EE2 Mathematics – Probability & Statistics

Solution 7

1. (a) We have

$$\sum_{\forall x,y} f_{X,Y}(x,y) = a\left(\frac{1}{1}+\frac{2}{1}+\frac{3}{1}+\frac{1}{2}+\frac{2}{2}+\frac{3}{2}+\frac{1}{3}+\frac{2}{3}+\frac{3}{3}\right) = 11a\,,$$

so $a = 1/11$.

(b)

x \ y	1	2	3	$f_X(x)$
1	a	a/2	a/3	11a/6
2	2a	a	2a/3	11a/3
3	3a	3a/2	a	11a/2
$f_Y(y)$	6a	3a	2a	

Marginals are:

x	1	2	3
$f_X(x)$	1/6	1/3	1/2

and

y	1	2	3
$f_Y(y)$	6/11	3/11	2/11

(c) Working directly with the joint PMF:

$$\begin{aligned} E(XY) &= \sum_{\forall x,y} xy f_{X,Y}(x,y) \\ &= f_{X,Y}(1,1) + 2f_{X,Y}(2,1) + \ldots + 9f_{X,Y}(3,3) \\ &= 42/11 \end{aligned}$$

The expectations are $\mathrm{E}(X) = \sum_x x f_X(x) = 7/3$ and $\mathrm{E}(Y) = \sum_y y f_Y(y) = 18/11$, so $\mathrm{Cov}(X,Y) = 0$. Thus, X and Y are uncorrelated. They are also independent, which we can check by factorising the joint PMF into the marginals. Indeed, $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ for all x,y.

(d) By independence:

$$\begin{aligned} P(X \le 2 | Y \le 2) &= \frac{P(X \le 2, Y \le 2)}{P(Y \le 2)} = \frac{P(X \le 2)P(Y \le 2)}{P(Y \le 2)} \\ &= P(X \le 2) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \end{aligned}$$

2. Consider the continuous random variables X and Y with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} k(x+y-2xy) & \text{if } 0 \le x, y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) The support is rectangular, so it is easy to integrate the PDF everywhere:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy &= k\int_0^1\int_0^1 (x+y-2xy)dxdy \\ &= k\int_0^1 \left[\frac{x^2}{2}+xy-x^2y\right]_0^1 dy \\ &= k\int_0^1 (1/2+y-y)dy = k/2\,,\end{aligned}$$

so we must have $k=2$.

(b) From the previous calculation, we already have $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = k/2 = 1$ if $y \in [0,1]$ and 0 otherwise. By symmetry, the other marginal is the same. These are continuous uniform distributions on $[0,1]$. The joint PDF is not equal to the product of the marginals, so X and Y are not independent.

(c) From standard distributional results, $\mathrm{E}(X) = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^1 xk/2dx = 1/2$. Similarly $\mathrm{E}(Y) = 1/2$. We then have

$$\begin{aligned}\mathrm{E}(XY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dxdy \\ &= \int_0^1\int_0^1 xy\,2(x+y-2xy)dxdy \\ &= 2\int_0^1 \frac{y}{3}-\frac{y^2}{6}dy \\ &= 2/9\,,\end{aligned}$$

so the covariance is $\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\,\mathrm{E}(Y) = 2/9 - (1/2)^2 = -1/36$.

3. (a) Note that the support is no longer rectangular. We have

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx &= c\int_0^1\int_{x^2}^1 x(y-x^2+1)dydx \\ &= c\int_0^1 \left[\frac{xy^2}{2}-x^3y+xy\right]_{x^2}^1 dx \\ &= c\int_0^1 (\frac{3x}{2}-2x^3+\frac{x^5}{2})dx = c/3\,,\end{aligned}$$

so $c=3$.

(b) We already have $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = 9x/2 - 6x^3 + 3x^5/2$ for $x \in [0,1]$ and 0 otherwise. For the other marginal we have

$$
\begin{aligned}
f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx &= \int_0^{\sqrt{y}} 3x(y - x^2 + 1)dx \\
&= \left[\frac{3x^2y}{2} - \frac{3x^4}{4} + \frac{3x^2}{2}\right]_0^{\sqrt{y}} \\
&= \frac{3y^2}{4} + \frac{3y}{2},
\end{aligned}
$$

so $f_Y(y) = 3y^2/4 + 3y/2$ for $y \in [0,1]$ and 0 otherwise.
Directly from its marginal,

$$
\begin{aligned}
\mathrm{E}(X) &= \int_{-\infty}^{\infty} x f_X(x)dx \\
&= \int_0^1 x\left(9x/2 - 6x^3 + 3x^5/2\right)dx \\
&= 18/35.
\end{aligned}
$$

(c) We have

$$
f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{3x(y - x^2 + 1)}{3y^2/4 + 3y/2} = \frac{x(y - x^2 + 1)}{y^2/4 + y/2}
$$

for x,y such that $0 \leq x, y \leq 1$ and $y \geq x^2$.
We can write

$$
\begin{aligned}
\mathrm{E}(X|Y = y) &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx \\
&= \int_0^{\sqrt{y}} \frac{x^2(y - x^2 + 1)}{y^2/4 + y/2}dx \\
&= \frac{1}{y^2/4 + y/2}\left[y\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^3}{3}\right]_0^{\sqrt{y}} \\
&= \frac{\frac{y^{5/2}}{3} - \frac{y^{5/2}}{5} + \frac{y^{3/2}}{3}}{y^2/4 + y/2}.
\end{aligned}
$$

Note that $\mathrm{E}(X|Y = y)$ is a random variable, contrary to $\mathrm{E}(X)$ that is a number.

(d)

$$\begin{aligned}
\mathrm{E}\left[\mathrm{E}(X|Y)\right] &= \int_{-\infty}^{\infty} \mathrm{E}(X|Y=y) f_Y(y)\, dy \\
&= \int_0^1 \frac{\frac{y^{5/2}}{3} - \frac{y^{5/2}}{5} + \frac{y^{3/2}}{3}}{y^2/4 + y/2} \left(3y^2/4 + 3y/2\right)\, dy \\
&= \int_0^1 3\left(\frac{2y^{5/2}}{15} + \frac{y^{3/2}}{3}\right)\, dy \\
&= \frac{18}{35} = \mathrm{E}(X).
\end{aligned}$$

4. We already know that $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$, and similarly for Y. From the definition of covariance we have

$$\begin{aligned}
\mathrm{Cov}(aX+b, U) &= \mathrm{E}[\{aX + b - \mathrm{E}(aX+b)\}\{U - \mathrm{E}(U)\}] \\
&= a\,\mathrm{E}[\{X - \mathrm{E}(X)\}\{U - \mathrm{E}(U)\}] = a\,\mathrm{Cov}(X, U)\,.
\end{aligned}$$

We can now set $U = cY + d$ and apply the same result to find

$$\mathrm{Cov}(aX+b, cY+d) = ac\,\mathrm{Cov}(X,Y)\,.$$

Finally, we have

$$\begin{aligned}
\mathrm{Corr}(aX+b, cX+d) &= \frac{\mathrm{Cov}(aX+b, cY+d)}{\sqrt{\mathrm{Var}(aX+b)\,\mathrm{Var}(cY+d)}} \\
&= \frac{ac\,\mathrm{Cov}(X,Y)}{\sqrt{a^2\,\mathrm{Var}(X) c^2\,\mathrm{Var}(Y)}} \\
&= \frac{ac\,\mathrm{Cov}(X,Y)}{|a||c|\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{ac}{|ac|}\,\mathrm{Corr}(X,Y)\,,
\end{aligned}$$

which is equal to $\pm\,\mathrm{Corr}(X,Y)$, depending on the sign of ac.

5. (a) X is Poisson so $f_X(x) = e^{-\lambda}\lambda^x/x!$, for $x = 0, 1, 2, \ldots$. Given the number of faults, the number of severe faults is binomial, that is,

$$Y|X = x \sim \mathrm{Bin}(x,p) \quad \Rightarrow \quad f_{Y|X}(y|x) = \binom{x}{y} p^y (1-p)^{x-y}\,.$$

Putting the two together, we find

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}\binom{x}{y} p^y (1-p)^{x-y}\,,$$

for $y = 0, 1, \ldots, x$ and $x = 0, 1, 2, \ldots$.

(b) To find the marginal PMF, we start with the joint PMF and sum out x. Notice that we need $x \geq y$.

$$f_Y(y) = \sum_{\forall x} f_{X,Y}(x,y) = \sum_{x=y}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} \frac{x!}{y!(x-y)!} p^y (1-p)^{x-y}$$
$$= \frac{e^{-\lambda}p^y}{y!} \sum_{x=y}^{\infty} \frac{\lambda^x (1-p)^{x-y}}{(x-y)!} = \frac{e^{-\lambda}p^y}{y!} \lambda^y \sum_{x=y}^{\infty} \frac{[\lambda(1-p)]^{x-y}}{(x-y)!}$$

Now consider the final summation; changing variables from x to $z = x - y$ yields

$$\sum_{x=y}^{\infty} \frac{[\lambda(1-p)]^{x-y}}{(x-y)!} = \sum_{z=0}^{\infty} \frac{[\lambda(1-p)]^{z}}{z!} = e^{\lambda(1-p)} .$$

Substituting this back into the joint PMF gives

$$f_Y(y) = \frac{e^{-\lambda}(\lambda p)^y}{y!} e^{\lambda(1-p)} = \frac{e^{-\lambda+\lambda-\lambda p}(\lambda p)^y}{y!} = \frac{e^{-\lambda p}(\lambda p)^y}{y!} ,$$

for $y = 0, 1, 2, \ldots$. Thus, the distribution of Y is Poisson(λp).

6. (a) $X \sim \text{Bin}(n,p)$
Directly from the PMF:

$$M_X(t) = \mathrm{E}(e^{tX}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$
$$= \sum_{x=0}^{n} \binom{n}{x} (p\,e^t)^x (1-p)^{n-x} = (p\,e^t + 1 - p)^n .$$

(Recall $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$)

The first two derivatives of this function are

$$M_X'(t) = n(p\,e^t + 1 - p)^{n-1} p\,e^t ,$$
$$M_X''(t) = n(n-1)(p\,e^t + 1 - p)^{n-2} p^2\,e^{2t} + n(p\,e^t + 1 - p)^{n-1} p\,e^t ,$$

from which we obtain

$$\mathrm{E}(X) = M_X'(0) = np$$
$$\mathrm{Var}(X) = M_X''(0) - M_X'(0)^2 = n(n-1)p^2 + np - (np)^2 = np(1-p) .$$

(b) $X \sim \text{Gamma}(k,r)$

$$f_X(x) = e^{-kx} x^{r-1} k^r / (r-1)! \qquad (x > 0,\, r \in \mathbb{N}^*,\, k > 0)$$

$$
\begin{aligned}
M_X(t) = \mathrm{E}(e^{tX}) &= \int_0^\infty e^{tx} e^{-kx} x^{r-1} k^r/(r-1)!\, dx \\
&= \int_0^\infty e^{(t-k)x} x^{r-1} k^r/(r-1)!\, dx \\
&= \int_0^\infty k^r/(r-1)! \frac{e^{-u} u^{r-1}}{(k-t)^r}\, du \\
&\qquad\qquad\qquad (u = x(k-t),\, du = (k-t)dx) \\
&= \frac{k^r}{(r-1)!(k-t)^r} \underbrace{\int_0^\infty e^{-u} u^{r-1}\, du}_{=(r-1)!} \\
&\qquad\qquad\qquad (f_U(u) \text{ is a valid Gamma pdf}) \\
&= \frac{k^r}{(k-t)^r} = \left(1 - \frac{t}{k}\right)^{-r}
\end{aligned}
$$

The first two derivatives of this function are

$$
\begin{aligned}
M_X'(t) &= \frac{r}{k}\left(1 - \frac{t}{k}\right)^{-(r+1)}, \\
M_X''(t) &= \frac{r(r+1)}{k^2}\left(1 - \frac{t}{k}\right)^{-(r+2)},
\end{aligned}
$$

from which we obtain

$$
\begin{aligned}
\mathrm{E}(X) &= M_X'(0) = \frac{r}{k} \\
\mathrm{Var}(X) &= M_X''(0) - M_X'(0)^2 = \frac{r(r+1)}{k^2} - \frac{r^2}{k^2} = \frac{r}{k^2}\,.
\end{aligned}
$$

This confirms results obtained in Exercise 4.