EE2 Mathematics – Probability & Statistics

Solution 4

1. If X is the number of particles emitted in a half-second interval, then $X \sim \text{Poisson}(1.6)$, so

$$\begin{aligned} P(X \le 2) &= P(X=0) + P(X=1) + P(X=2) \\ &= \frac{e^{-1.6}1.6^0}{0!} + \frac{e^{-1.6}1.6^1}{1!} + \frac{e^{-1.6}1.6^2}{2!} \\ &\approx 0.202 + 0.323 + 0.258 = 0.783 \end{aligned}$$

2. If $T \sim \text{Exp}(2)$, we know that $F_T(t) = 1 - e^{-2t}$ for $t > 0$.

   (a) $P(T \le 1) = F_T(1) = 1 - e^{-2} \approx 0.865$

   (b) $P(T > 3) = 1 - F_T(3) = e^{-6} \approx 0.0025$

   (c) $P(T > 3 | T > 2) = \frac{P(T>3 \cap T>2)}{P(T>2)} = \frac{1-F_T(3)}{1-F_T(2)} = \frac{e^{-6}}{e^{-4}} \approx 0.135$.
   This is the same as $P(T > 1)$, because the exponential is memoryless.

3. Consider the random variable X with probability density function

$$f_X(x) = k \cos x\,, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

   and zero otherwise.

   (a) $\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\pi/2}^{\pi/2} k \cos x dx = k[\sin x]_{-\pi/2}^{\pi/2} = 2k$, so $k = 1/2$ in order to guarantee $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

   (b) $\mathrm{E}(X) = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\pi/2}^{\pi/2} x \frac{\cos x}{2} dx = 0$, by symmetry.

$$\begin{aligned} \mathrm{Var}(X) = \mathrm{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f_X(x)dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} x^2 \cos x\, dx \\ &= \frac{1}{2}\left[\left[x^2 \sin x\right]_{-\pi/2}^{\pi/2} - 2\int_{-\pi/2}^{\pi/2} x \sin x\, dx \right] \\ &\qquad\qquad (\text{using } u = x^2 \text{ and } dv = \cos x\, dx) \\ &= \frac{1}{2}\left[\left[x^2 \sin x + 2x\cos x\right]_{-\pi/2}^{\pi/2} - 2\int_{-\pi/2}^{\pi/2} \cos x\, dx \right] \\ &\qquad\qquad (\text{using } u = x \text{ and } dv = \sin x\, dx) \\ &= \frac{1}{2}[x^2 \sin x + 2x \cos x - 2\sin x]_{-\pi/2}^{\pi/2} = \frac{\pi^2}{4} - 2 \approx 0.467. \end{aligned}$$

(c) Start by finding the CDF. For $t \in [-\pi/2, \pi/2]$ we have

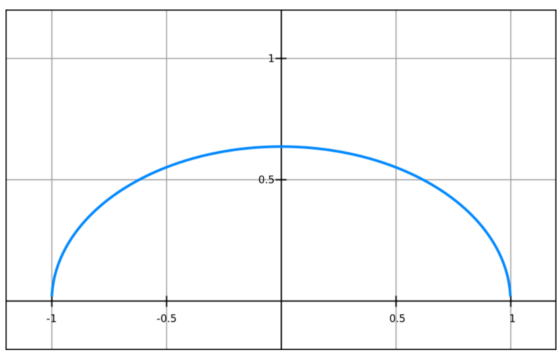
$$F_X(x) = \int_{-\infty}^{x} f_X(u)du = \int_{-\pi/2}^{x} \frac{\cos u}{2} du = \left[\frac{\sin u}{2}\right]_{-\pi/2}^{x} = \frac{\sin x + 1}{2}.$$

The full CDF is, thus,

$$F_X(x) = \begin{cases} 0 & x < -\pi/2 \\ (\sin x + 1)/2 & -\pi/2 \le x \le \pi/2 \\ 1 & x > \pi/2 \end{cases}$$

We then have $F_X(q_u) = 0.75 \Leftrightarrow q_u = \arcsin(1/2) = \pi/6 \approx 0.523$. By symmetry, $F_X(q_l) = 0.25 \Leftrightarrow q_l = \arcsin(-1/2) = -\pi/6 \approx -0.523$, and IQR $= \pi/3 \approx 1.046$.

4. $\int_{-\infty}^{\infty} f_X(x)dx = K\int_{-1}^{1} \sqrt{1-x^2}dx = K\pi/2$, because the integrand is the unit semicircle. Thus, $K = 2/\pi$ in order to guarantee $\int_{-\infty}^{\infty} f_X(x)dx = 1$.



Alternatively, make the change of variable $x = \sin u$ with $-\pi/2 \le u \le \pi/2$ such that

$$\begin{aligned} K\int_{-1}^{1} \sqrt{1-x^2}dx &= K\int_{-\pi/2}^{\pi/2} \cos^2 u \, du \\ &= \frac{K}{2}\int_{-\pi/2}^{\pi/2} (1 + \cos 2u) \, du \\ &= \frac{K}{2}\left[u + \frac{1}{2}\sin 2u\right]_{-\pi/2}^{\pi/2} = K\frac{\pi}{2}. \end{aligned}$$

Thus, $K = 2/\pi$.

5. $f(y)$ is always positive, so we just need to check that it integrates to 1,

$$\int_{-\infty}^{\infty} f(y)dy = \int_{0}^{\infty} \frac{y f_X(y)}{\mu} dy = \frac{1}{\mu} \underbrace{\int_{0}^{\infty} y f_X(y)dy}_{=\mathrm{E}(X)=\mu} = 1.$$

Now let Y be a random variable with this density function, and notice that

$$\mathrm{E}(Y^k) = \int_{-\infty}^{\infty} y^k f(y) dy = \int_0^{\infty} \frac{y^{k+1} f_X(y)}{\mu} dy = \frac{\mathrm{E}(X^{k+1})}{\mu},$$

which implies $\mathrm{E}(X^{k+1}) = \mathrm{E}(X)\,\mathrm{E}(Y^k)$. We can now prove the inequality directly:

$$\begin{aligned}
\mathrm{E}(X^3)\,\mathrm{E}(X) &= \mathrm{E}(X)\,\mathrm{E}(Y^2)\,\mathrm{E}(X)\,\mathrm{E}(Y^0) = \mathrm{E}(X)^2\,\mathrm{E}(Y^2) \\
&\geq \mathrm{E}(X)^2\,\mathrm{E}(Y)^2 \qquad (\text{recall that } \mathrm{Var}(Y) = \mathrm{E}(Y^2) - \mathrm{E}(Y)^2 \geq 0) \\
&= \left\{\mathrm{E}(X^2)\right\}^2 \qquad (\text{from above, recall that } \mathrm{E}(X^2) = \mathrm{E}(X)\,\mathrm{E}(Y))
\end{aligned}$$

6. (a) $F_X(u) = \int_{-\infty}^{u} f_X(x)\,dx = \frac{1}{\pi}\int_{-\infty}^{u} \frac{1}{(1+x^2)}\,dx$. Make the substitution $x = \tan\theta$ (with $dx = \sec^2\theta\,d\theta$) such that

$$\begin{aligned}
\int_{-\infty}^{u} \frac{1}{(1+x^2)}\,dx &= \int_{\theta_l}^{\theta_u} \frac{1}{\underbrace{(1+\tan^2\theta)}_{=\sec^2\theta}} \sec^2\theta\,d\theta \\
&= \int_{\theta_l}^{\theta_u} d\theta = \theta_u - \theta_l = \arctan u + \frac{\pi}{2}
\end{aligned}$$

where $\theta_u = \arctan u$ and $\theta_l = -\pi/2$. We finally get $F_X(u) = \frac{1}{\pi}\arctan u + \frac{1}{2}$, for all u.

(b) $F_X(u) = \int_{-\infty}^{u} f_X(x)\,dx = \int_{-\infty}^{u} \frac{e^{-x}}{(1+e^{-x})^2}\,dx = \left[\frac{1}{1+e^{-x}}\right]_{-\infty}^{u} = \frac{1}{1+e^{-u}}$, for all u.

(c) $F_X(u) = \int_{-\infty}^{u} f_X(x)\,dx = \int_0^u \frac{(a-1)}{(1+x)^a}\,dx = \left[\frac{-1}{(1+x)^{a-1}}\right]_0^u = 1 - \frac{1}{(1+u)^{a-1}}$, for $u > 0$.

(d) $F_X(u) = \int_{-\infty}^{u} f_X(x)\,dx = \int_0^u c\tau x^{\tau-1} e^{-cx^\tau}\,dx = \left[-e^{-cx^\tau}\right]_0^u = 1 - e^{-cu^\tau}$, for all $u > 0$.

7. (a) $f_X(x) = e^{-kx} x^{r-1} k^r/(r-1)!$, $x > 0$, $r = 1, 2, 3, ...$, $k > 0$.

$$\mathrm{E}(X) = \int_0^{\infty} x e^{-kx} x^{r-1} \frac{k^r}{(r-1)!}\,dx = \frac{k^r}{(r-1)!}\int_0^{\infty} e^{-kx} x^r\,dx.$$

Making use of integration by parts with $u = x^r$ and $dv = e^{-kx}\,dx$, we get

$$\begin{aligned}
\mathrm{E}(X) &= \frac{k^r}{(r-1)!}\left[\frac{-x^r e^{-kx}}{k}\right]_0^{\infty} + \frac{k^r}{(r-1)!}\frac{r}{k} \underbrace{\int_0^{\infty} e^{-kx} x^{r-1}\,dx}_{=\frac{(r-1)!}{k^r}\ (f_X(x) \text{ is a valid pdf})} \\
&= \frac{k^r}{(r-1)!}[0-0] + \frac{k^r}{(r-1)!}\frac{r}{k}\frac{(r-1)!}{k^r} = \frac{r}{k}.
\end{aligned}$$

$$\mathrm{E}(X^2) = \int_0^\infty x^2 e^{-kx} x^{r-1} \frac{k^r}{(r-1)!}\, dx = \frac{k^r}{(r-1)!} \int_0^\infty e^{-kx} x^{r+1}\, dx.$$

Here again, we can use integration by parts

$$\begin{aligned}
\int e^{-kx} x^{r+1}\, dx &= \frac{-x^{r+1} e^{-kx}}{k} + \frac{r+1}{k} \int e^{-kx} x^r\, dx \\
&= \frac{-x^{r+1} e^{-kx}}{k} + \frac{r+1}{k} \left[\frac{-x^r e^{-kx}}{k} + \frac{r}{k} \int e^{-kx} x^{r-1}\, dx \right]
\end{aligned}$$

such that

$$\begin{aligned}
&\int_0^\infty e^{-kx} x^{r+1}\, dx \\
&\quad = \left[\frac{-x^{r+1} e^{-kx}}{k} - \frac{r+1}{k} \frac{x^r e^{-kx}}{k} \right]_0^\infty + \frac{r+1}{k} \frac{r}{k} \underbrace{\int_0^\infty e^{-kx} x^{r-1}\, dx}_{= \frac{(r-1)!}{k^r}} \\
&\quad = [0 - 0] + \frac{r+1}{k} \frac{r}{k} \frac{(r-1)!}{k^r}.
\end{aligned}$$

We finally get

$$\mathrm{E}(X^2) = \frac{k^r}{(r-1)!} \frac{r+1}{k} \frac{r}{k} \frac{(r-1)!}{k^r} = \frac{r(r+1)}{k^2}$$

and

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = \frac{r(r+1)}{k^2} - \frac{r^2}{k^2} = \frac{r}{k^2}$$

(b) $f_X(x) = \binom{x+a-1}{x} p^x (1-p)^a$, $x \in \mathbb{N}_0$, $a \in \mathbb{N}^*$, $p \in (0,1)$.

$$
\begin{aligned}
\mathrm{E}(X) &= \sum_{x=0}^{\infty} x \binom{x+a-1}{x} p^x (1-p)^a \\
&= \sum_{x=1}^{\infty} x \frac{(x+a-1)!}{x!(a-1)!} p^x (1-p)^a && \text{(ignore the term for } x = 0\text{)} \\
&= \sum_{x=1}^{\infty} \frac{(x+a-1)!}{(x-1)!(a-1)!} p^x (1-p)^a && \text{(cancel out } x\text{)} \\
&= \sum_{y=0}^{\infty} \frac{(y+a)!}{y!(a-1)!} p^{y+1} (1-p)^a && \text{(set } y = x-1\text{)} \\
&= \sum_{y=0}^{\infty} \frac{(y+b-1)!}{y!(b-2)!} p^{y+1} (1-p)^{b-1} && \text{(set } b = a+1\text{)} \\
&= \frac{(b-1)p}{1-p} \underbrace{\sum_{y=0}^{\infty} \frac{(y+b-1)!}{y!(b-1)!} p^y (1-p)^b}_{=1} \\
& && \text{(} f_Y(y) \text{ is a valid neg. binomial pmf)} \\
&= \frac{ap}{1-p}
\end{aligned}
$$

Following similar derivations, we compute

$$\begin{aligned}
\mathrm{E}(X^2) &= \sum_{x=0}^{\infty} x^2 \binom{x+a-1}{x} p^x (1-p)^a \\
&= \sum_{x=1}^{\infty} x^2 \frac{(x+a-1)!}{x!(a-1)!} p^x (1-p)^a \quad \text{(ignore the term for } x=0) \\
&= \sum_{x=1}^{\infty} x \frac{(x+a-1)!}{(x-1)!(a-1)!} p^x (1-p)^a \qquad \text{(cancel out } x) \\
&= \sum_{y=0}^{\infty} (y+1) \frac{(y+a)!}{y!(a-1)!} p^{y+1} (1-p)^a \qquad \text{(set } y = x-1) \\
&= \sum_{y=0}^{\infty} (y+1) \frac{(y+b-1)!}{y!(b-2)!} p^{y+1} (1-p)^{b-1} \qquad \text{(set } b = a+1) \\
&= \frac{(b-1)p}{1-p} \left[\underbrace{\sum_{y=0}^{\infty} y \frac{(y+b-1)!}{y!(b-1)!} p^y (1-p)^b}_{=\mathrm{E}(Y)=\frac{bp}{1-p}} + \underbrace{\sum_{y=0}^{\infty} \frac{(y+b-1)!}{y!(b-1)!} p^y (1-p)^b}_{=1} \right] \\
&\qquad\qquad (f_Y(y) \text{ is a valid neg. binomial pmf}) \\
&= \frac{ap}{1-p} \left[\frac{(a+1)p}{1-p} + 1 \right] = \frac{ap(ap+1)}{(1-p)^2}
\end{aligned}$$

such that

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2 = \frac{ap(ap+1)}{(1-p)^2} - \left(\frac{ap}{1-p} \right)^2 = \frac{ap}{(1-p)^2}$$

8. (a) i. Exponential

$$\begin{aligned}
f_X(x) &= \begin{cases} \theta e^{-\theta x} & x > 0 \\ 0 & \textit{otherwise} \end{cases} \\
F_X(x) &= \begin{cases} 1 - e^{-\theta x} & x > 0 \\ 0 & \textit{otherwise} \end{cases} \\
\bar{F}_X(x) = 1 - F_X(x) &= \begin{cases} e^{-\theta x} & x > 0 \\ 1 & \textit{otherwise} \end{cases} \\
\lambda(x) = \frac{f_X(x)}{\bar{F}_X(x)} &= \begin{cases} \theta & x > 0 \\ 0 & \textit{otherwise} \end{cases}
\end{aligned}$$

The hazard rate is constant for all $x > 0$.

ii. Weibull

$$f_X(x) = c\tau x^{\tau-1}e^{-cx^\tau} \qquad (0 < x < \infty)$$
$$F_X(x) = 1 - e^{-cx^\tau} \qquad (0 < x < \infty)$$
$$\bar{F}_X(x) = 1 - F_X(x) = e^{-cx^\tau} \qquad (0 < x < \infty)$$
$$\lambda(x) = \frac{f_X(x)}{\bar{F}_X(x)} = c\tau x^{\tau-1} \qquad (0 < x < \infty)$$

Increasing hazard

iii. Pareto

$$f_X(x) = \frac{(a-1)}{(1+x)^a} \qquad (0 < x < \infty)$$
$$F_X(x) = 1 - \frac{1}{(1+x)^{(a-1)}} \qquad (0 < x < \infty)$$
$$\bar{F}_X(x) = 1 - F_X(x) = \frac{1}{(1+x)^{(a-1)}} \qquad (0 < x < \infty)$$
$$\lambda(x) = \frac{f_X(x)}{\bar{F}_X(x)} = \frac{(a-1)}{(1+x)} \qquad (0 < x < \infty)$$

Decreasing hazard

(b) Using the hint

$$\begin{aligned}\frac{d}{dx}\log\frac{\bar{F}(x+y)}{\bar{F}(x)} &= \frac{d}{dx}\left[\log\bar{F}(x+y) - \log\bar{F}(x)\right]\\ &= \frac{-f_X(x+y)}{\bar{F}(x+y)} + \frac{f_X(x)}{\bar{F}(x)} = \lambda(x) - \lambda(x+y)\end{aligned}$$

Hence the hazard rate does not decrease (i.e. $\lambda(x) - \lambda(x+y) \leq 0$) if $\frac{d}{dx}\log\frac{\bar{F}(x+y)}{\bar{F}(x)} \leq 0$, i.e. $\frac{\bar{F}(x+y)}{\bar{F}(x)}$ does not increase as x increases.