Imperial College London

Mathematics: Year 2

# Vectors (Field Theory)

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**Abstract**

In engineering, many applications uses functions of the space variable $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ as models for quantities in the 3-D space. Being able to manipulate and model quantities effectively is a critical skill in any engineering field.

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# 1 Revision

Given the vector notation for a 3-D space: $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \equiv (a_1, a_2, a_3)$, the critical values are as follows:

1. Magnitude/Length of vector:
$$|\mathbf{a}| = (a_1^2 + a_2^2 + a_3^2)^{\frac{1}{2}}$$

2. Scalar/Dot product of two vectors $a$ and $b$:
$$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

3. Angle between two vectors $a$ and $b$:
$$\mathbf{a}.\mathbf{b} = ab\cos(\theta)$$
where $\mathbf{a}.\mathbf{b} = 0$ means it is perpendicular

4. Vector/Cross product between two vectors $a$ and $b$:
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = [ab\sin(\theta)]\hat{\mathbf{n}}$$
where $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $a$ and $b$. If $a \times b = 0$ then both vectors are parallel if neither are null.

5. Scalar triple product between three vectors $a$, $b$ and $c$:
$$\mathbf{a}(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$\mathbf{a}(\mathbf{b} \times \mathbf{c}) \equiv \mathbf{b}(\mathbf{c} \times \mathbf{a}) \equiv \mathbf{c}(\mathbf{a} \times \mathbf{b})$$
where if any two of three vectors are equal then scalar product is zero.

6. Vector triple product between three vectors $a$, $b$ and $c$:
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a}.\mathbf{c}) - \mathbf{c}(\mathbf{a}.\mathbf{b})$$

More detailed notes can be found in Year 1 notes.

# 2 Scalar fields

The essence of field theory is the concept of "action by continuous contact" by which the contact is provided by a "stress" or "field" induced in the space between the objects by their presence. This is different form the older view of "action by contact" where it was thought there could only be action when two objects are touching each other. This older view does not account for other fields such as gravity and electromagnetism. The modern view developed by James Clerk Maxwell noted that a force can be exerted on each other *through the presence of an intervening medium or mechanism existing in the space between objects.*

Most applications of vectors in engineering requires dealing with two or more variables. Four variable functions are commonly encountered e.g. electromagnetism requires three variables for the 3-D space and a variable for time.

There are two different types of four variable functions: **Scalar Fields** and **Vector Fields**. Vector fields will be covered in the next chapter.

**Scalar field** or **Scalar function**:

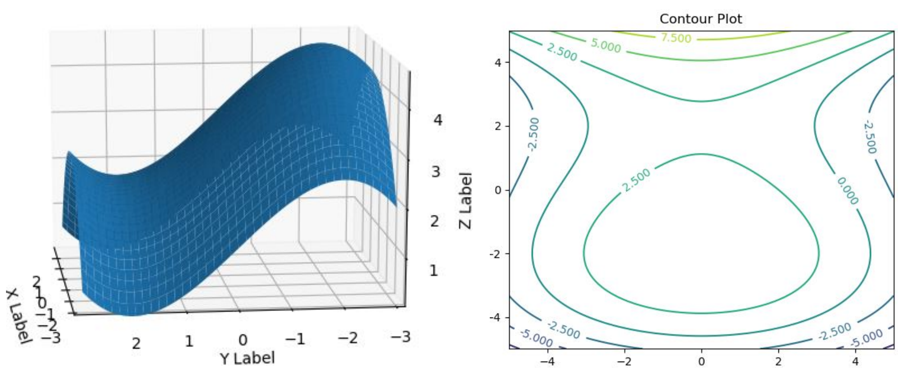
$$\psi = \psi(x, y, z, t)$$

A function that returns one value of some variable for every point in space. The term scalar indicates that variable at any point is a number rather than a vector (both magnitude and direction).

## 2.1 Visualization: Surface plots and Contour plots

The most common way to visualize is to plot a **surface plot**. Alternatively a **contour plot** given by $z = c = constant$ is also common.

**Example 1**: Given $\phi(x, y) = \frac{1}{12}y^3 - y - \frac{1}{4}x^2 + \frac{7}{2}$, plot the graph $z = \phi(x, y)$.




(a) Surface plot (b) Contour plot

Figure 1

Higher dimensions are harder to visualize. One approach is to fix one of the independent variables e.g. $z$ and then create a contour map for the two remaining dimensions. Each point in a 3D space is assigned one value ($\phi$) and for each $c$ can be equated

$$\phi(x, y, z) = c$$

This was seen in previous chapter where each $c$ is a contour line. In a function of three variables, the contours become **3-D** contour surfaces by changing the fixed variable. The contour surfaces become **level surfaces**.

**Level surfaces** of $(\phi(x, y, z))$ of level $c$: The set of all points in $R^3$ which are solutions to:

$$\phi(x, y, z) = c$$

An important detail to note is that each value of $c$ gives the level surface: $x + y + z = c$ which is **a series of parallel planes with common normal**. It is due to this detail that scalar fields can represent an electric potential since it is derived from an energy - it is a scalar field.

**Equipotential**: Region in space where every point is at the same potential, usually referring to the scalar potential.

For a function of two variables $(\phi(x, y) = c)$, the field gives equipotential **lines**. For a function of three variables $(\phi(x, y, z = c))$, the field gives equipotential **surfaces**.

**Example 2**: Given $\phi(x, y, z) = x^2 + y^2 + \frac{z^2}{4}$, plot a surface plot showing the equipotentials.

Equipotentials are given by:

$$\phi(x, y, z) = x^2 + y^2 + \frac{z^2}{4} = c$$

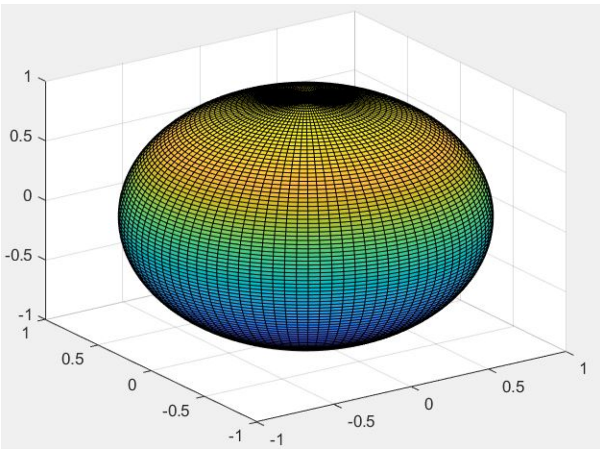



Figure 2: Ellipsoid showing the equipotential surfaces

## 2.2 Scalar fields varying with time

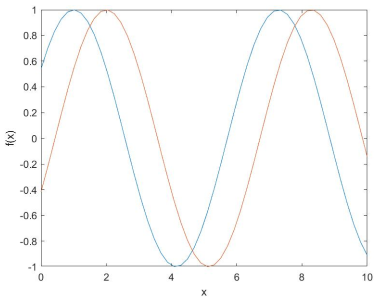
Certain precautions need to be made to display a scalar field that varies with time. Common ways to show the change is to overlay two graphs at different times or to plot a surface plot. The methodology is still the same with time $t$ being a variable.

**Example 3**: Given the travelling wave in 1D with temporal frequency $F$ and wavelength $\lambda$:

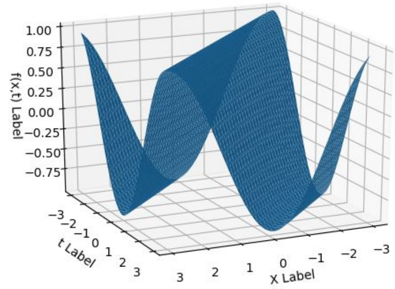
$$f(x,t) = A\cos\left[2\pi\left(Ft - \frac{1}{\lambda}x\right)\right]$$

Note the common equations: Angular Frequency - $\omega = 2\pi F$ and Propagation Constant - $k = \frac{2\pi}{\lambda}$:

$$f(x,t) = A\cos(\omega t - kx)$$




(a) Same graphs at different time




(b) Surface plot of graph with time varying

Figure 3

For higher variable functions, the only common approach is to take snapshots of the surface plot and varying the time variable $t$.

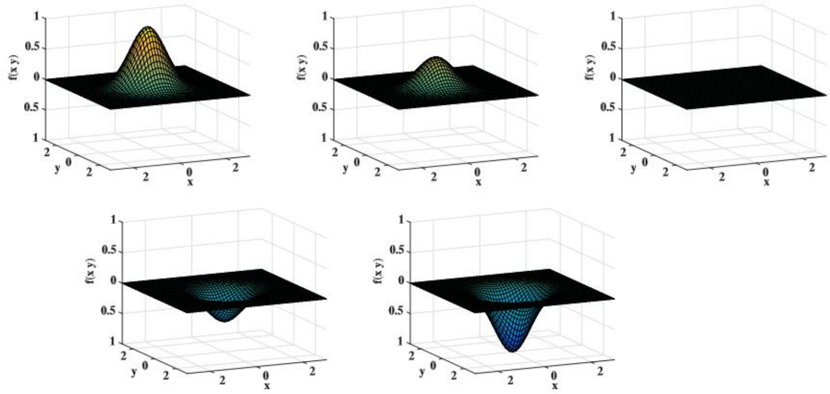



Figure 4: Snapshots of the same function at different times

# 3 Vector fields

A vector field has components in terms of the three unit vectors ($\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}}$).

$$\underline{\mathbf{B}} = \underline{\mathbf{i}}B_1(x, y, z, t) + \underline{\mathbf{j}}B_2(x, y, z, t) + \underline{\mathbf{k}}B_3(x, y, z, t)$$

where $B_n$ are all scalar fields.

## 3.1 Visualization of vector fields

The most common are **field plots** where an arrow represents the direction/-magnitude of the field - the vector at the point.

**Example 1**: Given $\underline{\mathbf{F}}(x, y) = y\underline{\mathbf{i}} - x\underline{\mathbf{j}}$

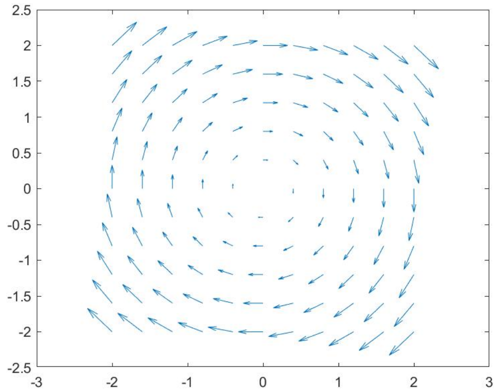



Figure 5: Vector field

**Field lines**: Graphical aid for visualizing vector fields. Consists of a directed line which is parallel to the vector field. Note it does not give the direction of the flow.

On vector fields, **field lines** can be drawn by following the gradient relationship:

Given a 2-D field $\underline{\mathbf{F}} = F_x\underline{\mathbf{i}} + F_y\underline{\mathbf{j}}$ and the field lines $y = y(x)$, the relationship between the vector field and field lines are:

$$\frac{dy}{dx} = \frac{F_y}{F_x}$$

Following **Example 1**, the relationship is defined as:

$$\frac{dy}{dx} = -\frac{x}{y} \rightarrow y\ dy = -x\ dx$$

Through integration, the field line equation is $x^2 + y^2 = c$ which is evident in Figure 5.

**Example 2**: Given a line of electric charge parallel to the $z$-axis, the electric field $\underline{\mathbf{E}}$ is radial, confirming with electric charge theory from physics.

$$\underline{\mathbf{E}} = \frac{q}{2\pi\epsilon_0 r}\underline{\hat{\mathbf{r}}}$$

where:

- $r = \sqrt{x^2 + y^2}$ - Distance from the origin
- $q$ - Charge per unit Length
- $\epsilon_0$ - Permittivity of free space
- $\underline{\hat{\mathbf{r}}}$ - Unit vector in radial direction

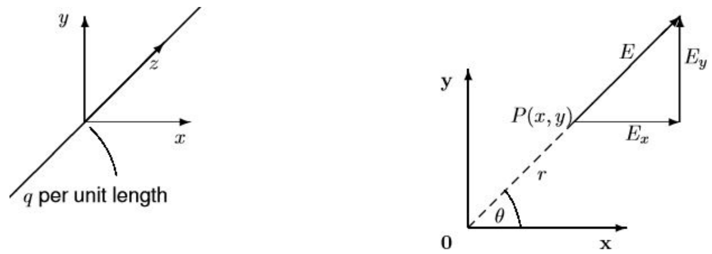



Figure 6: Graphical representation of line of charge on the $z$-axis

This allows an equation to be formed in terms of Cartesian components:

$$\begin{aligned}\underline{\mathbf{E}} &= E_x\underline{\mathbf{i}} + E_y\underline{\mathbf{j}} \\ &= \frac{k}{r}\cos(\theta)\underline{\mathbf{i}} + \frac{k}{r}\sin(\theta)\underline{\mathbf{j}}\end{aligned}$$

where $k = \frac{q}{2\pi\epsilon_0}$

Given that $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$), the equation simplifies to:

$$\underline{\mathbf{E}} = \frac{kx}{x^2 + y^2}\underline{\mathbf{i}} + \frac{ky}{x^2 + y^2}\underline{\mathbf{j}}$$

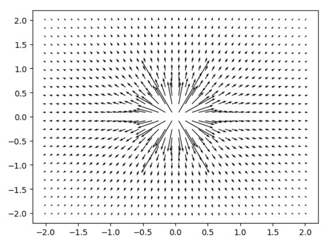


Figure 7: Vector field of electric charge

By shifting the charge from the origin to $(-d, 0)$, the equation becomes $r^2 = (x+d)^2 + y^2$:

$$\underline{\mathbf{E}} = \frac{k(x+d)}{(x+d)^2 + y^2}\underline{\mathbf{i}} + \frac{ky}{(x+d)^2 + y^2}\underline{\mathbf{j}}$$

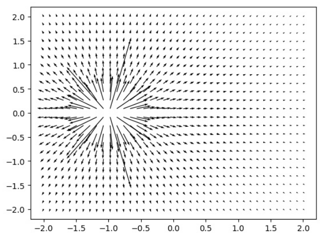


Figure 8: Shifted vector field of electric charge

By taking two line charges at $(\pm d, 0)$ with strength $\mp q$ per unit length, superposition can be used to draw the field where the total electric field is the sum of individual fields $\underline{\mathbf{E}} = \underline{\mathbf{E}}_1 + \underline{\mathbf{E}}_2$:

$$\underline{\mathbf{E}}_1 = \frac{q}{2\pi\epsilon_0 r_1}\hat{\underline{\mathbf{r}}}_1 = \frac{k}{r_1}\hat{\underline{\mathbf{r}}}_1 \quad \text{and} \quad \underline{\mathbf{E}}_2 = -\frac{q}{2\pi\epsilon_0 r_2}\hat{\underline{\mathbf{r}}}_2 = -\frac{k}{r_2}\hat{\underline{\mathbf{r}}}_2$$

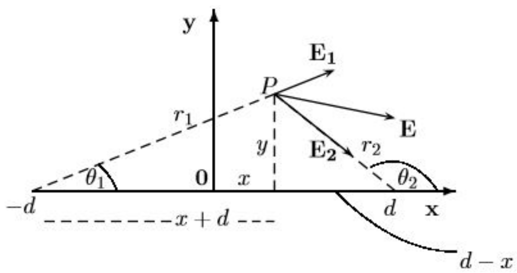



Figure 9: Superposition of two vectors resulting in a resultant vector

Simplifying:

$$E_x = \frac{k}{r_1}\frac{x+d}{r_1} - \frac{k}{r_2}\frac{x-d}{r_2} = \left(\frac{x+d}{(x+d)^2 + y^2 - \frac{x-d}{(x-d)^2+y^2}}\right)$$

$$E_y = \frac{k}{r_1}\frac{y}{r_1} - \frac{k}{r_2}\frac{y}{r_2} = \left(\frac{y}{(x+d)^2 + y^2 - \frac{y}{(x-d)^2+y^2}}\right)$$

Resulting in the following vector field:

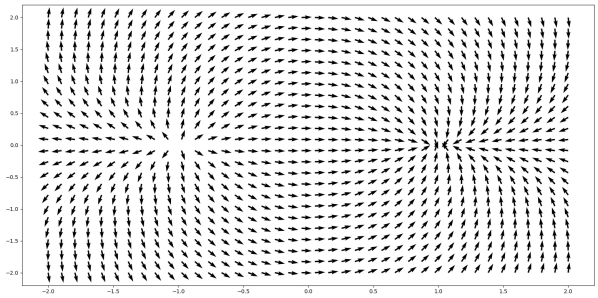


Figure 10: Graphical representation of the superimposed vector field

# 4 Vector operators: Grad, Div and Curl

## 4.1 Definition of the Gradient Operator ($\nabla$)

In a scalar field $(\phi) = \phi(x, y, z)$, how quickly the scalar field varies in the $x$, $y$ and $z$ direction is defined as:

$$\frac{\partial \phi}{\partial x} \quad \text{or} \quad \frac{\partial \phi}{\partial y} \quad \text{or} \quad \frac{\partial \phi}{\partial z}$$

How quickly the potential varies at any point - the **gradient** - is defined as:

$$grad\ \phi = \nabla\phi = \underline{\mathbf{i}}\frac{\partial \phi}{\partial x} + \underline{\mathbf{j}}\frac{\partial \phi}{\partial y} + \underline{\mathbf{k}}\frac{\partial \phi}{\partial z}$$

**Note: $\phi$ is a scalar field but $\nabla\phi$ is a vector field.**

## 4.2 Directional derivatives

When considering the rate of change of scalar $\phi$ **in a particular direction**, denote the direction by unit vector $\hat{\mathbf{m}}$ i.e. if direction $\hat{\mathbf{m}} = \underline{\mathbf{i}}$ then $\hat{\mathbf{m}} = \frac{\partial \phi}{\partial x}$.

Recall the component of a vector in direction of a unit vector is given by the **scalar product**. Hence, rate of change of $\phi$ in direction of a unit vector $\hat{\mathbf{m}}$ is defined as (the directional derivative of $\phi$ in the direction given by $\hat{\mathbf{m}}$):

$$\frac{\partial \phi}{\partial \hat{\mathbf{m}}} = \nabla\phi.\hat{\mathbf{m}}$$

**Note**: $\frac{\partial \phi}{\partial \hat{\mathbf{m}}}$ is a scalar

## 4.3 Gradient of a scalar field

The gradient of a scalar field can represent physical aspects in a scalar field.

For example, consider the electric potential (scalar field - $\phi$) with curves (contour lines) drawn representing where potential is constant.

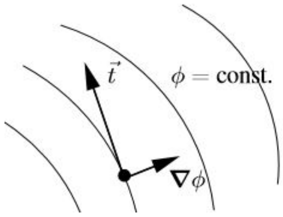



Figure 11

By considering how the potential varies in different directions, if the direction goes along a level surface (the contour lines) then the potential does not change. This is defined as if $\hat{\mathbf{m}}$ points along the level surface:

$$\nabla\phi.\hat{\mathbf{m}} = 0$$

or where $\overrightarrow{t}$ is tangent to a level surface:

$$\nabla\phi.\overrightarrow{t} = 0$$

It follows that the vector field ($\nabla\phi$) is **always** perpendicular to the tangent thus perpendicular to the level surface $\phi(x, y, z)$. This means that ($\nabla\phi$) points to the direction where $\phi$ varies the most rapidly.

$$\begin{aligned}\nabla\phi &= \left(\hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial\phi}{\partial z}\right) \\ &= \frac{\partial\phi}{\partial\overrightarrow{n}}\hat{\mathbf{n}}\end{aligned}$$

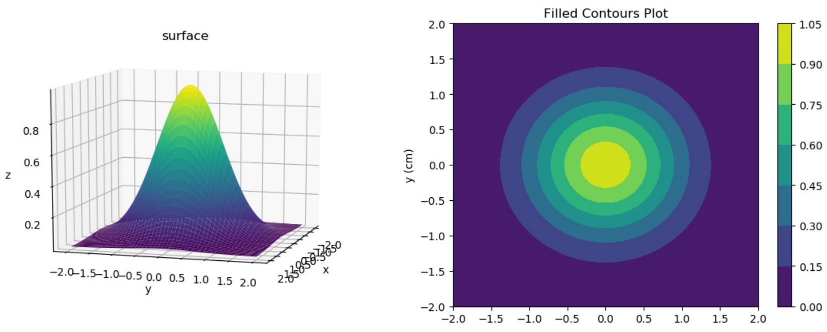
where the normal vector in direction of greatest increase of $\phi(\overrightarrow{r})$ ($\hat{\mathbf{n}}$):

$$\hat{\mathbf{n}} = \frac{\nabla\phi}{|\nabla\phi|}$$

where $\frac{\partial\phi}{\partial\overrightarrow{n}}$ is the normal derivative to surface

**Example 1**: Consider the 2-D scalar field $\phi(x, y) = e^{-x^2-y^2}$.

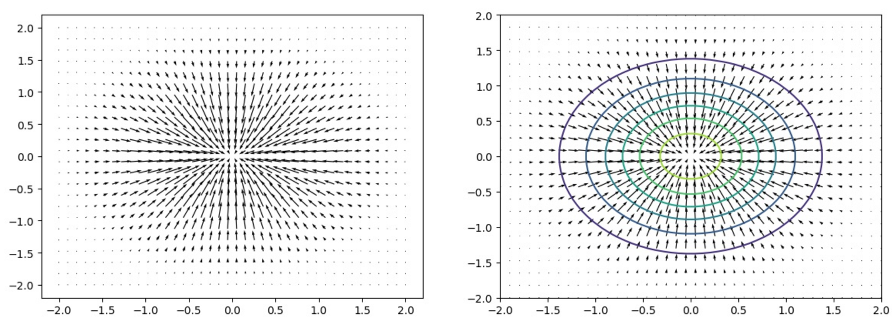
Note that $\phi = C$ implies $x^2 + y^2 = constant$. The surface plot and contour plot of $\phi$:




(a) Surface plot (b) Contour plot

Obtain the $\nabla\phi$ and plot.

$$\nabla\phi = \left(-2xe^{-x^2-y^2}\right)\underline{\mathbf{i}} - \left(2ye^{-x^2-y^2}\right)\underline{\mathbf{j}}$$



(a) Gradient field (b) Equipotential lines with gradient field

Figure 13

**Note**: Equipotential lines and gradient field are clearly perpendicular

## 4.4 Derivatives of Vector Point Function

When considering the vector field $\phi$, there are two ways of combining the vector operator $\nabla$ with $\phi$: $\nabla\ .\ \phi$ and $\nabla \times \phi$ - Note that *grad* $\phi = \nabla\phi$ is a different concept. Both derivatives have physical meanings.

Vector algebra allows two types of products: **Scalar** and **Vector**.

- Scalar products:
  - Results in a scalar (a numberer)
  - Used to define work and energy relationships.
- Vector products:
  - Results in another vector
  - Mainly used to derive other vector quantities e.g. vector "torque" defined as vector product of an applied force (vector) and distance from pivot (force).

The relation between these two are given in a vector field example with fluid flow. At every point in the flow, the **rate at which the field is flowing away from that point** (Scalar Product) and the **amount of spin possessed by the particle of fluid at that point** (Vector Product).

#### 4.4.1 Divergence of a vector field (div $B$)

Given vector field:

$$B = \underline{\mathbf{i}}B_1(x, y, z, t) + \underline{\mathbf{j}}B_2(x, y, z, t) + \underline{\mathbf{k}}B_3(x, y, z, t)$$

The scalar (dot) product:

$$\begin{aligned}\textbf{div B} &= \nabla \ . \ \mathbf{B} \\ &= \left(\underline{\mathbf{i}}\frac{\partial}{\partial x} + \underline{\mathbf{j}}\frac{\partial}{\partial y} + \underline{\mathbf{k}}\frac{\partial}{\partial z}\right) \ . \ \left(\underline{\mathbf{i}}B_1 + \underline{\mathbf{j}}B_2 + \underline{\mathbf{k}}B_3\right)\end{aligned}$$

Remember that $\underline{\mathbf{i}}.\underline{\mathbf{i}} = \underline{\mathbf{j}}.\underline{\mathbf{j}} = \underline{\mathbf{k}}.\underline{\mathbf{k}} = 1$ and $\underline{\mathbf{i}}.\underline{\mathbf{j}} = \underline{\mathbf{i}}.\underline{\mathbf{k}} = \underline{\mathbf{k}}.\underline{\mathbf{j}} = 0$

$$\textbf{div B} = \nabla \ . \ B = \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}$$

**Note**:

- div $B$ is a scalar field since div is formed through dot product.
- $B.\nabla \neq \nabla.B$

div $B$ is a measure of the compression/expansion of a vector field in cube (sphere in the case of liquid).

- div $B = 0$: Vector field $B$ is incompressible.
- div $B > 0$: Vector field $B$ is expanding.
- div $B < 0$: Vector field $B$ is compressing.

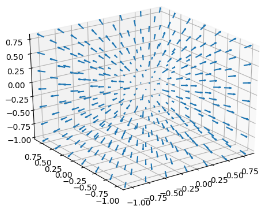
**Example 1**: Consider vector field that is the radial vector in 3-D:

$$\overrightarrow{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

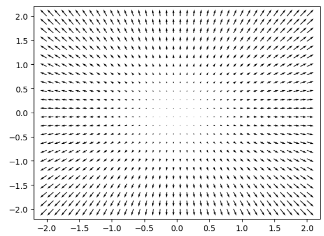
Finding the divergence:

$$\begin{aligned}\nabla.\overrightarrow{F} &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \\ &= 1 + 1 + 1 \\ &= 3\end{aligned}$$

This shows that the divergence is pointing outward (div $B > 0$). **Example 2**:




(a) Cube showing the circular flow (no flow in the center)




(b) Arrows pointing into the center

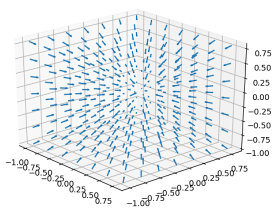
Consider vector field that is the radial vector in 3-D:

$$\overrightarrow{F} = -(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$$

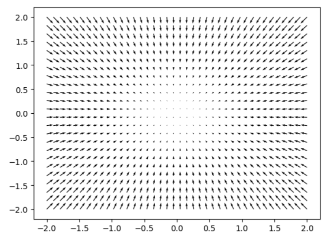
Finding the divergence:

$$\begin{aligned}\nabla . \overrightarrow{F} &= \frac{\partial - x}{\partial x} + \frac{\partial - y}{\partial y} + \frac{\partial - z}{\partial z} \\ &= -1 - 1 - 1 \\ &= -3\end{aligned}$$

This shows that the divergence is pointing inward (div $B < 0$).




(a) Cube showing the circular flow (no flow in the center)




(b) Arrows pointing into the center

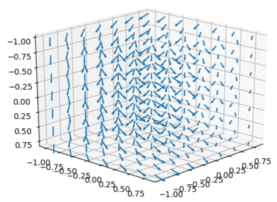
**Example 3**: Consider vector field that is the radial vector in 3-D:

$$\overrightarrow{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

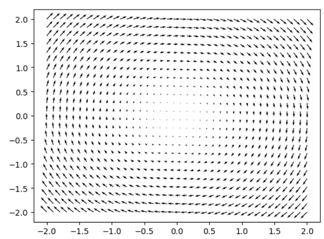
Finding the divergence:

$$\begin{aligned}\nabla.\overrightarrow{F} &= \frac{\partial y}{\partial x} + \frac{\partial - x}{\partial y} + \frac{\partial(1)}{\partial z} \\ &= 0 - 0 + 0 \\ &= 0\end{aligned}$$

This shows that there is no flow into or out of the center (div $B = 0$).



(a) Cube showing the circular flow (no flow in the center)




(b) No flow into or out of the center

The three examples show the three possible cases for divergence of a vector field.

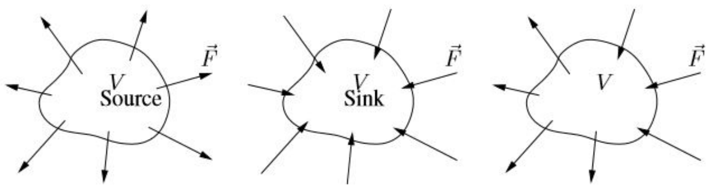



Figure 17: Three types of possible divergence

- div $B = 0$: Vector field $B$ is incompressible - no net outflow or inflow into region V.
- div $B > 0$: Vector field $B$ is expanding - net outflow from region V.
- div $B < 0$: Vector field $B$ is compressing - net inflow in region V.

#### 4.4.2 Derivatives of Vector Point Function: Curl of a vector field ($B$)

The second type of vector multiplication is Cross Product.

$$\textbf{curl } B = \nabla \times B = \begin{bmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{bmatrix}$$

From observations of corks in water (fluid), many types of fluid flow involve rotational motion of the fluid particles. Rotational motion requires knowledge of:

- Axis of rotation.
- Rate of rotation.
- The **sense** (clockwise or anti-clockwise).

The measure of rotation is thus a **vector quantity**.

**Example 1**: Straight line vector from origin to a point $(x, y, z)$ denoted $\mathbf{r} = \underline{\mathbf{i}}x + \underline{\mathbf{j}}y + \underline{\mathbf{k}}z$. Find the curl.

$$\begin{aligned} \textbf{curl } \mathbf{r} &= \nabla \times \mathbf{r} \\ &= \begin{bmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{bmatrix} \\ &= \underline{\mathbf{i}} \left[ \frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} \right] - \underline{\mathbf{j}} \left[ \frac{\partial B_3}{\partial x} - \frac{\partial B_1}{\partial z} \right] + \underline{\mathbf{k}} \left[ \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \right] \\ &= 0 \end{aligned}$$

**Example 2**: Find the curl of vector $\mathbf{v} = (2x - y^2, 3z + x^2, 4y - z^2)$ at point $(1, 2, 3)$.

Assign: $B_1 = 2x - y^2$, $B_2 = 3z + x^2$ and $B_3 = 4y - z^2$

$$\begin{aligned} \textbf{curl } \mathbf{v} = &\begin{bmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y^2 & 3z + x^2 & 4y - z^2 \end{bmatrix} \\ &= \underline{\mathbf{i}} \left[ \frac{\partial B_3}{\partial y} - \frac{\partial B_2}{\partial z} \right] - \underline{\mathbf{j}} \left[ \frac{\partial B_3}{\partial x} - \frac{\partial B_1}{\partial z} \right] + \underline{\mathbf{k}} \left[ \frac{\partial B_2}{\partial x} - \frac{\partial B_1}{\partial y} \right] \\ &= \underline{\mathbf{i}} \left[ \frac{\partial}{\partial y}(4y - z^2) - \frac{\partial}{\partial z}(3z + x^2) \right] - \underline{\mathbf{j}} \left[ \frac{\partial}{\partial x}(4y - z^2) - \frac{\partial}{\partial z}(2x - y^2) \right] \\ &+ \underline{\mathbf{k}} \left[ \frac{\partial}{\partial x}(3z + x^2) - \frac{\partial}{\partial y}(2x - y^2) \right] \\ &= \underline{\mathbf{i}}(4 - 3) - \underline{\mathbf{j}}(0 - 0) + \underline{\mathbf{k}}(2x + 2y) \\ &= \underline{\mathbf{i}} + 2(x + y)\underline{\mathbf{k}} \end{aligned}$$

At the point $(1, 2, 3)$: $\nabla \times \mathbf{v} = (1, 0, 6)$

### 4.4.3 Repeated use of $\nabla$ operator

$$\begin{aligned}\nabla.(\nabla\phi) &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right).\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\\ &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\\ &= \nabla.(\nabla\phi)\\ &= \nabla^2\phi\end{aligned}$$

Curl of the gradient of a scalar field is always zero:

$$\text{curl}[\text{grad}f(\mathbf{r})] = \nabla \times \nabla f(\mathbf{r}) \equiv 0$$

### 4.4.4 Further properties of the vector operator $\nabla$

Three main ways vector operators are used:

1. $\nabla f = \textbf{grad } f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
2. $\nabla.\mathbf{F} = \textbf{div } \mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$
3. $\nabla \times \mathbf{F} = \textbf{curl } \mathbf{F} = \underline{\mathbf{i}}\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right] - \underline{\mathbf{j}}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + \underline{\mathbf{k}}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$

Five useful identities:

1. Gradient of product of two scalars $\psi$ and $\phi$:

$$\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$$

2. Divergence of the product of a scalar $\psi$ with a vector $b$:

$$\text{div}(\psi\mathbf{B}) = \psi \text{ div } \mathbf{B} + (\nabla\psi).\mathbf{B}$$

3. Curl of product of a scalar $\psi$ with vector $\mathbf{B}$:

$$\text{curl } (\psi\mathbf{B}) = \psi \text{ curl } \mathbf{B} + (\nabla\psi) \times \mathbf{B}$$

4. Curl of gradient of any scalar $\psi$:

$$\text{curl}(\nabla\psi) = \nabla \times \nabla\psi = 0$$

5. Divergence of the curl of any vector $\mathbf{B}$:

$$\text{div}(\text{curl } \mathbf{B}) = \nabla.(\nabla \times \mathbf{B}) = 0$$

## 4.5 Irrotational and solenoidal vector fields

$$\text{curl}(\nabla\phi) = \nabla \times \nabla\phi = 0$$

This equation shows that if any vector $\mathbf{B}(x, y, z)$ can be written as the gradient of a scalar $\phi(x, y, z)$ then:

$$\text{curl } B = 0$$

The vector fields are called: **curl-free** or **irrotational** vector fields.

If it is found that curl $\mathbf{B} = 0$ for a given field $\mathbf{B}$, then:

$$\mathbf{B} = \pm\nabla\phi$$

where $\phi$ is the **scalar potential -** only curl-free vector fields has a corresponding scalar potential.

**Example 1**: Considering vector field $\underline{\mathbf{F}} = 2xyz^3\underline{\mathbf{i}} + x^2z^3\underline{\mathbf{j}} + 3x^2yx^2\underline{\mathbf{k}}$, is this the gradient of the scalar field $\phi$? If so, reconstruct $\phi$ from $\underline{\mathbf{F}}$.

1. Require $\nabla \times \underline{\mathbf{F}} = 0$:

$$\begin{aligned}
\textbf{curl } B &= \nabla \times B \\
&= \begin{bmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{bmatrix} \\
&= (3x^2z^2 - 3x^2z^2)\underline{\mathbf{i}} - (6xyz^2 - 6xyz^2)\underline{\mathbf{j}} + (2xz^3 - 2xz^3)\underline{\mathbf{k}} \\
&= 0
\end{aligned}$$

2. Once $\phi$ proved to exist:

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= F_x = 2xyz^3 \\
\frac{\partial\phi}{\partial y} &= F_y = x^2z^3 \\
\frac{\partial\psi}{\partial z} &= F_z = 3x^2yz^2
\end{aligned}$$

3. Integrating with respect to $F_x$, $F_y$ and $F_z$

$$\begin{aligned}
F_x = \phi &= \int 2xyz^3 \quad dx = x^2yz^3 + f(y, z) \\
F_y = \phi &= \int x^2z^3 \quad dy = x^2yz^3 + g(x, z) \\
F_z = \phi &= \int 3x^2yz^2 \quad dz = x^2yz^3 + h(x, y)
\end{aligned}$$

4. Equating all three equations:

$$f(y, z) = g(x, z) = h(x, y) = \text{constant}$$

hence:

$$\phi = x^2yz^3 + c$$

**Solenoidal of divergence-free**: Vector fields $\mathbf{A}$ for which div $\mathbf{A} = 0$

$$\mathbf{A} = \text{curl } \mathbf{B}$$

where vector $\mathbf{B}$ is the **vector potential**.

**Example 2**: Given the Newtonian gravitational force between masses $m$ and $M$ with gravitational constant $G$:

$$\underline{\mathbf{F}} = -GmM\frac{\mathbf{r}}{r^3}$$

where $\mathbf{r} = \underline{\mathbf{i}}x + \underline{\mathbf{j}}y + \underline{\mathbf{k}}z$ and $r^2 = x^2 + y^2 + z^2$

Calculate the curl $\underline{\mathbf{F}}$:

1. curl $\underline{\mathbf{F}}$ defined as:
$$\text{curl } \underline{\mathbf{F}} = -GmM\text{curl}(\psi\mathbf{r})$$
where $\psi = \frac{1}{r^3}$

2. The 3rd entry from list of vector identities:
$$\text{curl}(\psi\mathbf{r}) = \psi\text{curl } \mathbf{r} + (\nabla\psi) \times \mathbf{r}$$

3. Since curl $\mathbf{r} = 0$, calculate $\nabla\psi$:
$$\begin{aligned}\nabla\psi &= \nabla\left\{(x^2 + y^2 + z^2)^{\frac{3}{2}}\right\} \\ &= -\frac{3(\underline{\mathbf{i}}x + \underline{\mathbf{j}}y + \underline{\mathbf{k}}z)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} \\ &= -\frac{3\mathbf{r}}{r^5}\end{aligned}$$
Thus:
$$\text{curl } \underline{\mathbf{F}} = -GmM\left(0 - \frac{3\mathbf{r}}{2r^5} \times \mathbf{r}\right) = 0$$

4. Newton gravitational force field is curl-free hence why a gravitational potential exists.
$$\phi = -\frac{GmM}{r}$$

Calculate div $\underline{\mathbf{F}}$:

1. div $\underline{\mathbf{F}}$ defined as:
$$\text{div } \underline{\mathbf{F}}(\psi\mathbf{r}) = \psi\text{div } \mathbf{r} + (\nabla\psi).\mathbf{r}$$

2. div $\mathbf{r} = 3$ and $\nabla\psi$ is calculated already:
$$\text{div } \underline{\mathbf{F}} = -GmM\left(\frac{3}{r^3} - \frac{3\mathbf{r}}{r^5}.\mathbf{r}\right) = 0$$

# 5 Line (path) Integration

In the past, the fundamental idea of an integral is that of summing all the values of the function $f(x_i)$ at points $x_i$ by the area of small strips $\delta x_i$:

$$\int_a^b f(x)\, dx = \sum_{i=1}^{N} f(x_i)\delta x_i$$

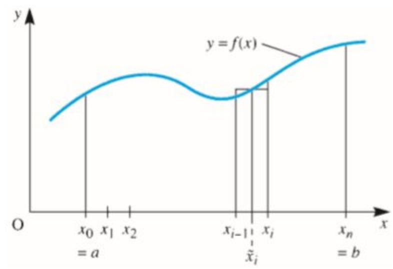



Figure 18: Definite integral as an area

But that view has to be changed when **line integration** is considered since the curve $C$ is now in a 3-D space where a scalar field $\psi(x, y, z)$ or vector field $\underline{\mathbf{F}}(x, y, z)$ takes values at **every point** in this space. Rather, consider a **specified continuous curve** $C$ in 3-D space (**path of integration**) and its associated methods for summing the values that either $\psi$ or $\underline{\mathbf{F}}$ take on that curve.

A curve in 3-D space starts at the point $A$ and ends at point $B$. On the curve, small elements of arc length $\delta s$ and the chord $\delta r$ where $O$ is the origin.

Pythagoras' Theorem in 2-D is a right-angled triangle and, in 3-D, the hypotenuse $\delta s$ in terms of $\delta x$, $\delta y$ and $\delta z$:

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2 + (\delta z)^2$$

There are two classified types of line integral:


(a) Curve in 3-D

(b) Small elements of arc length and the chord

Figure 19

1. Integration of a scalar field $\psi$ along a path $C$:

$$\int_C \psi(x, y, z)\ ds$$

2. Integration of a vector field $\underline{\mathbf{F}}$ along a path $C$:

$$\int_C \underline{\mathbf{F}}(x, y, z)\ .\ d\mathbf{r}$$

If the curve $C$ is closed then use notation:

$$\oint_C \psi(x, y, z)\ ds$$

$$\oint_C \underline{\mathbf{F}}(x, y, z)\ .\ d\mathbf{r}$$

## 5.1 Integration of scalar field

The case is based in the arc length formula. Consider that $ds = \sqrt{dx^2 + dy^2}$ is a infinitesimal element of length then $\int_C 1\ ds$ gives the length of the curve $C$.

Given the parametrization $x(t)$ and $y(t)$ of the curve where $a \leq t \leq b$ are the endpoints.

$$\frac{ds}{dt} = \sqrt{\dot{x}^2 + \dot{y}^2} \Rightarrow ds = \sqrt{\dot{x}^2 + \dot{y}^2}\ dt$$

The arc length is given by:

$$\int_C 1\ ds = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}\ dt$$

**Example 1**: Show that $\int_C x^2 y\ ds = \frac{1}{3}$ where $C$ is the circular arc in the first quadrant of the unit circle.

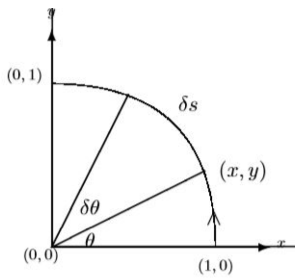



Figure 20: Graphical representation of the problem

1. Note that unit circle defined: $x^2 + y^2 = 1$ with parameters in polar notation:

$$x = cos(t) \Rightarrow \dot{x} = -\sin(t)$$
$$y = sin(t) \Rightarrow \dot{y} = cos(t)$$

2. Given the small element of arc length:

$$\delta s = \sqrt{\sin^2(t) + \cos^2(t)}\delta t = \delta t$$

3. Thus:

$$\int_C x^2 y \; ds = \int_0^{\frac{\pi}{2}} \cos^2(t)\sin(t) \; dt$$
$$= \frac{1}{3}$$

**Example 2**: Show that $\int_C xy^3 \; ds = -\frac{54\sqrt{10}}{5}$ where $C$ is the line $y = -3x$ from $x = -1 \to 1$

1. Note that $ds$ is an element on the line $y = -3x$, thus parametrize the segment with $x = t$ and $y = -3t$ with $-1 \le t \le 1$:

$$ds = \sqrt{\dot{x}^2 + \dot{y}^2} \; dt = \sqrt{10} \; dt$$

2. Thus:

$$\int_C xy^3 \; ds = \sqrt{10}\int_{-1}^{1} t(-3t)^3 \; dt = -\frac{54\sqrt{10}}{5}$$

## 5.2 Integration of vector field

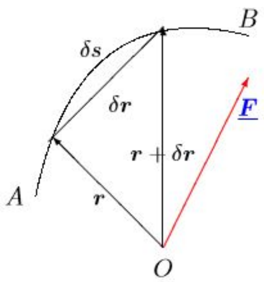



Figure 21: Vector $\underline{\mathbf{F}}$ and a curve $C$ with chord $\delta r$

$$\int_C \underline{\mathbf{F}}(x, y, z) \; . \; d\mathbf{r}$$

**Description**:

- $\underline{\mathbf{F}}$ is a force on a particle begin drawn through the path of the curve $C$.
- Work done $\delta W$ is the pulling of the particle along curve with arc length $\delta s$
- Chord $\delta \mathbf{r}$ is $\delta W = \underline{\mathbf{F}}.\delta \mathbf{r}$

$$W = \int_C \underline{\mathbf{F}} \, . \, d\mathbf{r}$$

**Example 1**: Evaluate the path of $\int_C \underline{\mathbf{F}} \, . \, d\mathbf{r}$ given that $\underline{\mathbf{F}} = \underline{\mathbf{i}}x^2y + \underline{\mathbf{j}}(x-z) + \underline{\mathbf{k}}xyz$ and two paths are

- Path $C_1$ is the parabola $y = x^2$
- Path $C_2$ is the straight line $y = x$

where both are in the plane where $z = 2$ from $(0, 0, 2)$ to $(1, 1, 2)$

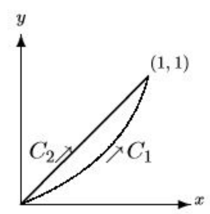



Figure 22

1. Note:
   - $C_1$ is along the curve in plane $z = 2$, thus $dz = 0$ and $dy = 2x \; dx$.
   - $C_2$ is along the line $y = x$, this $dy = dx$.
2. Express as $\int_C \underline{\mathbf{F}}.d\mathbf{r}$

$$\begin{aligned}\int_{C_1} \underline{\mathbf{F}}.d\mathbf{r} &= \int_{C_1} (F_1 \; dx + F_2 \; dy + F_3 \; dz) \\ &= \int_{C_1} (x^2y \; dx + (x - z) \; dy + xyz \; dz) \\ &= \int_{C_1} (x^2y \; dx + (x - 2) \; dy)\end{aligned}$$

3. Path along $C_1$:

- Note $y = x^2$, $dy = 2x\ dx$ and $dz = 0$:

$$\begin{aligned} Path_{(C_1)} &= \int_0^1 (x^4)\ dx + (x-2)2x\ dx \\ &= \int_0^1 (x^4 + 2x^2 - 4x)dx \\ &= -\frac{17}{15} \end{aligned}$$

4. Path along $C_2$:

$$\begin{aligned} Path_{(C_2)} &= \int_0^1 (x^3\ dx + (x-2)\ dx) \\ &= \frac{1}{4} + \frac{1}{2} - 2 \\ &= -\frac{5}{4} \end{aligned}$$

5. Note that with the same start/end points and same integrand, the value of integral can differ with a different route.

**Example 2**: Evaluate $I = \int_C (y^2\ dx - 2x^2\ dy)$ given that $\underline{\mathbf{F}} = (y^2 - 2x^2)$ with the path $C$ taken as $y = x^2$ in the plane $z = 0$ from $(0,0,0)$ to $(2,4,0)$.

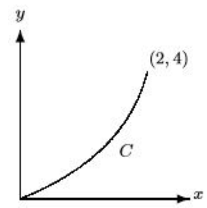



Figure 23

1. Note: Curve $C$ is $y = x^2$, thus $dz = 0$ and $dy = 2x\ dx$

2. Evaluate:

$$\begin{aligned} Path_{(C)} &= \int_C (y^2\ dx - 2x^2\ dy) \\ &= \int_0^2 (x^4 - 4x^3)dx \\ &= -\frac{48}{5} \end{aligned}$$

### 5.2.1 Fundamental Theorem for line integrals

There are circumstances where a line integral $\int_C \underline{\mathbf{F}}.d\mathbf{r}$ takes values which are independent of the path $C$.

**Theorem**: Suppose that $C$ is a smooth curve given by $r(t)$ - $a \leq t \leq b$. Also suppose that $f$ is a function whose gradient vector *nabla* $f$ is continuous on $C$:

$$\int_C \nabla f \ . \ dr = f[r(b)] - f[r(a)]$$

**Proof**:

1. Compute the line integral:

$$\begin{aligned}\int_C \nabla f \ . \ d\mathbf{r} &= \int_a^b \nabla f[r(t)].r(t) \ dt \\ &= \int_a^b \left( \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt\end{aligned}$$

2. Use Chain Rule to simplify the integrand:

$$\begin{aligned}\int_C \nabla f \ . \ d\mathbf{r} &= \int_a^b \left( \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt \\ &= \int_a^b \frac{d}{dt}[f[r(t)]]dt\end{aligned}$$

3. Fundamental Theorem of Calculus for single integrals:

$$\int_C \nabla f \ . \ dr = f[r(b)] - f[r(a)]$$

**Definitions**:

- **Conservative** (vector field): There is a function $f$ such that $F = \nabla f$ where $f$ is the **potential function** for the vector field - A curl-free vector field $\underline{\mathbf{F}}$.
- **Independent of path**: If $\int_{(C_1)} F \ . \ dr = \int_{(C_2)} F \ . \ dr$ for any two paths $C_1$ and $C_2$ in $D$ with the same initial and final points.
- **Curl-free** (vector field): Irrotational flow occurs when curl of velocity of fluid is 0 everywhere:
$$\nabla \times \nabla \phi = 0$$

Integral $\int_C \underline{\mathbf{F}} \ . \ d\mathbf{r}$ is independent of path only if curl $\underline{\mathbf{F}} = 0$.
If $C$ is closed (Start Point and End Point are the same) then:

$$\text{curl } \underline{\mathbf{F}} = 0$$

**Example 1**: Evaluate if line integral $\int_C (2xy^2 \ dx + 2x^2 y \ dy)$ is independent of path.

1. $\underline{\mathbf{F}}$ is expressed as $\underline{\mathbf{F}} = 2xy^2\underline{\mathbf{i}} + 2x^2y\underline{\mathbf{j}} + 0\underline{\mathbf{k}}$:

$$
\begin{aligned}
\text{curl } \underline{\mathbf{F}} &= \begin{bmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ 2xy^2 & 2x^2y & 0 \end{bmatrix} \\
&= (4xy - 4xy)\underline{\mathbf{k}} \\
&= 0
\end{aligned}
$$

2. Integral is proved to be independent of path and able to calculate $\phi$ from $\underline{\mathbf{F}} = -\nabla\phi$:

$$
\begin{aligned}
-\frac{\partial\phi}{\partial x} &= 2xy^2 \\
-\frac{\partial\phi}{\partial y} &= 2x^2y \\
-\frac{\partial\phi}{\partial z} &= 0
\end{aligned}
$$

3. Partial integration of the three equations gives:

$$
\begin{aligned}
\phi &= -x^2y^2 + A(y) \\
\phi &= -x^2y^2 + B(x)
\end{aligned}
$$

4. Since $A(y) = B(x) = \text{const} = C$

$$\phi = -x^2y^2 + C$$

**Example 2**: Find work done $\int_C \underline{\mathbf{F}} \, . \, d\mathbf{r}$ by the force $\underline{\mathbf{F}} = (yz, xz, xy)$ moving from $(1, 1, 1) \to (3, 3, 2)$.

1. Check is line integral independent of path (if curl $\underline{\mathbf{F}} = 0$):

$$\text{curl}\underline{\mathbf{F}} = \begin{bmatrix} \underline{\mathbf{i}} & \underline{\mathbf{j}} & \underline{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ yz & xz & xy \end{bmatrix} = 0$$

2. Integral is proved to be independent of path and able to calculate $\phi$ from $\underline{\mathbf{F}} = -\nabla\phi$:

$$
\begin{aligned}
-\frac{\partial\phi}{\partial x} &= yz \\
-\frac{\partial\phi}{\partial y} &= xz \\
-\frac{\partial\phi}{\partial z} &= xy
\end{aligned}
$$

3. Partial integration of the three equations gives:

$$
\begin{aligned}
\phi &= -xyz + A(y, z) \\
\phi &= -xyz + B(x, z) \\
\phi &= -xyz + C(x, y)
\end{aligned}
$$

4. Since $A(y,z) = B(x,z) = C(x,y) = \text{const}$

$$\phi = -xyz + C$$

5. Calculate the work:

$$\begin{aligned} W &= -\int_{(1,1,1)}^{(3,3,2)} d\phi \\ &= [xyz]_{(1,1,1)}^{3,3,2} \\ &= 18 - 1 \\ &= 17 \end{aligned}$$

# 6 Double and Multiple Integration

In the previous sections, the definite integral of a function $f(x)$ of one variable by the limit:

$$\int_b^a f(x)\,dx = \lim_{\substack{n\to\infty \\ \text{all } \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

where

- $a = x_0 < x_1 < x_2 < \cdots < x_n = b$
- $\Delta x_i = x_i - x_{i-1}$

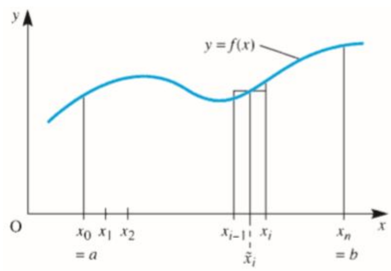



Figure 24: Graphical definition of definite integral

Considering the region $R$ in the $x-y$ plane with boundary curve $C$, the integral of $f(x,y)$ is defined as **"the double integral of $\psi$ over the region $R$"**:

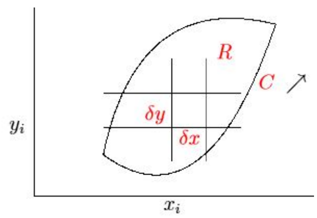
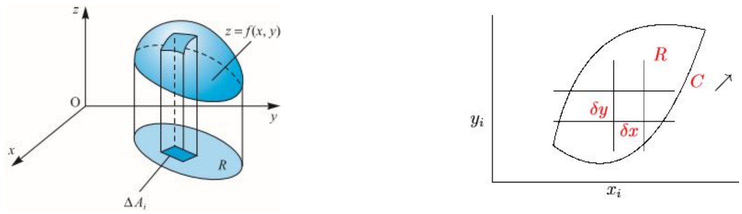



Figure 25




(a) Double integral of $\psi$ over the region $R$

(b) Area of $R$

Figure 26: Difference between the two statements as aforementioned

$$\sum_{i=1}^{N}\sum_{j=1}^{M}\psi(x_i, y_i)\delta A_i \rightarrow \int\int_R \psi(x, y)\ dx\ dy$$

as $\delta x \rightarrow 0$ and $\delta y \rightarrow 0$

where:

- $\psi(x_i, y_i)$ - The value of a scalar function at the point $(x_i, y_i)$
- The area defined as $\delta A_i = \delta x_i \delta y_i$

**Note**: There is a fundamental difference between "Double integral of $\psi$ over the region $R$" and "Area of $R$" which is defined as Area of R $= \int\int_R dxdy$

**Working example**:

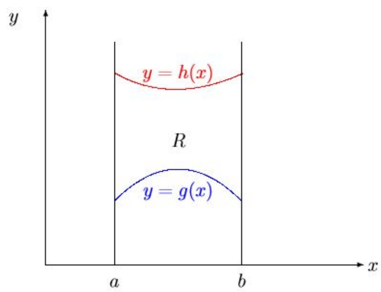



Figure 27

Region $R$ is bounded between the upper curve ($y = h(x)$), lower curve ($y = g(x)$), line $x = a$ and line $x = b$.

$$\int\int_R \psi(x, y)\ dxdy = \int_a^b \left\{\int_{y=g(x)}^{y=h(x)} \psi(x, y)\ dy\right\} dx$$

Given that the inner integral is a partial integral over $y$ holding $x$ constant - the inner integral is a function of $x$:

$$\int_{y=g(x)}^{y=h(x)} \psi(x, y) dy = P(x)$$

Thus can be simplified into:

$$\int\int_R \psi(x,y)\ dxdy = \int_a^b P(x)\ dx$$

Area of $R$:

$$\text{Area of R} = \int_a^b \left\{ \int_{y=g(x)}^{y=h(x)} dy \right\} dx = \int_a^b \{h(x) - g(x)\}\ dx$$

## 6.1 Application

1. Area under a curve For a function of a single variable $y = f(x)$ between $x = a$ and $x = b$:

$$\text{Area} = \int_a^b \left\{ \int_0^{f(x)} dy \right\} dx = \int_a^b f(x)\ dx$$

2. Volume under a surface A surface is 3-D space may be expressed as $z = f(x,y)$:

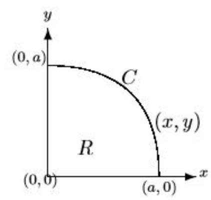
$$\begin{aligned}\text{Volume} &= \int\int\int_V dx\ dy\ dz \\ &= \int_R \left\{ \int_0^{f(x,y)} dz \right\} dxdy \\ &= \int\int_R f(x,y) dx\ dy\end{aligned}$$

This process reduces a 3-integral into a double integral

3. Mass of a solid body Let $p(x,y,z)$ be the variable density of the material in a solid body. Mass $\delta M$ of a small volume $\delta V = \delta x \delta y \delta z$ is $\delta M = \rho \delta V$:

$$\text{Mass of body} = \int\int\int_V p(x,y,z) dV$$

**Example 1**: Consider the first quadrant circle of radius $a$:




Show that:

1. Area of $R = \frac{\pi a^2}{4}$

   The area of $R$ is given by:

$$\begin{aligned} A &= \int_0^a \left\{ \int_0^{\sqrt{a^2-x^2}} dy \right\} dx \\ &= \int_0^a \sqrt{a^2-x^2} dx \end{aligned}$$

   Let $x = a\cos(\theta)$ and $dx = -a\sin(\theta)d\theta$:

$$\begin{aligned} A &= \frac{1}{2}a^2 \int_0^{\frac{\pi}{2}} (1-\cos(2\theta))d\theta \\ &= \frac{\pi a^2}{4} \end{aligned}$$

2. $\int\int_R xy\ dxdy = \frac{a^4}{8}$

$$\begin{aligned} \int\int_R xy\ dxdy &= \int_0^a x \left( \int_0^{\sqrt{a^2-x^2}} y\ dy \right) dx \\ &= \frac{1}{2}\int_0^a x(a^2-x^2)dx \\ &= \frac{1}{2}[\frac{1}{2}x^2a^2 - \frac{1}{4}x^4]_0^a \\ &= \frac{a^4}{8} \end{aligned}$$

3. $\int\int_R x^2y^2\ dxdy = \frac{\pi a^6}{96}$

$$\begin{aligned} \int\int_R x^2y^2\ dxdy &= \int_0^a x^2 \left( \int_0^{\sqrt{a^2-x^2}} y^2\ dy \right) dx \\ &= \frac{1}{3}\int_0^a x^2(a^2-x^2)^{\frac{3}{2}} dx \\ &= \frac{1}{3}a^6 \int_0^{\frac{\pi}{2}} \cos^2(\theta)\sin^4(\theta)d\theta \\ &= \frac{1}{3}a^6(I_4 - I_6) \end{aligned}$$

   where $I_n = \int_0^{\frac{\pi}{2}} \sin^n(\theta)d\theta$ which is an integral recurrence relation:

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2} \dots I_2 = \frac{\pi}{4} \dots I_4 = \frac{3\pi}{16} \dots I_6 = \frac{5\pi}{32}$$

   Thus substituting in the values of $I_n$:

$$\frac{1}{3}a^6(I_4 - I_6) = \frac{\pi a^6}{96}$$

## 6.2 Changing order of double integration

Sometimes changing the order of integration makes evaluations much easier. For example: $I = \int_0^a \left( \int_y^a \frac{x^2\, dx}{x^2+y^2} \right) dy$.

Process of changing order of double integration:

1. Deduce the area of integration from the internal limits:

   The internal integral (from the above example):

$$\int_{x=y}^{x=a} f(x,y)\, dx$$

   The **left limit** is $x = y$ and the **right limit** is $x = a$.

2. Draw the graph showing the **region of integration** $R$:

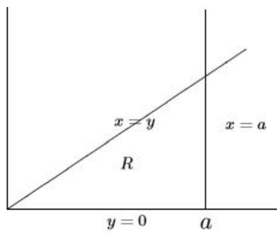



Figure 28: Region of Integration labelled $R$

3. Perform integration over $R$ but in reverse order: Integrate **vertically** then **horizontally**

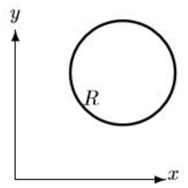
   - Lower vertical limit: $y = 0$
   - Upper vertical limit: $y = x$
   - Lower horizontal limit: $x = 0$
   - Upper horizontal limit: $x = a$

$$I = \int_{x=0}^{x=a} \left( \int_{y=0}^{y=x} \frac{x^2\, dy}{x^2+y^2} \right) dx$$

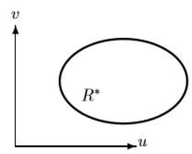
4. Inner integrate before using the result to outer integrate:

$$\text{Inner integration: } \int_0^x \frac{x^2\, dy}{x^2+y^2} = x \int_0^1 \frac{d\theta}{1+\theta^2} = \frac{\pi x}{4}$$

$$\text{Outer integration: } I = \frac{\pi}{4} \int_0^a x\, dx = \frac{\pi a^2}{8}$$




(a) Region $R$ in $x - y$ plane




(b) Distorted region $R*$ in plane of new co-ordinates $u = u(x, y)$ and $v = v(x, y)$

## 6.3 Change of variable and Jacobian

Sometimes it is easier to with the natural circular symmetry in the problem to change the order of variables. This requires expressions in polar co-ordinates.

**Theorem 1**:
The transformation relates the two small areas $\delta x \delta y$ and $\delta u \delta v$:

$$dxdy = |J_{u,v}(x, y)|dudv$$

where the Jacobian $J_{u,v}(x, y)$ is defined:

$$J_{u,v}(x, y) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Note:
- $\frac{\partial x}{\partial u} \neq \left(\frac{\partial u}{\partial x}\right)^{-1}$
- $J_{x,y}(u, v) = [J_{u,v}(x, y)]^{-1}$
- $dudv = |J_{x,y}(u, v)|dxdy$

**Theorem 2**:
Inverse relationship can be proved in a similar manner:

$$\int \int_R f(x, y)\ dxdy = \int \int_{R*} f[x(u, v), y(u, v)]|J_{u,v}(x, y)|\ dudv$$

**Example 1**: For polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$, take $u = r$ and $v = \theta$:

$$J_{r,\theta}(x, y) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix} = r$$

Thus:

$$dxdy = r\ drd\theta$$

**Example 2**: Calculate the volume of a sphere of radius $a$, note that $z = \pm\sqrt{a^2 - x^2 - y^2}$.

1. Double up the two hemisphere:

$$\text{Volume} = 2\int\int_R \sqrt{a^2 - x^2 - y^2}dxdy$$

where $R$ is the disc $x^2 + y^2 \leq a^2$ in the $z = 0$ plane

2. Use a change of variable:

$$\begin{aligned}
\text{Volume} &= 2\int\int_R \sqrt{a^2 - x^2 - y^2}dxdy \\
&= 2\int\int_R \sqrt{a^2 - r^2}r\ drd\theta \\
&= 2\int_0^a \sqrt{a^2 - r^2}r\ dr \int_0^{2\pi} d\theta \\
&= \frac{4\pi a^3}{3}
\end{aligned}$$

**Example 3**: Show that $\int\int_R (x^2 + y^2)dxdy = \frac{8}{3}$ using $u = x + y$ and $v = x - y$:

- Where $R$ has corners at $(0,0)$, $(1,1)$, $(2,0)$ and $(1,-1)$.
- Rotates to a square with corners at $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$.

Process:

1. From $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$:

$$J_{u,v}(x,y) = -\frac{1}{2}$$

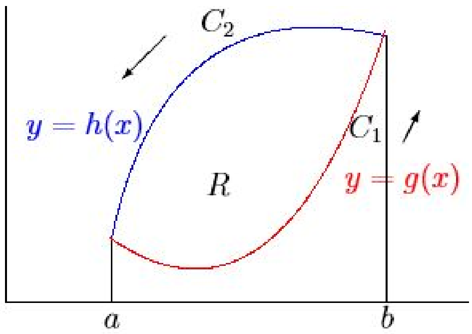
2. Solve:

$$\begin{aligned}
I &= \frac{1}{4}\int\int_{R*} 2(u^2 + v^2)| - \frac{1}{2}|\ dudv \\
&= \frac{1}{4}\left\{[\frac{1}{3}u^3]_0^2[v]_0^2 + [\frac{1}{3}v^3]_0^2[u]_0^2\right\} \\
&= \frac{8}{3}
\end{aligned}$$

# 7 Green's Theorem (in a plane)

Green's Theorem in a plane describes how a line integral **on the boundary** of a closed curve $C$ (that encloses region $R$) is related to the double integral over the region $R$.

Let $R$ be a closed region that is bounded in the $x - y$ plane with piecewise smooth boundary $C$. $P(x,y)$ and $Q(x,y)$ are continuous functions **within** $R$ that has continuous partial derivatives $Q_x$ and $P_y$ respectively.




**Green's Theorem**

$$\oint_C (P\ dx + Q\ dy) = \int\int_R (Q_x - P_y)\ dxdy$$

or

$$\oint_C (P\ dx + Q\ dy) = \int\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dxdy$$

**Proof**: With reference to the diagram, $R$ is bounded by upper and lower boundaries: $g(x) \leq y \leq h(x)$:

1. Perform double integration for $P$:

$$\begin{aligned}
-\int\int_R \frac{\partial P}{\partial y}\ dxdy &= -\int_a^b \left\{\int_{y=g(x)}^{y=h(x)} \frac{\partial P}{\partial y}\ dy\right\}\ dx \\
&= -\int_a^b \{P(x,\ h(x)) - P(x,\ g(x))\}\ dx \\
\text{(Switch limits and signs)} &= \int_a^b P(x,\ g(x))\ dx + \int_b^a P(x,\ h(x))\ dx \\
&= \int_{C1} P(x,\ y)\ dx + \int_{C2} P(x,\ y)\ dx \\
&= \oint_C P(x,\ y)\ dx
\end{aligned}$$

2. Perform the same double integration for $Q$

3. Combine both results:

$$\int\int_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dxdy = \oint_C [P(x,\ y)\ dx + Q(x,\ y)\ dy]$$

**Example 1**: Use Green's Theorem to evaluate the line integral:

$$\oint_C \{(x-y)dx - x^2 dy\}$$

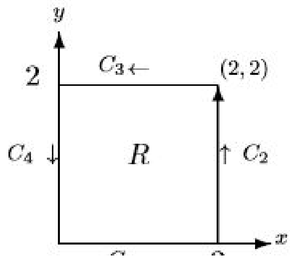



Figure 30

where $R$ and $C$ are given by:

1. Use Green's Theorem with $P = x - y$ and $Q = -x^2$ over region $R$:

$$\begin{aligned}\oint_C = (x-y)dx - x^2 dy &= \int\int_R (1-2x)\ dxdy \\ &= \int_0^2 dy \int_0^2 (1-2x)\ dx \\ &= -4\end{aligned}$$

2. Direct valuations results in:

$$\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$$

where:

$$\begin{aligned}C_1 &: y = 0 \\ C_2 &: x = 2 \\ C_3 &: y = 2 \\ C_4 &: x = 0\end{aligned}$$

3. Combine:

$$\begin{aligned}\oint_C &= \int_0^2 x\ dx - 4\int_0^2 dy + \int_2^0 (x-2)\ dx + 0 \\ &= 2 - 8 + 2 \\ &= -4\end{aligned}$$

**Example 2**: Use Green's Theorem to show that:

$$\oint_C \{y^3\ dx + (x^3 + 3xy^2)\ dy\} = \frac{3}{20}$$

Process:

$$\oint_C \{y^3\ dx + (x^3 + 3xy^2)\ dy\} = 3 \int\int_R x^2\ dxdy$$
$$= 3\int_0^1 x^2 \left(\int_{y=x^2}^{y=x} dy\right) dx$$
$$= 3\int_0^1 (x^3 - x^4)\ dx$$
$$= \frac{3}{20}$$

# 8 Surface Integrals

The extension of the idea of computing the integral of a vector field on a surface.

$$\int\int_S \mathbf{F}(\mathbf{r})\ .\ d\mathbf{S} = \int\int_S \mathbf{F}(\mathbf{r})\ .\ \hat{\mathbf{n}}\ dS$$

**Note**:

- $d\mathbf{S} = \hat{\mathbf{n}} dS$ - The vector element of area
- $\hat{\mathbf{n}}$ - Unit outward-drawn normal vector to the element $dS$
- $dS$ - Infinitesimal area patch on the surface

Such integrals are called **flux integrals** and gives the flux of the vector field through the surface $S$ e.g. if the vector field is the flow of water then the flux is the **volume** of water flowing through $S$ per unit time.

## 8.1 Guass' (Divergence) Theorem

In the same way the Green's Theorem relates surface and line integrals, Gauss's theorem relates surface and volume integral. It allows the conversion of **surface integrals** into **volume integrals** thus simplifying the evaluation.

**Green's Theorem as a 2-D version of Gauss's Theorem**:

$$\int\int_R \text{div}\ \mathbf{u}\ dxdy = \oint_C \mathbf{u}\ .\ \hat{\mathbf{n}}\ ds$$

The line integral expresses the sum of the normal component of $\mathbf{u}$ around the boundary. If $\mathbf{u}$ is a solenoid vector (div $\mathbf{u} = 0$) then $\oint_C \mathbf{u}\ .\ \hat{\mathbf{n}}\ ds = 0$

**Gauss's Theorem (3-D)**:

$$\int\int\int_V \text{div}\ \mathbf{u}\ dV = \int\int_S \mathbf{u}\ .\ \hat{\mathbf{n}}\ dS$$

or

$$\int\int\int_V \nabla \,.\, \mathbf{u}\, dV = \int\int_S \mathbf{u} \,.\, d\overrightarrow{S}$$

The theorem uses a 3-D vector field $\mathbf{u}(x, y, z)$ that exists in some finite volume $V$ whose surface is $S$ - $dS$ is some small element of are on the curved, closed surface $S$.

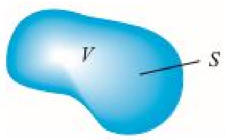



Figure 31: Closed volume $V$ with surface $S$

#### 8.1.1 Explanation using liquid flow

Vector fields are represented using the velocity field of a fluid as previously mentioned - a moving liquid has a velocity (speed and direction) at each point represented by a vector thus forming a vector field.

The divergence theorem is employed in any conservation law which states:

> The total volume of all sinks and sources (the volume integral of the divergence) is equal to the net flow across the volume's boundary.

Detailed explanation: Consider an imaginary closed surface $S$ inside a body of liquid and itself enclosing a volume of liquid.

1. The **flux** of liquid out of the volume is equal to the volume rate of fluid cross the surface $S$ - the **surface integral** of velocity over the surface.

2. As liquids are incompressible, the amount of liquid within the closed surface is constant. If there are no **sinks** or **source**, the **flux** is zero - the water moving in and out are equal so **net flux** is zero.

   - If a **source** (of liquid) e.g. pipe is within the closed surface, the additional introduced liquid will exert pressure on the surrounding liquid and cause an outflow in all directions.

     – Flux outward through $S$ equals volume rate of flow of fluid into $S$ from the pipe.

   - The same logic can be applied if there is a **sink** within the closed surface. The volume rate of flow of liquid inward through the surface $S$ is equal to the rate of flow of liquid removed by the pipe.

3. When there are multiple sources and sinks inside $S$, the flux is calculated by adding up the volume rate of liquid added by sources and subtracting

the volume rate of liquid removed by sinks.

4. The **volume rate of flow** of liquid through a source or sink is equal to the **divergence** of the velocity field at the pipe mouth.

5. Integrating (adding up) the divergences of the liquid throughout the enclosed volume $S$ equals the volume of rate of flux through $S$.

**Example 1**: A vector field $\mathbf{F}(\mathbf{r})$ is given by:

$$\mathbf{F}(\mathbf{r}) = x^3y\mathbf{i} + x^2y^2\mathbf{j} + x^2yz\mathbf{k}$$

Find $\int\int_S \mathbf{F} \, . \, d\mathbf{S}$ where $S$ is the surface of the region in the first octant for which $x + y + z \leq 1$

Process:

1. Sketch the region $V$ enclosed by $S$:

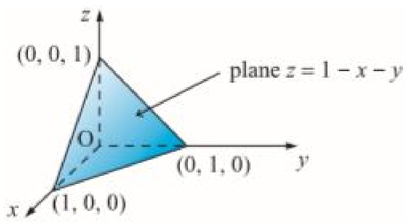



Figure 32: Region $V$ and surface $S$

2. Note evaluating surface integral is complicated due to the four faces each required separate integrals.

3. Transform into volume integral using divergence theorem:

$$\int\int_S \mathbf{F} \, . \, d\mathbf{S} = \int\int\int_V \text{div } \mathbf{V} \, dV$$

4. Find div $\mathbf{F}$:

$$\begin{aligned} \text{div } \mathbf{F} &= 3x^2y + 2x^2y + x^2y \\ &= 6x^2y \end{aligned}$$

5. Solve using the theorem:

$$\int\int_S \mathbf{F} \,.\, d\mathbf{S} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} 6x^2 y \, dz$$
$$= 6\int_0^1 x^2 \, dx \int_0^{1-x} y \, dy \int_0^{1-x-y} dz$$
$$= 6\int_0^1 x^2 \, dx \int_0^{1-x} [(1-x)y - y^2] \, dy$$
$$= \int_0^1 x^2(1-x)^3 \, dx$$
$$= \frac{1}{60}$$

**Example 2**:

## 8.2 Stokes' Theorem

Stokes' theorem is the generalization of Green's theorem relating line integrals in 3-D with surface integrals.

**Stokes' theorem**[a]:

$$\oint_C \mathbf{v} \,.\, d\mathbf{r} = \int\int_S \text{curl } \mathbf{v} \,.\, \hat{\mathbf{n}} \, dS = \int\int_S (\nabla \times \mathbf{v}) \,.\, d\overrightarrow{S}$$

- $\hat{\mathbf{n}}$ - Unit normal vector to surface $S$ of $V$
- $C$ - The boundary of the surface $S$
- $dS$ - An element of area

[a] The line integral $\oint_C \mathbf{v} \,.\, d\mathbf{r}$ is called the **circulation**. If $\mathbf{v}$ is an irrotational vector then $\oint_C \mathbf{v} \,.\, d\mathbf{r} = 0$ - there is no rotation.

There are several wats that the result can be used:

1. Allows the evaluation of the surface integral in terms of the simpler line integral:
   - Requires that the field can be written as $\nabla \times \overrightarrow{F}$
   - Find $\overrightarrow{F}$
2. Allows the evaluation of closed line integrals in terms of any surface bounded by the line $C$, whichever is the most convenient:
   - For any two surfaces $S_1$ and $S_2$ with the same boundary $C$:

$$\oint_C \overrightarrow{F} \,.\, d\overrightarrow{r} = \int\int_{S_1} \left(\nabla \times \overrightarrow{F}\right) \,.\, d\overrightarrow{S}$$
$$= \int\int_{S_2} \left(\nabla \times \overrightarrow{F}\right) \,.\, d\overrightarrow{S}$$

3. Implies that the flux of $\left(\nabla \times \overrightarrow{F}\right)$ through surface $S_j$ is **independent** of which surface $S_j$ is chosen, provided all the surfaces are bounded by the same curve $C$.

**Example 1**: If $\mathbf{v} = \underline{\mathbf{i}}y^2 + \underline{\mathbf{j}}x^2$ and $R$ is bounded as shown below, evaluate the line integral to show that:

$$\oint_C \mathbf{v} \, . \, d\mathbf{r} = \frac{5}{48}$$

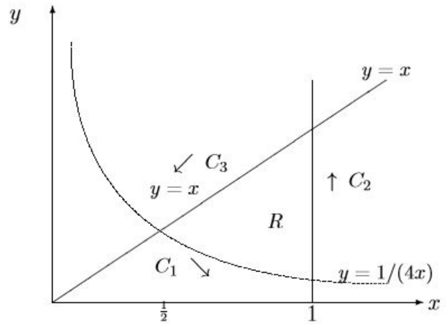



Figure 33

Process:

1. Since $\mathbf{v} = \underline{\mathbf{i}}y^2 + \underline{\mathbf{j}}x^2$, express as:

$$\oint_C \mathbf{v} \, . \, d\mathbf{r} = \oint_C (y^2 \, dx + x^2 \, dy)$$

2. On $C_1$, $y = \frac{1}{4x}$ thus $dy = \frac{-dx}{4x^2}$:

$$\begin{aligned}\int_{C_1} \mathbf{v} \, . \, d\mathbf{r} &= \int_{\frac{1}{2}}^{1} \left(\frac{1}{16x^2} - \frac{1}{4}\right) dx \\ &= -\frac{1}{16}\end{aligned}$$

3. On $C_2$, $x = 1$ this $dx = 0$:

$$\begin{aligned}\int_{C_2} \mathbf{v} \, . \, d\mathbf{r} &= \int_{\frac{1}{4}}^{1} dy \\ &= \frac{3}{4}\end{aligned}$$

4. On $C_3$, $y = x$ this $dy = dx$:

$$\begin{aligned}\int_{C_3} \mathbf{v} \, . \, d\mathbf{r} &= 2\int_{1}^{\frac{1}{2}} x^2 dx \\ &= -\frac{7}{12}\end{aligned}$$

5. Sum the results:
$$-\frac{1}{16}+\frac{3}{4}-\frac{7}{12}=\frac{5}{48}$$

**Example 2**: If $\mathbf{u}=\underline{\mathbf{i}}\left(\frac{x^2y}{1+y^2}\right)+\underline{\mathbf{j}}\left(x\ln(1+y^2)\right)$ and $R$ bounded as shown below, show that:
$$\int\int_R \text{div }\mathbf{u}\; dxdy = 2\ln(2)-1$$

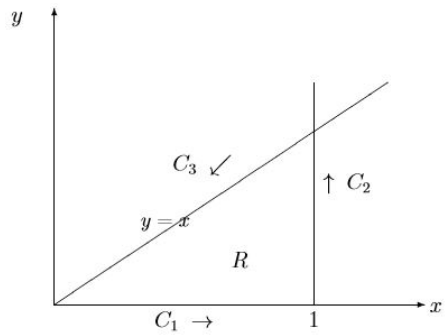



Figure 34

1. Calculate div $\mathbf{u}$:
$$\text{div }\mathbf{u}=\frac{4xy}{1+y^2}$$

2. Solve as normal:
$$\begin{aligned}\int\int_R \text{div }\mathbf{u}\; dxdy &= 4\int\int_R \frac{xy}{1+y^2}\; dxdy\\ &= 4\int_0^1 x\left(\int_{y=0}^{y=x}\int\frac{y\; dy}{1+y^2}\right)dx\\ &= 4\int_0^1 x\left[\frac{1}{2}\ln(1+y^2)\right]_0^x\; dx\\ &= 2\int_0^1 x\ln(1+x^2)\; dx\\ &= 2\ln(2)-1\end{aligned}$$

# A Python and MatLab codes

## A.1 2: Scalar fields

**Figure 1.a - Surface plot (Python):**

```
import numpy as np
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
import random

def fun(x, y):
    return (1/12)*y**3 - y - (1/4)*x**2 + (7/2)

fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
x = y = np.arange(-3.0, 3.0, 0.05)
X, Y = np.meshgrid(x, y)
zs = np.array(fun(np.ravel(X), np.ravel(Y)))
Z = zs.reshape(X.shape)

ax.plot_surface(X, Y, Z)

ax.set_xlabel('X Label')
ax.set_ylabel('Y Label')
ax.set_zlabel('Z Label')

plt.show()
```

**Figure 1.b - Contour plot (Python):**

```
import numpy as np
import matplotlib.pyplot as plt

fig = plt.figure(figsize=(6,5))
left, bottom,width,height = 0.1,0.1,0.8,0.8
ax = fig.add_axes([left, bottom, width, height])

xlist = np.linspace(-5.0,5.0,100)
ylist = np.linspace(-5.0,5.0,100)

X, Y = np.meshgrid(xlist, ylist)

Z = (1/12)*(Y**3) - Y - (1/4)*(X**2) + (7/2)
cp = ax.contour(X, Y, Z)
ax.clabel(cp, inline=True, fontsize=10)

ax.set_title('Contour Plot')
plt.show()
```

**Figure 2 - Ellipsoid (MatLab):**

```
[theta,phi] = ndgrid(linspace(0,pi),linspace(0,2*pi));
x = sin(theta).*cos(phi);
y = sin(theta).*sin(phi);
z = cos(theta);

figure
surf(x,y,z)
axis equal
```

**Figure 3.a - Same graphs at different time (MatLab)**

```
x = linspace(0,10,50);
y1 = cos(1-x);
plot(x,y1)
hold on
y2 = cos(2-x);
plot(x,y2)
hold off
xlabel('x')
ylabel('f(x)')
```

**Figure 3.b - Surface plot of graph with time varying (Python)**

```
import numpy as np
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
import random

def fun(x, t):
    return np.cos(t - x)

fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
x = y = np.arange(-3.0, 3.0, 0.05)
X, Y = np.meshgrid(x, y)
zs = np.array(fun(np.ravel(X), np.ravel(Y)))
Z = zs.reshape(X.shape)

ax.plot_surface(X, Y, Z)

ax.set_xlabel('X Label')
ax.set_ylabel('t Label')
ax.set_zlabel('f(x,t) Label')

plt.show()
```

## A.2 3: Vector fields

**Figure 5 - Vector field (MatLab)**

```
g = -2:0.4:2;
[x,y]=meshgrid(g);
figure;
u=y;
v=-x;
quiver(x,y,u,v);
```

**Figure 7 - Vector field of electric charge (Python)**

```
from pylab import *

X=linspace(-2,2,40)
Y=linspace(-2,2,24)
X,Y=meshgrid(X, Y)


v = (Y/((X+0)**2 + Y**2))
u = ((X+0)/((X+0)**2 + Y**2))

Q  = quiver(X,Y,u,v)
show()
```

**Figure 8 - Shifted vector field of electric charge (Python)**

```
from pylab import *

X=linspace(-2,2,40)
Y=linspace(-2,2,24)
X,Y=meshgrid(X, Y)


v = (Y/((X+1)**2 + Y**2))
u = ((X+1)/((X+1)**2 + Y**2))

Q  = quiver(X,Y,u,v)
show()
```

**Figure 10 - Graphical representation of the superimposed vector field (Python)**

```
from pylab import *

X=linspace(-2,2,40)
Y=linspace(-2,2,24)
X,Y=meshgrid(X, Y)

u1 = ((X+1)/((X+1)**2 + Y**2))-((X-1)/((X-1)**2 +
Y**2))
v1 = (Y/((X+1)**2 + Y**2))-(Y/((X-1)**2 + Y**2))

U=u1/np.sqrt(u1**2+v1**2)
V=v1/np.sqrt(u1**2+v1**2)

Q  = quiver(X,Y,U,V)
show()
```

## A.3 4: Vector operators: Grad, Div and Curl

**Figure 13.a - Surface plot (Python)**

```
from mpl_toolkits import mplot3d
import numpy as np
import matplotlib.pyplot as plt

# Function definition
def func(x,y):
    return np.exp(-x**2 -y**2)

# Define axes boundaries
x = np.linspace(-2,2, 20)
y = np.linspace(-2,2, 20)

# Create grid and populate
X, Y = np.meshgrid(x, y)
Z = func(X, Y)

# Define type of plot
fig = plt.figure()
ax = plt.axes(projection='3d')
ax.plot_surface(X, Y, Z, rstride=1, cstride=1,cmap
='viridis', edgecolor='none')

# Label plots
ax.set_xlabel('x')
ax.set_ylabel('y')
ax.set_zlabel('z')

plt.show()
```

**Figure 13.b - Surface plot (Python)**

```
import numpy as np
import matplotlib.pyplot as plt
xlist = np.linspace(-2,2,50)
ylist = np.linspace(-2,2,50)
X, Y = np.meshgrid(xlist, ylist)
Z = np.exp(-X**2 - Y**2)
fig,ax=plt.subplots(1,1)
cp = ax.contour(X, Y, Z)

fig.colorbar(cp) # Add a colorbar to a plot
ax.set_title('Filled Contours Plot')
#ax.set_xlabel('x (cm)')
ax.set_ylabel('y (cm)')
plt.show()
```

**Figure 13.d - Equipotential lines with gradient field (Python)**

```
from pylab import *
import numpy as np
import matplotlib.pyplot as plt

xlist = np.linspace(-2,2,50)
ylist = np.linspace(-2,2,50)
X, Y = np.meshgrid(xlist, ylist)
Z = np.exp(-X**2 - Y**2)
fig,ax=plt.subplots(1,1)
cp = ax.contour(X, Y, Z)

X=linspace(-2,2,40)
Y=linspace(-2,2,24)
X,Y=meshgrid(X, Y)

u1 = -2*X * np.exp(-X**2 - Y**2)
v1 = -2*Y * np.exp(-X**2 - Y**2)


Q  = quiver(X,Y,u1,v1)
show()
```

## A.4 4: Definition of divergence of a vector field $(B)$

**Figure 14 - Cube showing the arrows pointing outward from center (Python)**

```
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
import numpy as np

fig = plt.figure()
ax = fig.gca(projection='3d')

# Make the grid
x, y, z = np.meshgrid(np.arange(-1, 1, 0.3),
                      np.arange(-1, 1, 0.3),
                      np.arange(-1, 1, 0.3))

# Make the direction data for the arrows
u = x
v = y
w = z

ax.quiver(x, y, z, u, v, w, length=0.1, normalize=True)

plt.show()
```