EE2-08 Mathematics

Solutions to Example Sheet 5: Functions of a complex variable

1. Verify that the following satisfy the Cauchy-Riemann equations:

   a) $u = x; \quad v = y\,,$

   b) $u = e^x \cos y; \quad v = e^x \sin y\,,$

   c) $u = x^3 - 3xy^2; \quad v = 3x^2y - y^3$

   Solution:

   To verify that the following satisfy the Cauchy-Riemann equations $u_x = v_y \;\; v_x = -u_y$:

   a) $u_x = 1 \;\; v_y = 1$; $u_y = v_x = 0$. $\therefore$ CR equations satisfied.

   b) $u = e^x \cos y \;\Rightarrow\; u_x = e^x \cos y, \quad u_y = -e^x \sin y$.
   $v = e^x \sin y \;\Rightarrow\; v_y = e^x \cos y, \;\; v_x = e^x \sin y$. $\therefore$ CR equations satisfied.

   c) $u = x^3 - 3xy^2 \;\Rightarrow\; u_x = 3x^2 - 3y^2, \quad u_y = -6xy$.
   $v = 3x^2y - y^3 \;\Rightarrow\; v_x = 6xy, \;\; v_y = 3x^2 - 3y^2$. $\therefore$ CR equations satisfied.

2. Show that the following functions $u(x, y)$ each satisfy Laplace's equation and then use the Cauchy-Riemann equations to determine the conjugate function v. Find also $f(z) = u+iv$.

   $$a) \;\; u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1\,, \qquad\qquad b) \;\; u = xy\,.$$

   Solution:

   (a) With $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ we have $u_x = 3x^2 - 3y^2 + 6x$ and $u_y = -6xy - 6y$. Therefore $u_{xx} = 6x + 6$ and $u_y = -6x - 6$ and so $u_{xx} + u_{yy} = 0$. Because u satisfies Laplace's equation, there exists a conjugate function $v(x, y)$ that satisfies the CR equations: $u_x = v_y, \;\; v_x = -u_y$. To find v we integrate these

   $$v_y = u_x = 3x^2 - 3y^2 + 6x \quad\Rightarrow\quad v = \int (3x^2 - 3y^2 + 6x)\,dy + A(x)$$
   $$v_x = -u_y = 6xy + 6y \quad\Rightarrow\quad v = \int (6xy + 6y)\,dx + B(y)$$

   where $A(x)$ and $B(y)$ are arbitrary functions of x and y respectively. The solution(s) for v must be the same from each equation; together we find that $v = 3x^2y - y^3 + 6xy + c$ where $A = c$ and $B = c - y^3$ with c as an arbitrary constant. In combination $f(z) = u + iv = z^3 + 3z^2 + 1 + ci$.

   (b) $u = xy$ we have $u_x = y$ and $u_y = x$. Therefore $u_{xx} = 0$ and $u_{yy} = 0$ and so $u_{xx} + u_{yy} = 0$. Because u satisfies Laplace's equation, there exists a conjugate function $v(x, y)$ that satisfies the CR equations: $u_x = v_y, \;\; v_x = -u_y$. To find v we integrate these

   $$v_y = u_x = y \quad\Rightarrow\quad v = \int y\,dy + A(x)$$
   $$v_x = -u_y = -x \quad\Rightarrow\quad v = -\int x\,dx + B(y)$$

Together we find that $v = \frac{1}{2}(y^2 - x^2) + c$ where $A(x) = -\frac{1}{2}x^2 + c$ and $B(y) = \frac{1}{2}y^2 + c$. In combination we find that $f(z) = u + iv = -\frac{1}{2}iz^2$+const.

3. Show that the function

$$u(x, y) = (x \cos y - y \sin y)\, e^x$$

satisfies Laplace's equation. Find the conjugate function $v(x, y)$ for which u and v together satisfy the Cauchy-Riemann equations and hence find in its simplest form $w = u+iv = f(z)$ where $z = x + iy$.

Solution:

To show that the function $u(x, y) = e^x\,(x \cos y - y \sin y)$ satisfies Laplace's equation:

$$\begin{aligned} u_x &= e^x\,(\cos y + x \cos y - y \sin y) \quad \Rightarrow \quad u_{xx} = e^x\,(2\cos y + x \cos y - y \sin y) \\ u_y &= -e^x\,(x \sin y + \sin y + y \cos y) \quad \Rightarrow \quad u_{yy} = -e^x\,(x \cos y + 2\cos y - y \sin y) \end{aligned}$$

Thus Laplace's equation $u_{xx} + u_{yy} = 0$ is satisfied and we can find a conjugate function v:

$$\begin{aligned} v_y = u_x \quad &\Rightarrow \quad v = \int \left[e^x\,(\cos y + x \cos y - y \sin y)\right]\, dy + A(x) \\ v_x = -u_y \quad &\Rightarrow \quad v = \int \left[e^x\,(x \sin y + \sin y + y \cos y)\right]\, dx + B(y) \end{aligned}$$

The (partial) integrations are messy but give

$$v = e^x\,(x \sin y + y \cos y) + C$$

where $A = B = C =$ const. For $f(z) = u + iv$ together we have

$$\begin{aligned} f(z) &= e^x\,(x \cos y - y \sin y + ix \sin y + iy \cos y) + c \\ &= e^x z\,(\cos y + i \sin y) + c \\ &= e^z z + c \end{aligned}$$

having used $e^{iy} = \cos y + i \sin y$.

4. Consider the mapping $w = \frac{1}{z-1}$ from the z-plane to the w-plane.
a) Show that in the z-plane, the circle

$$(x - 1)^2 + y^2 = 4$$

maps to a circle in the w-plane. What is the radius of this circle and where is its centre?

Solution:

The mapping $w = \frac{1}{z-1}$ from the z-plane to the w-plane can be written as

$$w = u + iv = \frac{1}{x - 1 + iy} = \frac{(x - 1) - iy}{(x - 1)^2 + y^2}$$

$$u = \frac{x-1}{(x-1)^2+y^2} \qquad v = -\frac{y}{(x-1)^2+y^2} \qquad \Rightarrow \qquad u^2+v^2 = \frac{1}{(x-1)^2+y^2}$$

Then the circle $(x-1)^2+y^2=4$ maps to $u^2+v^2=\frac{1}{4}$, which is a circle in the w-plane, of radius $\frac{1}{2}$ centred at $(0,0)$.

b) To what curve does the line $x=0$ in the z-plane map in the w-plane?

Solution:

The line $x=0$ in the z-plane gives values of u, v

$$u = -\frac{1}{1+y^2} \qquad v = -\frac{y}{1+y^2} \qquad \Rightarrow \qquad u^2+v^2 = \frac{1}{1+y^2}$$

Hence $u^2+v^2=-u$ which, on completing the square, becomes $(u+\frac{1}{2})^2+v^2=\frac{1}{4}$. This is a circle in the w-plane, of radius $\frac{1}{2}$ centred at $(-\frac{1}{2},0)$.

5. a) Fixed points of a map $w=f(z)$ occur when $w=z$. Show that the fixed points of $w=\frac{4z-2}{z+1}$ occur at $z=1$ and $z=2$.

Solution:

For fixed points of $w=\frac{4z-2}{z+1}=z$ solve $z(z+1)=4z-2$. Roots occur at $z=1$ and $z=2$.

b) For $w=u+iv=\frac{4z-2}{z+1}$ show that the image in the w-plane of the line $x=0$ is the circle $(u-1)^2+v^2=9$. What is the image in the w-plane of the unit circle $|z|=1$?

Solution:

For $w=u+iv=\frac{4z-2}{z+1}$ we have

$$u+iv = \frac{4z-2}{z+1} = \frac{4x-2+4iy}{x+1+iy}$$

Thus solving for u and v through rationalisation of the denominator

$$u = \frac{4(x^2+y^2)+2(x-1)}{(x+1)^2+y^2} \qquad v = \frac{6y}{(x+1)^2+y^2} \qquad \Rightarrow \qquad (u-1)^2+v^2 = \frac{9[x^2+y^2-1]^2+36y^2}{[(x+1)^2+y^2]^2}$$

(i) When $x=0$ in the z-plane then this reduces to $(u-1)^2+v^2=9$. This is a circle in the w-plane of radius 3 centred at $(1,0)$.

(ii) For the circle $|z|=1$ in the z-plane we have $x^2+y^2=1$ which means that

$$u = \frac{2x+2}{2x+2} = 1 \qquad v = \frac{6y}{2x+2}$$

Hence in the w-plane we have the vertical line $u=1$ for all values of v.