

EE2 Mathematics – Probability & Statistics

Exercise 7

1. The discrete random variables X and Y have joint PMF

$$f_{X,Y}(x,y) = \begin{cases} a \frac{x}{y} & \text{if } x, y \in \{1, 2, 3\} \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Find the value of a that makes this a valid joint PMF.
- (b) Construct a table of joint probabilities and include the marginal distributions.
- (c) Compute $E(XY)$, $\text{Cov}(X, Y)$ and $\text{Corr}(X, Y)$. Are X and Y uncorrelated? Are they independent?
- (d) Evaluate $P(X \leq 2 | Y \leq 2)$.

2. Consider the continuous random variables X and Y with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} k(x+y-2xy) & \text{if } 0 \leq x, y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Which value of k makes this a valid PDF?
- (b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$. What are these distributions? Are X and Y independent?
- (c) Evaluate $E(X)$, $E(Y)$, $E(XY)$ and $\text{Cov}(X, Y)$.

3. Consider the continuous random variables X and Y with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx(y-x^2+1) & \text{if } 0 \leq x, y \leq 1 \text{ and } y \geq x^2 \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Which value of c makes this a valid PDF?
- (b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$. Compute $E(X)$.
- (c) Write down the conditional PDF $f_{X|Y}(x|y)$ and use it to compute $E(X|Y=y)$. What is $E(X|Y)$?
- (d) Verify that $E[E(X|Y)] = E(X)$. (This is known as the law of iterated expectation.)

4. Suppose that X, Y are random variables and a, b, c, d are constants, with $a, c \neq 0$. Find an expression for $\text{Corr}(aX+b, cY+d)$ in terms of $\text{Corr}(X, Y)$ and the constants.

5. Over the course of its operation a system may develop faults. Suppose that the distribution of the number of faults in one month is Poisson(λ). Each of these faults is either severe (with probability p) or not severe (with probability $1-p$), independently of all other faults.

- (a) Let X denote the number of all faults and Y the number of severe faults in a one-month period. Write down $f_X(x)$ and $f_{Y|X}(y|x)$, and hence find an expression for the joint PDF $f_{X,Y}(x,y)$.
 - (b) Find the marginal PDF $f_Y(y)$. What is the distribution of Y ?
6. Find the moment generating function of X and use it to compute the mean and variance of the binomial distribution.
- (a) $X \sim \text{Bin}(n, p)$
 - (b) $X \sim \text{Gamma}(k, r)$ such that $f_X(x) = e^{-kx} x^{r-1} k^r / (r-1)!$, $x > 0$, $r \in \mathbb{N}^*$, $k > 0$