EE2-08 Mathematics

Example Sheet 5: Complex Integration

The residue of a complex function $F(z)$ at a pole $z = a$ of multiplicity m is given by

$$\lim_{z \to a} \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m F(z) \right\} \right] .$$

1. By taking the contour C as the unit circle $|z| = 1$ (positive is anti-clockwise), evaluate the following contour integrals $\oint_C F(z)dz$:

   (a) $F(z) = (z^2 - 2z)^{-1}$,
   (b) $F(z) = (z+1)(4z^3 - z)^{-1}$,
   (c) $F(z) = z(1 + 9z^2)^{-1}$.

   Remember to include only those poles which lie inside C. *Answers:* a) $-\pi i$, b) 0, c) $2\pi i/9$.

2. Use the Residue Theorem to show that

   $$\oint_C \frac{z\,dz}{(z-i)^2} = 2\pi i .$$

   where the contour C is the rectangle with vertices at $\pm\frac{1}{2} + 2i$ and $\pm\frac{1}{2} - 2i$.

3. Show that

   $$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \tfrac{1}{2}\pi .$$

4. Given the real integral

   $$I = \int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2} \qquad\qquad (|p| \neq 1)$$

   show that the substitution $z = e^{i\theta}$ converts it into

   $$I = \frac{i}{p} \oint_C \frac{dz}{(z-p)(z-p^{-1})},$$

   where C is the unit circle $|z| = 1$. Evaluate the residues at the poles and hence show that

   (i) $I = -2\pi \left(p^2 - 1\right)^{-1}$ when $|p| < 1$,
   (ii) $I = +2\pi \left(p^2 - 1\right)^{-1}$ when $|p| > 1$.

5. By choosing a suitable contour in the upper half of the complex plane, use the Residue Theorem & Jordan's Lemma to show that for $a > b > 0$

   $$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right) .$$