EE2-08 Mathematics : Laplace Transforms

D. Nucinkis

d.nucinkis@ic.ac.uk

These notes are not identical word-for-word with my lectures which will be given on a BB/WB. Some of these notes may contain more examples than the corresponding lecture while in other cases the lecture may contain more detailed working. I will not be handing out copies of these notes – **you are therefore advised to attend lectures and take your own**.

There will be one **Handout** on Laplace Transforms.

1 Laplace Transforms

1.1 Introduction

For a function $f(t)$ uniquely defined on $0 \leq t \leq \infty$, its Laplace transform (LT) is defined as

$$\mathcal{L}\left[f(t)\right] = \overline{f}(s) = \int_0^\infty e^{-st} f(t)\, dt\,, \tag{1.1}$$

where s may be complex. The LT may not exist if $f(t)$ becomes singular in $[0\,\infty]$. The LT is a **one-sided transform** in that it operates on $[0\,\infty]$ and not, like the FT, on $[-\infty\,\infty]$. For this reason, LTs are useful for initial value problems, such as circuit theory, where a function switches on at $t = 0$ and where $f(0)$ has been specified.

Because s is a complex variable the inverse transform

$$f(t) = \mathcal{L}^{-1}\left[\overline{f}(s)\right] = \oint_C e^{st}\,\overline{f}(s)\, ds \tag{1.2}$$

is more difficult to handle because the contour C is a tricky infinite rectangle in the right-hand-half of the s-plane. Referred to as 'Bromwich integrals' the evaluation of these is beyond our present course. To circumvent this difficulty we resort firstly to a **library of transforms** (see Handout 7) for the standard functions and secondly to ways of piecing combinations of these together for those not in the list.

1.2 Library of Laplace Transforms

1. **The constant function** $f(t) = 1$ **:**

$$\boxed{f(t) = 1\,; \qquad \overline{f}(s) = \frac{1}{s} \qquad \mathrm{Re}\, s > 0} \tag{1.3}$$

Proof :

$$\overline{f}(s) = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}\,, \tag{1.4}$$

provided $\mathrm{Re}\, s > 0$.

2. **The exponential-function** $f(t) = e^{at}$ **:**

$$\boxed{f(t) = \exp(at)\,; \qquad \overline{f}(s) = \frac{1}{s-a}\,; \qquad \text{Re}\, s > a} \tag{1.5}$$

Proof :

$$\overline{f}(s) = \int_0^\infty e^{-(s-a)t}\,dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}\,, \tag{1.6}$$

provided Re$(s-a) > 0$.

3. **The sine function :**

$$\boxed{f(t) = \sin(at)\,; \qquad \overline{f}(s) = \frac{a}{s^2+a^2}\,; \qquad \text{Re}\, s > 0} \tag{1.7}$$

Proof : Take both the sine and cosine functions in combination : $\cos at + i \sin at = e^{iat}$

$$\mathcal{L}\big(e^{iat}\big) = \int_0^\infty e^{-(s-ia)t}dt = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} \tag{1.8}$$

provided Re$(s) > 0$. Then the imaginary (real) part gives the result for sine (cosine).

4. **The cosine function**

$$\boxed{f(t) = \cos(at)\,; \qquad \overline{f}(s) = \frac{s}{s^2+a^2}\,; \qquad \text{Re}\, s > 0} \tag{1.9}$$

5. **The polynomial function** $f(t) = t^n$ **:**

$$\boxed{f(t) = t^n; \quad \overline{f}(s) = \frac{n!}{s^{n+1}}\,; \qquad (n \geq 0)\,; \qquad \text{Re}\, s > 0} \tag{1.10}$$

Proof : Define the LT as $\overline{f}(s) = I_n$ as

$$\begin{aligned} I_n &= \int_0^\infty e^{-st}t^n\,dt = -\frac{1}{s}\int_0^\infty t^n\,d\big[e^{-st}\big] \\ &= \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt = \frac{n}{s}I_{n-1} \end{aligned} \tag{1.11}$$

provided Re$(s) > 0$. With $n = 0$ and $\mathcal{L}[1] = s|^{-1}$ we obtain $I_1 = s^{-2}$ and end up with

$$\overline{f}(s) = I_n = \frac{n!}{s^{n+1}}\,. \tag{1.12}$$

6. **The Heaviside function :**

$$\boxed{f(t) = H(t-t_0)\,; \qquad \overline{f}(s) = \frac{\exp(-st_0)}{s}\,; \qquad \text{Re}\, s > 0} \tag{1.13}$$

Proof : For Re $s > 0$

$$\begin{aligned} \mathcal{L}\big[H(t-t_0)\big] &= \int_0^\infty e^{-st}H(t-t_0)\,dt \\ &= \int_{t_0}^\infty e^{-st}dt = \frac{e^{-st_0}}{s}\,. \end{aligned} \tag{1.14}$$

7. **The Dirac δ-function:**

$$\boxed{f(t) = \delta(t - t_0)\,; \qquad \overline{f}(s) = \exp(-st_0)\,; \qquad t_0 \geq 0} \tag{1.15}$$

Proof: t_0 needs to reside within the positive range of t

$$\int_0^\infty e^{-st}\delta(t - t_0)\,dt = \begin{cases} e^{-st_0} & t_0 \geq 0\,, \\ 0 & t_0 < 0\,. \end{cases} \tag{1.16}$$

8. **Shift theorem:**

$$\boxed{\mathcal{L}\,[\exp(at)f(t)] = \overline{f}(s - a)} \tag{1.17}$$

Proof: Provided Re$(s - a) > 0$

$$\begin{aligned} \mathcal{L}\,[\exp(at)f(t)] &= \int_0^\infty e^{-(s-a)t}f(t)\,dt \\ &= \overline{f}(s - a)\,. \end{aligned} \tag{1.18}$$

9. **Second shift theorem:**

$$\boxed{\mathcal{L}\,[H(t - a)f(t - a)] = \exp(-sa)\,\overline{f}(s)} \tag{1.19}$$

Proof: let $\tau = t - a$. Then

$$\begin{aligned} \mathcal{L}\,[H(t - a)f(t - a)] &= \int_0^\infty e^{-st}H(t - a)f(t - a)\,dt \\ &= e^{-sa}\int_{-a}^\infty e^{-s\tau}H(\tau)f(\tau)\,d\tau \\ &= e^{-sa}\int_0^\infty e^{-s\tau}f(\tau)\,d\tau = e^{-sa}\overline{f}(s)\,. \end{aligned} \tag{1.20}$$

10. **Convolution theorem:**

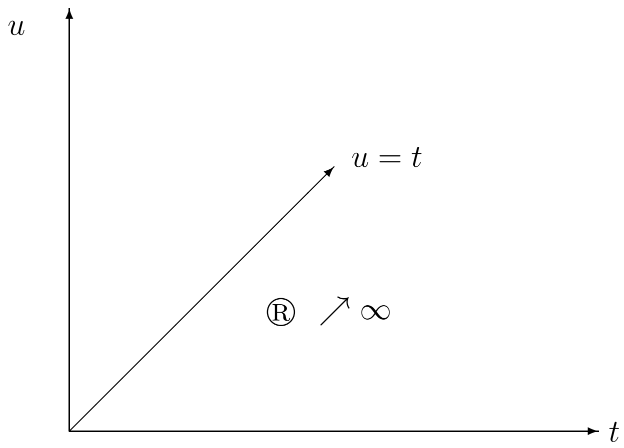
$$\boxed{\mathcal{L}\,\{f \star g\} = \overline{f}(s)\,\overline{g}(s)} \tag{1.21}$$

where the convolution between two functions $f(t)$ and $g(t)$ is defined as

$$f \star g = \int_0^t f(u)g(t - u)\,du\,. \tag{1.22}$$

Note that the convolution is over $[0,\, t]$ and not $[-\infty,\, \infty]$ as for the FT. The convolution integral on the RHS can also be written with f and g reversed: that is $\int_0^t f(t-u)g(u)\,du$.

Proof: The LT of the convolution product in (??) is written down and then the order of the integrals is exchanged, as in the figure, using $\tau = t - u$



The region of integration (Ⓡ in the figure) is obtained from the original integral: for the inner integral, the u-integration is taken in the vertical direction in Ⓡ, with $u = 0 \ldots t$ and then the outer integral has $t = 0 \ldots \infty$. To cover the area Ⓡ in the reverse order, the inner, t-integration is taken in the horizontal direction, with $t = u \ldots \infty$ and the outer integral has $u = 0 \ldots \infty$.

$$
\begin{aligned}
\mathcal{L}\big(f \star g\big) &= \int_0^\infty e^{-st} \left(\int_0^t f(u) g(t-u)\, du \right) dt \\
&= \int_0^\infty \left(\int_{t=u}^{t=\infty} e^{-st} g(t-u)\, dt \right) f(u)\, du \\
&\quad \text{now make the change of variables: } \tau = t - u \\
&= \int_0^\infty e^{-su} \left(\int_{\tau=0}^{\tau=\infty} e^{-s\tau} g(\tau)\, d\tau \right) f(u)\, du \\
&= \overline{f}(s)\, \overline{g}(s)\,. \qquad (1.23)
\end{aligned}
$$

11. **Integral :**

$$\boxed{\mathcal{L}\left(\int_0^t f(u)\, du \right) = \frac{\overline{f}(s)}{s}} \qquad (1.24)$$

Proof : The integral in (??) is a convolution product between $f(t)$ and $g(t) = 1$. Thus $\overline{g}(s) = 1/s$, giving the result from (??).

12. **Derivative :**

$$\boxed{\mathcal{L}\left[f'(t)\right] = s\overline{f}(s) - f(0)} \qquad (1.25)$$

Proof: Noting that $f(0)$ means $f(t=0)$

$$\begin{aligned}
\mathcal{L}[\dot{f}] &= \int_0^\infty e^{-st}\dot{f}\,dt \\
&= \int_0^\infty e^{-st}\,df = \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f\,dt \\
&= s\overline{f}(s) - f(0) \qquad (1.26)
\end{aligned}$$

provided $\mathrm{Re}\,s > 0$.

13. **Second derivative:** Noting that $f'(0)$ means $f'(t=0)$

$$\boxed{\mathcal{L}\left[f''(t)\right] = s^2\overline{f}(s) - sf(0) - f'(0)} \qquad (1.27)$$

Proof:

$$\begin{aligned}
\mathcal{L}\left[f''\right] &= \int_0^\infty e^{-st}f''\,dt \\
&= \int_0^\infty e^{-st}\,df' = \left[e^{-st}f'(t)\right]_0^\infty + s\int_0^\infty e^{-st}f'\,dt \\
&= s\mathcal{L}\left[f'\right] - f'(0) \\
&= s^2\overline{f}(s) - sf(0) - f'(0)\,, \qquad (1.28)
\end{aligned}$$

provided $\mathrm{Re}\,s > 0$.

1.3 Using the Convolution Theorem to find inverses

If we are given an inverse LT as a function $\overline{F}(s)$ which is too complicated to appear in the Library above but can be split into composite functions $\overline{F}(s) = \overline{f}(s)\,\overline{g}(s)$ where $\overline{f}(s)$ and $\overline{g}(s)$ do belong to the Library, then the Convolution Theorem allows us to write

$$F(t) = \mathcal{L}^{-1}\left(\overline{f}(s)\,\overline{g}(s)\right) = f(t)\star g(t)\,. \qquad (1.29)$$

Example 1: Find $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right]$. We identify $f(s) = s^{-1}$ and $g(s) = (s^2+1)^{-1}$. The Library tell us that $f(t) = 1$ and $g(t) = \sin t$. Thus

$$F(t) = 1\star\sin t = \int_0^t \sin u\,du = 1 - \cos t\,. \qquad (1.30)$$

Example 2: Find $\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$. Identify

$$\overline{f}(s) = \frac{s}{s^2+a^2} \qquad\qquad \overline{g}(s) = \frac{1}{s^2+a^2}\,. \qquad (1.31)$$

The Library tell us that $f(t) = \cos at$ and $g(t) = a^{-1}\sin at$, and so

$$F(t) = a^{-1}\sin at\star\cos at = a^{-1}\int_0^t \sin(au)\cos a(t-u)\,du\,. \qquad (1.32)$$

Using $\sin(A+B)+\sin(A-B) = 2\sin A\cos B$ we find

$$\sin(au)\cos a(t-u) = \tfrac{1}{2}\left[\sin at + \sin a(2u-t)\right] \tag{1.33}$$

and so from (??)

$$\begin{aligned} F(t) &= \frac{1}{2a}\int_0^t \left[\sin(at) + \sin a(2u-t)\right]\,du \\ &= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}\big\{\cos at - \cos at\big\}\right] \\ &= \frac{t}{2a}\sin at\,. \end{aligned} \tag{1.34}$$

Example 3: Find $\mathcal{L}^{-1}\left[\frac{a^2}{(s^2+a^2)^2}\right]$. Identify $F(s) = |\overline{f}(s)|^2$ where

$$\overline{f}(s) = \frac{a}{s^2+a^2} \qquad\qquad \overline{g}(s) = \overline{f}(s)\,. \tag{1.35}$$

The Library tell us that $f(t) = g(t) = \sin at$ so $\sin at$ is convolved with itself

$$\begin{aligned} F(t) &= \sin at \star \sin at \\ &= \int_0^t \sin au \sin a(t-u)\,du \\ &= \frac{1}{2a}\left[\sin at - at\cos at\right]\,. \end{aligned} \tag{1.36}$$

having used the trig-identity $\cos(A-B) - \cos(A+B) = 2\sin A\sin B$.

1.4 Examples involving partial fractions and the Shift theorem

Example 1: Find $f(t)$ when

$$\overline{f}(s) = \frac{6s^2+10s+2}{s(s^2+3s+2)}\,. \tag{1.37}$$

Noting that $s^2+3s+2 = (s+1)(s+2)$ (??) can be split by Partial Fractions (PFs) into

$$\overline{f}(s) = \frac{6s^2+10s+2}{s(s+1)(s+2)} = \frac{1}{s} + \frac{2}{s+1} + \frac{3}{s+2}\,. \tag{1.38}$$

Thus, using the Library

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{2}{s+1} + \frac{3}{s+2}\right) \\ &= 1 + 2e^{-t} + 3e^{-2t}\,. \end{aligned} \tag{1.39}$$

Example 2: Find $f(t)$ when

$$\overline{f}(s) = \frac{2}{s(s-2)}\,. \tag{1.40}$$

in which case

$$\overline{f}(s) = -\frac{1}{s} + \frac{1}{s-2}\,, \tag{1.41}$$

and so

$$f(t) = -1 + e^{2t}\,. \tag{1.42}$$

Example 3 : Find $f(t)$ when $\overline{f}(s) = (s-1)^{-4}$. From the Library,

$$\mathcal{L}\left[t^3\right] = \frac{3!}{s^4} \tag{1.43}$$

therefore $\mathcal{L}^{-1}\left[s^{-4}\right] = t^3/6$. With the application of the Shift Theorem with $a = 1$ we have

$$\mathcal{L}^{-1}\left[(s-1)^{-4}\right] = \frac{1}{6}t^3e^t\,. \tag{1.44}$$

1.5 Solving ODEs using Laplace Transforms

Many textbook methods are given to solve 2nd order ODEs of the type

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x = f(t)\,, \tag{1.45}$$

but only the LT-method can handle those cases when the forcing function is not smooth. Examples might be voltage inputs of the square wave or saw-tooth type. To approach this using LTs, the transform is taken of (??)

$$\left(s^2\overline{x}(s) - sx_0 - \dot{x}_0\right) + \alpha\left(s\overline{x}(s) - x_0\right) + \omega_0^2\,\overline{x}(s) = \overline{f}(s)\,. \tag{1.46}$$

where $x_0 = x(0)$ and $\dot{x}_0 = \dot{x}(0)$. This re-organizes into

$$\left(s^2 + \alpha s + \omega_0^2\right)\overline{x}(s) = \overline{f}(s) + (s+\alpha)x_0 + \dot{x}_0\,. \tag{1.47}$$

Note that the final expression for $\overline{x}(s)$ divides conveniently into two parts corresponding to the Complementary Function and the Particular Integral

$$\overline{x}(s) = \underbrace{\frac{\overline{f}(s)}{s^2 + \alpha s + \omega_0^2}}_{P.I.} + \underbrace{\frac{(s+\alpha)x_0 + \dot{x}_0}{s^2 + \alpha s + \omega_0^2}}_{C.F.} \tag{1.48}$$

The initial conditions appear in x_0 and $\dot{x}_0$ as part of the Complementary Function. How to take the inverse depends on whether the denominator has real or complex roots. These we consider by example.

Example 1 : Solve $\ddot{x} + \dot{x} - 2x = e^t$ with $x_0 = 3$ and $\dot{x}_0 = 0$.

(??) becomes

$$\left(s^2 + s - 2\right)\overline{x}(s) = \frac{1}{s-1} + 3(s+1)\,. \tag{1.49}$$

Noting that $s^2 + s - 2 = (s-1)(s+2)$ we have

$$\overline{x}(s) = \underbrace{\frac{1}{(s-1)^2(s+2)}}_{P.I.} + \underbrace{\frac{3(s+1)}{(s-1)(s+2)}}_{C.F.} \tag{1.50}$$

Using PFs

$$\overline{x}(s) = \frac{1}{3(s-1)^2} + \frac{17}{9(s-1)} + \frac{10}{9(s+2)} \tag{1.51}$$

and so the Library gives us

$$x(t) = \frac{1}{3}te^t + \frac{17}{9}e^t + \frac{10}{9}e^{-2t}\,. \tag{1.52}$$

Example 2 : Solve $\ddot{x} + 16x = \sin 2t$ with $x_0 = 0$ and $\dot{x}_0 = 1$.
(??) becomes

$$\left(s^2+16\right)\overline{x}(s) = 1 + \frac{2}{s^2+4}\,, \tag{1.53}$$

and so

$$\begin{aligned} \overline{x}(s) &= \frac{1}{s^2+16} + \frac{2}{(s^2+4)(s^2+16)} \\ &= \frac{5}{6(s^2+16)} + \frac{1}{6(s^2+4)} \\ &= \frac{5}{24}\left(\frac{4}{s^2+4^2}\right) + \frac{1}{12}\left(\frac{2}{s^2+2^2}\right)\,. \end{aligned} \tag{1.54}$$

Therefore, from the Library

$$x(t) = \frac{5}{24}\sin 4t + \frac{1}{12}\sin 2t\,. \tag{1.55}$$

Example 3 (real roots) : Solve $\ddot{x} + 3\dot{x} + 2x = f(t)$ with $x_0 = 1$ and $\dot{x}_0 = -2$. In this example $f(t)$ has not been specified although it must be assumed that its LT exists.
We obtain

$$\overline{x}(s) = \frac{\overline{f}(s)}{s^2+3s+2} + \frac{x_0(s+3)+\dot{x}_0}{s^2+3s+2} \tag{1.56}$$

so from (??) with the specified initial conditions

$$\overline{x}(s) = \frac{\overline{f}(s)}{(s+1)(s+2)} + \frac{1}{s+2}\,. \tag{1.57}$$

Using PFs we find

$$\begin{aligned} \overline{x}(s) &= \frac{\overline{f}(s)}{s+1} - \frac{\overline{f}(s)}{s+2} + \frac{1}{s+2} \\ &\equiv \overline{f}(s)\overline{g}_1(s) - \overline{f}(s)\overline{g}_2(s) + \overline{g}_2(s) \end{aligned} \tag{1.58}$$

where $\overline{g}_1(s) = (s+1)^{-1}$ and $\overline{g}_2(s) = (s+2)^{-1}$. From these definitions it is clear that $g_1(t) = e^{-t}$ and $g_2(t) = e^{-2t}$. From the Convolution Theorem we have

$$x(t) = \underbrace{\int_0^t \left[e^{-(t-u)} - e^{-2(t-u)}\right] f(u)\, du}_{P.I.} + \underbrace{e^{-2t}}_{C.F.}\,. \tag{1.59}$$

The power of the LT-method can be seen here in that it solves, in principle, an ODE with any forcing, provided $\overline{f}(s)$ exists.

Example 4 (complex roots) : Solve $\ddot{x} + 2\dot{x} + 2x = f(t)$ with $x_0 = 1$ and $\dot{x}_0 = 0$. In this example $f(t)$ has not been specified although it must be assumed that its LT exists.
From (??) the next step comes out to be

$$\overline{x}(s) \quad = \quad \frac{\overline{f}(s)}{(s+1)^2+1} + \frac{s+2}{(s+1)^2+1} \tag{1.60}$$

where it has been noted that $s^2 + 2s + 2$ does not have real roots. Now define

$$\overline{g}_1(s) = \frac{1}{(s+1)^2+1} \qquad\qquad \overline{g}_2(s) = \frac{s+1}{(s+1)^2+1}\,. \tag{1.61}$$

Therefore $\overline{x}(s)$ can be re-expressed as

$$\overline{x}(s) = \overline{f}(s)\overline{g}_1(s) + \overline{g}_2(s) + \overline{g}_1(s)\,. \tag{1.62}$$

Inverse transforms can be found from the Shift Theorem and the Library

$$g_1(t) = e^{-t}\sin t \qquad\qquad g_2(t) = e^{-t}\cos t\,. \tag{1.63}$$

The Convolution Theorem gives the final result

$$x(t) = \int_0^t f(t-u)e^{-u}\sin u\, du + e^{-t}\left[\cos t + \sin t\right]\,. \tag{1.64}$$