Laplace Transforms Solutions to Exercises

- Exercises have been classified according to the topics:

- Linearity, Shifting Theorems, Derivatives & Integrals
- Convolution
- Laplace Transforms in ODEs

Before attempting these exercises, you are strongly encouraged to go through the relevant portion of the notes and be familiar with the solutions to the examples presented in the notes.

1. Linearity, Shifting Theorems, Derivatives & Integrals

Question 1: Evaluate the Laplace Transform for each of the following functions: (i) $t^3 e^{-3t}$, (ii) $e^{-t}\cos(2t)$, (iii) $2e^{3t}\sin(4\mathrm{t})$, (iv) $(t+2)^2 e^t$, (v) $e^{-t}(3\sinh(2t) - 5\cosh(2t))$

(i) $f(t) = t^3$ so $F(s) = \frac{3!}{s^4}$. Then, the LT of $e^{-3t}f(t) = F(s+3) = \frac{3!}{(s+3)^4} = \frac{6}{(s+3)^4}$

(ii) $f(t) = \cos(2t)$ so $F(s) = \frac{s}{s^2+4}$. Then, the LT of $e^{-t}f(t) = F(s+1) = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$

(iii) $f(t) = 2\sin(4t)$ so $F(s) = 2 \cdot \frac{4}{s^2+16}$. Then, the LT of $e^{3t}f(t) = F(s-3) = \frac{8}{(s-3)^2+16} = \frac{8}{s^2-6s+25}$

(iv) $(t+2)^2 e^t = e^t(t^2+4t+4)$. So $f(t) = t^2+4t+4 \;\rightarrow\; F(s) = \frac{2!}{s^3} + \frac{4}{s^2} + \frac{4}{s}$.

Then, the LT of $e^t f(t) = \frac{2}{(s-1)^3} + \frac{4}{(s-1)^2} + \frac{4}{s-1} = \frac{2+4(s-1)+4(s-1)^2}{(s-1)^3} = \frac{4s^2-4s+2}{(s-1)^3}$

(v) $f(t) = 3\sinh(2t) - 5\cosh(2t) \;\rightarrow\; F(s) = 3 \cdot \frac{2}{s^2-4} - 5 \cdot \frac{s}{s^2-4}$. Then, LT of $e^{-t}f(t) = \frac{6}{(s+1)^2-4} - \frac{5(s+1)}{(s+1)^2-4} = \frac{1-5s}{(s+1)^2-4}$

2. Convolution

Question 1: Evaluate the inverse Laplace Transform for the following functions: (a) $s/(s^2+a^2)^2$ (b) $1/s^2(s+1)^2$

(a) We can write $\frac{s}{(s^2+a^2)^2} = \frac{s}{(s^2+a^2)} \cdot \frac{1}{s^2+a^2}$. $F(s) = \frac{s}{(s^2+a^2)} \rightarrow f(t) = \cos(at)$ and $G(s) = \frac{1}{(s^2+a^2)} = \frac{1}{a} \cdot \frac{a}{(s^2+a^2)} \rightarrow g(t) = \frac{\sin(at)}{a}$

The convolution theorem gives us the Laplace Transform pair: $\int_0^t f(\tau)g(t-\tau)\,d\tau \quad \leftrightarrow \quad F(s)G(s)$

So the inverse Laplace Transform of $F(s)G(s)$ is given by

$$= \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t \frac{\cos(a\tau)\sin(at-a\tau)}{a}\,d\tau = \int_0^t \frac{\cos(a\tau)\,[\sin(at)\cos(a\tau) - \cos(at)\sin(a\tau)]}{a}\,d\tau$$

$$= \frac{1}{a}\int_0^t [\sin(\mathrm{at}) \cdot \cos^2(a\tau) - \cos(at) \cdot \cos(a\tau)\sin(a\tau)]d\tau = \frac{\sin(at)}{a}\int_0^t \frac{1+\cos(2a\tau)}{2}\,d\tau - \frac{\cos(at)}{a}\int_0^t \cos(a\tau)\sin(a\tau)d\tau$$

$$= \frac{\sin(at)}{a}\left[\frac{t}{2} + \frac{\sin(2at)}{4a}\right] - \frac{\cos(at)}{a}\left[\frac{1}{2a}\sin^2(at)\right] = \frac{\sin(at)}{a}\left[\frac{t}{2} + \frac{\sin(at)\cos(at)}{2a}\right] - \frac{\cos(at)}{a}\left[\frac{1}{2a}\sin^2(at)\right]$$

$$= \frac{t\sin(at)}{2a}$$

2. Convolution

Question 1: Evaluate the inverse Laplace Transform for the following functions: (a) $s/(s^2+a^2)^2$ (b) $1/s^2(s+1)^2$

(b) We can write $\frac{1}{s^2(s+1)^2} = \frac{1}{s^2} \cdot \frac{1}{(s+1)^2}$. $F(s) = \frac{1}{s^2} \rightarrow f(t) = t$ and $G(s) = \frac{1}{(s+1)^2} \rightarrow g(t) = te^{-t}$

The convolution theorem gives us the Laplace Transform pair: $\int_0^t f(\tau)g(t-\tau)\,d\tau \leftrightarrow F(s)G(s)$

So the inverse Laplace Transform of $F(s)G(s)$ is given by

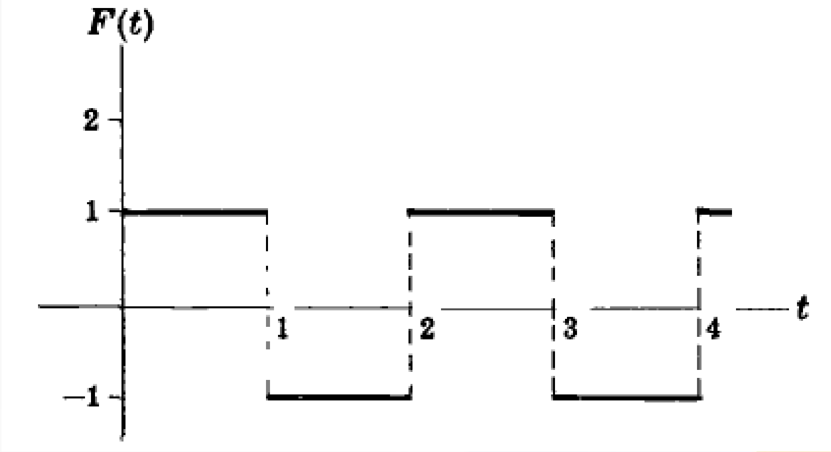
$$= \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t \tau(t-\tau)e^{-(t-\tau)}\,d\tau = e^{-t}\int_0^t \left(t\tau - \tau^2\right)e^{\tau}\,d\tau$$

$$= te^{-t}\int_0^t \tau e^{\tau} d\tau - e^{-t}\int_0^t \tau^2 e^{\tau} d\tau$$

$$= te^{-t}[te^t - e^t + 1] - e^{-t}[t^2e^t - 2te^t + 2e^t - 2]$$

$$= t^2 - t + te^{-t} - (t^2 - 2t + 2 - 2e^{-t}) = t + te^{-t} - 2 + 2e^{-t}$$

3. Laplace Transform of Special Functions



Question 1: Obtain, from first principles, the Laplace Transform of the function $F(t)$ given on the right.

We first recognize that $F(t)$ is a periodic function with period $T = 2$. Using first principles means that we start with the definition and work our way. So the Laplace Transform is then

$$\int_0^{\infty} e^{-st} F(t)\, dt = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\, dt$$

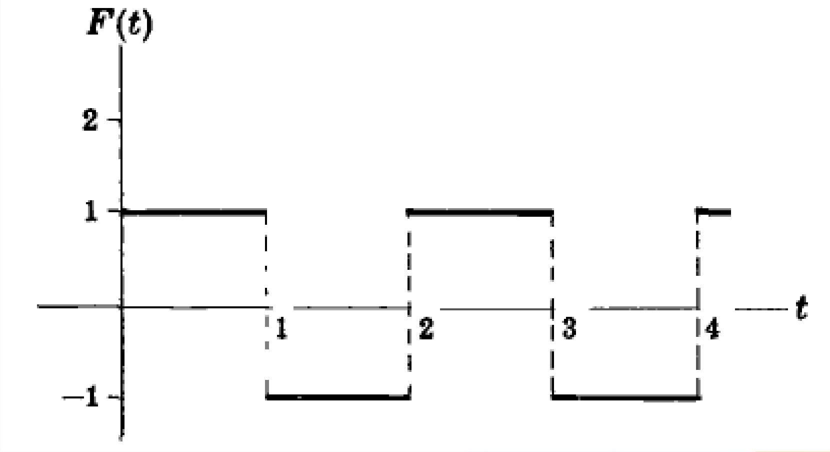
Apply a change of variables: $u = t - nT$

$$\sum_{n=0}^{\infty} \int_0^{T} e^{-s(u+nT)} f(u+nT)\, du = \sum_{n=0}^{\infty} e^{-n(sT)} \int_0^{T} e^{-su} f(u+nT)\, du = \sum_{n=0}^{\infty} e^{-n(sT)} \int_0^{T} e^{-su} f(u)\, du$$

where we have exploited the periodicity of the function: $f(u) = f(u+nT)$ for $n = 0,1,2,\ldots$

3. Laplace Transform of Special Functions

Question 1: Obtain, from first principles, the Laplace Transform of the function $F(t)$ given on the right.



Further, we see that

$$\sum_{n=0}^{\infty} e^{-n(sT)} = \sum_{n=0}^{\infty} (e^{-sT})^n = \frac{1}{1-e^{-sT}}$$

is an infinite geometric series where $e^{-sT} < 1$ and $s > 0$. Thus we have

$$\int_0^{\infty} e^{-st} f(t)\, dt = \frac{\int_0^T e^{-su} f(u)\, du}{1-e^{-sT}} = \frac{\int_0^1 e^{-su}\,(1) du + \int_1^2 e^{-su}(-1)\, du}{1-e^{-2s}} = -\frac{1}{s(1-e^{-2s})}\left[e^{-s\cdot 1} - e^{-s\cdot 0} - e^{-s\cdot 2} + e^{-s\cdot 1}\right]$$

$$\int_0^{\infty} e^{-st} f(t)\, dt = \frac{(1-e^{-s})^2}{s(1-e^{-s})(1+e^{-s})} = \frac{1-e^{-s}}{s(1+e^{-s})} = \frac{e^{-\frac{s}{2}}\left(e^{\frac{s}{2}} - e^{-\frac{s}{2}}\right)}{se^{\left(-\frac{s}{2}\right)}\left(e^{\frac{s}{2}} + e^{-\frac{s}{2}}\right)} = \frac{1}{s}\tanh\left(\frac{s}{2}\right)$$

We can leave it as the above (in blue) or “simplify” it further but it is really not necessary unless requested in question.

4. Laplace Transforms in ODEs

Question 1: Use Laplace Transforms to solve $y'' + 9y = \cos(2t)$ where $y(0) = 1$ and $y\left(\frac{\pi}{2}\right) = -1$

Let the Laplace Transform of $y(t)$ be $Y(s)$. Then, the Laplace Transform of $y'' = s^2Y(s) - sy(0) - y'(0)$.

Note that $y'(0)$ is not known at this time. All we know is that it is a constant, so let us call it c.

So, the Laplace Transform of the ODE is $s^2Y(s) - sy(0) - c + 9Y(s) = \frac{s}{s^2+4}$

$$\rightarrow \quad (s^2 + 9)Y(s) - s - c = \frac{s}{s^2 + 4}$$

$$\rightarrow \quad Y(s) = \frac{s}{(s^2 + 9)} + \frac{c}{(s^2 + 9)} + \frac{s}{(s^2+4)(s^2 + 9)}$$

Perform partial fraction decomposition for the 3rd term on the right hand side.

Let $\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$ and find A, B, C and D.

4. Laplace Transforms in ODEs

Question 1: Use Laplace Transforms to solve $y'' + 9y = \cos(2t)$ where $y(0) = 1$ and $y\left(\frac{\pi}{2}\right) = -1$

$$\frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$\rightarrow \quad s = (As+B)(s^2+9) + (Cs+D)(s^2+4) = (A+C)s^3 + (B+D)s^2 + (9A+4C)s + (9B+4D)$$

This implies $A + C = 0,\ \ B + D = 0,\ \ 9B + 4D = 0,\ \ 9A + 4C = 1$. So, we get $B = D = 0$ and $A = -C$ and $-5C = 1 \ \rightarrow \ C = 1/5$

$$\rightarrow \ Y(s) = \frac{s}{(s^2+9)} + \frac{c}{(s^2+9)} + \frac{s}{(s^2+4)(s^2+9)} = \frac{s}{(s^2+9)} + \frac{c}{(s^2+9)} + \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)}$$

Taking the inverse Laplace Transform, we have $y(t) = \frac{4}{5}\cos(3t) + \frac{c}{3}\sin(3t) + \frac{1}{5}\cos(2t)$.

Using the initial and boundary conditions, we finally obtain $y(t) = \frac{4}{5}\cos(3t) + \frac{4}{5}\sin(3t) + \frac{1}{5}\cos(2t)$.

4. Laplace Transforms in ODEs

Question 2: Use Laplace Transforms to solve $y'' + a^2 y = f(t)$ where $y(0) = 1$ and $y'(0) = -2$

Let the Laplace Transform of $y(t)$ be $Y(s)$. Then, the Laplace Transform of $y'' = s^2 Y(s) - sy(0) - y'(0)$.

So, the Laplace Transform of the ODE is $s^2 Y(s) - s + 2 + a^2 Y(s) = F(s)$

$$\rightarrow \quad (s^2 + a^2)Y(s) - s + 2 = F(s)$$

$$\rightarrow \quad Y(s) = \frac{s}{(s^2 + a^2)} - \frac{2}{(s^2 + a^2)} + \frac{F(s)}{(s^2 + a^2)} = \frac{s}{(s^2 + a^2)} - \frac{2}{(s^2 + a^2)} + F(s) \cdot \frac{1}{(s^2 + a^2)}$$

We recall the definition of convolution and so the inverse Laplace Transform is $f(t) * g(t)$

Taking the inverse Laplace Transform, we have $y(t) = \cos(at) - \frac{2}{a}\sin(at) + f(t) * \frac{\sin(at)}{a}$.

$$\therefore \quad y(t) = \cos(at) - \frac{2}{a}\sin(at) + \int_0^t \frac{1}{a} f(\tau) \sin[a(t - \tau)]\, d\tau$$

4. Laplace Transforms in ODEs

Question 3: Solve the coupled system of first order ODE $x' = 2x - 3y$ and $y' = y - 2x$, subject to $x(0) = 8$ and $y(0) = 3$.

Let the Laplace Transform of $x(t)$ and $y(t)$ be given by $X(s)$ and $Y(s)$ respectively. Then, the ODE transforms to

$$sX(s) - x(0) = sX(s) - 8 = 2X(s) - 3Y(s)$$

$$sY(s) - y(0) = sY(s) - 3 = -2X(s) + Y(s)$$

We see that what we have is simultaneous equations in $X(s)$ and $Y(s)$ which need to be solved in terms of s.

Solving It algebraically or by matrix method is acceptable. In any case, we should obtain

$$(s-2)X(s) + 3Y(s) = 8$$

$$2X(s) + (s-1)Y(s) = 3$$

$$X(s) = \frac{8s-17}{s^2-3s-4} = \frac{8s-17}{(s-4)(s+1)} \quad , \quad Y(s) = \frac{3s-22}{s^2-3s-4} = \frac{3s-22}{(s-4)(s+1)}$$

4. Laplace Transforms in ODEs

Question 3: Solve the coupled system of first order ODE $x' = 2x - 3y$ and $y' = y - 2x$, subject to $x(0) = 8$ and $y(0) = 3$.

We need to now apply partial fraction decomposition for $X(s)$:

$$\frac{8s - 17}{(s-4)(s+1)} = \frac{A}{s-4} + \frac{B}{s+1} \quad \rightarrow \quad 8s - 17 = (A+B)s + (A - 4B) \quad \rightarrow \quad A = 3, \qquad B = 5$$

We need to now apply partial fraction decomposition for $Y(s)$:

$$\frac{3s - 22}{(s-4)(s+1)} = \frac{A}{s-4} + \frac{B}{s+1} \quad \rightarrow \quad 3s - 22 = (A+B)s + (A - 4B) \quad \rightarrow \quad A = -2, \qquad B = 5$$

Applying the inverse Laplace Transform, we then obtain

$$x(t) = 5e^{-t} + 3e^{4t}, \qquad y(t) = 5e^{-t} - 2e^{4t}$$

4. Laplace Transforms in ODEs

Question 4: Solve the ODE $y'' + 9y = \begin{cases} 8\cos(t), 0 \le t < \pi \\ 0 \quad , t > \pi \end{cases}$ subject to the initial conditions $y(0) = 0,\ y'(0) = 4.$

The first thing to do is to recognize the right hand side of the ODE as $8\cos(t)H(t) - 8\cos(t)\,H(t-\pi)$.

Now we can apply the Laplace Transform to the entire equation: $y'' + 9y = 8\cos(t)H(t) - 8\cos(t)\,H(t-\pi)$

In anticipation of the application of the t-shift theorem (ref slide 7 of notes), we modify the second term on the right hand side with $8\cos(t) = -8\cos(t-\pi)$. So the right hand side becomes $8\cos(t)H(t) - 8\cos(t-\pi)\,H(t-\pi)$

If the Laplace Transform of $y(t)$ is given by $Y(s)$, then we obtain

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = s^2Y(s) - 4 + 9Y(s) = \frac{8s}{s^2+1}e^{-0\cdot s} + \frac{8s}{s^2+1}e^{-\pi\cdot s}$$

$$\rightarrow \quad Y(s) = \frac{4}{s^2+9} + \frac{8s}{(s^2+9)(s^2+1)} + \frac{8s}{(s^2+9)(s^2+1)}e^{-\pi\cdot s}$$

4. Laplace Transforms in ODEs

Question 4: Solve the ODE $y'' + 9y = \begin{cases} 8\cos(t), 0 \le t < \pi \\ 0 \quad , t > \pi \end{cases}$ subject to the initial conditions $y(0) = 0,\ y'(0) = 4.$

Applying the method of partial fractions, we consider

$$\frac{8s}{(s^2+9)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9} \quad \rightarrow \quad 8s = (As+B)(s^2+9) + (Cs+D)(s^2+1)$$

$$\rightarrow \quad 8s = s^3(A+C) + s^2(B+D) + s(9A+C) + (9B+D)$$

This means $A = -C,\ B = -D,\ 8A = 8$ and $8D = 0$. So $\boldsymbol{B = D = 0, A = 1}$ and $\boldsymbol{C = -1}$

$$Y(s) = \frac{4}{s^2+9} + \frac{8s}{(s^2+9)(s^2+1)} + \frac{8s}{(s^2+9)(s^2+1)} e^{-\pi\cdot s} = \frac{4}{s^2+9} + \left(\frac{s}{(s^2+1)} - \frac{s}{(s^2+9)}\right)(1+e^{-\pi s})$$

$$\rightarrow \quad Y(s) = \frac{4}{s^2+9} + \left(\frac{s}{(s^2+1)} - \frac{s}{(s^2+9)}\right) + \left(\frac{s}{(s^2+1)} - \frac{s}{(s^2+9)}\right)e^{-\pi s}$$

4. Laplace Transforms in ODEs

Question 4: Solve the ODE $y'' + 9y = \begin{cases} 8\cos(t), & 0 \le t < \pi \\ 0, & t > \pi \end{cases}$ subject to the initial conditions $y(0) = 0,\ y'(0) = 4$.

Now we refer to the table and see that the inverse transform of the following functions:

$$\frac{s}{(s^2+1)} \to \cos(t)H(t), \qquad \frac{s}{(s^2+9)} \to \cos(3t)\,H(t),$$

$$e^{-\pi s}\frac{s}{(s^2+1)} \to \cos(t-\pi)H(t-\pi), \qquad e^{-\pi s}\frac{s}{(s^2+9)} \to \cos[3(t-\pi)]\,H(t-\pi)$$

$$Y(s) = \frac{4}{s^2+9} + \left(\frac{s}{(s^2+1)} - \frac{s}{(s^2+9)}\right) + \left(\frac{s}{(s^2+1)} - \frac{s}{(s^2+9)}\right)e^{-\pi s}$$

$$\to\ y(t) = \frac{4}{3}\sin(3\mathrm{t}) + \big(\cos(\mathrm{t}) - \cos(3\mathrm{t})\big) + \{\cos(\mathrm{t}-\pi)H(\mathrm{t}-\pi) + \cos[3(\mathrm{t}-\pi)]\,H(\mathrm{t}-\pi)\}$$

$$\to\ y(t) = \frac{4}{3}\sin(3\mathrm{t}) + [\cos(t) - \cos(3t)]\,[H(t) - H(t-\pi)]$$