Complex Variables Solutions to Exercises

- Exercises have been classified according to the topics:

- Harmonic Functions and Cauchy-Riemann Equations
- Functions and Transformations in the Complex Plane
- Complex Integration & Residues

Before attempting these exercises, you are strongly encouraged to go through the relevant portion of the notes and be familiar with the solutions to the examples presented in the notes.

1. Harmonic Functions and Cauchy-Riemann Equations

Question 1: Prove that $u = e^{-x}(x\sin(y) - y\cos(y))$ is harmonic and find v such that $f(z) = u + jv$ is analytic.

For u to be harmonic, it must satisfy $u_{xx} + u_{yy} = 0$.

So, $u_x = e^{-x}(\sin(y) - x\sin(y) + y\cos(y))$

$\rightarrow\ \ u_{xx} = -e^{-x}(\sin(y) - x\sin(y) + y\cos(y)) - e^{-x}\sin(y) = e^{-x}(-2\sin(y) + x\sin(y) - y\cos(y))$

$u_y = e^{-x}(x\cos(y) - \cos(y) + y\sin(y))$

$\rightarrow\ \ u_{yy} = e^{-x}(-x\sin(y) + \sin(y) + \sin(y) + y\cos(y)) = e^{-x}(2\sin(y) - x\sin(y) + y\cos(y))$

Then, $u_{xx} + u_{yy} = 0$, as required. The harmonic conjugate v is obtained from the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow e^{-x}(\sin(y) - x\sin(y) + y\cos(y)) = \frac{\partial v}{\partial y} \rightarrow v = e^{-x}(-\cos(y) + x\cos(y) + y\sin(y) + \cos(y)) + a(x)$$
$$\rightarrow v = e^{-x}(x\cos(y) + y\sin(y)) + a(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow -e^{-x}(x\cos(y) - \cos(y) + y\sin(y)) = \frac{\partial v}{\partial x} \rightarrow\ v = \cos(y)\left(e^{-x}(x+1)\right) + e^{-x}(-\cos(y) + y\sin(y)) + b(y)$$
$$\rightarrow v = e^{-x}(x\cos(y) + y\sin(y)) + b(y)$$

So, $v = e^{-x}(x\cos(y) + y\sin(y))$+constant is the required harmonic conjugate.

1. Harmonic Functions and Cauchy-Riemann Equations

Question 2: Determine a and b such that the following functions are harmonic and find the corresponding harmonic conjugate.

(a) $u = ax^3 + bxy$

For u to be harmonic, it must satisfy $u_{xx} + u_{yy} = 0$.

So $u_x = 3ax^2 + by \;\rightarrow\; u_{xx} = 6ax$ and $u_y = bx \;\rightarrow\; u_{yy} = 0$

Then, $u_{xx} + u_{yy} = 6ax + 0 = 0 \;\rightarrow\; a = 0$. All values of b are acceptable.

The harmonic conjugate v is obtained from the Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\rightarrow\; by = \frac{\partial v}{\partial y} \;\rightarrow\; v = \frac{b}{2}y^2 + p(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \;\rightarrow\; -bx = \frac{\partial v}{\partial x} \;\rightarrow\; v = -\frac{b}{2}x^2 + q(y)$$

So, $v = \frac{b}{2}(y^2 - x^2)$+constant is the required harmonic conjugate.

1. Harmonic Functions and Cauchy-Riemann Equations

Question 2: Determine a and b such that the following functions are harmonic and find the corresponding harmonic conjugate.

(b) $u = \cos(ax)\cosh(by)$

For u to be harmonic, it must satisfy $u_{xx} + u_{yy} = 0$.

So $u_x = -a\sin(ax)\cosh(by) \;\rightarrow\; u_{xx} = -a^2\cos(ax)\cosh(by)$ and $u_y = \cos(ax)\,b\sinh(by) \;\rightarrow\; u_{yy} = \cos(ax)\,b^2\cosh(by)$

Then, $u_{xx} + u_{yy} = -a^2\cos(ax)\cosh(by) + \cos(ax)\,b^2\cosh(by) = 0 \;\rightarrow\; b = \pm a$ and all values of a are acceptable.

The harmonic conjugate v is obtained from the Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\rightarrow -a\sin(ax)\cosh(by) = \frac{\partial v}{\partial y} \;\rightarrow\; v = \mp\sin(ax)\sinh(by) + p(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \;\rightarrow\; -\cos(ax)\,b\sinh(by) = \frac{\partial v}{\partial x} \;\rightarrow\; v = \mp\sin(ax)\sinh(by) + q(y)$$

So, $v = \mp\sin(ax)\sinh(by)$+constant is the required harmonic conjugate.

2. Functions and Transformations in the Complex Plane

Question 1: Let the rectangular region R in the z-plane be bounded by $x = 0$, $y = 0$, $x = 2$ and $y = 1$.

(a) Determine the region R' of the w-plane into which R is mapped under the transformation: $w = z + (1 - 2j)$. Describe the nature of the transformation.

Given $w = z + (1 - 2j)$, then $u + jv = (x + jy) + (1 - 2j) = (x + 1) + j(y - 2)$.

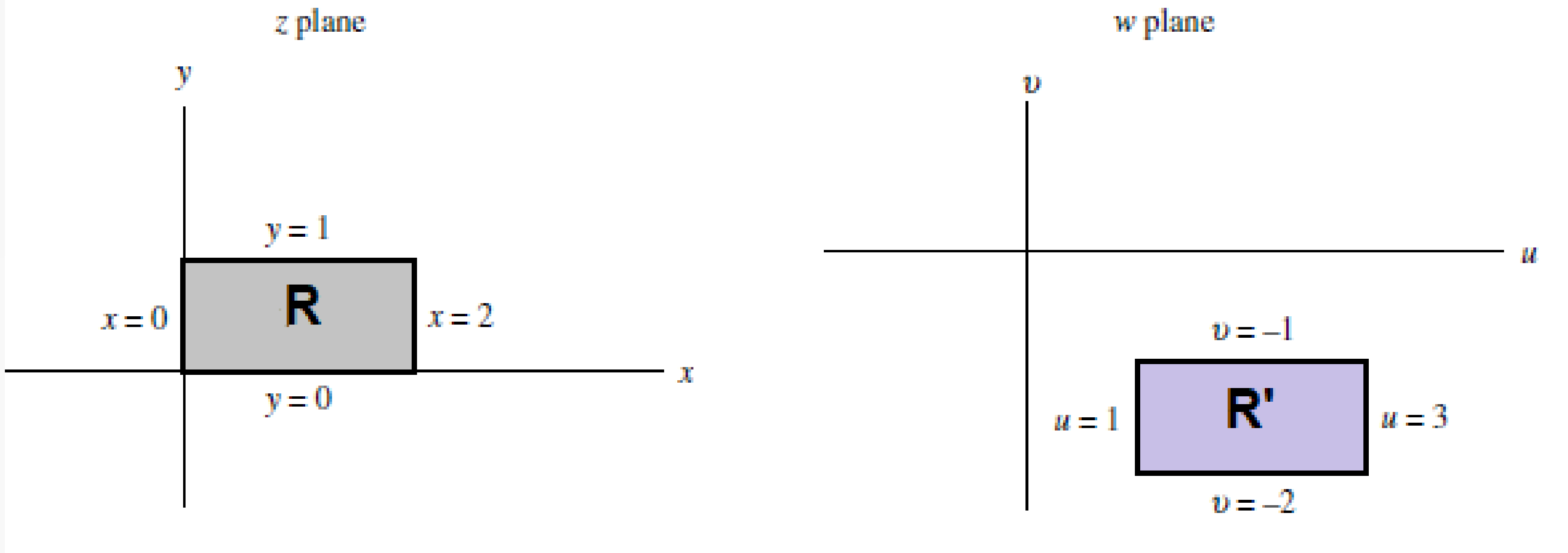
So we have

$x = 0,\ y \in [0,1] \rightarrow\ u = 1,\ v \in [-2, -1]$

$y = 0,\ x \in [0,2] \rightarrow\ v = -2,\ u \in [1,3]$

$x = 2,\ y \in [0,1] \rightarrow\ u = 3,\ v \in [-2,1]$

$y = 1,\ x \in [0,2] \rightarrow\ v = -1,\ u \in [1,3]$



Plotting these we get the region R' in the w-plane.

We observe that what we have done is translate the region R.

2. Functions and Transformations in the Complex Plane

Question 1: Let the rectangular region R in the z-plane be bounded by $x = 0$, $y = 2$, $x = 1$ and $y = 1$.

(b) Determine the region R' of the w-plane into which R is mapped under the transformation: $w = \sqrt{2}e^{j\pi/4}z$

$$w = \sqrt{2}\left(e^{\frac{j\pi}{4}}\right)z = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right)(x + jy) = (1 + j)(x + jy) = (x - y) + j(x + y) = u + jv$$

Then, we have

$x = 0 \quad y \in [0,1] \rightarrow \quad u = -y, \qquad v = y \qquad \rightarrow \quad v = -u, u \in [-1,0]$

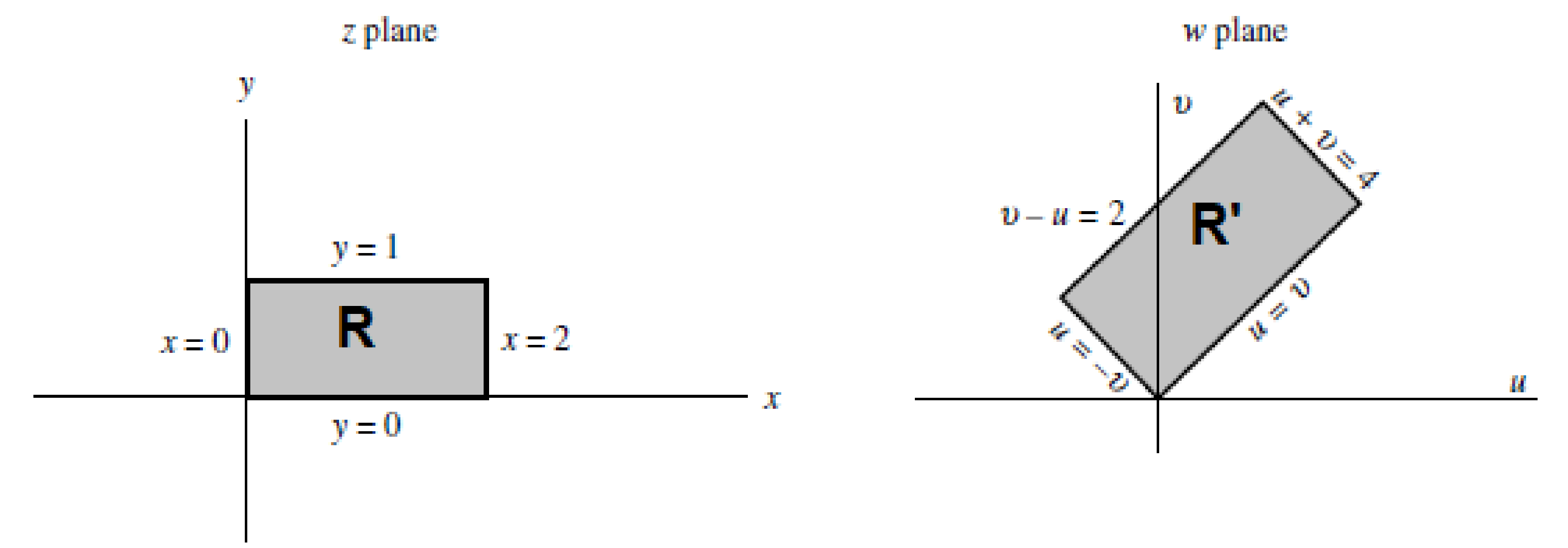
$y = 0, \ x \in [0,2] \rightarrow \quad v = x, \qquad u = x \qquad \rightarrow \quad v = u, \quad u \in [0,2]$,

$x = 2, \ y \in [0,1] \rightarrow \quad u = 2 - y, \quad v = 2 + y \quad \rightarrow v = -u + 4, \ u \in [1,2]$

$y = 1, \ x \in [0,2] \rightarrow \quad u = x - 1, \quad v = x + 1 \quad \rightarrow v = u + 2, \ u \in [-1,1]$.

Plotting these we get the region R' in the w-plane.

We observe that what we have done is rotate the region R.



3. Complex Integration & Residues

Question 1: Find the residues of (a) $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$ (b) $\mathrm{f(z)} = e^z \csc^2(z)$

(a) $f(z)$ has a double pole at $z = -1$ and a simple pole at $z = \pm 2i$.

The residue at $z = -1$ is given by

$$Res(f(z), -1) = \frac{1}{(2-1)!} \lim_{z\to -1} \frac{d^{2-1}}{dz^{2-1}}\left[(z+1)^2 f(z)\right] = \lim_{z\to -1} \frac{d}{dz}\left[\frac{z^2-2z}{(z^2+4)}\right] = \lim_{z\to -1} \frac{(2z-2)(z^2+4)-(z^2-2z)(2z)}{(z^2+4)^2} = -\frac{14}{25}$$

The residue at $z = 2j$ is given by

$$Res(f(z), 2j) = \frac{1}{(1-1)!} \lim_{z\to 2j} \frac{d^{1-1}}{dz^{1-1}}\left[(z-2j) f(z)\right] = \lim_{z\to 2j}\left[\frac{z^2-2z}{(z+1)^2(z+2j)}\right] = \frac{(2j)^2-2(2j)}{(2j+1)^2(2j+2j)} = \frac{7+j}{25}$$

The residue at $z = -2j$ is given by

$$Res(f(z), -2j) = \frac{1}{(1-1)!} \lim_{z\to -2j} \frac{d^{1-1}}{dz^{1-1}}\left[(z+2j) f(z)\right] = \lim_{z\to -2i}\left[\frac{z^2-2z}{(z+1)^2(z-2j)}\right] = \frac{(-2j)^2-2(-2j)}{(-2j+1)^2(-2j-2j)} = \frac{7-j}{25}$$

3. Complex Integration & Residues

Question 1: Find the residues of (a) $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$ (b) $f(z) = e^z \csc^2(z)$

(b) We can simplify $f(z) = \frac{e^z}{\sin^2(z)}$ and there are double poles at $z = 0, \pm\pi, \pm 2\pi$ etc. Thus, the pole can be generalized to $z = k\pi$ for $k = 0, \pm 1, \pm 2$ etc.

The residue at $z = k\pi$ is given by

$$Res(f(z), k\pi) = \frac{1}{(2-1)!} \lim_{z \to k\pi} \frac{d^{2-1}}{dz^{2-1}}[(z-k\pi)^2 f(z)] = \lim_{z \to k\pi} \frac{d}{dz}\left[(z-k\pi)^2 \frac{e^z}{\sin^2(z)}\right]$$

$$= \lim_{z \to k\pi} \frac{[2(z-k\pi)e^z + (z-k\pi)^2 e^z]\sin^2(z) - (z-k\pi)^2 e^z \cdot 2\sin(z)\cos(z)}{\sin^4(z)}$$

Apply a change of variable: $u = z - k\pi$. So, we get the following:

$$= \lim_{u \to 0} e^{u+k\pi} \frac{[2u + u^2]\sin(u+k\pi) - u^2 \cdot 2\cos(u+k\pi)}{\sin^3(u+k\pi)}$$

$$= 1 \cdot e^{k\pi} \lim_{u \to 0} \frac{[2u + u^2]\sin(u)\cos(k\pi) - u^2 \cdot 2\cos(u)\cos(k\pi)}{\sin^3(u)\cos^3(k\pi)}$$

3. Complex Integration & Residues

Question 1: Find the residues of (a) $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$ (b) $f(z) = e^z \csc^2(z)$

$$= e^{k\pi} \lim_{u\to 0} \frac{[2u+u^2]\sin(u) - u^2 \cdot 2\cos(u)}{\sin^3(u)\cos^2(k\pi)} = e^{k\pi} \lim_{u\to 0} \frac{[2u+u^2]\sin(u) - u^2 \cdot 2\cos(u)}{\sin^3(u)} \cdot \frac{\sin^3(u)}{u^3}$$

We did this because we want to use $\lim_{u\to 0} \frac{\sin(u)}{u} = 1 \;\to\; \lim_{u\to 0} \frac{\sin^3(u)}{u^3} = 1$. We then swap the u^3 and $\sin^3(u)$.

$$Res(f(z), k\pi) = e^{k\pi}\left[\lim_{u\to 0} \frac{[2u+u^2]\sin(u) - 2u^2\cos(u)}{u^3}\right]$$

Now we can use L'Hospital's rule three times since the denominator is order 3. This means we differentiate the numerator and the denominator separately three times:

$$Res(f(z), k\pi) = e^{k\pi}\left[\lim_{u\to 0} \frac{-2u^2\sin(u) - u^2\cos(u) - 6u\sin(u) + 10u\cos(u) + 6\sin(u) + 6\cos(u)}{6}\right]$$

$$= e^{k\pi}\left[\frac{6}{6}\right] = e^{k\pi} \text{ for } k = 0, \pm 1, \pm 2 \text{ etc.}$$

3. Complex Integration & Residues

Question 2: Find the residue of $f(z) = \frac{\cot(z)\coth(z)}{z^3}$ at $z = 0$.

We can express $f(z)$ as follows:

$$f(z) = \frac{\cot(z)\coth(z)}{z^3} = \frac{\cos(z)\cosh(z)}{z^3\sin(z)\sinh(z)} = \frac{\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)}{z^3\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)} = \frac{\left(1 - \frac{z^4}{6} + \cdots\right)}{z^3\left(z^2 - \frac{z^6}{90} + \cdots\right)} = \frac{\left(1 - \frac{z^4}{6} + \cdots\right)}{z^5\left(1 - \frac{z^4}{90} + \cdots\right)}$$

This in turn simplifies to

$$f(z) = \frac{1}{z^5}\left(1 - \frac{14}{90}z^4 + \cdots\right)$$

So we see that the pole $z = 0$ is of order five. This means that the residue at $z = 0$ is obtained after we differentiate $\left(1 - \frac{14}{90}z^4 + \cdots\right)$ four times. Thus, $Res(f(z), 0) = -\frac{7}{45}$.

So we see that we can also use the series expansion method as well.

3. Complex Integration & Residues

Question 3: Evaluate the following complex integral around the circle C with radius $|z| = 3$

$$\oint_C \frac{e^{zt}}{z^2\,(z^2+2z+2)}dz$$

The first thing we do is to determine the poles of $f(z) = \frac{e^{zt}}{z^2\,(z^2+2z+2)}$.

We see a double pole at $z = 0$ and a simple pole at $z = -1 \pm j$. All of them are within the circle C. So we must find the residues of all 3 poles.

$$\oint_C \frac{e^{zt}}{z^2\,(z^2+2z+2)}dz = 2\pi j(\text{sum of residues with poles in } C)$$

The residue at $z = 0$ is given by

$$Res(f(z),0) = \frac{1}{(2-1)!}\lim_{z\to 0}\frac{d^{2-1}}{dz^{2-1}}[z^2 f(z)] = \lim_{z\to 0}\frac{d}{dz}\left[\frac{e^{zt}}{(z^2+2z+2)}\right]$$

$$= \lim_{z\to 0}\frac{te^{zt}(z^2+2z+2) - e^{zt}(2z+2)}{(z^2+2z+2)^2} = \frac{2t-2}{4} = \frac{t-1}{2}$$

3. Complex Integration & Residues

Question 3: Evaluate the following complex integral around the circle C with radius $|z| = 3$

$$\oint_C \frac{e^{zt}}{z^2\,(z^2+2z+2)}\,dz$$

The residue at $z = -1 + j$ is given by

$$Res(f(z), -1+j) = \frac{1}{(1-1)!}\lim_{z\to -1+j}\frac{d^{1-1}}{dz^{1-1}}[(z+1-j)f(z)] = \lim_{z\to -1+j}\left[\frac{e^{zt}}{z^2(z+1+j)}\right] = \frac{e^{t(-1+j)}}{(-2j)(2j)} = \frac{e^{t(-1+j)}}{4}$$

The residue at $z = -1 - j$ is given by

$$Res(f(z), -1-j) = \frac{1}{(1-1)!}\lim_{z\to -1-i}\frac{d^{1-1}}{dz^{1-1}}[(z+1+j)f(z)] = \lim_{z\to -1-i}\left[\frac{e^{zt}}{z^2(z+1-j)}\right] = \frac{e^{t(-1-j)}}{(2j)(-2j)} = \frac{e^{t(-1-j)}}{4}$$

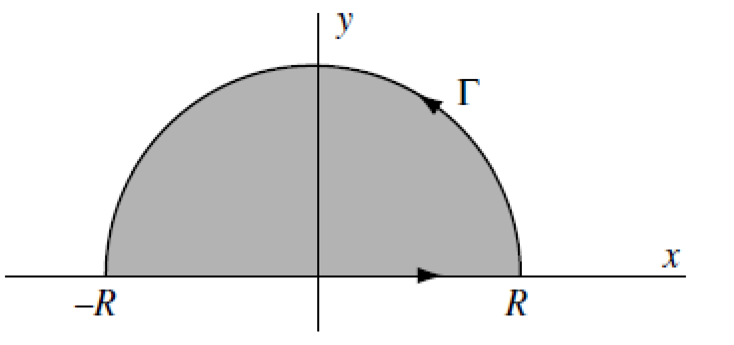
$$\oint_C \frac{e^{zt}}{z^2\,(z^2+2z+2)}\,dz = 2\pi j(\text{sum of residues with poles in } C)$$

$$= 2\pi j\left(\frac{t-1}{2} + \frac{e^{t(-1+j)}}{4} + \frac{e^{t(-1-j)}}{4}\right) = 2\pi j\left(\frac{t-1}{2} + \frac{e^{-t}}{4}\left(e^{jt}+e^{-jt}\right)\right) = 2\pi j\left(\frac{t-1}{2} + \frac{e^{-t}}{4}2\cos(t)\right)$$

$$= \pi j(t - 1 + e^{-t}\cos(t))$$

Question 4: Evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)}$$



Construct the following complex integral:

$$\oint_C \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} = \int_{-R}^{R} \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} + \int_{\Gamma} \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)}$$

where the closed contour C is given in the diagram above. We tackle the left hand side and recognize that we are to use residues whose poles are in the upper half plane. So the poles of interest are $z = j$ (double) and $z = -1 + j$ (simple).

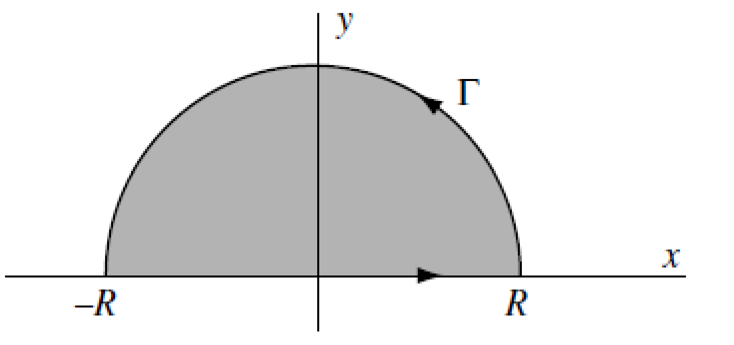
$$Res(f(z), j) = \frac{1}{(2-1)!} \lim_{z \to j} \frac{d^{2-1}}{dz^{2-1}} [(z-j)^2 f(z)] = \lim_{z \to j} \frac{d}{dz} \left[\frac{z^2}{(z+j)^2(z^2+2z+2)} \right]$$

$$= \lim_{z \to j} \frac{2z[(z+j)^2(z^2+2z+2)] - z^2[2(z+j)(z^2+2z+2) + (z+j)^2(2z+2)]}{(z+j)^4(z^2+2z+2)^2} = \frac{9j-12}{100}$$

3. Complex Integration & Residues

Question 4: Evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)}$$



$$Res(f(z), -1+j) = \frac{1}{(1-1)!} \lim_{z\to -1+j} \frac{d^{1-1}}{dz^{1-1}}[(z+1-j)\, f(z)] = \lim_{z\to -1+j} \left[\frac{z^2}{(z^2+1)^2(z+1+j)}\right] = \frac{3-4j}{25}$$

$$\oint_C \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi j\left(\frac{9j-12}{100} + \frac{3-4j}{25}\right) = \frac{7\pi}{50} = \int_{-R}^{R} \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)} + \int_\Gamma \frac{z^2 dz}{(z^2+1)^2(z^2+2z+2)}$$

Now we turn to the right hand side, we take the limit as $R \to \infty$ and apply Jordan's lemma, noting that on the horizontal line $z = x$,

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}$$

Moreover, the residue calculation stays the same as we are still in the upper half plane in the complex plane.