EE2-08 Mathematics

Example Sheet 4: Functions of a complex variable

Recall that for a complex function $f(z) = u(x, y) + iv(x, y)$ the Cauchy-Riemann equations are $u_x = v_y$ and $u_y = -v_x$.

1. Verify that the following satisfy the Cauchy-Riemann equations:

   a) $u = x; \quad v = y\,,$

   b) $u = e^x \cos y; \quad v = e^x \sin y\,,$

   c) $u = x^3 - 3xy^2; \quad v = 3x^2y - y^3$

2. Show that the following functions $u(x, y)$ each satisfy Laplace's equation and then use the Cauchy-Riemann equations to determine the conjugate function v. Find also $f(z) = u + iv$.

   a) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1\,,$ *b)* $u = xy\,.$

3. Show that the function
   $$u(x, y) = (x \cos y - y \sin y)\, e^x$$
   satisfies Laplace's equation. Find the conjugate function $v(x, y)$ for which u and v together satisfy the Cauchy-Riemann equations and hence find in its simplest form $w = u + iv = f(z)$ where $z = x + iy$.

4. Consider the mapping $w = \frac{1}{z-1}$ from the z-plane to the w-plane.

   a) Show that in the z-plane, the circle
   $$(x - 1)^2 + y^2 = 4$$
   maps to a circle in the w-plane. What is the radius of this circle and where is its centre?

   b) To what curve does the line $x = 0$ in the z-plane map in the w-plane?

5. a) Fixed points of a map $w = f(z)$ occur when $w = z$. Show that the fixed points of $w = \frac{4z-2}{z+1}$ occur at $z = 1$ and $z = 2$.

   b) For $w = u + iv = \frac{4z-2}{z+1}$ show that the image in the w-plane of the line $x = 0$ is the circle $(u - 1)^2 + v^2 = 9$. What is the image in the w-plane of the unit circle $|z| = 1$?

Answers:

2a) $v = 3x^2y - y^3 + 6xy + c; \quad f(z) = z^3 + 3z^2 + c.$
2b) $v = \frac{1}{2}(y^2 - x^2) + c; \quad f(z) = -\frac{i}{2}z^2 + c.$
3) $v = e^x\,(x \sin y + y \cos y) + c; \quad f(z) = z\,e^z + c.$
4) a) $u^2 + v^2 = \frac{1}{4}$; b) $\left(u + \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}.$
5b) The line $u = 1$.