EE2-08 Mathematics

Example Sheet 6: LAPLACE TRANSFORMS

1. a) For the coupled ODEs

$$2\dot{x} + \dot{y} + x + 6 = 0 \qquad \dot{x} + 2\dot{y} + y = 0$$

where $x(0) = y(0) = 1$, show that

$$\overline{y}(s) = \frac{3(s+3)}{(s+1)(3s+1)} \qquad \overline{x}(s) = \frac{3(s^2 - 3s - 2)}{s(s+1)(3s+1)}.$$

Split these expressions into partial fractions & invert to find $x(t)$ and $y(t)$.

(b) In the same manner as part a), use Laplace transforms to solve

$$\dot{x} + 5x + 2y = e^{-t}, \qquad \dot{y} + 2x + 2y = 0, \qquad x(0) = 1, \quad y(0) = 0.$$

2. A function $f(t)$ has a Laplace transform $\mathcal{L}\{f(t)\} \equiv \overline{f}(s)$. Use the 'shift property' $\mathcal{L}\left\{e^{at}f(t)\right\} = \overline{f}(s-a)$, where a is a constant, and the 'second shift property'; $\mathcal{L}\left\{H(t-a)f(t-a)\right\} = e^{-sa}\overline{f}(s)$ to show that the solution of the SHM equation with discontinous driving terms

$$\ddot{x} + x = H(t-\pi) - H(t-2\pi)$$

and with initial conditions $x(0) = \dot{x}(0) = 0$, is

$$\begin{array}{ll} x = 0 & 0 \le t \le \pi \\ x = 1 + \cos t & \pi \le t \le 2\pi \\ x = 2\cos t & 2\pi \le t \end{array}$$

where H(t) is the Heaviside step function.

3. If a function $f(t)$ is periodic in time t with fixed period T such that $f(t) = f(t-T)$ with $T > 0$ show that for $s > 0$

$$\overline{f}(s) = \frac{1}{1 - e^{-sT}} \int_0^T f(t)e^{-st}\, dt.$$

Note that this enables a Laplace Transform to be found by performing the integral only over the period $(0, T)$ for which $f(t)$ is defined.

4. Use the result of Q3 to show that the Laplace transform of the 'saw-tooth' function

$$\begin{array}{ll} f(t) = t & 0 \le t \le 1 \\ f(t) = f(t-1) & 1 \le t \end{array}$$

is given by

$$\overline{f}(s) = s^{-2} - s^{-1}\left(e^{-s} + e^{-2s} + e^{-3s} \ldots\right).$$

The function $f(t)$ is often used in electronics for representing discontinuous voltages.

Answers:

1a) $x = -6 + 3e^{-t} + 4e^{-\frac{1}{3}t}$; $\quad y = -3e^{-t} + 4e^{-\frac{1}{3}t}$

1b) $x = \frac{16}{25}e^{-6t} + \frac{1}{5}(t + \frac{9}{5})e^{-t}$; $\quad y = \frac{8}{25}e^{-6t} - \frac{1}{5}(\frac{8}{5} + 2t)e^{-t}$