EE2-08 Mathematics:
Solutions to Example Sheet 7: Laplace Transforms

1. a) For the coupled ODEs

$$2\dot{x} + \dot{y} + x + 6 = 0 \qquad \dot{x} + 2\dot{y} + y = 0$$

where $x(0) = y(0) = 1$, show that

$$\overline{y}(s) = \frac{3(s+3)}{(s+1)(3s+1)} \qquad \overline{x}(s) = \frac{3(s^2-3s-2)}{s(s+1)(3s+1)}.$$

Split these expressions into partial fractions & invert to find $x(t)$ and $y(t)$.

Solution: Recalling[1] that $\mathcal{L}(\dot{x}) = s\overline{x}(s) - x(0)$, Laplace Transform the pair of ODEs using the initial conditions $x(0) = y(0) = 1$ to get

$$2(s\overline{x} - 1) + (s\overline{y} - 1) + \overline{x} = -6/s \qquad (s\overline{x} - 1) + 2(s\overline{y} - 1) + \overline{y} = 0$$

Solve these simultaneous equations in $\overline{x}$ and $\overline{y}$ to get

$$\overline{x}(s) = \frac{3(s^2-3s-2)}{s(s+1)(3s+1)} \qquad \overline{y}(s) = \frac{3(s+3)}{(s+1)(3s+1)}$$

Split these expressions into partial fractions

$$\overline{x}(s) = -\frac{6}{s} + \frac{3}{s+1} + \frac{4}{s+\frac{1}{3}} \qquad \overline{y}(s) = -\frac{3}{s+1} + \frac{4}{s+\frac{1}{3}}$$

and then invert to find the solutions from the tables

$$x(t) = -6 + 3e^{-t} + 4e^{-\frac{1}{3}t} \qquad y(t) = -3e^{-t} + 4e^{-\frac{1}{3}t}$$

(b) In the same manner as part a), use Laplace transforms to solve

$$\dot{x} + 5x + 2y = e^{-t}, \quad \dot{y} + 2x + 2y = 0, \quad x(0) = 1, \quad y(0) = 0.$$

Solution: In the same manner as part a), use Laplace transforms on the ODEs to get

$$(s\overline{x} - 1) + 5\overline{x} + 2\overline{y} = \frac{1}{s+1} \qquad s\overline{y} + 2\overline{x} + 2\overline{y} = 0$$

Solving these simultaneous equations we obtain

$$\overline{x}(s) = \frac{(s+2)^2}{(s+1)^2(s+6)} \qquad \overline{y}(s) = -\frac{2(s+2)}{(s+1)^2(s+6)}$$

which split into partial fractions thus

$$\overline{x}(s) = \frac{9}{25(s+1)} + \frac{1}{5(s+1)^2} + \frac{16}{25(s+6)} \qquad \overline{y}(s) = -\frac{8}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{8}{25(s+6)}$$

which invert to

$$x(t) = \frac{1}{5}\left(\frac{9}{5} + t\right)e^{-t} + \frac{16}{25}e^{-6t} \qquad y(t) = -\frac{2}{5}\left(\frac{4}{5} + t\right)e^{-t} + \frac{8}{25}e^{-6t}$$

[1] In the Formula Sheet.

2. A function $f(t)$ has a Laplace transform $\mathcal{L}\{f(t)\} \equiv \overline{f}(s)$. Use the 'first shift property' $\mathcal{L}\left\{e^{at}f(t)\right\} = \overline{f}(s-a)$, where a is a constant, and the 'second shift property'; $\mathcal{L}\left\{H(t-a)f(t-a)\right\} = e^{-sa}\overline{f}(s)$ to show that the solution of the SHM equation with discontinous driving terms

$$\ddot{x} + x = H(t-\pi) - H(t-2\pi)$$

and with initial conditions $x(0) = \dot{x}(0) = 0$, is

$$\begin{array}{ll} x = 0 & 0 \le t \le \pi \\ x = 1 + \cos t & \pi \le t \le 2\pi \\ x = 2\cos t & 2\pi \le t \end{array}$$

where H(t) is the Heaviside step function.

Solution: Laplace transforming the ODE and using the shift theorem, we get

$$\left(s^2+1\right)\overline{x}(s) = \frac{e^{-s\pi}}{s} - \frac{e^{-2s\pi}}{s} \qquad \Rightarrow \qquad \overline{x}(s) = \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-s\pi} - \left(\frac{1}{s} - \frac{s}{s^2+1}\right)e^{-2s\pi}$$

Noting that the $\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$, and using the second shift theorem (formula sheet) which says that $\mathcal{L}\left[H(t-a)f(t-a)\right] = e^{-sa}\,\overline{f}(s)$, we find that the inversion becomes

$$x(t) = H(t-\pi)\left[1 - \cos(t-\pi)\right] - H(t-2\pi)\left[1 - \cos(t-2\pi)\right]$$

Noting that $H(t-\pi) = 1$ for $t > \pi$ but is zero for $t < \pi$ (with equivalent results for $H(t-2\pi)$), we obtain

$$\begin{array}{ll} x = 0 & 0 \le t \le \pi \\ x = 1 + \cos t & \pi \le t \le 2\pi \\ x = 2\cos t & 2\pi \le t \end{array}$$

Note that when $t \ge 2\pi$ the two cosines add.

3. If a function $f(t)$ is periodic in time t with fixed period T such that $f(t) = f(t-T)$ with $T > 0$ show that for $s > 0$

$$\overline{f}(s) = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}\,dt.$$

Note that this enables a Laplace Transform to be found by performing the integral only over the period $(0, T)$ for which $f(t)$ is defined.

Solution: The function $f(t)$ is periodic in time t with fixed period T such that $f(t) =$

$f(t-T)$ with $T > 0$. Laplace transform (for $s > 0$) and split the domain up into an infinite set of successive intervals

$$\overline{f}(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^T f(t)e^{-st}\,dt + \int_T^{2T} f(t)e^{-st}\,dt + \int_{2T}^{3T} f(t)e^{-st}\,dt + \ldots$$

Now consider the integrals on the RHS: typically they all have the form $\int_{nT}^{(n+1)T} f(t)e^{-st}\,dt$ on the time interval $[nT,\,(n+1)T]$. Use a substitution $\tau_n = t - nT$ and appeal to the fact that $f(t)$ is periodic $f(\tau_n + nT) = f(\tau_n)$ to obtain

$$\int_{nT}^{(n+1)T} f(t)e^{-st}\,dt = e^{-snT}\int_0^T f(\tau_n)e^{-s\tau_n}\,d\tau_n$$

The labels on the dummy variables τ_n within the integrals don't matter if the limits are the same; that is $\int_0^T f(\tau_n)e^{-s\tau_n}\,d\tau_n = \int_0^T f(t)e^{-st}\,dt$. Hence we have

$$\overline{f}(s) = \left(1 + e^{-sT} + e^{-2sT} + \ldots\right)\int_0^T f(t)e^{-st}\,dt$$

For $s > 0$ the series sums to $\left(1 - e^{-sT}\right)^{-1}$ to give the answer.

4. Use the result of Q3 to show that the Laplace transform of the 'saw-tooth' function

$$\begin{array}{ll} f(t) = t & 0 \le t \le 1 \\ f(t) = f(t-1) & 1 \le t \end{array}$$

is given by

$$\overline{f}(s) = s^{-2} - s^{-1}\left(e^{-s} + e^{-2s} + e^{-3s}\ldots\ \right).$$

The function $f(t)$ is often used in electronics for representing discontinuous voltages.

Solution: This example uses $T = 1$ and the results of Q3 on the sawtooth function, which is a piece-wise linear periodic function

$$\overline{f}(s) = \left(1 - e^{-s}\right)^{-1}\int_0^1 te^{-st}\,dt$$

Now it is easily shown that

$$\int_0^1 te^{-st}\,dt = \frac{1 - e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

and so

$$\overline{f}(s) = \left(1 - e^{-s}\right)^{-1}\left[\frac{1 - e^{-s}}{s^2} - \frac{e^{-s}}{s}\right] = \left[\frac{1}{s^2} - \frac{1}{s}\left(\frac{e^{-s}}{1 - e^{-s}}\right)\right]$$

On expanding $(1 - e^{-s})^{-1}$ as a series $(s > 0)$ we have

$$\overline{f}(s) = \frac{1}{s^2} - \frac{1}{s}\left(\frac{e^{-s}}{1 - e^{-s}}\right) = \frac{1}{s^2} - \frac{1}{s}\left(e^{-s} + e^{-2s} + e^{-3s} + \ldots\right)$$