Vector Calculus Solutions to Exercises

- Exercises have been classified according to the topics:
- Gradient, Divergence and Curl of a Vector Field
- Line/Path Integrals & Path Independence
- Green's Theorem in the Plane
- Surface Integrals & Fluxes
- Divergence & Stokes' Theorem

Before attempting these exercises, you are strongly encouraged to go through the relevant portion of the notes and be familiar with the solutions to the examples presented in the notes.

1. Gradient, Divergence and Curl

Question 1: If $f(x,y) = e^{xy}\sin(x+y)$, then

i. in what direction, starting at $\left(0,\frac{\pi}{2}\right)$, is the function $f(x,y)$ changing the fastest?

ii. in what directions, starting at $\left(0,\frac{\pi}{2}\right)$, is the function $f(x,y)$ changing at 50% of its maximum rate?

iii. find the directional derivative of $f\left(0,\frac{\pi}{2}\right)$ in the direction of vector $\underline{i} - 2\underline{j}$.

i. Recall that the fastest/maximum rate of change occurs in the direction of $\underline{\nabla f}\left(0,\frac{\pi}{2}\right)$. Then, we have

$$\underline{\nabla f} = [ye^{xy}\sin(x+y) + e^{xy}\cos(x+y)]\,\underline{i} + [x\sin(x+y) + e^{xy}\cos(x+y)]\,\underline{j} \quad \rightarrow \quad \underline{\nabla f}\left(0,\frac{\pi}{2}\right) = \frac{\pi}{2}\underline{i}$$

ii. We need to find unit direction vector $\underline{b}$. Then, recall that the directional derivative $D_b f = \underline{\nabla f} \cdot \underline{b} = \left|\underline{\nabla f}\right| \cos(\theta)$ where θ is the angle between $\underline{\nabla f}$ and $\underline{b}$. So, $D_b f\left(0,\frac{\pi}{2}\right) = \frac{\pi}{2}\cos(\theta)$. We have maximum rate of change when $\theta = 0$ (as expected). So, 50% of maximum rate of change means $\cos(\theta) = \frac{1}{2} \quad \rightarrow \quad \theta = \pm\frac{\pi}{3}$.

So this means that the angle that the unit direction vector $\underline{b}$ makes with $\underline{\nabla f}$ is $= \pm\frac{\pi}{3}$.

1. Gradient, Divergence and Curl

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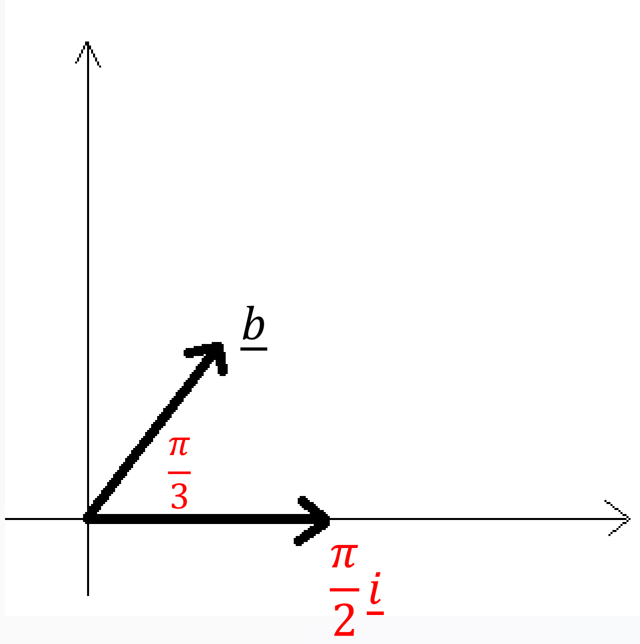
iii. find the directional derivative of $f\left(0,\frac{\pi}{2}\right)$ in the direction of vector $\underline{i} - 2\underline{j}$.

Since $\underline{\nabla f}\left(0,\frac{\pi}{2}\right) = \frac{\pi}{2}\underline{i}$, one of the unit vectors $\underline{b}$ will look like the diagram on the right. So the required vectors are $\cos(\frac{\pi}{3})\underline{i} \pm \sin(\frac{\pi}{3})\underline{j}$.

Note that these are already of unit length. Thus, we have $\frac{1}{2}\underline{i} \pm \frac{\sqrt{3}}{2}\underline{j}$.



iii. The required unit direction vector is then $\underline{b} = \frac{1}{\sqrt{5}}\ \underline{(i - 2j)}$.

Then, $D_b f = \underline{\nabla f} \cdot \underline{b} = \frac{\pi}{2\sqrt{5}}$.

1. Gradient, Divergence and Curl

Question 2: A solenoidal vector field is defined as one which has zero divergence.

(a) For what v_3 will the vector field $\underline{v} = e^x\cos(y)\underline{i} + e^x\sin(y)\underline{j} + v_3\underline{k}$ be solenoidal? Explain your reasoning.

(b) For what v_3 will the vector field $\underline{v} = e^x\sin(y)\underline{i} + e^x\cos(y)\underline{j} + v_3\underline{k}$ be irrotational? Explain your reasoning.

(a) solenoidal means $\nabla \cdot \underline{v} = e^x\cos(y) + e^x\cos(y) + \frac{\partial v_3}{\partial z} = 0 \rightarrow \quad \frac{\partial v_3}{\partial z} = -[e^x\cos(y) + e^x\cos(y)]$

$$\rightarrow v_3 = -2ze^x\cos(y) + \text{constant}$$

The constant featured above can either be a real number or a function of x and/or y.

(b) irrotational means $\nabla \times \underline{v} = \left[\frac{\partial v_3}{\partial y} - 0\right]\underline{i} - \left[\frac{\partial v_3}{\partial x} - 0\right]\underline{j} + [e^x\cos(y) - e^x\cos(y)]\underline{k} = \underline{0}$

$$\rightarrow \frac{\partial v_3}{\partial y} = \frac{\partial v_3}{\partial x} = 0 \quad \rightarrow \quad v_3 = f(z)$$

v_3 can be any function of z.

1. Gradient, Divergence and Curl

Question 3: Evaluate $\nabla \cdot (\nabla \times \underline{F})$ for $\underline{F} = x^2\, \underline{i} + y^2\, \underline{j} - 2xz\underline{k}$. For the vector fields $\underline{F}$ and $\nabla \times \underline{F}$, do we have dispersion, concentration or neither of the two? Explain your answer.

$$\nabla \cdot \underline{F} = 2x + 2y - 2x = 2y$$

$$\nabla \times \underline{F} = (0)\underline{i} - (-2z - 0)\underline{j} + (0)\underline{k} = 2z\,\underline{j}$$

$$\rightarrow \quad \nabla \cdot \left(\nabla \times \underline{F}\right) = \frac{\partial}{\partial y}(2z) = 0$$

For $\underline{F}$, we have dispersion for $y > 0$ and concentration for $y < 0$. For $\nabla \times \underline{F}$, we have neither dispersion nor concentration.

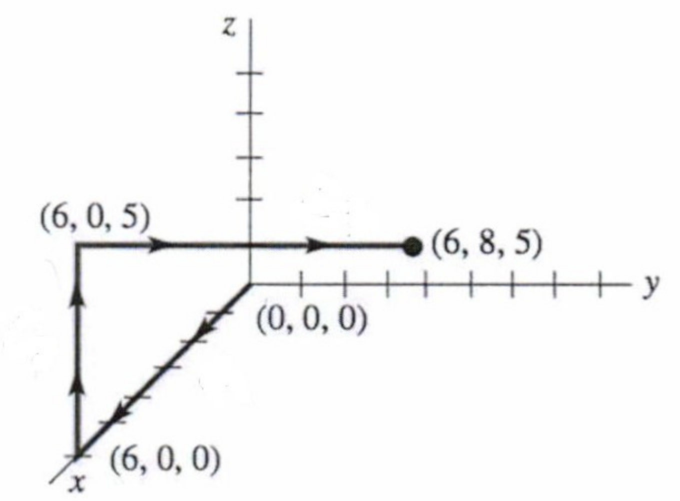
Question 4: Obtain $\varphi(x, y, z)$ which is the potential associated with the vector field $\underline{F} = -yz\,\underline{i} - xz\,\underline{j} - xy\,\underline{k}$.

$$\underline{F} = \underline{\nabla\varphi} = \frac{\partial\varphi}{\partial x}\,\underline{i} + \frac{\partial\varphi}{\partial y}\,\underline{j} + \frac{\partial\varphi}{\partial z}\,\underline{k} = -yz\,\underline{i} - xz\,\underline{j} - xy\,\underline{k}$$

Then this leads to the equations after integrating: $\varphi = -xyz + a(y, z)$; $\varphi = -xyz + b(x, z)$; $\varphi = -xyz + c(x, y)$

2. Line/Path Integrals & Path Independence

Question 1: Evaluate the line integral $I = \int_{(0,0,0)}^{(6,8,5)} ydx + zdy + xdz$ along the arrowed path shown between (0,0,0) and (6,8,5).



path is given by $C = C_1 \cup C_2 \cup C_3$ where $C_1: x = 0$ to 6 $(y = z = 0)$,

$C_2: z = 0$ to 5 $(x = 6, y = 0)$, $C_3: y = 0$ to 8 $(x = 6, z = 5)$

$$I = \int_{C_1} ydx + zdy + xdz + \int_{C_2} ydx + zdy + xdz + \int_{C_3} ydx + zdy + xdz$$

$$I = \int_0^6 (0)dx + (0)0 + (x)0 + \int_0^5 (0)0 + (z)0 + (6)dz + \int_0^8 (y)0 + (5)dy + (6)0 = 0 + (6)5 + (5)8 = 70$$

Question 2: Find the work done by the force field $\underline{F} = yz\,\underline{i} + xz\,\underline{j} + xy\,\underline{k}$ acting along the curve given by $\underline{r}(t) = t^3\underline{i} + t^2\underline{j} + t\underline{k}$ from $t = 1$ to $t = 3$.

We should parametrize everything in terms of t. So what is needed is to convert the force field appropriately.

The position vector is always $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$. Thus, $x = t^3$, $y = t^2$ and $z = t$ and $\underline{F} = t^3\,\underline{i} + t^4\,\underline{j} + t^5\,\underline{k}$

$$W = \int_A^B \underline{F} \cdot \underline{dr} = \int_A^B \underline{F} \cdot \frac{\underline{dr}}{dt} dt = \int_1^3 [t^3 \cdot 3t^2 + t^4 \cdot 2t + t^5 \cdot 1]dt = [t^6]_{t=1}^{t=3} = 728$$

2. Line/Path Integrals & Path Independence

Question 3: Given the force field $\underline{F} = -yz\,\underline{i} - xz\,\underline{j} - xy\,\underline{k}$, find the work done by this force field to bring a particle from $A(1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$ to $B(1,0,0)$. Explain your reasoning.

The “problem” with this question is that we are not given the path. So we want to check if the force is conservative. If it is, then the work done is not dependent on the path but only on the two end points.

A conservative force means that we can write $\underline{F} = \underline{\nabla f}$ for some potential function f.

So, the equations are $\frac{\partial f}{\partial x} = -yz$, $\frac{\partial f}{\partial y} = -xz$ and $\frac{\partial f}{\partial z} = -xy$. These lead to $f = -xyz$ which means that the force field is conservative.

$$W = \int_A^B \underline{F} \cdot \underline{dr} = \int_{(1/\sqrt{2},1/\sqrt{2},1/\sqrt{2})}^{(1,0,0)} \underline{\nabla f} \cdot \underline{dr} = f(1,0,0) - f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}$$

2. Line/Path Integrals & Path Independence

Question 4: Show that the path integral $\int_{(1,0,2)}^{(2,\pi,3)}(2x\cos(y)+2e^{3z})dx+(-x^2\sin(y)+1)dy+(6xe^{3z}+2)dz$ is path independent of path and evaluate the integral.

First, we recognize that $P=2x\cos(y)+2e^{3z},\ Q=-x^2\sin(y)+1,\ R=6xe^{3z}+2$. To prove that the integral is path independent (or not), we need to see if we can find $f(x,y,z)$ such that

$$P=\frac{\partial f}{\partial x}=2x\cos(y)+2e^{3z}\ \rightarrow\ f=x^2\cos(y)+2xe^{3z}+a(y,z)$$

$$Q=\frac{\partial f}{\partial y}=-x^2\sin(y)+1\ \rightarrow\ f=x^2\cos(y)+y+b(x,z)$$

$$R=\frac{\partial f}{\partial z}=6xe^{3z}+2\ \rightarrow\ f=2xe^{3z}+2z+c(x,y)$$

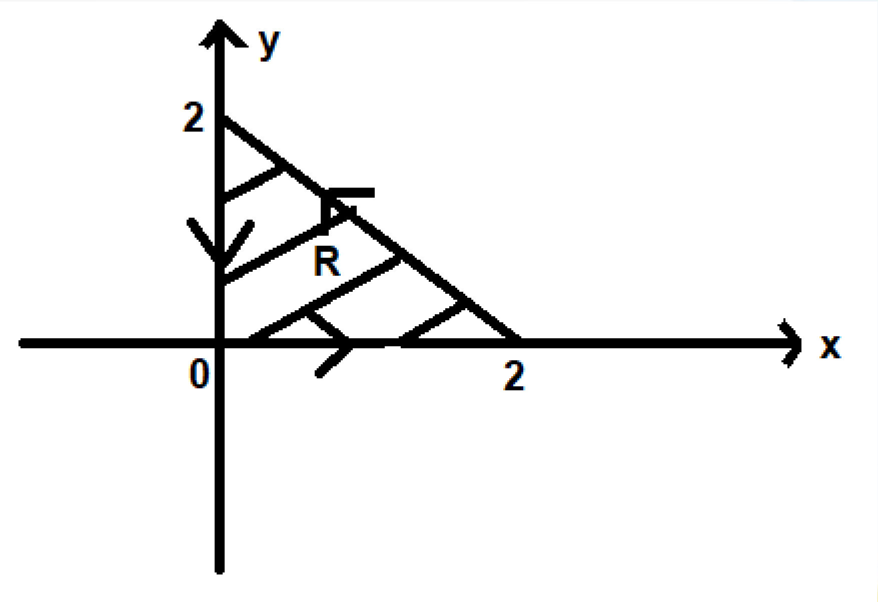
If $a(y,z)=y+2z$, $b(x,z)=2xe^{3z}+2z$ and $c(x,y)=x^2\cos(y)+y$, then $f=x^2\cos(y)+2xe^{3z}+y+2z$. We then have

$$\int_{(1,0,2)}^{(2,\pi,3)}(2x\cos(y)+2e^{3z})dx+(-x^2\sin(y)+1)dy+(6xe^{3z}+2)dz=\int_{(1,0,2)}^{(2,\pi,3)}\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=\int_{(1,0,2)}^{(2,\pi,3)}df$$

$$=f(2,\pi,3)-f(1,0,2)=4e^9-2e^6+\pi-3$$

3. Green's Theorem in the Plane

Question 1: Use Green's Theorem to evaluate $\oint 2y^2dx + x^2ydy$ where C comprises the boundary of the shaded region R with a counter-clockwise orientation. What is the work done by the force field $\underline{F} = 2y^2\,\underline{i} + x^2y\,\underline{j}$ over the path C and clockwise orientation.



$$\oint Pdx + Qdy = \oint 2y^2dx + x^2ydy = \int_0^2 \int_0^{-x+2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dydx$$

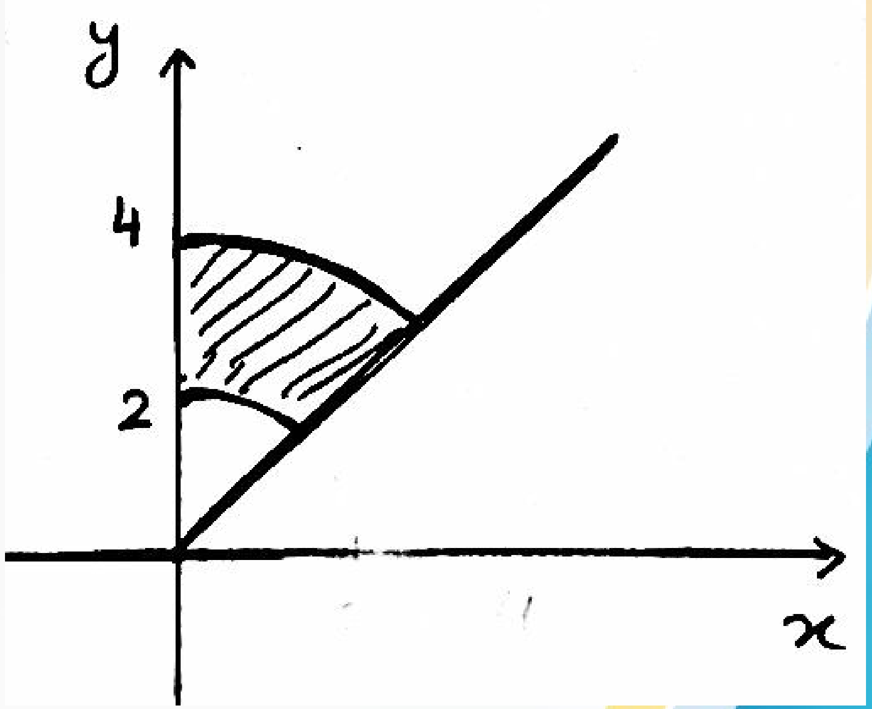
$$= \int_0^2 \int_0^{-x+2} (2xy - 4y)dydx = \int_0^2 [xy^2 - 2y^2]_{y=0}^{y=-x+2} dx$$

$$= \int_0^2 [x(x-2)^2 - 2(x-2)^2]dx = \int_0^2 (x-2)^3 dx = \left[\frac{1}{4}(x-2)^4\right]_0^2 = -4$$

Since the work done by the force field is the same as path integral (except for orientation), all we need to do is to just reverse the sign. Thus, the required work done by the force field is +4.

3. Green's Theorem in the Plane

Question 2: Use Green's Theorem to evaluate $\oint(2y^2+\tan(2x^2))dx+(3x^2+4xy)dy$ where C is the positively oriented boundary of the region bounded by $x=0,\ y=x,\ x^2+y^2=4\ ,\ x^2+y^2=16$.



$$\oint Pdx+Qdy=\oint(2y^2+\tan(2x^2))dx+(3x^2+4xy)dy$$

$$=\iint_R(6x+4y-4y)dydx=\int_{\pi/4}^{\pi/2}\int_2^4(6r\cos(\theta))rdrd\theta$$

You can see that we can convert to polar coordinates: $dA=rdrd\theta$

The line $y=x$ forms an angle of $\pi/4$ with respect to the x-axis, so the angle range is from $\pi/4$ to $\pi/2$.

$$\oint Pdx+Qdy=\int_{\pi/4}^{\pi/2}\int_2^4(6r\cos(\theta))rdrd\theta=[\sin(\theta)]_{\theta=\pi/4}^{\theta=\pi/2}\cdot[2r^3]_{r=2}^{r=4}=56(2-\sqrt{2})$$

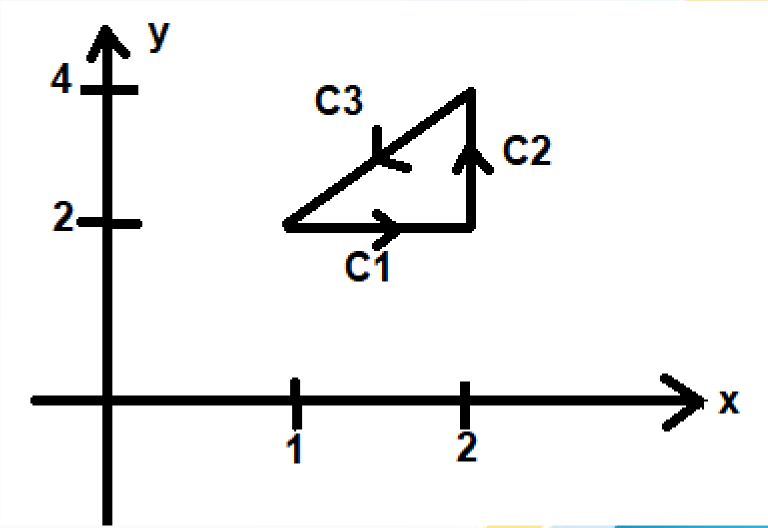
3. Green's Theorem in the Plane

Question 3: Evaluate $\oint_C \; 2xydx + 3xy^2dy$ as a line integral where C is the counter-clockwise triangular path formed by the vertices (1,2), (2,2) and (2,4). Confirm your computed result using Green's Theorem.

$$\oint_C \; 2xydx + 3xy^2dy = \int_{C1} 2xydx + 3xy^2dy + \int_{C2} 2xydx + 3xy^2dy + \int_{C3} 2xydx + 3xy^2dy$$

$$= \int_1^2 2x(2)dx + \int_2^4 3(2)y^2dy + \int_2^1 2x(2x)dx + 3x(2x)^2(2dx)$$

C1: $y = 2,\ x = 1 \to 2$. C2: $x = 2,\ y = 2 \to 4$. C3: $y = 2x \to dy = 2dx,\ \ x = 2 \to 1$



$$\oint_C \; 2xydx + 3xy^2dy = [2x^2]_{x=1}^{x=2} + [2y^3]_{y=2}^{y=4} + \left[\frac{4}{3}x^3 + 6x^4\right]_{x=2}^{x=1} = \frac{56}{3}$$

Now we recall Green's theorem shown below. The enclosed region R is the triangle.

$$\oint_C \; Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dydx = \int_1^2 \int_2^{2x} (3y^2 - 2x)dydx = \int_1^2 [y^3 - 2xy]_{y=2}^{y=2x} dx$$

$$= \int_1^2 (8x^3 - 4x^2 + 4x - 8)dx = \left[2x^4 - \frac{4}{3}x^3 + 2x^2 - 8x\right]_{x=2}^{x=1} = \frac{56}{3}$$

4. Surface Integrals & Fluxes

Question 1: Find the surface area of the portion of the cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 1$.

The surface is given by $z = f(x, y) = \sqrt{x^2 + y^2}$. Surface Area is given by

$$\iint_S dS = \iint_D \left[\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\right] dxdy$$

where D is the projection of the surface on the xy-plane. The partial derivatives are given by

$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x$, $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y$. So, $1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 1 + \frac{x^2+y^2}{x^2+y^2} = 2$. So, $dS = \sqrt{2} dA$.

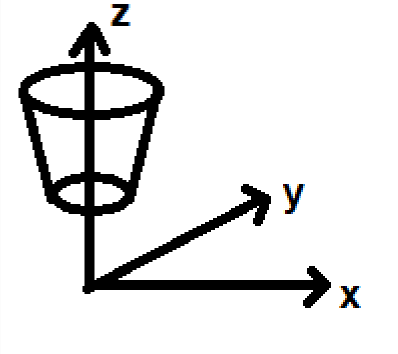
The projection of the relevant portion of the cone is just a circle of radius 1.

$$\iint_S dS = \iint_D \left[\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\right] dxdy = \sqrt{2} \cdot \pi(1)(1) = \pi\sqrt{2}$$

4. Surface Integrals & Fluxes

Question 2: Evaluate the surface integral $\iint_S y^2 z dS$ where S is the portion of the cone $z = \sqrt{x^2+y^2}$ that lies between the planes $z = 1$ and $z = 2$.

The surface is given by $z = f(x,y) = \sqrt{x^2+y^2}$ between 1 and 2, as shown on the right.



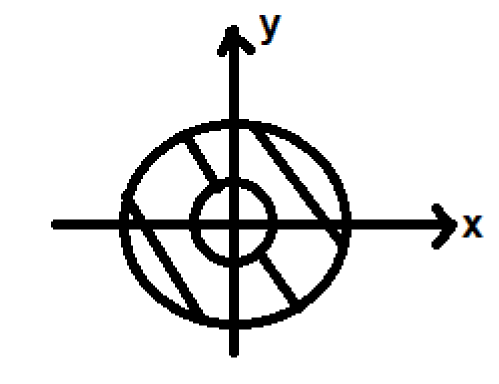
Surface integral is given by

$$\iint_S y^2 z dS = \iint_D y^2 z \left[\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\right] dxdy$$

where D is the projection of the surface on the xy-plane. The partial derivatives are given by

$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x$, $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y$. So, $1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 1 + \frac{x^2+y^2}{x^2+y^2} = 2$. So, $dS = \sqrt{2} dA$.

The projection of the relevant portion of the surface is the area between two circles of radius 1 and 2, as illustrated on the right. Use polar coordinates: $y = r\sin(\theta)$ where $r = 1 \to 2$ and $\theta = 0 \to 2\pi$.

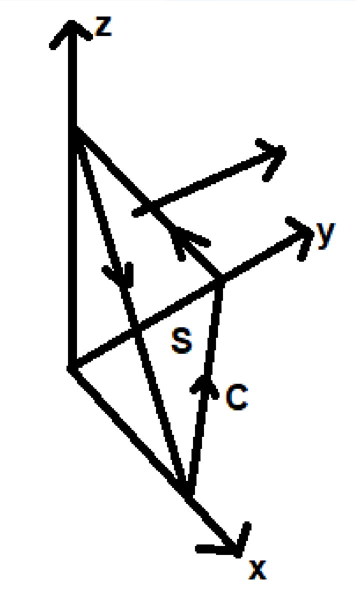


$$\iint_S y^2 z dS = \iint_D y^2 \sqrt{x^2+y^2}\sqrt{2} dxdy = \int_0^{2\pi}\int_1^2 (r\sin(\theta))^2 \cdot r \cdot \sqrt{2} \cdot r dr d\theta = \frac{31\sqrt{2}}{5}\pi$$

4. Surface Integrals & Fluxes

Question 3: Calculate the flux of the vector field $\underline{F} = 2yz\,\underline{i} + xy\,\underline{j} + xy\underline{k}$ through the surface S given by the plane

$x + y + z = 2$, found only in the first octant. Assume that the surface S has a positive orientation.



The surface is given by $z = f(x, y) = 2 - x - y$ and its unit normal $\hat{n} = \frac{1}{\sqrt{3}}(\underline{i} + \underline{j} + \underline{k})$ The flux is given by

$$\iint_S \underline{F}\cdot\hat{n}\,dS = \iint_S \frac{1}{\sqrt{3}}(2yz + xy + xy)\,dS = \iint_S \frac{1}{\sqrt{3}}[2y(2-x-y) + 2xy]dS = \iint_S \frac{1}{\sqrt{3}}[4y - 2y^2]dS$$

Now we see that we have a surface integral.

$$\iint_S \frac{1}{\sqrt{3}}[4y - 2y^2]dS = \iint_D \frac{1}{\sqrt{3}}[4y - 2y^2]\left[\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\right]dxdy$$

where D is the projection of the surface on the xy-plane and this is the a triangle: $x = 0 \to -y + 2$ and $y = 0 \to 2$.

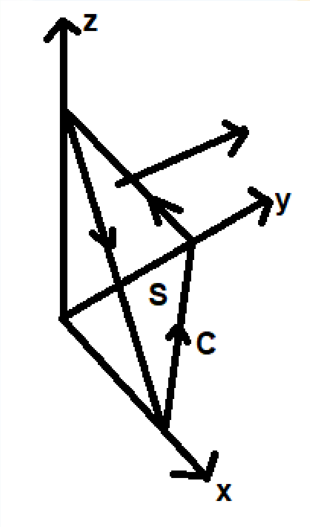
The partial derivatives are given by $\frac{\partial f}{\partial x} = -1$, $\frac{\partial f}{\partial y} = -1$. So, $1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 1 + 1 + 1 = 3$. So, $dS = \sqrt{3}dA$.

$$\iint_S \frac{1}{\sqrt{3}}[4y - 2y^2]dS = \iint_D \frac{1}{\sqrt{3}}[4y - 2y^2]\left[\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\right]dxdy = \int_0^2\int_0^{-x+2}[4y - 2y^2]dydx$$

$$= \int_0^2\int_0^{-y+2}[4y - 2y^2]dxdy = \int_0^2[4y(2-y) - 2y^2(2-y)]dy = \int_0^2[8y - 8y^2 + 2y^3]dy = \frac{8}{3}$$

5. Divergence & Stokes' Theorem

Question 1: Use Stokes Theorem to determine the work done by the force field $\underline{F} = 2yz\,\underline{i} + xy\,\underline{j} + xy\underline{k}$ along the path C traced out by the intersection of the plane $x + y + z = 2$ with the coordinate planes and oriented positively.



Stokes theorem states that

$$\oint_C \underline{F} \cdot \underline{dr} = \iint_S (\nabla \times \underline{F}) \cdot \hat{n}\, dS$$

First, $\nabla \times \underline{F} = x\underline{i} + y\,\underline{j} + (y - 2z)\underline{k}$. Next, the surface is given by $z = f(x, y) = 2 - x - y$ and its unit normal given by $\hat{n} = \frac{1}{\sqrt{3}}(\underline{i} + \underline{j} + \underline{k})$.

$$\oint_C \underline{F} \cdot \underline{dr} = \iint_S (\nabla \times \underline{F}) \cdot \hat{n}\, dS = \iint_S \frac{1}{\sqrt{3}}(x + y + y - 2z)\, dS = \iint_D \frac{1}{\sqrt{3}}(x + 2y - 2z)\left[\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\right] dxdy$$

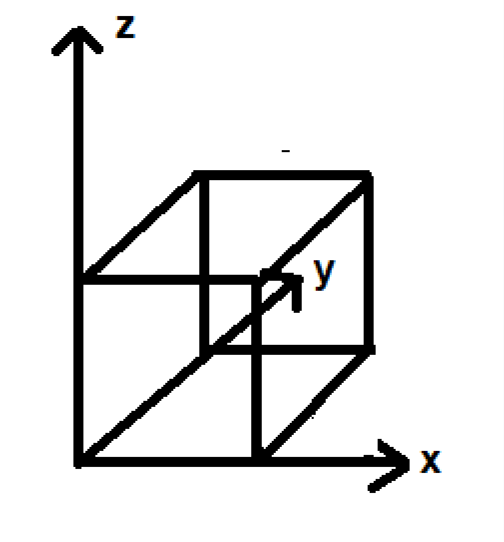
where D is the projection of the surface on the xy-plane and this is the a triangle: $y = 0 \to -x + 2$ and $x = 0 \to 2$.

The partial derivatives are given by $\frac{\partial f}{\partial x} = -1$, $\frac{\partial f}{\partial y} = -1$. So, $1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 1 + 1 + 1 = 3$. So, $dS = \sqrt{3}dA$.

$$\int_0^2 \int_0^{-x+2} [x + 2y - 2(2 - x - y)]dydx = \int_0^2 \int_0^{-x+2} [3x + 4y - 4]dydx = \frac{4}{3}$$

5. Divergence & Stokes' Theorem

Question 2: Let S be the surface of the unit cube D defined by $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$. If we have a force field $\underline{F} = x^2 y\, \underline{i} + xz\, \underline{j} + xyz\, \underline{k}$ then calculate the surface flux of this force field through the surface of the entire cube.



We can use the divergence theorem which states that

$$\iint_S \underline{F} \cdot \hat{n}\, dS = \iiint_V (\nabla \cdot \underline{F}) dV$$

where S is the bounding surface for the entire volume V. Then we have

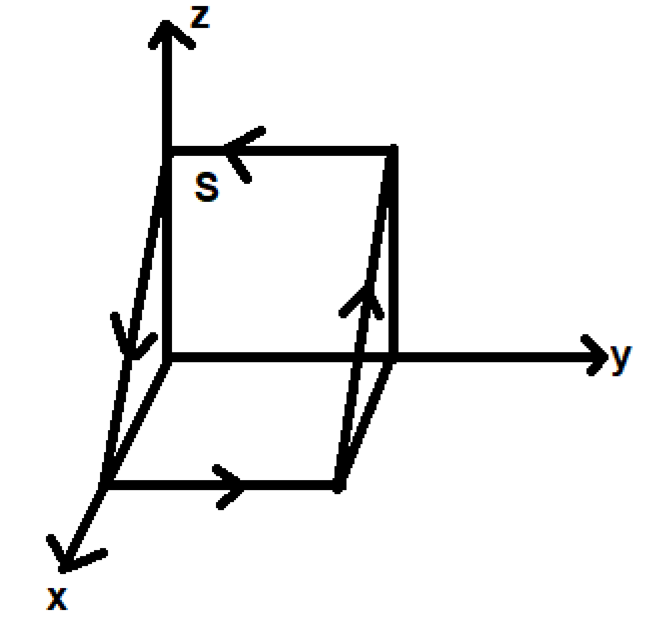
$$\iiint_V (\nabla \cdot \underline{F}) dV = \int_0^1 \int_0^1 \int_0^1 (2xy + 0 + xy) dzdydx = \int_0^1 \int_0^1 \int_0^1 (3xy) dzdydx = \int_0^1 \int_0^1 3xydydx = \frac{3}{4}$$

5. Divergence & Stokes' Theorem

Question 3: Given a force field $\underline{F} = xy\,\underline{i} + 2yz\,\underline{j} + xz\,\underline{k}$ and a surface given by $z = 1 - x$ in the first octant and $y = 1$.

(a) Determine the work done by the force field around a path C which describes the boundary of the surface and oriented counter-clockwise from above.

(b) Determine the flux of the force field through the entire surface.



(a) We can use Stokes theorem which states that

$$\oint_C \underline{F} \cdot \underline{dr} = \iint_S (\nabla \times \underline{F}) \cdot \hat{n}\, dS$$

We have $\nabla \times \underline{F} = -2y\,\underline{i} - z\,\underline{j} - x\,\underline{k}$ and $\hat{n} = \frac{1}{\sqrt{2}}(1,0,1)$.

$$\oint_C \underline{F} \cdot \underline{dr} = \iint_S (\nabla \times \underline{F}) \cdot \hat{n}\, dS = \iint_S \frac{1}{\sqrt{2}}(-2y - x)dS = \iint_D \frac{1}{\sqrt{2}}(-2y - x)\left[\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\right] dxdy = \iint_D (-2y - x)dxdy$$

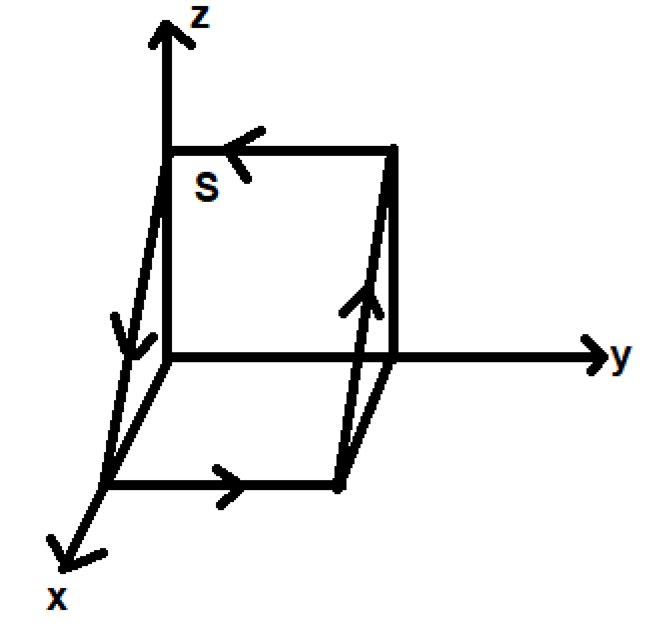
where D is the projection of surface S on the xy-plane and D is the square given by [0,1] x [0,1].

5. Divergence & Stokes' Theorem

Question 3: Given a force field $\underline{F} = xy\,\underline{i} + 2yz\,\underline{j} + xz\,\underline{k}$ and a surface given by $z = 1 - x$ in the first octant and $y = 1$.

(a) Determine the work done by the force field around a path C which describes the boundary of the surface and oriented counter-clockwise from above.

(b) Determine the flux of the force field through the entire surface.



$$\oint_C \underline{F} \cdot \underline{dr} = \iint_D (-2y - x)dxdy = \int_0^1 \int_0^1 (-2y - x)dxdy = -\frac{3}{2}$$

(b) We can employ the Divergence Theorem:

$$\iint_S \underline{F} \cdot \hat{n}\, dS = \iiint_V (\nabla \cdot \underline{F})dV = \iiint_V \big((y + 2z + x)\big)dV$$

where S is the bounding surface for the entire volume V.

$$\iint_S \underline{F} \cdot \hat{n}\, dS = \int_0^1 \int_0^1 \int_0^{1-x} (y + 2z + x)dzdydx = \int_0^1 \int_0^1 [y(1-x) + (1-x)^2 + x(1-x)]dydx = \frac{3}{4}$$