Imperial College London

Mathematics: Year 2

Complex Variables

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Abstract

Complex variables, just like Real and Natural numbers, were invented as a way to make calculations simpler. While numerical analysis is used in modern day, complex variables still plays an important role in Electrical Engineering as means to simplify calculations as seen with Fourier and Laplace.

Many of the theory builds on existing Complex Numbers covered previously - a revision topic focuses on the abstract meaning of complex numbers.

Contents

1 Revision

1.1 Complex Numbers

Complex numbers $\mathbb{C}$ are defined as points in a plane meaning that it is 2-D number system compared to real numbers $\mathbb{R}$ which is a 1-D number system i.e. a line. The 2-D system allows it to be easily described through the use of matrices.

Complex plane ($\mathbb{C}$): A geometric representation of the complex numbers established by the **real axis** and the **imaginary axis**.

Complex analysis: The branch of mathematical analysis that investigates functions of complex numbers.

Complex numbers: A number that can be expressed in the form $a + bi$ where a and b are real numbers and i represents the imaginary unit satisfying $i^2 = -1$.

The matrix form can be compressed into the form $a + ib$ usually seen. It should be noted that *real* and *imaginary* are purely just terminology used only due to historical reasons. Imaginary numbers are just as real as Real numbers as it **is only a system to address certain equations that have no solutions**.

1.2 Complex Functions

The basic concept of a function involves two sets X and Y and each element (x - **independent variable**) in X maps to an element (y - **dependent variable**) in Y:

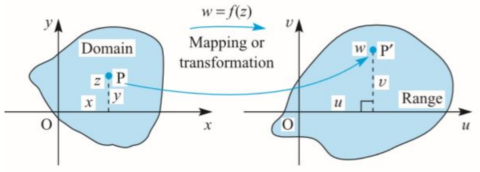
$$y = f(x)$$

where $(x \in X)$

A **complex function** is a function such that the independent variable z is a complex variable $z = x + iy$ where x and y are real numbers. If the independent variable is a complex variable, the dependent variable w is usually also a complex number:

$$w = f(z) = u + jv$$

All the complex function details cannot be plotted like a normal Cartesian $y = f(x)$ plot. The values are instead plotted **on two separate planes**: z-plane (independent variable) and w-plane (dependent variable)



(a) z plane (b) w plane

2 Analyticity and Cauchy-Riemann equations

As encountered previously, an analytic function is a function that can be approximated by a Taylor power series. The relation between analyticity, Taylor series and complex variables are rooted in understanding if a function can be evaluated. It is a common problem in mathematics when dealing with functions.

The fundamental basic tools mathematicians have for function evaluation are the four arithmetic operations $+, -, \times, \div$ In summary, the study of polynomials is basically the study of *"what can we do with basic arithmetic?"* So when it is proven that a function can be approximated by a Taylor series i.e. being analytic, it means that the function can be evaluated to a desired precision via basic arithmetic.

So in complex analysis, analyticity and its associated properties are critical.

Suppose that a function of a complex number z

$$f(z) = u(z) + \mathrm{i}v(z)$$

Then the complex derivative of f at a point z_0 is defined by:

$$f'(z_0) = \lim_{h \to 0} \left[\frac{f(z_0 + h) - f(z_0)}{h} \right]$$

provided this limit exists.

If this limit does exists, then it may be computed by taking the limit as $h \to 0$ along the real axis or the imaginary axis. Either case **it should give the same result**. Approaching along the real axis x:

$$\lim_{h \to 0} \left[\frac{f(z_0 + h) - f(z_0)}{h} \right] = \frac{\partial f}{\partial x}(z_0)$$

On the other hand, approaching along the imaginary axis y:

$$\lim_{h \to 0} \left[\frac{f(z_0 + \mathrm{i}h) - f(z_0)}{\mathrm{i}h} \right] = \frac{1}{\mathrm{i}} \frac{\partial f}{\partial y}(z_0)$$

The equality of the derivative of f taken along the two axis is:

$$\frac{\partial f}{\partial y}(z_0) = \mathrm{i}\frac{\partial f}{\partial x}(z_0)$$

which are the Cauchy–Riemann equations at the point z_0.

The difference compared to real variables is that the point z_0 is on a plane and can be approached along an infinite number of curves in the z plane. This means that the existence of a unique limit is not easily verified but it is still an important requirement to prove a function can be evaluated.

2.1 Cauchy-Riemann equations

As noted above, any analytic function (real or complex) is **infinitely differentiable** i.e. smooth or C^∞.

> **Analytic *in* R**: The function $f(z)$ has a derivative $f'(z)$ that exists at all points of a region R of the z plane. Can also be defined as: **Regular** or **Holomorphic**.

Though the term **analytic function** is often used interchangeably with **holomorphic function**, the word "analytic" is defined in a broader sense to denote any function (real, complex, or of more general type) that can be written as a convergent power series in a neighbourhood of each point in its domain i.e. all points can be approximated. The fact that all holomorphic functions are complex analytic functions, and vice versa, is a major theorem in complex analysis.

As mentioned earlier that the existence of an **unique limit** is required and, in order for there to be an **unique** limit, the limit needs to be independent of the direction in which the limit is taken.

The easiest way to check whether it is independent of the direction of the limit is to establish the **Cauchy-Riemann equations**.

> If $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, and $f(z)$ is analytic in some region R of the z-plane then:
>
> $$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \equiv u_x = v_y \tag{1}$$
>
> $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \equiv u_y = -v_x \tag{2}$$
>
> where (1) and (2) are the **Cauchy-Riemann equations** and holds throughout R

Some functions are analytic everywhere in the complex plane except at certain points i.e. **singularities** - where the limit not does exist.

2.1.1 Derivation of Cauchy-Riemann equations

A general test is performed on functions to determine if the function is independent of the direction of the limit:

Given:

$$f(z) = u(x, y) + iv(x, y)$$

Take the limit in the horizontal direction ($\partial z = \partial x$):

$$\frac{df(z)}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \equiv u_x + iv_x$$

Take the limit in the vertical direction ($\partial z = i\partial y$):

$$\begin{aligned}\frac{df(z)}{dz} &= \frac{\partial u}{\partial(iy)} + i\frac{\partial v}{\partial(iy)} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \equiv -iu_y + iv_y\end{aligned}$$

Example 1: Given $f(z) = z^{-1}$, determine the analyticity of the equation.

Expressing z as:

$$z^{-1} = \frac{(x - iy)}{x^2 + y^2}$$

Two equations obtained are:

$$u(x, y) = \frac{x}{x^2 + y^2}$$

$$v(x, y) = -\frac{y}{x^2 + y^2}$$

It is easy to see that Cauchy-Riemann equations hold everywhere except at origin $z = 0$ where the limit is indeterminate - the point (Singularity) at where it fails to be differentiable.

Example 2: Given $f(z) = z^2$, determine the analyticity of the equation.

Expressing z^2 as:

$$z^2 = x^2 - y^2 + 2ixy$$

Two equations obtained are:

$$u(x, y) = x^2 - y^2$$

$$v(x, y) = 2xy$$

The four partial derivatives are:

$$u_x = 2x$$

$$u_y = -2y$$

$$v_x = 2y$$

$$v_y = 2x$$

This indicates that the equation is differentiable at all points in the z-plane - analytic everywhere.

2.2 Properties of analytic functions

If functions satisfies the Cauchy-Riemann equations, these functions are also:

1. Harmonic functions

2. Conjugate functions

2.2.1 Conjugate functions

Conjugate functions satisfy the **orthogonality property**. The curves in the x, y-plane defined by $u(x, y) =$ constant and $v(x, y) =$ constant are orthogonal:

$$\left[\frac{dy}{dx}\right]_u = -\frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} \text{ and } \left[\frac{dy}{dx}\right]_v = -\frac{\frac{\partial v}{\partial y}}{\frac{\partial v}{\partial x}}$$

$$\left[\frac{dy}{dx}\right]_u \times \left[\frac{dy}{dx}\right]_v = -1$$

Conjugate functions form the basis of forming contour maps.

2.2.2 Harmonic functions

Harmonic functions satisfies **Laplace's equation** in two dimension:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

By satisfying Laplace's equation, it shows that function v exists and can be found.

As functions satisfying the Cauchy-Riemann equations are analytic hence the functions are also harmonic functions.

The name harmonic function originates from a point on a taut string which is undergoing harmonic motion. The solution to the differential equation for this type of motion can be written in terms of sines and cosines, functions which are thus referred to as harmonics. Harmonic functions have applications in areas such as stress analysis and fluid flow.

Example 1: Given the function $u = x^2 - y^2$, show

- Function is harmonic:

  From the equation above: $u_x = 2x$, $u_y = -2y$, $u_{xx} = 2$ and $u_{yy} = -2$:

  $$u_{xx} + u_{yy} = 0$$

  hence satisfying the Laplace equation, indicating it is a Harmonic function.

- Find $v(x, y)$:

  As v exists, it is defined by $v_y = u_x = 2x$ and $v_x = -u_y = 2y$. Using integration, v is found:

  $$v = 2xy + A(x) \text{ and } v = 2xy + B(y)$$

  $$v = 2xy + c$$

  where c is a constant to be defined by conditions set out initially

- Construct the corresponding complex function $f(z)$:

$$\begin{aligned} f(z) &= x^2 - y^2 \\ &= 2ixy + ic \\ &= z^2 + ic \end{aligned}$$

Example 2: Given that $u = x^3 - 3xy^2$, find:

- Find conjugate function $v(x, y)$:

  1. Check that $u = x^3 - 3xy^2$ satisfies Laplace's equation to ensure v exists.

     $$\begin{aligned} u_x &= 2x^2 - 3y^2 \quad \text{and} \quad u_{xx} = 6x \\ u_y &= -6xy \quad \text{and} \quad u_{yy} = -6x \end{aligned}$$

     Equate as $u_{xx} + u_{yy}$:

     $$u_{xx} + u_{yy} = 6x + (-6x) = 0$$

     Satisfies Laplace's equation hence v exists.

  2. Find v from Cauchy-Riemann equations:

     $$\begin{aligned} v_y &= 3x^2 - 3y^2 \\ v_x &= 6xy \end{aligned}$$

     Integrate:

     $$\begin{aligned} v &= 3x^2y - y^3 + A(x) \\ v &= 3x^2y + B(y) \end{aligned}$$

     Choose $A(x) = c$ and $B(y) = -y^3 + c$:

     $$v = 3x^2y - y^3 + c$$

- Corresponding complex function $f(z)$:

$$\begin{aligned} f(z) &= x^3 - 3xy^2 + i(2x^2y - y^3 + c) \\ &= z^3 + ic \end{aligned}$$

3 Mappings

As with functions, it is very useful to be able to graphically express a function. However, complex functions cannot be plotted on the complex plane since one would need four dimensions:

- Two inputs to plot: Real and Imaginary
- Two outputs to plot: Real and Imaginary

But, there are several possible ways to get around the problem:

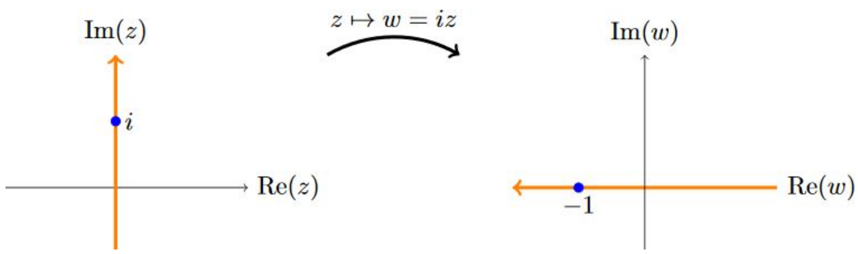
- Separately plot the **real** and **imaginary** components.
- Express complex variables as an **argument** and **absolute** value.

However, more commonly, a more primitive method called **mappings** or **transformations** is used. The z-plane and its points are inputted into a specified complex function and resulting values are plotted in the w-plane.

Commonly described as *mapping of z to w* or *w is the image of z*.

$$w = f(z)$$

For example, under the mapping $z \mapsto iz$, the image of the **imaginary** z-axis is the **real** w-axis



This change of shape during course of the transition gives mathematicians details on how the fundamental function behaves. It is fundamentally this feature of complex mappings that allows complex functions to have so much potential applications in various engineering fields.

Common notations used:

- Domain $z = x + iy$: The initial z-plane.
- Range $w = u + iv$: The mapped w-plane.
- $w = f(z)$: The equation indicating z-plane is mapped to w-plane.

3.1 Conformal Mapping

As mentioned earlier, studying this transformation process allows engineers to find potential applications to engineering problems. One of the aspects of the mapping process is that **the mapping may not be unique** i.e. given the transformation $w = z^2$, the two values $\pm z_0$ maps to only one value w_0 thus showing that the function is not a one-to-one.

This aspect has potential ramifications since one-to-one functions are a common condition in engineering systems so there must be a condition to ensure the mapping is one-to-one. Recall the concept of **analytic functions** - an important property of an analytic function is its uniqueness i.e. a value in the z-domain maps to a value in the w-plane.

Using the concepts of analytic functions, an important property within mappings is developed called **conformality**.

Conformal mapping: A mapping $w = f(z)$ that preserves angles in both magnitude and sense.

- **Magnitude**: Angle between two intersecting curves in z-plane is the same as angle between corresponding intersecting curves in the w-plane.
- **Sense**: Given θ, the angle between curves take in the anti-clockwise sense in z-plane then θ is also the angle between image of curves in the w-plane also taken in anti-clockwise sense.

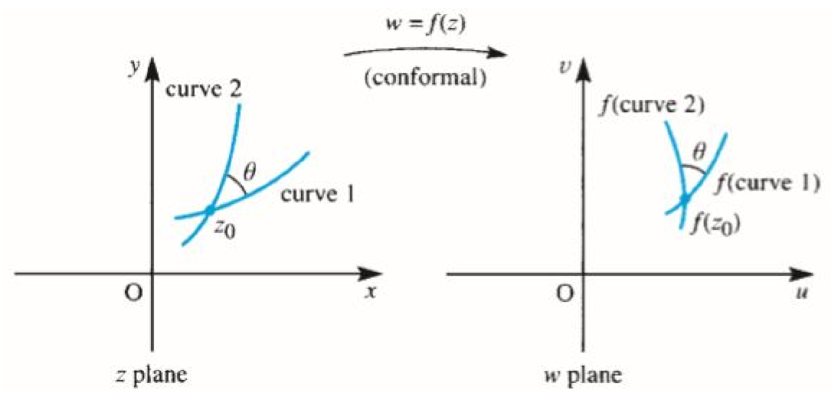


Figure 1: Conformal mapping showing magnitude and sense

Conformal mappings are invaluable for solving problems in engineering and physics that can be expressed in terms of functions of a complex variable yet exhibit inconvenient geometries. By choosing an appropriate mapping, the analyst can transform the inconvenient geometry into a much more convenient one while maintaining the critical details of the problem.

A mapping has a **fixed point** i.e. a point that remains the same throughout the transformation process:

$$w = f(z) = z$$

Note that though a mapping defined by an analytic function, the resulting transformation $w = f(z)$ is **not conformal everywhere**. There are points where the analytic characteristic fails called **singularities**:

$$f'(z) = 0$$

Example 1: Given $w = z^2$ and $f' = 2z$, describe the mapping.

The mapping is conformal everywhere except at $z = 0$ because $f'(0) = 2(0) = 0$, conformality fails at the origin which means the orthogonality property also fails.

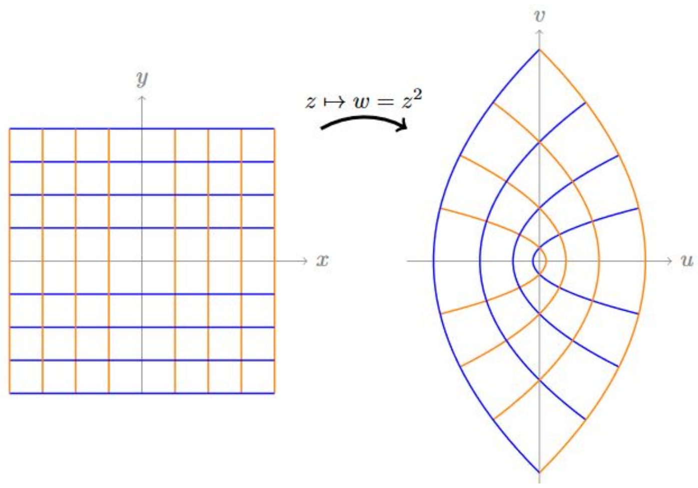


Figure 2: Transformation $w = z^2$

This transformation maps the horizontal and vertical grid lines to mutually orthogonal parabolas. $f(z)$ is conformal so the orthogonality of the parabolas is no accident. This shows that the conformal map preserves the right angles between the grid lines.

3.2 Linear transformation

The general complex linear function

$$w = \alpha z + \beta$$

where:

- w and z are complex-valued variables
- α and β are complex constants

It is of interest to study the mappings of the z plane onto the w plane for different choices of the constants α and β. In so doing, some general properties of mappings are revealed.

3.2.1 $\alpha = 0$

Letting $\alpha = 0$

$$w = \beta$$

This mapping implies that $w = \beta$ no matter what the value of z. This is obviously a **degenerate mapping** i.e. the entire z plane is being mapped onto the one point $w = \beta$ in the w plane.

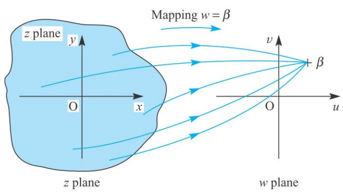


Figure 3: Degenerate mapping

The point β is a fixed point in this mapping. Note that this mapping has no inverse since it is not analytic.

3.2.2 $\beta = 0$ and $\alpha \neq 0$

Letting $\beta = 0$ and $\alpha \neq 0$

$$w = \alpha z$$

Under this mapping, the origin is the only fixed point, there being no other fixed points that are finite. Also, in this case there exists an inverse mapping

$$z = \frac{1}{\alpha} w$$

This mapping has the following effects:

- Maps the origin in the z plane to the origin in the w plane i.e. fixed point
- **Magnification**: Expands mapping by $|\alpha|$
- **Rotation**: Rotates anticlockwise by $arg(\alpha)$

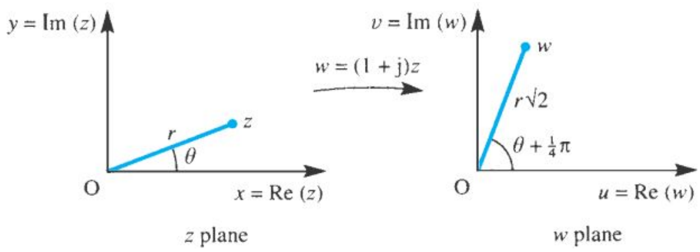


Figure 4: Magnification and rotation

Certain properties in linear mappings are preserved, the most important being that straight lines in the z plane will be transformed to straight lines in the w plane. Likewise, circles are mapped onto circles.

Note the above example does not include any translation since $\beta = 0$

$$z \xrightarrow[\text{rotation}]{} e^{j\theta} z \xrightarrow[\text{magnification}]{} |\alpha| e^{j\theta} z \xrightarrow[\text{translation}]{} |\alpha| e^{j\theta} z + \beta = \alpha z + \beta = w$$

Figure 5: Summary of the effects of different variables

3.3 Reciprocal transformation: $w = \frac{1}{z}$

As mentioned previously, studying a transformation gives clues on how transformations can be used. This transformation is also a one-to-one correspondence between the nonzero points of the z and w planes.

An interesting property of this mapping $w = \frac{1}{z}$ is considering the image of circles and straight lines in the z plane under such a mapping.

Every circle or line in the z-plane can be described by the general equation:

$$Ax + By + C(x^2 + y^2) = D$$

where if $C = 0$ then it is a straight line and if $C \neq 0$ then it is a circle.

This general equation in the z-plane can be converted into another equation in the w-plane:

$$Au - Bv + C = D(u^2 + v^2)$$

This shows that a line or circle in x, y is transformed to a line or circle in u, v, meaning that inversion maps lines and circles to lines and circles.

Several properties of this transformation:

1. Any line not through the origin is mapped to a circle through the origin.
2. Any line through the origin is mapped to a line through the origin.
3. Any circle not through the origin is mapped to a circle not through the origin.
4. Any circle through the origin is mapped to a line not through the origin.

The value of α and Δ affects the mapping of lines and circles:

- $C = 0$: Transformation results in lines
- $C \neq 0$: Transformation results in circles

3.4 Fractional linear transformation

A fractional linear transformation or **Mobius** transformation is a function of the form:

$$w = \frac{az+b}{cz+d}$$

where a, b, c, d are complex constants and $ad - bc \neq 0$

Simple point: If $ad - bc = 0$ then w is a constant function.

A Mobius transformation maps lines/circles to lines/circle with contraction, rotation and translation.

This transformation includes cases such as:

1. $w = \frac{1}{z}$ when $a = d = 0$ and $\frac{b}{c} = 1$
2. $w\frac{1}{z-1}$ when $a = 0$, $b = 1$, $c = 1$ and $d = -1$

The above equation can be written as:

$$w = c^{-1}\left\{a + \frac{bc-ad}{cz+d}\right\}$$

The special cases are defined as:

1. $w = z + b$ where $a = d = 1$ and $c = 0$: **Translation**
2. $w = az$ where $b = c = 0$ and $d = 1$: **Contraction + Rotation**
3. $w = \frac{1}{z}$ where $a = d = 0$ and $b = c$: **Lines/Circles to Lines/Circles**

3.5 Example Questions

Example 1: Given the complex mapping from $z = x + iy$ to $w = u + iv$:

$$w = \frac{1}{z + i}$$

- Show that circles $x^2 + (y+1)^2 = a^2$ in the z-plane map to circles in the w-plane, and give the equation of the circles in terms of u, v.

$$\begin{aligned} u + iv &= \frac{1}{x + iy + i} \\ &= \frac{x - i(y+1)}{x^2 + (y+1)^2} \end{aligned}$$

  Splitting between real and imaginary:

$$u = \frac{x}{x^2 + (y+1)^2} \quad \text{and} \quad v = \frac{-y+1}{x^2 + (y+1)^2}$$

  Circle formula:

$$\begin{aligned} u^2 + v^2 &= \left[\frac{x}{x^2 + (y+1)^2}\right]^2 + \left[\frac{-y+1}{x^2 + (y+1)^2}\right]^2 \\ &= \frac{1}{x^2 + (y+1)^2} \\ &= \frac{1}{a^2} \end{aligned}$$

- Show that the axes in the z-plane map to an axis and a circle in the w-plane. Obtain the axis and circle.

  y-axis: $x = 0$

$$\begin{aligned} w &= \frac{1}{x + iy + i} \\ &= \frac{1}{0 + iy + i} \\ &= \frac{1}{i(y+1)} \\ v &= \frac{1}{y+1} \end{aligned}$$

  x-axis: $y = 0$

$$\begin{aligned} w &= \frac{1}{x + iy + i} \\ &= \frac{1}{x + 0 + i} \\ &= \frac{1}{x + i} \\ &= \frac{x - i}{x^2 + 1} \end{aligned}$$

Use the circle formula:

$$\begin{aligned} u^2 + v^2 &= \left[\frac{x}{x^2+1}\right]^2 + \left[\frac{-1}{x^2+1}\right]^2 \\ &= \frac{1}{x^2+1} \\ &= -v \end{aligned}$$

Simplify:

$$\begin{aligned} u^2 + v^2 + v &= 0 \\ u^2 + \left(v + \frac{1}{2}\right)^2 &= \frac{1}{4} \end{aligned}$$

- Obtain the images in w of the lines $y = x - 1$ and $y = -1$.
  - $y = x - 1$ and $x = y + 1$:

    $$z = x + iy$$

    Split:

    $$\begin{aligned} u = \frac{x}{x^2 + (y+1)^2} \quad &\text{and} \quad v = \frac{-y+1}{x^2+(y+1)^2} \\ u = \frac{1}{2(y+1)} \quad &\text{and} \quad v = \frac{-1}{2(y+1)} \end{aligned}$$

  - $y = -1$:

    $$\begin{aligned} w &= \frac{1}{x + i(y+1)} \\ &= \frac{1}{x} \end{aligned}$$

4 Singularities, Zeroes and Residues

In order to see how singularities contribute to complex integrals, methods to find their nature and classification is required.

4.1 Singularities and Zeroes

Singularity (of a complex function $f(z)$): A point of the z-plane where $f(z)$ ceases to be analytic - a point where the mathematical object ceases to be well-behaved, such as the lack of differentiability or analyticity e.g. $f(z) = \frac{1}{x}$ where a singularity is $x = 0$.

Zero (of a complex function $f(z)$): A point in the z-plane at which $f(z) = 0$.

Poles (of a complex function $f(z)$): A zero of $\frac{1}{f}$.

Singularities of complex functions has many classifications and the simplest class is: **Simple Poles** (Order of 1).

A function has a simple pole at a point a if it can be written in the form:

$$f(z) = \frac{g(z)}{z-a}$$

where $g(z)$ is an analytic function

4.2 Residue

Meromorphic: A function that is analytic on except for a set of poles of finite order.

The residue $Res(f, c)$ (of a function $f(z)$ at c): The coefficient of $(z-c)^{-1}$ resulting from the Laurent series expansion off at point c.

Residue of a pole is important in integration, once known, allow the determination of general contour integrals via the residue theorem. Contour integration is a method of evaluating certain integrals along paths in the complex plane and it is closely related to residue of a complex function.

There are more efficient ways of calculating the residue than finding the Laurent series expansion.

1. Residue of $f(z)$ at a simple pole at $z = a$:

$$\text{Residue of } f(z) = \lim_{z \to a} (z-a) f(z)$$

2. Residue of $f(z)$ at a pole of multiplicity at $z = a$: (Formula provided)

$$\text{Residue of } f(z) = \lim_{z \to a} \left[\frac{1}{(m-1)!} \times \frac{d^{m-1}}{dz^{m-1}} \times [z-a]^m \times f(z) \right]$$

Example 1: Given $f(z) = \frac{2z}{(z-1)(z-2)}$ which has poles at $z = 1$ and $z = 2$, find the residues.

- Residue at $z = 1$: $\lim_{z \to 1} [(z-1)f(z)] = -2$
- Residue at $z = 2$: $\lim_{z \to 1} [(z-2)f(z)] = 4$

Example 2: Given $f(z) = \frac{2z}{(z-1)^2(z+4)}$ which has double pole at $z = 1$ and a simple pole at $z = -4$, find the residues.

- Residue at double pole $z = 1$:

$$\begin{aligned} \lim_{z \to 1} \frac{1}{1!} \left\{ \frac{d}{dz}[(z-1)^2 f(z)] \right\} &= \lim_{z \to 1} \frac{d}{dz} \left[\frac{2z}{z+4} \right] \\ &= 2 \lim_{z \to 1} \left[\frac{(z+4) - z}{(z+4)^2} \right] \\ &= \frac{8}{25} \end{aligned}$$

- Residue at simple pole $z = -4$:

$$\lim_{z \to -4} \{(z+4)f(z)\} = -\frac{8}{25}$$

Example 3: Evaluate the residue from the function $F(z) = (z+1)\frac{1}{(4z^3 - z)}$ given the poles must lie within $|z| = 1$

1. Change format of the function:

$$F(z) = (z+1)\frac{1}{(4z^3 - z)} = \frac{(z+1)}{z(2z+1)(2z-1)}$$

2. Identify the poles:

$$\begin{aligned} z &= 0 \\ z &= \pm\frac{1}{2} \end{aligned}$$

3. Note all poles lie within the provided limit.

4. Find residue at each poles:

   - $z = 0$

$$lim_{z \to 0} \left[\frac{z(z+1)}{z.(2z+1).(2z-1)} \right] = \left[\frac{(z+1)}{(2z+1).(2z-1)} \right] = -1$$

- $z = +\frac{1}{2}$

$$lim_{z\to\frac{1}{2}}\left[\frac{(z+\frac{1}{2}).(z+1)}{z.(2z+1).(2z-1)}\right] = \left[\frac{(z+\frac{1}{2}).(z+1)}{z.2(z+\frac{1}{2}).(2z-1)}\right] = \frac{1}{4}$$

- $z = -\frac{1}{2}$

$$lim_{z\to\frac{1}{2}}\left[\frac{(z-\frac{1}{2}).(z+1)}{z.(2z+1).(2z-1)}\right] = \left[\frac{(z-\frac{1}{2}).(z+1)}{z(2z+1).2(z-\frac{1}{2})}\right] = \frac{3}{4}$$

4.2.1 Residue of $f(z) = \frac{h(z)}{g(z)}$ when $g(z)$ is 0 at a point a

Residue at $z = a : \frac{h(a)}{g'(a)}$

Proof:

Expand $g(z)$ about 0 at $z = a$ in a Taylor series:

$$g(z) = g(a) + (z-a)g'(a) + \frac{1}{2}(z-a)^2 g''(a) + ...$$

Noting that $g(a) = 0$:

$$\begin{aligned}
\lim_{z\to a}\left\{\frac{(z-a)h(z)}{g(z)}\right\} &= \lim_{z\to a}\left\{\frac{(z-a)h(z)}{(z-a)g'(a)+\frac{1}{2}(z-a)^2g''(a)+...}\right\} \\
&= \lim_{z\to a}\left\{\frac{h(z)}{g'(a)+\frac{1}{2}(z-a)g''(a)+...}\right\} \\
&= \frac{h(a)}{g'(a)}
\end{aligned}$$

5 Contour Integration

Contour integrals are very useful tools to evaluate integrals, there are many functions whose indefinite integrals cannot be written in terms of elementary functions but their contour integrals can be used to derive the definite integrals.

The basic definite integral with a complex function $f(z)$ is defined as:

$$\int_{z_1}^{z_2} f(z)dz$$

where complex variable is z and pair z_1 and z_2 are complex numbers

The above integral definition means that the integral with z is taken from point z_1 to point z_2 in the z-plane - this means that a path from z_1 to z_2 must be defined.

5.1 Contour integrals

Complex function $f(z)$ needs to satisfy certain characteristics just like normal integrals:

- Continuous at all points in the z-plane
- Finite length
- Contains the two points a and b

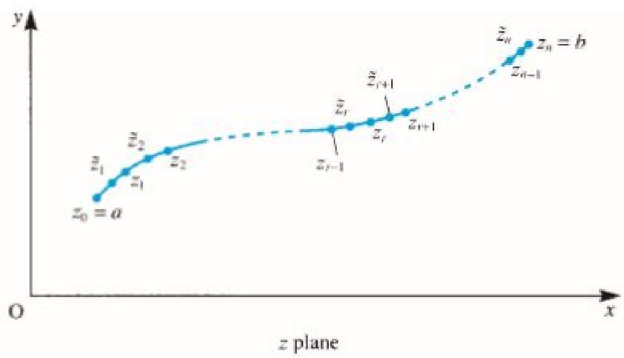


Figure 6: Graphical representation of integration

By dividing the curve from a to b into n parts with points z_{n-1}, the sum of the points is:

$$S_n = f(\hat{z}_1)(z_1 - z_0) + f(\hat{z}_2)(z_2 - z_1) + ... + f(\hat{z}_n)(z_n - z_{n-1})$$

The equation can be simplified into:

$$S_n = \sum_{k=1}^{n} f(\hat{z}_k)\Delta z_k$$

The **Contour Integral** is found when:

- n is the largest possible value such that difference between two points $|\Delta z_k|$ approaches 0
- The sum S_n approaches a limit (constant value)

Contour Integral:

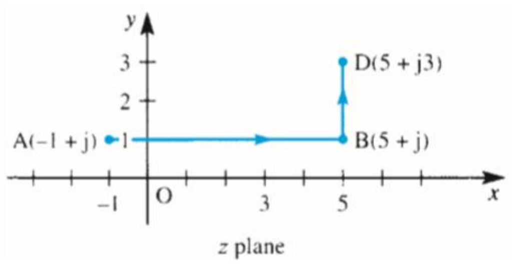
$$\int_C f(z)dz = \lim_{|\Delta z_k| \to 0} \sum_{k=1}^{n} f(\hat{z}_k)\Delta z_k$$

where z is defined as $z = x + \mathrm{i}\ y$ thus $f(z) = u(x,y) + \mathrm{i}\ v(x,y)$

$$\int_C f(z)dz = \int_C [u(x,y) + \mathrm{i}\ v(x,y)](dx + \mathrm{i}\ dj)$$

$$\int_C f(z)dz = \int_C [u(x,y)dx - v(x,y)dy] + \mathrm{i} \int_C [v(x,y)dx - u(x,y)dy]$$

Example 1: Evaluate contour integral $\int_C z^2$ along path C listed below.



1. Find z^2:

$$\begin{aligned} z^2 &= (x + jy)^2 \\ &= (x^2 - y^2) + j2xy \end{aligned}$$

2. Define integral I:

$$\begin{aligned} z^2 &= \int_C z^2 dz \\ &= \int_C [x^2 - y^2]\ \mathrm{dx} - 2xy\ \mathrm{dy}] + j \int_C [2xy\ \mathrm{dx} + (x^2 - y^2)\ \mathrm{dy}] \end{aligned}$$

3. The integral of AB ($y = 1$ and $dy = 0$):

$$\begin{aligned} I_{AB} &= \int_{-1}^{5} (x^2 - 1)\ \mathrm{dx} + \mathrm{j} \int_{-1}^{5} 2x\ \mathrm{dx} \\ &= [\frac{1}{3}x^3 - x]_{-1}^{5} + \mathrm{j}[x^2]_{-1}^{5} \\ &= 36 + 24\mathrm{j} \end{aligned}$$

4. The integral of BD ($x = 5$ and $dx = 0$):

$$\begin{aligned} I_{BD} &= \int_1^3 (-10y)\ \mathrm{dy} + \mathrm{j}\int_1^3 (25 - y^2)\ \mathrm{dy} \\ &= [-5y^2]_1^3 + \mathrm{j}[25y - \frac{1}{3}y^3]_1^3 \\ &= 40 + \mathrm{j}\frac{124}{3} \end{aligned}$$

5. Combine the results:

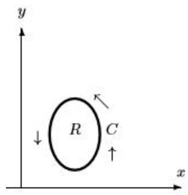
$$\begin{aligned} \int_C z^2\ \mathrm{dz} &= I_{AB} + I_{BD} \\ &= (36 + \mathrm{j}24) + (-40 + \mathrm{j}\frac{124}{3}) \\ &= -4 + \mathrm{j}\frac{196}{3} \end{aligned}$$

5.2 Cauchy's theorem

The most important theorem in complex variable theory and the foundation of complex variable integration.

If $f(z)$ has the following characteristics:

- An analytic function with derivative $f'(z)$
- Continuous at all points inside
- On a simple close curve C



then:

$$\oint_C f(z)\ \mathrm{dz} = 0$$

where $\oint_C$ denotes the integration **around a closed curve**.

The proof of Cauchy's Theorem makes use of Green's Theorem in a plane.

Green's Theorem: For differentiable functions $P(x, y)$ and $Q(x, y)$:

$$\oint_C (P\ \mathrm{dx} + Q\ \mathrm{dy}) = \int\int_R (Q_x - P_y)\mathrm{dx}\ \mathrm{dy}$$

Proof:

Given the definition:

$$\begin{aligned}\oint_C F(z)\ \mathrm{dz} &= \oint_C (u+iv)(dx+idy)\\ &= \oint_C (u\ \mathrm{dx} - x\ \mathrm{dv}) + i\oint_C (v\ \mathrm{dx} + u\ \mathrm{dy})\end{aligned}$$

With the application of Green's Theorem:

$$\begin{aligned}\oint (u\ \mathrm{dx} - v\ \mathrm{dy}) &= \int\int_R\int(-v_x - u_y)\mathrm{dx\ dy}\\ \oint (v\ \mathrm{dx} + u\ \mathrm{dy}) &= \int\int_R (u_x - v_y)\mathrm{dx\ dy}\end{aligned}$$

These two combines into the final form:

$$\oint_C F(z)\ \mathrm{dz} = -\int\int_R (v_x + u_y)\mathrm{dx\ dy} + i\int\int_R (u_x - v_y)\mathrm{dx\ dy}$$

If $F(z)$ is analytic everywhere on C then u and v satisfies Cauchy-Riemann equations. This means:

$$u_x = v_y \text{ and } v_x = -u_y$$

This simplifies to Cauchy's Theorem:

$$\oint_C F(z)\ \mathrm{dz} = 0$$

Cauchy's Theorem can be expressed alternatively with $z = a * e^{i\theta}$ for $\theta : 0 \to 2\pi$

$$\begin{aligned}\oint_C \frac{dz}{z} &= \int_0^{2\pi} \frac{iae^{i\theta}\ \mathrm{d}\theta}{ae^{i\theta}}\\ &= i\int_0^{2\pi} d\theta\\ &= 2\pi i\end{aligned}$$

5.3 Residue Theorem

The Residue Theorem is used to evaluate contour integrals and it draws from theories found in differentiation and integration of complex functions.

Residue Theorem: If $f(z)$ is an analytic function within and on a simple closed curve C, and the only singularities of $f(z)$ within C are poles then:

$$\oint_C f(z)\ \mathrm{dz} = 2\pi \mathrm{i} \times \{\text{Sum of residues of } f(z) \text{ at its poles within } C\}$$

Proof:

Example 1: Find $\oint_{C_i} \frac{2z\ \mathrm{dz}}{(z-1)(z-2)}$ where:

- C_1 is a circle centred at $(0,0)$ of radius 3
- C_2 is a circle centred at $(0,0)$ of radius $\frac{3}{2}$

Identifying the poles, $f(z)$ has two simple poles: $z=1$ and $z=2$

For C_1, both poles are inside C_1:

$$\begin{aligned}\oint_{C_1} f(z)\ \mathrm{dz} &= 2\pi \mathrm{i} \times (-2+4) \\ &= 4\pi \mathrm{i}\end{aligned}$$

For C_2, only pole inside C_2 is $z=1$:

$$\begin{aligned}\oint_{C_1} f(z)\ \mathrm{dz} &= 2\pi \mathrm{i} \times (-2) \\ &= -4\pi \mathrm{i}\end{aligned}$$

Example 2: Find $\oint_C \frac{2z\ \mathrm{dz}}{(z-1)^2(z+4)}$ where C circle has radius 5 centred at $z=0$.

Identifying the poles, $f(z)$ has two simple poles: $z=1$ and $z=-4$ and both lie within C

$$\begin{aligned}\oint_C \frac{2z\ \mathrm{dz}}{(z-1)^2(z+4)} &= 2\pi \mathrm{i} \times (-\frac{8}{25} + \frac{8}{25}) \\ &= 0\end{aligned}$$

Example 3: Find $\oint_C \frac{z^2\ \mathrm{dz}}{(z-i)^3}$ where C circle $|z|=2$ centred at $z=0$.

Identifying the poles, $f(z)$ has a triple pole: $z=i$

$$\lim_{z\to 1} \frac{1}{2!}\frac{d^2}{dz^2}\left\{\frac{z^2(z-i)^3}{(z-1)^3}\right\} = 1$$

Hence:

$$\oint_C \frac{z^2 \text{ dz}}{(z-i)^3} = 2\pi\text{i}$$

5.4 Improper integrals and Jordan's Lemma

The Residue Theorem is used to evaluate *real integrals* of the following:

$$\int_{-\infty}^{\infty} e^{imx} f(x) \text{ dx}$$

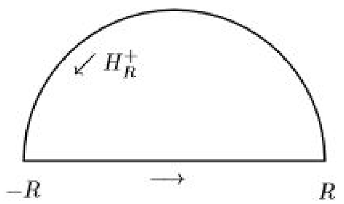
where $m \geq 0$

Improper Integrals: A definite integral that has either or both limit of an infinite nature.

$$\int_{-\infty}^{\infty} e^{imx} f(x) \text{ dx} = \lim_{R\to\infty} \int_{-R}^{R} e^{imx} f(x) \text{ dx}$$

Jordan's lemma is a result frequently used in conjunction with the residue theorem to evaluate contour integrals and improper integrals.

This is important as there is a class of Complex Integrals where C consists of a semi-circle arc:



where the semi-circle has radius R and arc denoted H_R. The real axis has range $[-R, R]$.

Building from the improper integral, it can be split into *Real integral* and *Complex integral* and form the following equation for evaluation:

$$\oint_C e^{imz} F(z)\, dz = \underbrace{\int_{-R}^{R} e^{imx} F(x)\, dx}_{real\ integral} + \underbrace{\int_{H_R} e^{imz} F(z)\, dz}_{complex\ integral}$$

- Complex integral: Evaluated with the Residue Theorem
- Real integral: Solving as $R \to \infty$ requires a result found in Jordan's Lemma.

5.4.1 Jordan's Lemma

Jordan's Lemma deals with how a contour integral behaves on a semi-circular arc H_R^+ of a closed contour C such as Figure **??**

> If the only singularities of $f(z)$ are poles then:
>
> $$\lim_{R\to\infty}\int_{H_R} e^{imz}f(z)\ \mathrm{dz} = 0$$
>
> provided that $m > 0$ and $|f(z)| \to 0$ as $R \to \infty$.
>
> *If $m = 0$, a faster convergence to 0 is required for $f(z)$.*

Proof:

Since H_R is a semi-circle:

$$z = Re^{i\theta} = R(\cos(\theta) + \mathrm{i}\sin(\theta))$$

and

$$\mathrm{dz} = \mathrm{i}Re^{i\theta}\ \mathrm{d}\theta$$

Recall the following:

- $|e^{i\alpha}| = 1$ for any real α
- $|\int f(z)\ \mathrm{dz}| \leq \int |f(z)|\ \mathrm{dz}$

Expand:

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{H_R} e^{imz}f(z)\ \mathrm{dz}\right| &= \lim_{R\to\infty}\left|\int_{H_R} e^{imR\cos\theta - mR\sin\theta}f(z)Re^{i\theta}\ \mathrm{d}\theta\right| \\ &\leq \lim_{R\to\infty}\int_{H_R} e^{-mR\sin\theta}|f(z)|R\ \mathrm{d}\theta\end{aligned}$$

Provided $m > 0$ as mentioned earlier, the exponential ensures that RHS is 0 in the limit $R \to \infty$ thus satisfying the proof.

Remarks:

1. If $m > 0$, forms of $f(z)$ will all converge fast enough due to all having simple poles and $|f(z) \to 0|$ as $R \to \infty$
2. If $m = 0$, alterations on the integer restrictions are required. E.g. $f(z) = \frac{1}{z}$ then $|f(z) \to 0|$ but $\lim_R \to \infty|f(z)| = 1$.
3. If $m < 0$, $\sin\theta < 0$ needs to ensure exponential is decreasing for $R \to \infty$. This is true in **lower half plane**, the contour taken in lower half plane H_r^- but will still be in an anti-clockwise direction.

If $f(z)$ satisfies the conditions for Jordan's Lemma:

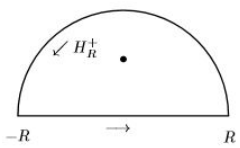
$$\int_{-\infty}^{\infty} e^{imx} f(x)\ \mathrm{dx} = 2\pi i \times \{\text{Sum of residues of poles of } e^{imzf(z)} \text{ in the upper } \frac{1}{2}\text{-plane}\}$$

Example 1: Show that $\int_{-\infty}^{\infty} \frac{\mathrm{dx}}{1+x^2} = \pi$

This problem can be solved simply with:

$$\begin{aligned} I &= \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{dx}}{1+x^2} \\ &= \lim_{R\to\infty} [\tan^{-1} x]_{-R}^{R} \\ &= \frac{\pi}{2} + \frac{\pi}{2} \\ &= \pi \end{aligned}$$

Using Jordan's Lemma and Residue Theorem:



C is comprised of a semi-circular arc H_R^+ and a section on real axis from $-R$ to R with the simple pole at $z = i$ lying within C.

Considering the complex integral over C in the upper half-plane with $m = 0$:

$$\oint_C \frac{\mathrm{dz}}{1+z^2}$$

The simple pole at $z = i$ and quadratic nature of denominator is enough for convergence thus Jordan's Lemma:

$$\lim_{R\to\infty} \int_{H_R} \frac{\mathrm{dz}}{1+z^2} = 0$$

Residue of $f(z)$ at the pole in upper-half plane at $z = i$ is $\frac{1}{2i}$ so Residue Theorem:

$$\oint_C \frac{\mathrm{dz}}{1+z^2} = 2\pi i \times \frac{1}{2i} = \pi$$

5.5 Integrals around the unit circle

The integrals considered here are the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\ \mathrm{d}\theta$

Working Example: $I = \int_0^{2\pi} \frac{\mathrm{d}\theta}{a+\cos\theta}$ where $a > 1$

1. Take C as unit circle $z = e^{i\theta}$ thus $\mathrm{dz} = ie^{i\theta}$

$$\begin{aligned} I &= \oint_C \frac{\mathrm{dz}}{iz(a + \frac{1}{2}(z + z^{-1}))} \\ &= -2i \oint_C \frac{\mathrm{dz}}{z^2 + 2az + 1} \end{aligned}$$

2. Determine the roots of $z^2 + 2az + 1 = 0$

$$z^2 + 2az + 1 = (z - \alpha^+)(z - \alpha^-)$$

   where $\alpha^\pm = -a \pm \sqrt{a^2 - 1}$
   When $a > 1$ and:
   - α^+: Lies within C
   - α^-: Lies outside C

3. Compute the residue of the integrand at $z = \alpha^-$

$$\frac{1}{\alpha^+ - \alpha^-}$$

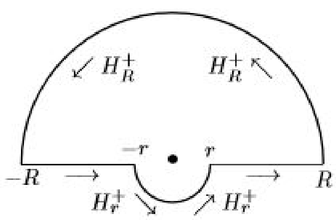
4. Use the Residue Theorem

$$\begin{aligned} T &= -2i \times 2\pi i \times \frac{1}{\alpha^+ - \alpha^-} \\ &= \frac{2\pi}{\sqrt{a^2 - 1}} \end{aligned}$$

5.6 Poles on Real axis

If an integrand has a pole on the real axis, certain precautions need to be taken.

Working Example:



Contour is deformed by small circle H_r of radius r at the origin excluding the pole at $z = 0$. The big semi-circle of radius R is H_R.

Given: $\int_{-\infty}^{\infty} \frac{\sin x\ \mathrm{dx}}{x}$, we consider the complex integral derived: $\oint_C \frac{e^{iz}\ \mathrm{dz}}{z}$

The integrand has no poles in C since $z = 0$ is excluded. Cauchy's Theorem is used:

$$\begin{aligned} 0 &= \oint_C \frac{e^{iz}\ \mathrm{dz}}{z} \\ &= \int_{-R}^{-r} \frac{e^{ix}\ \mathrm{dx}}{x} + \int_{H_r \hookleftarrow} \frac{e^{iz}\ \mathrm{dz}}{z} + \int_r^R \frac{e^{ix}\ \mathrm{dx}}{x} + \int_{H_R \hookrightarrow} \frac{e^{iz}\ \mathrm{dz}}{z} \end{aligned}$$

Notice the following:

- Take limit $R \to \infty$
- $m = 1$ and Jordan's Lemma indicating that $\int_{H_R} = 0$ (Only singularity is a pole and integrand decays to 0 as $R \to \infty$)
- Small circle has equation $z = r(\cos\theta + i\sin\theta)$ and $\sin\theta \geq 0$

$$\begin{aligned} \lim_{r\to 0} \int_{H_r} \frac{e^i z}{z}\ \mathrm{dz} &= i \lim_{r\to 0} \int_\pi^0 e^{-r\sin\theta} * e^{ir\cos\theta}\ \mathrm{d}\theta \\ &= -\pi i \end{aligned}$$

Taking the two limit $R \to \infty$ and $r \to 0$:

$$0 = \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\ \mathrm{dx} - \pi i + 0$$

Notice the real part of the integrand $\frac{\cos x}{x}$ is odd so contributions from $(-\infty, 0)$ and $(0, \infty)$ cancel out:

$$\int_\infty^\infty \frac{\sin x}{x}\ \mathrm{dx} = \pi$$