EE2 Mathematics – Probability & Statistics

Solution 8

1. Remember that we can ignore the terms which do not depend on the unknown parameter(s).

   (a) The likelihood is

   $$\begin{aligned} L(\lambda; \mathbf{x}) &= \prod_{i=1}^{n} f_{X_i}(x_i) \\ &= \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \\ &\propto e^{-n\lambda}\lambda^{n\bar{x}}\,. && \text{(recall } \bar{x} = 1/n \textstyle\sum_{i=1}^{n} x_i) \end{aligned}$$

   The log-likelihood is then

   $$\ell(\lambda; \mathbf{x}) = -n\lambda + n\bar{x}\log\lambda + C\,,$$

   for some constant C. Differentiate this with respect to λ and set it equal to zero to find

   $$-n + n\bar{x}\frac{1}{\hat{\lambda}} = 0 \Rightarrow \hat{\lambda} = \bar{x}\,,$$

   so the MLE is $\hat{\lambda} = \bar{X}$. This is unbiased, as we know that the sample mean is an unbiased estimator of the population mean (λ).

   (b) The likelihood is

   $$\begin{aligned} L(p; \mathbf{x}) &= \prod_{i=1}^{n} f_{X_i}(x_i; p) \\ &= \prod_{i=1}^{n} \binom{m}{x_i} p^{x_i}(1-p)^{m-x_i} \\ &\propto p^{n\bar{x}}(1-p)^{nm-n\bar{x}}\,. \end{aligned}$$

   The log-likelihood is then

   $$\ell(p; \mathbf{x}) = n\bar{x}\log p + n(m-\bar{x})\log(1-p) + C\,,$$

   for some constant C. Differentiate this with respect to p and set it equal to zero to find

   $$n\bar{x}\frac{1}{\hat{p}} - n(m-\bar{x})\frac{1}{1-\hat{p}} = 0 \Rightarrow \hat{p} = \frac{\bar{x}}{m}\,,$$

   so the MLE is $\hat{p} = \bar{X}/m$. This is unbiased, as $\mathrm{E}(\bar{X}) = \mathrm{E}(X_1) = mp$ ($X_1 \sim \mathrm{Bin}(m,p)$, hence $\mathrm{E}(X_1) = mp$), so $\mathrm{E}(\hat{p}) = mp/m = p$.

(c) The likelihood is

$$
\begin{aligned}
L(p;\mathbf{x}) &= \prod_{i=1}^{n} f_{X_i}(x_i;p) \\
&= \prod_{i=1}^{n} (1-p)^{x_i-1}p \\
&\propto (1-p)^{n\bar{x}-n}p^n\,.
\end{aligned}
$$

The log-likelihood is then

$$
\ell(p;\mathbf{x}) = n(\bar{x}-1)\log(1-p) + n\log p + C\,,
$$

for some constant C. Differentiate this with respect to p and set it equal to zero to find

$$
-n(\bar{x}-1)\frac{1}{1-\hat{p}} + n\frac{1}{\hat{p}} = 0 \Rightarrow \hat{p} = \frac{1}{\bar{x}}\,,
$$

so the MLE is $\hat{p} = 1/\bar{X}$. This is biased, as $\mathrm{E}(1/\bar{X}) \neq 1/\mathrm{E}(\bar{X}) = 1/p^{-1} = p$.

(d) The likelihood is

$$
\begin{aligned}
L(\beta;\mathbf{x}) &= \prod_{i=1}^{n} f_{X_i}(x_i;\beta) \\
&= \prod_{i=1}^{n} \frac{\beta^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\beta x_i} \\
&\propto \beta^{n\alpha} e^{-\beta n\bar{x}}\,.
\end{aligned}
$$

The log-likelihood is then

$$
\ell(\beta;\mathbf{x}) = n\alpha\log\beta - \beta n\bar{x} + C\,,
$$

for some constant C. Differentiate this with respect to β and set it equal to zero to find

$$
n\alpha\frac{1}{\hat{\beta}} - n\bar{x} = 0 \Rightarrow \hat{\beta} = \frac{\alpha}{\bar{x}}\,,
$$

so the MLE is $\hat{\beta} = \frac{\alpha}{\bar{X}}$. This is biased, as $\mathrm{E}(1/\bar{X}) \neq 1/\mathrm{E}(\bar{X}) = \alpha/\beta$, which implies that $\mathrm{E}(\hat{\beta}) \neq \beta$.

2. (a)

$$
\begin{aligned}
F_Y(y) = P(Y \le y) &= 1 - P(Y > y) \\
&= 1 - P(X_1 > y, X_2 > y, \ldots, X_n > y) \quad &\text{(all must be } > y) \\
&= 1 - \prod_{i=1}^{n} P(X_i > y) \quad &\text{(independent)} \\
&= 1 - (P(X_1 > y))^n \quad &\text{(identically distributed)} \\
&= 1 - (1 - F_{X_1}(y))^n \, .
\end{aligned}
$$

The CDF of X_1 is $F_{X_1}(x) = 1 - e^{-\lambda x}$, so we have

$$F_Y(y) = 1 - e^{-n\lambda y} \, ,$$

from which we deduce that $Y \sim \text{Exp}(n\lambda)$.

(b) We have $\text{E}(Y) = (n\lambda)^{-1} = \mu/n$, so $\hat{\mu}_u = nY$ is an unbiased estimator of μ.

(c) The likelihood is

$$L(\lambda; \mathbf{x}) = \prod_{i=1}^{n} f_{X_i}(x_i; \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \propto \lambda^n e^{-\lambda n \bar{x}} \, .$$

The likelihood with respect to μ is

$$L(\mu; \mathbf{x}) \propto \mu^{-n} e^{-n\bar{x}/\mu} \, .$$

The log-likelihood is then

$$\ell(\mu; \mathbf{x}) = -n \log \mu - \frac{n\bar{x}}{\mu} + C \, ,$$

for some constant C. Differentiate this with respect to μ and set it equal to zero to find

$$-\frac{n}{\hat{\mu}} + \frac{n\bar{x}}{\hat{\mu}^2} = 0 \Rightarrow \hat{\mu} = \bar{x} \, ,$$

so the MLE is $\hat{\mu} = \bar{X}$. This is unbiased, as $\text{E}(\bar{X}) = \text{E}(X_1) = \lambda^{-1} = \mu$.

(d) Both estimators are unbiased, so we need to compare their variances. The first estimator has variance

$$\text{Var}(\hat{\mu}_u) = \text{Var}(nY) = n^2 \,\text{Var}(Y) = n^2 (n\lambda)^{-2} = \lambda^{-2} = \mu^2 \, ,$$

while the second has variance

$$\text{Var}(\hat{\mu}) = \text{Var}(\bar{X}) = \frac{\sum_{k=1^n} \text{Var}(X_k)}{n^2} = \frac{n\lambda^{-2}}{n^2} = \frac{\mu^2}{n} \, ,$$

which is smaller. Thus, the MLE has smaller MSE for $n > 1$, and is the better estimator.