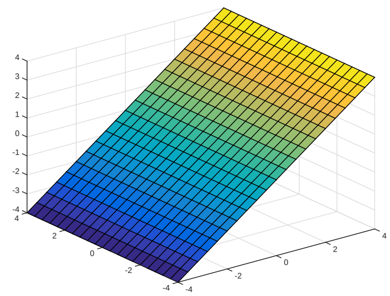
EE2 Mathematics

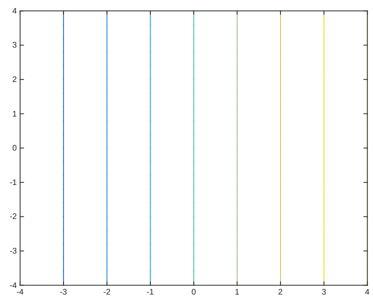
Solutions to Example Sheet 1: Fields, grad, div and curl.

1. Plot the following 2D scalar fields using Matlab, and sketch the equipotential lines:

   a)$\phi(x, y) = x$

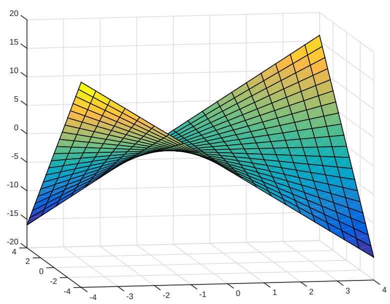
   Solution: The contour lines are $z = x = c$, vertical lines.

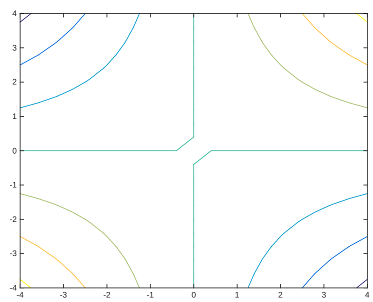




   (b) $\psi(x, y) = xy$

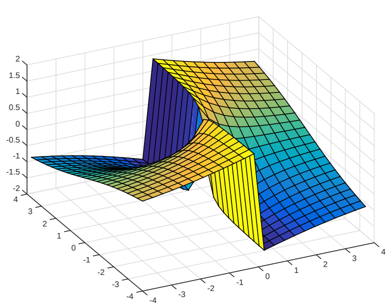
   Solution: Contour lines are $z = xy = c$: a family of hyperbolae, note the plotting failure at the origin.

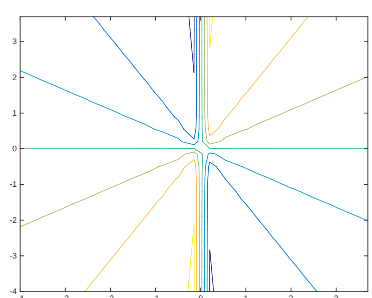




   (c) $\xi(x, y) = \tan^{-1}(y/x)$

   Solution: Contour lines are $z = \tan^{-1}(y/x) = c \Rightarrow y/x = tan(c) = C \Rightarrow y = Cx$, a family of lines through the origin. Note discontinuity along line $x = 0$.





   The matlab code is similar for all of them, here's (b) and (c):

```
v=-4:0.35:4;
[x,y]=meshgrid(v);
%z=x;
z=x.*y;
%z= atan(y./x);
figure;
surf(x,y,z);
figure;
contour(x,y,z);
```

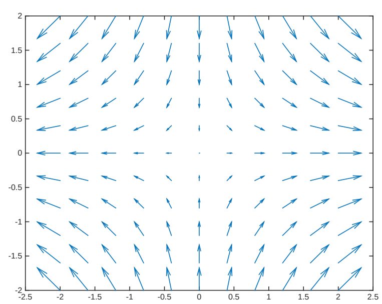
Why 0.35?

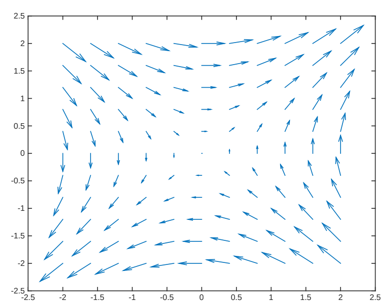
2. Draw by hand the vector fields given by

$$(a)\ \boldsymbol{F} = \boldsymbol{i}x - \boldsymbol{j}y \quad \text{and (b)}\ \boldsymbol{F} = \boldsymbol{i}y + \boldsymbol{j}x$$

and confirm your result by plotting these using matlab.

Solution: In both (a) and (b), drawing by hand should be done following the method outlined in class. Begin with vectors on the axes, then on the lines $y = \pm x$, if necessary, also on lines $y = \pm 2x$ and $y = \pm\frac{1}{2}x$ until you spot the pattern and complete. Here are the fields:

(a) 

(b) 

Here's the code for (b):

```
u=-2:0.4:2;
[x,y]=meshgrid(u);
figure;
u=x;
v=-y;
quiver(x,y,u,v);
```

3. Full notes, section 3.1, Example 2, of two parallel line charges at $\pm d$ with opposite charges $\mp q$, use matlab to plot the vector field. If both lines have charge q, obtain expressions for $\underline{\boldsymbol{E}}_x$ and $\underline{\boldsymbol{E}}_y$ and plot the vector field.

Solution: From notes:

$$E_x = k\left(\frac{x+d}{(x+d)^2+y^2} - \frac{x-d}{(x-d)^2+y^2}\right), \quad \text{and} \quad E_y = k\left(\frac{y}{(x+d)^2+y^2} - \frac{y}{(x-d)^2+y^2}\right).$$

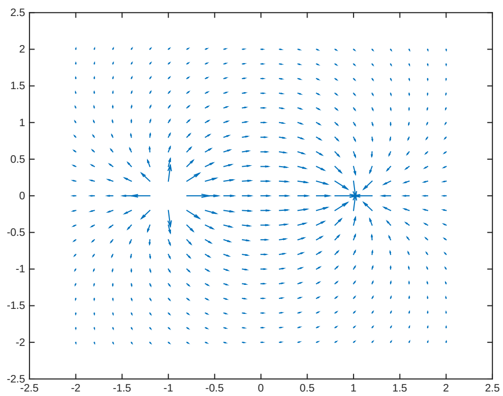
using $k = d = 1$ gives the plot in the notes.

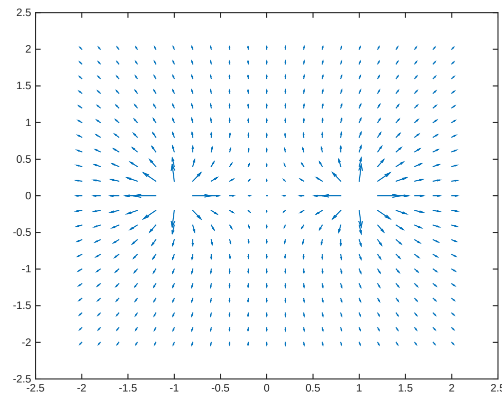
```
v=-2:0.2:2;
[x,y]=meshgrid(v);

u=(x+1)./((x+1).^2+y.^2)-(x-1)./((x-1).^2+y.^2);
v=y./((x+1).^2+y.^2)-y./((x-1).^2+y.^2);
figure;
quiver(x,y,u,v);
```

For equal charges, it's sufficient to realize that the only thing that changes is the sign of $E_{2,x}$ and $E_{2,y}$, so we have

$$E_x = k\left(\frac{x+d}{(x+d)^2+y^2} + \frac{x-d}{(x-d)^2+y^2}\right), \quad \text{and} \quad E_y = k\left(\frac{y}{(x+d)^2+y^2} + \frac{y}{(x-d)^2+y^2}\right).$$

Both plots:





4. Find $\nabla\phi$ where

   a) $\phi = x$,
   b) $\phi = x^3 + y^3 + z^3$,
   c) $\phi = \boldsymbol{r}\cdot\nabla(x+y+z)$,
   d) $\phi = (x^2y + 4z^2)$,

   and also find (e) $\operatorname{div}\Big(2xy\hat{\boldsymbol{\imath}} + 4yz\hat{\boldsymbol{\jmath}} - xz\hat{\boldsymbol{k}}\Big)$ and (f) $\operatorname{curl}\Big(y^2z\hat{\boldsymbol{\imath}} + 2xyz\hat{\boldsymbol{\jmath}} + xy^2\hat{\boldsymbol{k}}\Big)$.

   Solution: Because $\nabla\phi = \hat{\boldsymbol{\imath}}\,\frac{\partial\phi}{\partial x} + \hat{\boldsymbol{\jmath}}\,\frac{\partial\phi}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial\phi}{\partial z}$

   a) then with $\phi = x$ we have $\nabla\phi = \hat{\boldsymbol{\imath}}$.
   b) with $\phi = x^3+y^3+z^3$, $\nabla\phi = 3\left(\hat{\boldsymbol{\imath}}x^2 + \hat{\boldsymbol{\jmath}}y^2 + \hat{\boldsymbol{k}}z^2\right)$.
   c) with $\phi = \boldsymbol{r}\cdot\nabla(x+y+z)$, we have $\phi = \boldsymbol{r}\cdot(\hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}}) = (x+y+z)$. Hence $\nabla\phi = \hat{\boldsymbol{\imath}}+\hat{\boldsymbol{\jmath}}+\hat{\boldsymbol{k}}$.
   d) with $\phi = (x^2y+4z^2)$ then $\nabla\phi = 2xy\hat{\boldsymbol{\imath}} + x^2\hat{\boldsymbol{\jmath}} + 8z\hat{\boldsymbol{k}}$.
   e) $\operatorname{div}\Big(2xy\hat{\boldsymbol{\imath}} + 4yz\hat{\boldsymbol{\jmath}} - xz\hat{\boldsymbol{k}}\Big) = \frac{\partial(2xy)}{\partial x} + \frac{\partial(4yz)}{\partial y} - \frac{\partial(xz)}{\partial z} = 2y + 4z - x$.
   f) If $\boldsymbol{A} = \operatorname{curl}\Big(y^2z\hat{\boldsymbol{\imath}} + 2xyz\hat{\boldsymbol{\jmath}} + xy^2\hat{\boldsymbol{k}}\Big)$ then

$$\boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z & 2xyz & xy^2 \end{vmatrix} = (2xy-2xy)\hat{\boldsymbol{\imath}} - (y^2-y^2)\hat{\boldsymbol{\jmath}} + (2yz-2yz)\hat{\boldsymbol{k}} = 0\,.$$

5. Show that $(\boldsymbol{F}\cdot\nabla)\boldsymbol{r}=\boldsymbol{F}$ for *any* vector field $\boldsymbol{F}$ where $\boldsymbol{r}=(x,y,z)$.

Solution: For any vector field $\boldsymbol{F}$,

$$(\boldsymbol{F}\cdot\nabla)\boldsymbol{r}=\left(F_1\frac{\partial}{\partial x}+F_2\frac{\partial}{\partial y}+F_3\frac{\partial}{\partial z}\right)\left(\hat{\boldsymbol{\imath}}x+\hat{\boldsymbol{\jmath}}y+\hat{\boldsymbol{k}}z\right)=\hat{\boldsymbol{\imath}}F_1+\hat{\boldsymbol{\jmath}}F_2+\hat{\boldsymbol{k}}F_3=\boldsymbol{F}$$

6. Show that if $\boldsymbol{r}=(x,y,z)$

(a) $\operatorname{div}\boldsymbol{r}=3$ and $\operatorname{curl}\boldsymbol{r}=0$,

Solution: $\operatorname{div}\boldsymbol{r}=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=3$ and

$$\operatorname{curl}\boldsymbol{r}=\begin{vmatrix}\hat{\boldsymbol{\imath}}&\hat{\boldsymbol{\jmath}}&\hat{\boldsymbol{k}}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x&y&z\end{vmatrix}=0\,.$$

(b) if $\boldsymbol{V}=(\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{r}$ where $\boldsymbol{a}$ is a constant vector, then $\operatorname{div}\boldsymbol{V}=4(\boldsymbol{a}\cdot\boldsymbol{r})$,

Solution: If $\boldsymbol{V}=(\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{r}$, with $\boldsymbol{a}$ a constant vector, write this expression as $V=\phi\boldsymbol{A}$ where $\phi=\boldsymbol{a}\cdot\boldsymbol{r}=a_1x+a_2y+a_3z$ and $\boldsymbol{A}=\boldsymbol{r}$. Using the identity $\operatorname{div}(\phi\boldsymbol{A})=\phi\operatorname{div}\boldsymbol{A}+(\nabla\phi)\cdot\boldsymbol{A}$ we note that $\operatorname{div}\boldsymbol{r}=3$ and $\nabla\phi=a_1\hat{\boldsymbol{\imath}}+a_2\hat{\boldsymbol{\jmath}}+a_3\hat{\boldsymbol{k}}=\boldsymbol{a}$. Altogether we have $\operatorname{div}(\phi\boldsymbol{r})=4(\boldsymbol{a}\cdot\boldsymbol{r})$.

(c) $\operatorname{curl}\boldsymbol{V}=\boldsymbol{a}\times\boldsymbol{r}$ where $\boldsymbol{V}$ is given in (b)

Solution: To find $\operatorname{curl}(\phi\boldsymbol{r})$ where $V=\phi\boldsymbol{r}$ as above, we use the expression $\operatorname{curl}(\phi\boldsymbol{r})=\phi\operatorname{curl}\boldsymbol{r}+(\nabla\phi)\times\boldsymbol{r}$. Now $\operatorname{curl}\boldsymbol{r}=0$ and $\nabla\phi=\boldsymbol{a}$ so $\operatorname{curl}(\phi\boldsymbol{r})=\boldsymbol{a}\times\boldsymbol{r}$.

7. Obtain the curl of the following vector fields:

(i) $x\hat{\boldsymbol{\imath}}$;

Solution: For $x\hat{\boldsymbol{\imath}}$

$$\operatorname{curl}x\hat{\boldsymbol{\imath}}=\begin{vmatrix}\hat{\boldsymbol{\imath}}&\hat{\boldsymbol{\jmath}}&\hat{\boldsymbol{k}}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ x&0&0\end{vmatrix}=0\,.$$

(ii) $\boldsymbol{r}f(r)$ where $r^2=x^2+y^2+z^2$ and $f(r)$ is an arbitrary function of r

Solution: We use the expression $\operatorname{curl}(\phi\boldsymbol{r})=\phi\operatorname{curl}\boldsymbol{r}+(\nabla\phi)\times\boldsymbol{r}$ where $\phi=f(r)$. We know that $\operatorname{curl}\boldsymbol{r}=0$ and

$$\nabla\phi=(f_x,\,f_y,\,f_z)=f_r\left(\frac{\partial r}{\partial x},\,\frac{\partial r}{\partial y},\,\frac{\partial r}{\partial z}\right)$$

From $r^2=x^2+y^2+z^2$ we have $r\frac{\partial r}{\partial x}=x$ etc so

$$\nabla\phi=(f_x,\,f_y,\,f_z)=\frac{f_r}{r}\,(x,y,z)=\frac{f_r}{r}\boldsymbol{r}$$

in which case

$$\nabla\phi\times\boldsymbol{r}=\left(\frac{f_r}{r}\right)\boldsymbol{r}\times\boldsymbol{r}=0$$

Altogether $\operatorname{curl}(\phi\boldsymbol{r})=0$.

(iii) $(x\hat{\boldsymbol{i}} - y\hat{\boldsymbol{j}})/(x+y)$.

Solution: To find the curl of $\boldsymbol{A} = (x\hat{\boldsymbol{i}} - y\hat{\boldsymbol{j}})/(x+y)$ we write

$$\operatorname{curl}\boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x+y} & -\frac{y}{x+y} & 0 \end{vmatrix} = \hat{\boldsymbol{k}}\left[\frac{\partial}{\partial x}\left(-\frac{y}{x+y}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x+y}\right)\right] = \frac{\hat{\boldsymbol{k}}}{x+y}\,.$$

8. You are given the vector identity

$$\operatorname{div}(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{v}\cdot\operatorname{curl}\boldsymbol{u} - \boldsymbol{u}\cdot\operatorname{curl}\boldsymbol{v}\,.$$

Use this and one of the identities at the head of the sheet to verify that $\operatorname{div}(\nabla\phi\times\nabla\psi) = 0$ where ϕ and ψ are arbitrary scalar fields. Verify also that $\frac{1}{2}[\phi\nabla\psi - \psi\nabla\phi]$ is its vector potential; that is, show that

$$\operatorname{curl}\left[\tfrac{1}{2}(\phi\nabla\psi - \psi\nabla\phi)\right] = \nabla\phi\times\nabla\psi\,.$$

Hint: To do this recall from your notes that if a vector field $\boldsymbol{F}$ has the property that $\operatorname{div}\boldsymbol{F} = 0$, then $\boldsymbol{F}$ can be written as $\boldsymbol{F} = \operatorname{curl}\boldsymbol{A}$ where $\boldsymbol{A}$ is the vector potential. Note also that for arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$: $\frac{1}{2}\operatorname{curl}\boldsymbol{a} = \operatorname{curl}\frac{1}{2}\boldsymbol{a}$ and $\operatorname{curl}(\boldsymbol{a}+\boldsymbol{b}) = \operatorname{curl}\boldsymbol{a} + \operatorname{curl}\boldsymbol{b}$. This is easily provable from the determinantal definition of curl.

Solution: In the identity $\operatorname{div}(\boldsymbol{u}\times\boldsymbol{v}) = \boldsymbol{v}\cdot\operatorname{curl}\boldsymbol{u} - \boldsymbol{u}\cdot\operatorname{curl}\boldsymbol{v}$ identify $\boldsymbol{u}\equiv\nabla\phi$ and $\boldsymbol{v}\equiv\nabla\psi$. Hence

$$\operatorname{div}(\nabla\phi\times\nabla\psi) = \nabla\psi\cdot\operatorname{curl}\nabla\phi - \nabla\phi\cdot\operatorname{curl}\nabla\psi$$

However, $\operatorname{curl}(\nabla\phi) = \nabla\times(\nabla\phi) = 0$ and likewise $\operatorname{curl}(\nabla\psi) = 0$ so $\operatorname{div}(\nabla\phi\times\nabla\psi) = 0$.

Because $\operatorname{div}\boldsymbol{F} = 0$ where $\boldsymbol{F} = \nabla\phi\times\nabla\psi$ we know that a vector potential $\boldsymbol{A}$ exists such that $\boldsymbol{F} = \operatorname{curl}\boldsymbol{A}$. The question asks us to show that when $\boldsymbol{F} = \nabla\phi\times\nabla\psi$ then $\boldsymbol{A} = \frac{1}{2}[(\phi\nabla\psi - \psi\nabla\phi)]$ by verifying that

$$\operatorname{curl}\left[\tfrac{1}{2}(\phi\nabla\psi - \psi\nabla\phi)\right] = \nabla\phi\times\nabla\psi\,.$$

The definition of curl not only shows that $\operatorname{curl}\frac{1}{2}\boldsymbol{a} = \frac{1}{2}\operatorname{curl}\boldsymbol{a}$ but also shows that $\operatorname{curl}(\boldsymbol{a}+\boldsymbol{b}) = \operatorname{curl}\boldsymbol{a} + \operatorname{curl}\boldsymbol{b}$. To evaluate the LHS of our expression write

$$\operatorname{curl}\left[\tfrac{1}{2}(\phi\nabla\psi - \psi\nabla\phi)\right] = \tfrac{1}{2}\operatorname{curl}(\phi\nabla\psi) - \tfrac{1}{2}\operatorname{curl}(\psi\nabla\phi)$$

Using the identity $\operatorname{curl}(\phi\boldsymbol{u}) = \phi\operatorname{curl}\boldsymbol{u} + (\nabla\phi)\times\boldsymbol{u}$ we have

$$\operatorname{curl}\phi\nabla\psi = \phi\operatorname{curl}\nabla\psi + \nabla\phi\times\nabla\psi \qquad\qquad \operatorname{curl}\psi\nabla\phi = \psi\operatorname{curl}\nabla\phi + \nabla\psi\times\nabla\phi$$

Because $\operatorname{curl}\nabla\phi = 0$ and $\operatorname{curl}\nabla\psi = 0$ we have

$$\tfrac{1}{2}\operatorname{curl}\left[(\phi\nabla\psi - \psi\nabla\phi)\right] = \tfrac{1}{2}(\nabla\phi\times\nabla\psi - \nabla\psi\times\nabla\phi) = \nabla\phi\times\nabla\psi$$

9. Find a unit vector normal $\hat{\mathbf{n}}$ to the surface $\phi = x^2 + 2y^2 - 4z^2 = 5$ at the point $(1, 2, 1)$ and find the equation of the tangent plane there.

Solution: The gradient is perpendicular to the tangent vector, so the normal vector is obtained by evaluating the gradient at that point:

$$\nabla\phi = 2x\hat{\boldsymbol{\imath}} + 4y\hat{\boldsymbol{\jmath}} - 8z\hat{\boldsymbol{k}}$$

so that at $(1, 2, 1)$ we have $\phi(1, 2, 1) = 2\hat{\boldsymbol{\imath}} + 8\hat{\boldsymbol{\jmath}} - 8\hat{\boldsymbol{k}}$ and

$$\hat{\mathbf{n}} = (\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}})/\sqrt{33}$$

and the plane is $x + 4y - 4z = 5$.