Imperial College London

Mathematics: Year 2

Vectors (1st Year)

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Abstract

Vector notation is essential in the analysis of forces on a particular engineering system. This later grew to include modelling multi-dimensional simulations.

Contents

1 Vectors

1.1 Basic definitions of vectors

- Vector addition and scalar multiplication satisfies the following properties:
  - $\underline{\mathbf{x}} + \underline{\mathbf{y}} = \underline{\mathbf{y}} + \underline{\mathbf{x}}$
  - $(\underline{\mathbf{x}} + \underline{\mathbf{y}}) + \underline{\mathbf{z}} = \underline{\mathbf{x}} + (\underline{\mathbf{y}} + \underline{\mathbf{z}})$
  - $\lambda(\underline{\mathbf{x}} + \underline{\mathbf{y}}) = \lambda\underline{\mathbf{x}} + \lambda\underline{\mathbf{y}}$
  - $(\lambda + \mu)\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} + \mu\underline{\mathbf{x}}$
  - $\mu(\lambda\underline{\mathbf{x}}) = (\mu\lambda)\underline{\mathbf{x}}$
  - $1 \times \underline{\mathbf{x}} = \underline{\mathbf{x}}$ and $0 \times \underline{\mathbf{x}} = 0$

1.2 Dot product

- Magnitude: $|\underline{\mathbf{x}}| = \sqrt{\underline{\mathbf{x}}}$
- Properties:
  - $\underline{\mathbf{x}}.\underline{\mathbf{y}} = \underline{\mathbf{y}}.\underline{\mathbf{x}}$
  - $\lambda(\underline{\mathbf{x}}.\underline{\mathbf{y}}) = (\lambda\underline{\mathbf{x}}).\underline{\mathbf{y}} = \underline{\mathbf{x}}.(\lambda\underline{\mathbf{y}})$
  - $\underline{\mathbf{x}}.(\underline{\mathbf{y}} + \underline{\mathbf{z}}) = \underline{\mathbf{x}}.\underline{\mathbf{y}} + \underline{\mathbf{x}}.\underline{\mathbf{z}}$
- Orthogonality: $\underline{\mathbf{x}}.\underline{\mathbf{y}} = 0$
- Angles: $\underline{\mathbf{x}}.\underline{\mathbf{y}} = |\underline{\mathbf{x}}||\underline{\mathbf{y}}|\cos(\theta)$

1.2.1 Projection

- Vector $\lambda\underline{\mathbf{v}}$ is the **projection of $\underline{\mathbf{u}}$ along direction vector $\underline{\mathbf{v}}$**
- How much correlation λ: $\lambda = \frac{\underline{\mathbf{u}}.\underline{\mathbf{v}}}{\underline{\mathbf{v}}.\underline{\mathbf{v}}}$

1.2.2 Unit vector

- $\hat{\underline{\mathbf{x}}} = \frac{1}{|\underline{\mathbf{x}}|} = \underline{\mathbf{x}}$

1.3 Cross product

- Properties:
  - $\underline{\mathbf{a}} \times \underline{\mathbf{b}} = -\underline{\mathbf{b}} \times \underline{\mathbf{a}}$
  - $\underline{\mathbf{a}} \times \underline{\mathbf{a}} = 0$
  - $\lambda(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = (\lambda\underline{\mathbf{a}}) \times \underline{\mathbf{b}} = \underline{\mathbf{a}} \times (\lambda\underline{\mathbf{b}})$
  - $\underline{\mathbf{a}}.(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = \underline{\mathbf{b}}.(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = 0$
  - $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} + \underline{\mathbf{c}}) = \underline{\mathbf{a}} \times \underline{\mathbf{b}} + \underline{\mathbf{a}} \times \underline{\mathbf{c}}$

1.3.1 Shortest distance from point to plane

- Shortest distance is the perpendicular line between two vectors

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

where point $P = (x_1, y_1, z_1)$ and plane $Ax + By + Cz + D = 0$

1.3.2 Distance between two lines

- Given two lines: $Ax + By + Cz + D_1 = 0$ and $Ax + By + Cz + D_2 = 0$:

$$d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

1.3.3 Equation of a plane

- Given point x, normal n and point on plane r, the equation:

$$(\underline{\mathbf{r}} - \underline{\mathbf{x}}).\underline{\mathbf{n}} = 0$$

- $Ax + By + Cz = D$ where:
  - $\underline{\mathbf{r}} = (x, y, z)$
  - $\underline{\mathbf{n}} = (A, B, C)$
  - $D = \underline{\mathbf{x}}.\underline{\mathbf{n}}$

1.3.4 Triple cross product

- Properties:
  - Equal:

    $\underline{\mathbf{a}}.(\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = \underline{\mathbf{b}}.(\underline{\mathbf{c}} \times \underline{\mathbf{a}}) = \underline{\mathbf{c}}.(\underline{\mathbf{a}} \times \underline{\mathbf{b}}) = -\underline{\mathbf{a}}.(\underline{\mathbf{c}} \times \underline{\mathbf{b}}) = -\underline{\mathbf{b}}.(\underline{\mathbf{a}} \times \underline{\mathbf{c}}) = -\underline{\mathbf{c}}.(\underline{\mathbf{b}} \times \underline{\mathbf{a}})$
  - $\underline{\mathbf{a}} \times (\underline{\mathbf{b}} \times \underline{\mathbf{c}}) = (\underline{\mathbf{a}}.\underline{\mathbf{c}})\underline{\mathbf{b}} - (\underline{\mathbf{a}}.\underline{\mathbf{b}})\underline{\mathbf{c}}$

2 Matrix algebra

2.1 Matrix multiplication

- Properties:
  - $A(BC) = (AB)C$
  - $(\lambda A)B = A(\lambda B) = \lambda(AB)$
  - $(AB)^T = B^T A^T$

2.2 Inverse

- $\underline{\mathbf{x}} = A^{-1}\underline{\mathbf{b}}$

2.3 Determinants

- $\det(A) = ad - bc$
- 3×3 matrix:

$$\det(A) = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{13}\begin{vmatrix} a_{12} & a_{12} \\ a_{22} & a_{22} \end{vmatrix}$$

- Properties:
  - The value of a determinant remains unchanged under transposition:
  $$det(A) = det(A^T)$$
  - If B is obtained from A by exchanging exactly two rows (or two columns)
  $$det(A) = -det(B)$$
  - If the elements of any column (or row) are multiplied by a factor λ then the determinant is multiplied by λ.
  - If a multiple of one row (or column) is added to another row (or column), the determinant is unchanged.
  - $det(AB) = det(A)det(B)$