EE2 Mathematics – Probability & Statistics

Solution 9

1. Recall basics of change of variables. Suppose X_1, X_2 have joint pdf $f_{X_1,X_2}(x_1, x_2)$ with support $A = \{(x_1, x_2) : f_{X_1,X_2}(x_1, x_2) > 0\}$. We are interested in the random variable $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$. The transformation $y_1 = g_1(x_1, x_2)$ and $y_2 = g_2(x_1, x_2)$ is a 1-1 transformation of A onto B. So there is an inverse transformation $x_1 = g_1^{-1}(y_1, y_2)$ and $x_2 = g_2^{-1}(y_1, y_2)$. If the determinant of the Jacobian J

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix},$$

exists (functions are continuous and differentiable) and is non-zero for $(y_1, y_2) \in B$, then the joint pdf of $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ is given by

$$f_{Y_1,Y_2}(y_1, y_2) = |\det(J)|\, f_{X_1,X_2}(g_1^{-1}(y_1, y_2), g_2^{-1}(y_1, y_2)), \qquad (y_1, y_2) \in B.$$

Let us now consider the transformation $y_1 = x_1$ and $y_2 = x_1 + x_2$ with inverse $x_1 = y_1$, $x_2 = y_2 - y_1$. The set $B = \{(y_1, y_2) : 0 \le y_1 \le y_2 \le \infty\}$. We write

$$\det(J) = \det\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\right) = 1.$$

The joint pdf is given by

$$f_{Y_1,Y_2}(y_1, y_2) = 1 \times e^{-y_2} \qquad 0 \le y_1 \le y_2 \le \infty.$$

To find the pdf of $Y_2 = X_1 + X_2$, we compute the marginal of Y_2 by integrating the joint pdf out Y_1

$$f_{Y_2}(y_2) = \int_0^{y_2} e^{-y_2} dy_1 = y_2 e^{-y_2} \qquad 0 \le y_2 \le \infty.$$

2. We are interested in the pdf of the random variable $Y_1 = X_1/X_2$. Let us introduce another random variable $Y_2 = X_2$ such that that the inverse transformation can easily be found. The inverse is $x_1 = y_1 y_2$, $x_2 = y_2$. We then have

$$\det(J) = \det\left(\begin{bmatrix} y_2 & y_1 \\ 0 & 1 \end{bmatrix}\right) = y_2.$$

We have $A = \{(x_1, x_2) : 0 < x_1 < x_2 < 1\}$ which implies that $B = \{(y_1, y_2) : 0 < y_1 y_2 < y_2 < 1\}$, which further implies that $B = \{(y_1, y_2) : 0 < y_1 < 1, 0 < y_2 < 1\}$. So

$$f_{Y_1,Y_2}(y_1, y_2) = 8(y_1 y_2) y_2 \times y_2 = 8 y_1 y_2^3 \qquad (y_1, y_2) \in B.$$

The marginal of Y_1 is finally obtained as

$$f_{Y_1}(y_1) = \int_0^1 8y_1 y_2^3 dy_2 = 8y_1 \left[\frac{y_2^4}{4} \right]_0^1 = 2y_1 \qquad 0 < y_1 < 1.$$