J_{u,v}(x,y)|\;dudv\,. \tag{6.28}$$

Example 1 : For polar co-ordinates $x = r\cos\theta$ and $y = r\sin\theta$ we take $u = r$ and $v = \theta$

$$J_{r,\theta}(x,y) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \tag{6.29}$$

Thus $dxdy = r\,drd\theta\,.$

Example 2 : To calculate the volume of a sphere of radius a we note that $z = \pm\sqrt{a^2 - x^2 - y^2}$. Doubling up the two hemispheres we obtain

$$\text{Volume} = 2\int\!\!\int_R \sqrt{a^2 - x^2 - y^2}\,dxdy \tag{6.30}$$

where R is the disc $x^2 + y^2 \le a^2$ in the $z = 0$ plane. Using a change of variable

$$\begin{aligned} \text{Volume} &= 2\int\!\!\int_R \sqrt{a^2 - x^2 - y^2}\,dxdy \\ &= 2\int\!\!\int_R \sqrt{a^2 - r^2}\,rdrd\theta \\ &= 2\int_0^a \sqrt{a^2 - r^2}\,rdr \int_0^{2\pi} d\theta \\ &= 4\pi a^3/3\,. \end{aligned} \tag{6.31}$$

Example 3 : From the example in §6.3 over the quarter circle in the first quadrant it would be easier to compute in polars. Thus

$$\begin{aligned} \text{Area of R} &= \int\!\!\int_R dxdy = \int\!\!\int_R rdrd\theta \\ &= \int_0^a rdr \int_0^{\pi/2} d\theta = \pi a^2/4 \end{aligned} \tag{6.32}$$

$$\begin{aligned}\int\int_R xy\,dxdy &= \int\int_R r^3\cos\theta\sin\theta dr d\theta \\ &= \int_0^a r^3 dr \int_0^{\pi/2} \cos\theta\sin\theta d\theta \\ &= \tfrac{1}{4}a^4\int_0^{\pi/2}\tfrac{1}{2}\sin 2\theta d\theta \\ &= (a^4/16)[-\cos 2\theta]_0^{\pi/2} = a^4/8\,. \end{aligned} \tag{6.33}$$

Likewise one may show that

$$\begin{aligned}\int\int_R x^2y^2\,dxdy &= \int\int_R r^5\cos^2\theta\sin^2\theta\,drd\theta \\ &= \int_0^a r^5 dr\int_0^{\pi/2}\cos^2\theta\sin^2\theta d\theta = \pi a^6/96\,. \end{aligned} \tag{6.34}$$

Example 4 : Show that $\int\int_R(x^2+y^2)\,dxdy = 8/3$ using $u = x+y$ and $v = x-y$. where R has corners at $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$ and rotates to a square with corners at $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$. From $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$ it is found that

$$J_{u,v}(x,y) = -\tfrac{1}{2} \tag{6.35}$$

$$\begin{aligned} I &= \tfrac{1}{4}\int\int_{R^*} 2(u^2+v^2)\,|-\tfrac{1}{2}|\,dudv \\ &= \tfrac{1}{4}\left\{\left[\tfrac{1}{3}u^3\right]_0^2[v]_0^2 + \left[\tfrac{1}{3}v^3\right]_0^2[u]_0^2\right\} = 8/3\,. \end{aligned} \tag{6.36}$$

7 Green's Theorem in a plane

Green's Theorem in a plane tells us how a line integral on the boundary of a closed curve C, enclosing a region R, is related to the double integral over the region R.

Theorem 1 *Let R be a closed bounded region in the $x-y$ plane with a piecewise smooth boundary C. Let $P(x,y)$ and $Q(x,y)$ be arbitrary, continuous functions within R having continuous partial derivatives Q_x and P_y. Then*

$$\oint_C (Pdx + Qdy) = \int\int_R (Q_x - P_y)\,dxdy\,. \tag{7.1}$$

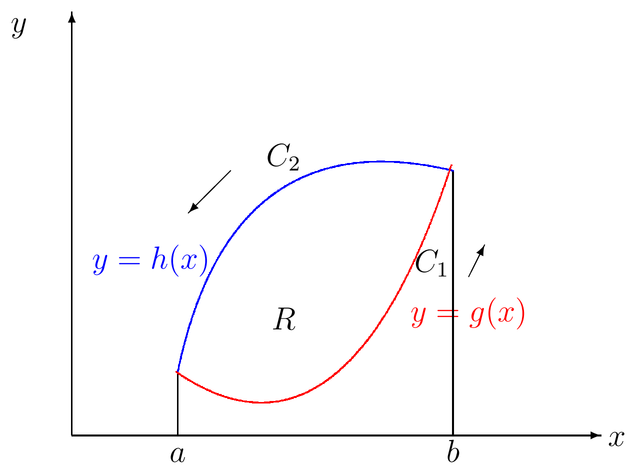


Figure : *In the $x-y$ plane the boundary curve C is made up from two counter-clockwise curves C_1 and C_2 : R denotes the region inside.*

Proof : R is represented by the upper and lower boundaries (as in the Figure above)

$$g(x) \le y \le h(x) \tag{7.2}$$

and so

$$\begin{aligned}
\int\int_R \frac{\partial P}{\partial y}\,dxdy &= \int_a^b \left\{ \int_{y=g(x)}^{y=h(x)} \frac{\partial P}{\partial y}\,dy \right\} dx \\
&= \int_a^b \left\{ P\big(x,\, h(x)\big) - P\big(x,\, g(x)\big) \right\} dx \\
&= -\int_a^b P\big(x,\, g(x)\big)\,dx - \underbrace{\int_b^a P\big(x,\, h(x)\big)\,dx}_{switched\ limits\ \&\ sign} \\
&= -\int_{C_1} P(x,y)\,dx - \int_{C_2} P(x,y)\,dx \\
&= -\oint_C P(x,y)\,dx\,.
\end{aligned} \tag{7.3}$$

The same method can be used the other way round to prove that (note the +ve sign)

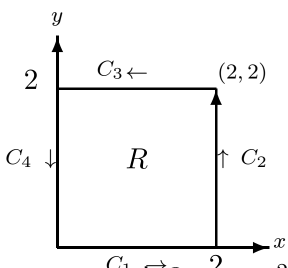
$$\int\int_R \frac{\partial Q}{\partial x}\,dxdy = \oint_C Q(x,y)\,dy \tag{7.4}$$

Both these results are true separately but can be pieced together to form the final result. ∎

Example 1 : Use Green's Theorem to evaluate the line integral

$$\oint_C \left\{ (x-y)\,dx - x^2 dy \right\}\,, \tag{7.5}$$

where R and C are given by



Using G.T. with $P = x - y$ and $Q = -x^2$ over the box-like region R we obtain

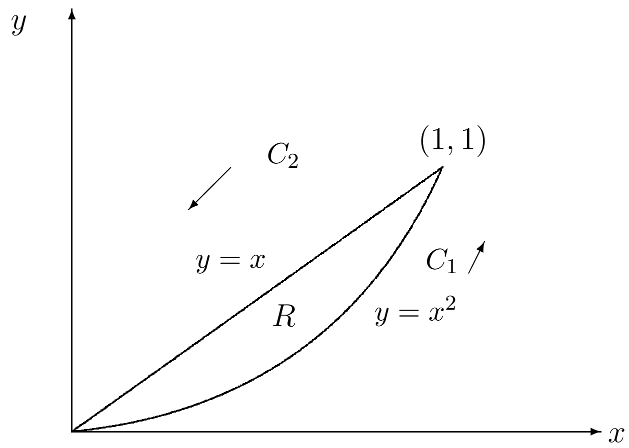
$$\begin{aligned}
\oint_C \left\{ (x-y)\,dx - x^2 dy \right\} &= \int\int_R (1-2x)\,dxdy \\
&= \int_0^2 dy \int_0^2 (1-2x)\,dx \\
&= -4\,.
\end{aligned} \tag{7.6}$$

Direct valuation gives $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$ where C_1 is $y = 0$; C_2 is $x = 2$; C_3 is $y = 2$ and C_4 is $x = 0$. Thus

$$\oint_C = \int_0^2 x\,dx - 4\int_0^2 dy + \int_2^0 (x-2)\,dx + 0 = 2 - 8 + 2 = -4\,. \tag{7.7}$$

Example 2 : Using Green's Theorem over R in the diagram, show that

$$\oint_C \left\{y^3\,dx + (x^3 + 3xy^2)\,dy\right\} = 3/20 \tag{7.8}$$

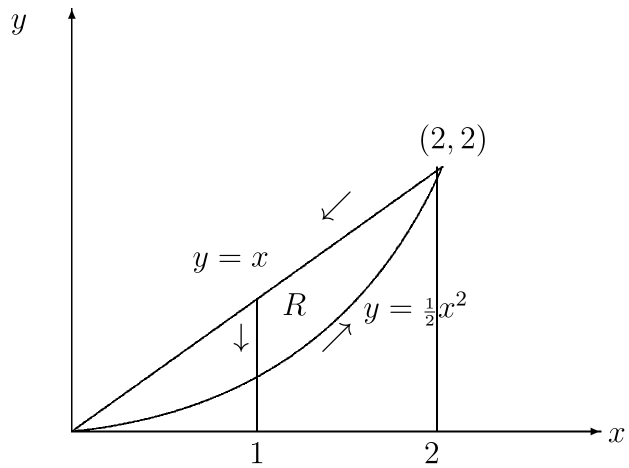


Using Green's Theorem

$$\begin{aligned}
\oint_C \left\{y^3\,dx + (x^3 + 3xy^2)\,dy\right\} &= 3\int\int_R x^2\,dxdy \\
&= 3\int_0^1 x^2\left(\int_{y=x^2}^{y=x} dy\right)\,dx \\
&= 3\int_0^1 \left(x^3 - x^4\right)\,dx = 3/20\,.
\end{aligned} \tag{7.9}$$

Example 3 : (Part of 2005 exam) : With a suitable choice of P and Q and R as in the figure, show that

$$\tfrac{1}{2}\oint_C (x\,dy - y\,dx) = \int\int_R dxdy = 1/3\,. \tag{7.10}$$



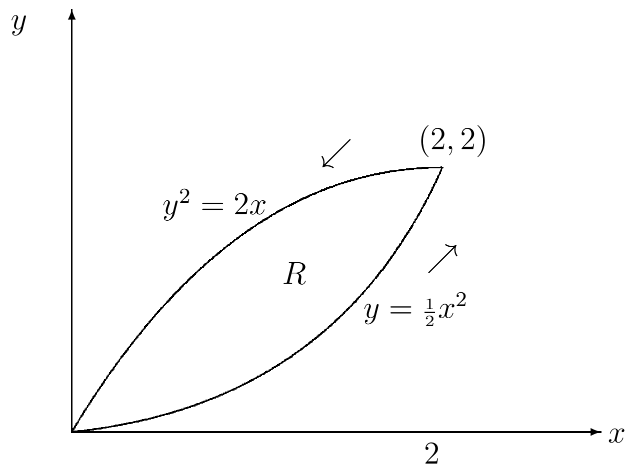
$$\int\int_R dxdy = \int_1^2 \left(\int_{\frac{1}{2}x^2}^{x} dy\right) dx = \int_1^2 \left(x - \tfrac{1}{2}x^2\right) dx = 1/3\,. \tag{7.11}$$

Around the line integral we have

$$\begin{aligned}\oint &= \tfrac{1}{2}\int_1^2 \left(x^2 - \tfrac{1}{2}x^2\right) dx + \tfrac{1}{2}\int_1^2 (xdx - xdx) + \tfrac{1}{2}\int_1^{1/2} dy \\ &= 7/12 - 1/4 = 1/3\,. \end{aligned} \tag{7.12}$$

Example 4 : (Part of 2004 exam) : If $Q = x^2$ and $P = -y^2$ and R is as in the figure below, show that

$$\oint_C (x^2dy - y^2dx) = 2\int\int_R (x+y)\,dxdy = 24/5\,. \tag{7.13}$$



The RHS is obvious in the sense that $Q_x - P_y = 2x + 2y$ and so

$$\begin{aligned} 2\int\int_R (x+y)\,dxdy &= 2\int_0^2 x\left(\int_{\frac{1}{2}x^2}^{\sqrt{2x}} dy\right) dx + 2\int_0^2 \left(\int_{\frac{1}{2}x^2}^{\sqrt{2x}} y\,dy\right) dx \\ &= 2\int_0^2 \left[x\left(\sqrt{2x} - \tfrac{1}{2}x^2\right) + \tfrac{1}{2}\left(2x - \tfrac{1}{4}x^4\right)\right] dx \\ &= 2\int_0^2 \left[\sqrt{2}x^{3/2} - \tfrac{1}{2}x^3 + x - \tfrac{1}{8}x^4\right] dx = 24/5\,. \end{aligned} \tag{7.14}$$

The sum of the two line integrals can be done to get the same answer.

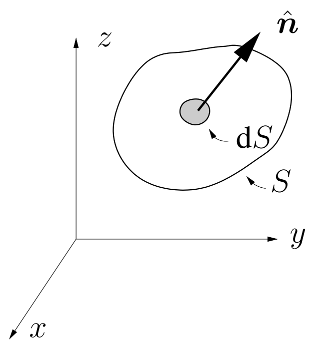
8 $2D$ Divergence and Stokes' Theorems

8.1 Surface Integrals

Having considered the line integral of a vector field we now look at computing the integral of a vector field on a surface. In particular we look at the integral of the component of the flux in the direction normal to the surface. The notation we shall use is illustrated below:

A surface integral of a 3-D vector field is written as

$$\iint_S \vec{F}\cdot \mathrm{d}\vec{S}. \tag{8.1}$$



and this is a scalar quantity. Here the vector field is

$$\vec{F} = F_x\,\hat{\boldsymbol{\imath}} + F_y\,\hat{\boldsymbol{\jmath}} + F_z\,\hat{\boldsymbol{k}} \tag{8.2}$$

and we wish to compute the integral on the surface

$$S = \text{a 2 dimensional surface in 3 dimensions} \tag{8.3}$$

We want to compute the component on the vector field in the direction normal to the surface so introduce the notation

$$\mathrm{d}\vec{S} = \hat{\boldsymbol{n}}\,\mathrm{d}S, \tag{8.4}$$

where

$$\hat{\boldsymbol{n}} = \text{unit normal to surface } S \text{ at each point.} \tag{8.5}$$

and this unit normal $\hat{\boldsymbol{n}}$ is defined to *point outwards* with

$$\mathrm{d}S = \text{infinitesimal area patch on surface.} \tag{8.6}$$

That is, the surface S is a “patchwork quilt” of the individual patches $\mathrm{d}S$.

From this definition we immediately see that we have

$$\iint\limits_S \vec{F}\cdot\mathrm{d}\vec{S} = \iint\limits_S \underbrace{\vec{F}\cdot\hat{\boldsymbol{n}}}_{\text{scalar field}}\;\mathrm{d}S. \tag{8.7}$$

Such integrals are also called “*flux*” integrals and are said to give the flux of the vector field $\vec{F}$ through the surface S. For example, if $\vec{F}$ is the velocity field of a fluid (in m s^{-1}) then $\iint_S \vec{F}\cdot\mathrm{d}\vec{S}$ gives the *volume* of water flowing through S per unit time.

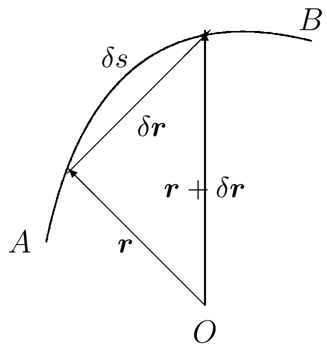


Figure 6.1 : *On a curve C, with starting and ending points A and B, small elements of arc length δs and the chord $\delta \boldsymbol{r}$, where O is the origin.*

The chord $\delta \boldsymbol{r}$ and the arc δs in the figure show us how to define a unit tangent vector $\hat{\boldsymbol{T}}$

$$\begin{aligned} \hat{\boldsymbol{T}} &= \lim_{\delta \boldsymbol{r} \to 0} \frac{\delta \boldsymbol{r}}{\delta s} = \frac{d\boldsymbol{r}}{ds} \\ &= \underline{\boldsymbol{i}}\frac{dx}{ds} + \underline{\boldsymbol{j}}\frac{dy}{ds}\,. \end{aligned} \tag{8.8}$$

The unit normal $\hat{\boldsymbol{n}}$ must be perpendicular to this : that is $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{T}} = 0$, giving

$$\hat{\boldsymbol{n}} = \pm \left(\underline{\boldsymbol{i}}\frac{dy}{ds} - \underline{\boldsymbol{j}}\frac{dx}{ds} \right)\,, \tag{8.9}$$

where $\pm$ refer to inner and outer normals.

8.2 Divergence (Gauss') Theorem

Define a $2D$ vector $\boldsymbol{u} = \underline{\boldsymbol{i}}Q - \underline{\boldsymbol{j}}P$. Then

$$\boldsymbol{u} \cdot \hat{\boldsymbol{n}} = P\frac{dx}{ds} + Q\frac{dy}{ds} \qquad\qquad \text{div}\, \boldsymbol{u} = Q_x - P_y \tag{8.10}$$

Thus Green's Theorem turns into a $2D$ version of **Divergence (Gauss') Theorem**

$$\int\!\!\int_R \text{div}\, \boldsymbol{u}\, dxdy = \oint_C \boldsymbol{u} \cdot \hat{\boldsymbol{n}}\, ds \tag{8.11}$$

This line integral simply expresses the sum of the normal component of $\boldsymbol{u}$ around the boundary. If $\boldsymbol{u}$ is a solenoidal vector (div $\boldsymbol{u} = 0$) then automatically $\oint_C \boldsymbol{u} \cdot \hat{\boldsymbol{n}}\, ds = 0$.

The $3D$ version uses an arbitrary $3D$ vector field $\boldsymbol{u}(x, y, z)$ that lives in some finite, simply connected volume V whose surface is S: dS is some small element of area on the curved, *closed* surface S

$$\int\!\!\int\!\!\int_V \text{div}\, \boldsymbol{u}\, dV = \oint\!\!\oint_S \boldsymbol{u} \cdot \hat{\boldsymbol{n}}\, dS \tag{8.12}$$

or equivalently

$$\int\int\int_V \nabla \cdot \boldsymbol{u}\, dV = \oiint_S \boldsymbol{u} \cdot \mathrm{d}\vec{S}\,. \tag{8.13}$$

Gauss's theorem has a simple physical interpretation: If $\boldsymbol{u}$ is the velocity field of a fluid, the volume of fluid flowing out of a closed surface in unit time is *equal* to the total amount of fluid pumped into the volume surrounded by S.

In electrostatic theory we consider a closed surface S and a magnetic field $\vec{B}$. Maxwell's equations imply that $\nabla \cdot \vec{B} = 0$. Hence we have that

$$\oiint_S \vec{B} \cdot \mathrm{d}\vec{S} = \iiint_V \left(\nabla \cdot \vec{B}\right) \mathrm{d}V \tag{8.14}$$

$$= 0. \tag{8.15}$$

This says that the total magnetic flux across any closed surface must be zero.

8.3 Stokes' Theorem

Now define a $2D$ vector $\boldsymbol{v} = \underline{\boldsymbol{i}}P + \underline{\boldsymbol{j}}Q$. Then

$$\boldsymbol{v} \cdot d\boldsymbol{r} = (\boldsymbol{v} \cdot \hat{\boldsymbol{T}})\, ds = P\, dx + Q\, dy\,. \tag{8.16}$$

Moreover

$$\mathrm{curl}\, \boldsymbol{v} = \begin{vmatrix} \underline{\boldsymbol{i}} & \underline{\boldsymbol{j}} & \underline{\boldsymbol{k}} \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = \underline{\boldsymbol{k}}\,(Q_x - P_y) \tag{8.17}$$

Thus Green's Theorem turns into a $2D$ version of **Stokes' Theorem**

$$\int\int_R (\underline{\boldsymbol{k}} \cdot \mathrm{curl}\, \boldsymbol{v})\, dxdy = \oint_C \boldsymbol{v} \cdot d\boldsymbol{r} \tag{8.18}$$

The line integral on the RHS is called the *circulation*. Note that if $\boldsymbol{v}$ is an irrotational vector then $\oint_C \boldsymbol{v} \cdot d\boldsymbol{r} = 0$ which means there is no circulation.

The 3D-version of Stokes' Theorem for an arbitrary $3D$ vector field $\boldsymbol{v}(x, y, z)$ in a volume V is given by

$$\oint_C \boldsymbol{v} \cdot d\boldsymbol{r} = \int\int_S \mathrm{curl}\, \boldsymbol{v}\; \cdot \hat{\boldsymbol{n}}\, dS \;=\; \int\int_S (\nabla \times \boldsymbol{v})\; \cdot \mathrm{d}\vec{S} \tag{8.19}$$

$\hat{\boldsymbol{n}}$ is the unit normal vector to the surface S of V, C is the boundary of the surface S, and dS is an element of area.

Stokes' Theorem allows us to either compute a surface integral directly or compute a line integral to find the answer. Stokes's theorem equates the surface integral of the *curl* of the vector field $\vec{F}$ to the *line* integral of the vector field $\vec{F}$ along the *boundary* of the surface. The curve C is the boundary of the surface S and is a *closed* curve (we use the notation $\oint$ for the integral, to emphasise that the start point and the end point are the same). Figure 9 indicates the notation used. When defining the curve C we must be very careful to get the direction along the curve correct. We do this by using the "right-hand corkscrew" rule with the corkscrew travelling in the direction of the outward normal to the surface S.

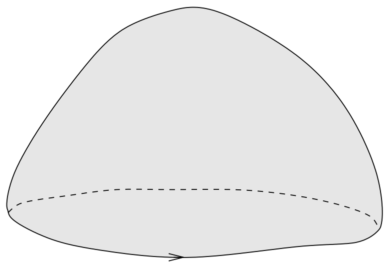


Figure 9: A surface with boundary as used in Stokes's Theorem.

There are three different ways that this result can be used:

1. It allows us to evaluate the surface integral in terms of the (much simpler) line integral, provided that we can recognise that the field that we want to integrate can be written as $\nabla \times \vec{F}$ and hence find $\vec{F}$.

2. It allows us to evaluate closed line integrals in terms of *any* surface bounded by the line C.

   That is, because for any two surfaces S_1, S_2 with the same boundary C we have that

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_{S_1} \left(\nabla \times \vec{F}\right) \cdot \mathrm{d}\vec{S} = \iint_{S_2} \left(\nabla \times \vec{F}\right) \cdot \mathrm{d}\vec{S}, \tag{8.20}$$

   we can pick the most convenient surface to perform the integral. This behaviour is illustrated in Figure 10

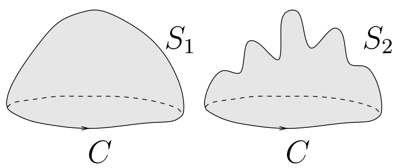


Figure 10: A single curve may be the boundary of many different surfaces.

3. The final result implies that the flux of $\left(\nabla \times \vec{F}\right)$ through a surface S_j is *independent* of which surface S_j you pick, provided that all the surfaces S_j are bounded by the same curve C.

8.4 Examples

Magnetic example 1:

We now consider a practical example where Stokes' theorem is applicable. The work done in moving a charged particle along the curve C within an electric field $\vec{E}$ is given by

$$\int_C \vec{E} \cdot \mathrm{d}\vec{r}. \tag{8.21}$$

But from Maxwell's equations we can relate the curl of the electric field to the magnetic field $\vec{B}$ by

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{8.22}$$

Therefore if we consider moving the charged particle around a *closed* path we have, from Stokes's Theorem,

$$\oint_C \vec{E} \cdot \mathrm{d}\vec{r} = \iint_S \left(\nabla \times \vec{E}\right) \cdot \mathrm{d}\vec{S} \tag{8.23}$$

$$= \iint_S -\frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{S}. \tag{8.24}$$

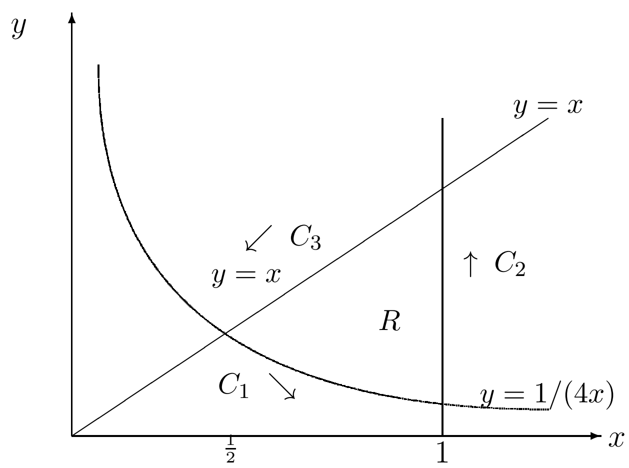
In the case where the magnetic field is *independent of time* it then follows that

$$\oint_C \vec{E} \cdot \mathrm{d}\vec{r} = 0. \tag{8.25}$$

That is, the work done in moving a charged particle around a closed path in a time independent electric field (a completely steady electric field) is identically zero.

Example 2 : (part of 2002 exam) If $\boldsymbol{v} = \underline{\boldsymbol{i}}y^2 + \underline{\boldsymbol{j}}x^2$ and R is as in the figure below, by evaluating the line integral, show that

$$\oint_C \boldsymbol{v} \cdot d\boldsymbol{r} = 5/48\,. \tag{8.26}$$



Firstly

$$\oint_C \boldsymbol{v}\cdot d\boldsymbol{r} = \oint_C \left(y^2dx + x^2dy\right). \tag{8.27}$$

(i) On C_1 we have $y = 1/(4x)$ and so $dy = -dx/(4x^2)$. Therefore

$$\int_{C_1} \boldsymbol{v}\cdot d\boldsymbol{r} = \int_{\frac{1}{2}}^{1} \left(\frac{1}{16x^2} - \frac{1}{4}\right) dx = -\frac{1}{16}\,. \tag{8.28}$$

(ii) On C_2 we have $x = 1$ and so $dx = 0$. Therefore

$$\int_{C_2} \boldsymbol{v}\cdot d\boldsymbol{r} = \int_{\frac{1}{4}}^{1} dy = \frac{3}{4}\,. \tag{8.29}$$

(iii) On C_3 we have $y = x$ and so $dy = dx$. Therefore

$$\int_{C_3} \boldsymbol{v}\cdot d\boldsymbol{r} = 2\int_{1}^{\frac{1}{2}} x^2\,dx = -\frac{7}{12}\,. \tag{8.30}$$

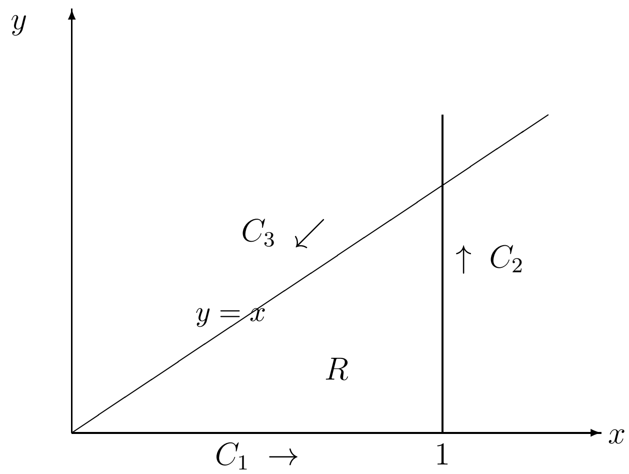
Summing these three results gives $-\frac{1}{16} + \frac{3}{4} - \frac{7}{12} = \frac{5}{48}$.

Example 3 : (part of 2003 exam) If

$$\boldsymbol{u} = \underline{\boldsymbol{i}}\frac{x^2y}{1+y^2} + \underline{\boldsymbol{j}}\left[x\ln(1+y^2)\right] \tag{8.31}$$

and R is as in the figure below, show that

$$\int\int_R \text{div}\,\boldsymbol{u}\,dxdy = 2\ln 2 - 1\,. \tag{8.32}$$



Firstly we calculate div $\boldsymbol{u}$

$$\text{div}\,\boldsymbol{u} = \frac{4xy}{1+y^2} \tag{8.33}$$

and so

$$\begin{aligned}
\int\int_R \mathsf{div}\, \boldsymbol{u}\, dxdy &= 4\int\int_R \frac{xy}{1+y^2}\, dxdy \\
&= 4\int_0^1 x \left(\int_{y=0}^{y=x} \int \frac{y\, dy}{1+y^2} \right) dx \\
&= 4\int_0^1 x \left[\tfrac{1}{2}\ln(1+y^2) \right]_0^x dx \\
&= 2\int_0^1 x\, \ln(1+x^2)\, dx \\
&= 2\ln 2 - 1 \qquad (8.34)
\end{aligned}$$

8.5 Long Example: Gauss' Theorem

We shall now examine how to evaluate surface and volume integrals. We start by considering the vector field

$$\vec{F} = x^2 y\, \hat{\boldsymbol{\imath}} + (y^2 - z^2)\, \hat{\boldsymbol{\jmath}} + xyz\, \hat{\boldsymbol{k}}. \qquad (8.35)$$

We now wish to calculate

$$\oiint \vec{F} \cdot \mathsf{d}\vec{S}$$

where S is the unit cube as illustrated in Figure 11.

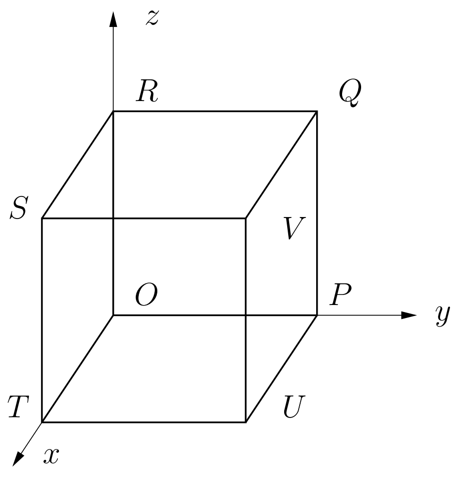


Figure 11: The unit cube used in the example of the Gauss divergence theorem.

The cube has six sides and so we compute the integral on each of these six surfaces and subsequently add them together to get the total integral. On each of the six surfaces we need to determine the outward normal and the infinitesimal area. The six parts of the surface integral are:

$$1) - OPQR \qquad (8.36)$$

This is the surface $x = 0$ with $0 \le y \le 1$, $0 \le z \le 1$. So we have the infinitesimal area $dy\, dz$.

$$x = 0 \qquad \mathsf{d}\vec{S} = -\hat{\boldsymbol{\imath}}\, \mathsf{d}y\, \mathsf{d}z \tag{8.37}$$

we evaluate the vector field on the surface to get

$$\vec{F} = (y^2 - z^2)\hat{\boldsymbol{\jmath}} \tag{8.38}$$

and hence the flux normal to the surface is

$$\vec{F} \cdot \mathsf{d}\vec{S} = 0. \tag{8.39}$$

$$2) - SVUT \tag{8.40}$$

This is the surface $x = 1$ with $0 \le y \le 1$, $0 \le z \le 1$. So we have the infinitesimal area $dy\, dz$.

$$: \qquad x = 1 \qquad \mathsf{d}\vec{S} = \hat{\boldsymbol{\imath}}\, \mathsf{d}y\, \mathsf{d}z \tag{8.41}$$

$$\vec{F} = y\hat{\boldsymbol{\imath}} + (y^2 - z^2)\hat{\boldsymbol{\jmath}} + yz\hat{\boldsymbol{k}} \tag{8.42}$$

$$\iint \vec{F}.\mathsf{d}\vec{S} = \int_0^1 \int_0^1 y\, \mathsf{d}y\, \mathsf{d}z \tag{8.43}$$

$$= \left[\frac{y^2}{2}\right]_0^1 \tag{8.44}$$

$$= \frac{1}{2}. \tag{8.45}$$

$$3) - ORST: \qquad y = 0 \qquad \mathsf{d}\vec{S} = -\hat{\boldsymbol{\jmath}}\,\mathsf{d}x\,\mathsf{d}z \tag{8.46}$$

$$\vec{F} = -z^2\hat{\boldsymbol{\jmath}} \tag{8.47}$$

$$\iint \vec{F}\cdot \mathsf{d}\vec{S} = \int_0^1\int_0^1 z^2\,\mathsf{d}x\,\mathsf{d}z \tag{8.48}$$

$$= \left[\frac{z^3}{3}\right]_0^1 \tag{8.49}$$

$$= \frac{1}{3}. \tag{8.50}$$

$$4) - PQVU: \qquad y = 1 \qquad \mathsf{d}\vec{S} = \hat{\boldsymbol{j}}\,\mathsf{d}x\,\mathsf{d}z \tag{8.51}$$

$$\vec{F} = x^2\hat{\boldsymbol{\imath}} + (1 - z^2)\hat{\boldsymbol{\jmath}} + xz\hat{\boldsymbol{k}} \tag{8.52}$$

$$\iint \vec{F} \cdot \mathsf{d}\vec{S} = \int_0^1 \int_0^1 (1 - z^2)\,\mathsf{d}x\,\mathsf{d}z \tag{8.53}$$

$$= \left[z - \frac{z^3}{3}\right]_0^1 \tag{8.54}$$

$$= \frac{2}{3} \tag{8.55}$$

$$5) - OTUP: \qquad z = 0 \qquad \mathsf{d}\vec{S} = -\hat{\boldsymbol{k}}\,\mathsf{d}x\,\mathsf{d}y \tag{8.56}$$

$$\vec{F} = x^2y\hat{\boldsymbol{\imath}} + y^2\hat{\boldsymbol{\jmath}} \tag{8.57}$$

$$\vec{F} \cdot \mathsf{d}\vec{S} = 0 \tag{8.58}$$

$$6) - QRSV: \qquad z = 1 \qquad \mathsf{d}\vec{S} = \hat{\boldsymbol{k}}\,\mathsf{d}x\,\mathsf{d}y \tag{8.59}$$

$$\vec{F} = x^2y\hat{\boldsymbol{\imath}} + (y^2 - 1)\hat{\boldsymbol{\jmath}} + xy\hat{\boldsymbol{k}} \tag{8.60}$$

$$\iint \vec{F} \cdot \mathsf{d}\vec{S} = \int_0^1 \int_0^1 xy\,\mathsf{d}x\,\mathsf{d}y \tag{8.61}$$

$$= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 \tag{8.62}$$

$$= \frac{1}{4}\,. \tag{8.63}$$

Hence the total integral is

$$\oiint \vec{F} \cdot \mathsf{d}\vec{S} = 0 + \frac{1}{2} + \frac{1}{3} + \frac{2}{3} + 0 + \frac{1}{4} \tag{8.64}$$

$$= \frac{7}{4}. \tag{8.65}$$

However, it is much simpler to calculate this integral by first using the Gauss divergence theorem.

$$\oiint_S \vec{F} \cdot \mathrm{d}\vec{S} = \iiint_V \left(\boldsymbol{\nabla} \cdot \vec{F}\right) \mathrm{d}V \,. \tag{8.66}$$

We must therefore compute the divergence of the vector field and find

$$\boldsymbol{\nabla} \cdot \vec{F} = 2xy + 2y + xy \tag{8.67}$$

$$= (3x+2)y. \tag{8.68}$$

We must compute the integral over the cube. We know that the cube has $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ and an infinitesimal part of the cube is dx dy dz so that

$$\iiint_V \boldsymbol{\nabla} \cdot \vec{F} \,\mathrm{d}V = \int_0^1 \int_0^1 \int_0^1 (3x+2)y \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \tag{8.69}$$

$$= \left[\frac{3x^2}{2} + 2x\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 \tag{8.70}$$

$$= \frac{7}{2} \cdot \frac{1}{2} \tag{8.71}$$

$$= \frac{7}{4}. \tag{8.72}$$

and this is in agreement with the calculation done using the surface integral.

9 Maxwell's Equations

Maxwell's equations are technically not in the syllabus but it's good to tie everything together in a physical context.

Consider a $3D$ volume with a closed circuit C drawn on its surface. Let's use the $3D$ versions of the Divergence (8.12) and Stokes' Theorems (8.19) to derive some relationships between various electro-magnetic variables.

1. Using the charge density ρ, the total charge within the volume V must be equal to surface area integral of the electric flux density $\boldsymbol{\mathcal{D}}$ through the surface S (recall that $\boldsymbol{\mathcal{D}} = \epsilon \underline{\boldsymbol{E}}$ where $\underline{\boldsymbol{E}}$ is the electric field).

$$\int\int\int_V \rho \, dV = \int\int_S \boldsymbol{\mathcal{D}} \cdot \hat{\boldsymbol{n}} \, dA \,. \tag{9.1}$$

Using a 3D version of the Divergence Theorem (8.12) above on the RHS of (9.1)

$$\int\int\int_V \rho\, dV = \int\int\int_V \text{div}\,\boldsymbol{\mathcal{D}}\, dV\,. \tag{9.2}$$

Hence we have the first of Maxwell's equations

$$\boxed{\text{div}\,\boldsymbol{\mathcal{D}} = \rho} \tag{9.3}$$

This is also known as Gauss's Law.

2. Following the above in the same manner for the magnetic flux density $\underline{\boldsymbol{B}}$ (recall that $\underline{\boldsymbol{B}} = \mu\underline{\boldsymbol{H}}$ where $\underline{\boldsymbol{H}}$ is the magnetic field) but noting that there are no magnetic sources (so $\rho_{mag} = 0$), we have the 2nd of Maxwell's equations

$$\boxed{\text{div}\,\underline{\boldsymbol{B}} = 0} \tag{9.4}$$

3. Faraday's Law says that the rate of change of magnetic flux linking a circuit C is proportional to the electromotive force (in the negative sense). Mathematically this is expressed as

$$\frac{d}{dt}\int\int_S \underline{\boldsymbol{B}}\cdot\hat{\boldsymbol{n}}\, dA = -\oint_C \underline{\boldsymbol{E}}\cdot d\boldsymbol{r} \tag{9.5}$$

Using a 3D version of Stokes' Theorem (8.19) on the RHS of (9.4), and taking the time derivative through the surface integral (thereby making it a partial derivative) we have the 3rd of Maxwell's equations

$$\boxed{\text{curl}\,\underline{\boldsymbol{E}} + \frac{\partial\underline{\boldsymbol{B}}}{\partial t} = 0} \tag{9.6}$$

4. Ampère's (Biot-Savart) Law expressed mathematically (the line integral of the magnetic field around a circuit C is equal to the current enclosed) is

$$\int\int_S \boldsymbol{J}\cdot\hat{\boldsymbol{n}}\, dA = \oint_C \underline{\boldsymbol{H}}\cdot d\boldsymbol{r}\,, \tag{9.7}$$

where $\boldsymbol{J}$ is the current density and $\underline{\boldsymbol{H}}$ is the magnetic field. Using 3D-Stokes' Theorem (8.19) on the RHS we find that we have curl $\underline{\boldsymbol{H}} = \boldsymbol{J}$ and therefore div $\boldsymbol{J} = 0$, which is inconsistent with the first three of Maxwell's equations. Why? The continuity equation for the total charge is

$$\frac{d}{dt}\int\int\int_V \rho\, dV = -\int\int_S \boldsymbol{J}\cdot\hat{\boldsymbol{n}}\, dA\,. \tag{9.8}$$

Using the Divergence Theorem on the LHS we obtain

$$\text{div}\,\boldsymbol{J} + \frac{\partial\rho}{\partial t} = 0\,. \tag{9.9}$$

If div $\boldsymbol{J} = 0$ then ρ would have to be independent of t. To get round this problem we use Gauss's Law $\rho = \text{div}\,\boldsymbol{\mathcal{D}}$ expressed above to get

$$\text{div}\left\{\boldsymbol{J} + \frac{\partial\boldsymbol{\mathcal{D}}}{\partial t}\right\} = 0 \tag{9.10}$$

and so this motivates us to replace $\boldsymbol{J}$ in curl $\underline{\boldsymbol{H}} = \boldsymbol{J}$ by $\boldsymbol{J} + \partial\boldsymbol{\mathcal{D}}/\partial t$ giving the 4th of Maxwell's equations

$$\boxed{\text{curl}\,\underline{\boldsymbol{H}} = \boldsymbol{J} + \frac{\partial\boldsymbol{\mathcal{D}}}{\partial t}} \tag{9.11}$$

5. We finally note that because div $\underline{\boldsymbol{B}} = 0$ then $\underline{\boldsymbol{B}}$ is a solenoidal vector field: there must exist a vector potential $\boldsymbol{A}$ such that $\underline{\boldsymbol{B}} = \text{curl}\, \boldsymbol{A}$. Using this in the 3rd of Maxwell's equations, we have
$$\text{curl}\, \left[\underline{\boldsymbol{E}} + \frac{\partial \boldsymbol{A}}{\partial t}\right] = 0\,. \tag{9.12}$$
This means that there must also exist a scalar potential ϕ that satisfies
$$\underline{\boldsymbol{E}} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi\,. \tag{9.13}$$