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Mathematics: Year 1

Laplace Transform

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Abstract

Laplace Transform methods play a important role Analysis and Design of engineering systems. Laplace Transform is concerned with the systematic solution of ordinary differential equations with constant coefficients.

Mathematical transformations are used to simplify solutions of problems by creating a new domain in which it is easier to handle the problem. The results are then inverse-transformed to give the desired results in the original domain.

Laplace Transform is an example of **Integral Transforms** like Fourier Transforms. Laplace Transforms turn *differential equations* in time domain t to *algebraic equations* in complex frequency domain s. Initial Conditions are essential in Laplace Transforms, making it perfect for solving initial-value problems e.g. electrical circuits and mechanical vibrations.

Contents

1 Introduction

Given a function defined uniquely on $0 \leq t \leq \infty$, the Laplace transform is defined as:

$$\mathcal{L}[f(t)] = \overline{f}(s) = \int_{\infty}^{0} e^{-st} * f(t)dt$$

where s can be complex

This definition shows that Laplace transform is a **one-sided transform** - it works on $[0, \infty]$ unlike Fourier transform $[-\infty, \infty]$. This suits its purpose for solving initial-value problems where a function switches on at $t = 0$ and where $f(0)$ is defined.

The inverse transform is defined as:

$$f(t) = \mathcal{L}^{-1}[\hat{f}(s)] = \oint_{C} e^{st} * \overline{f}(s)\, ds$$

It is very difficult to evaluate inverse transforms (Bromwich integrals), rather a Library of Transforms is defined and problems are broken down into those standard blocks.

2 Library of Laplace Transforms

2.1 Functions

1. Constant function - $f(t) = 1$:

$$f(t) = 1 \Leftrightarrow \overline{f}(s) = \frac{1}{s}$$

where $\Re(s) > 0$

Proof:

$$\begin{aligned} \overline{f}(s) &= \int_{0}^{\infty} e^{-s*t}dt \\ &= \left[\frac{e^{-s*t}}{s}\right]_{0}^{\infty} \\ &= \frac{1}{s} \end{aligned}$$

2. Exponential function - $f(t) = e^{at}$:

$$f(t) = e^{at} \Leftrightarrow \overline{f}(s) = \frac{1}{s-a}$$

where $\Re(s-a) > 0$

Proof:

$$\begin{aligned}\overline{f}(s) &= \int_0^\infty e^{-(s-a)*t}dt \\ &= \left[\frac{e^{-(s-a)*t}}{s-a}\right]_0^\infty \\ &= \frac{1}{s-a}\end{aligned}$$

3. Sine function - $f(t) = \sin(at)$:

$$f(t) = \sin(at) \Leftrightarrow \overline{f}(s) = \frac{a}{s^2+a^2}$$

where $\Re(s) > 0$

Proof:

$$\begin{aligned}\mathcal{L}(e^{iat}) &= \int_0^\infty e^{-(s-ia)*t}dt \\ &= \frac{1}{s-ia} \\ &= \frac{s+ia}{s^2+a^2} \\ &= \frac{a}{s^2+a^2}\end{aligned}$$

4. Cosine function - $f(t) = \cos(at)$:

$$f(t) = \cos(at) \Leftrightarrow \overline{f}(s) = \frac{s}{s^2+a^2}$$

where $\Re(s) > 0$

Proof:

$$\begin{aligned}\mathcal{L}(e^{iat}) &= \int_0^\infty e^{-(s-ia)*t}dt \\ &= \frac{1}{s-ia} \\ &= \frac{s+ia}{s^2+a^2} \\ &= \frac{s}{s^2+a^2}\end{aligned}$$

5. Polynomial function - $f(t) = t^n$:

$$f(t) = t^n \Leftrightarrow \overline{f}(s) = \frac{n!}{s^{n+1}}$$

where $n \geq 0$ and $\Re(s) > 0$

Proof:

$$\begin{aligned}
\overline{f}(s) &= \int_0^\infty e^{-st} * t^n dt \\
&= -\frac{1}{s} \int_0^\infty t^n d[e^{-st}] \\
&= \frac{n}{s} \int_0^\infty e^{-st} * t^{n-1} dt \\
&= \frac{n}{s} * \overline{f}(s)_{n-1}
\end{aligned}$$

6. Heaviside function - $f(t) = H(t - t_0)$:

$$f(t) = H(t - t_0) \Leftrightarrow \overline{f}(s) = \frac{e^{-s*t_0}}{s}$$

where $\Re(s) > 0$

Proof:

$$\begin{aligned}
\mathcal{L}[H(t - t_0)] &= \int_0^\infty e^{-st} * H(t - t_0) dt \\
&= \int_{t_0}^\infty e^{-st} dt \\
&= \frac{e^{-s*t_0}}{s}
\end{aligned}$$

7. Dirac (δ) function - $f(t) = \delta(t - t_0)$:

$$f(t) = \delta(t - t_0) \Leftrightarrow \overline{f}(s) = e^{-s*t_0}$$

where $t_0 \geq 0$

Proof:

$$\int_0^\infty e^{-s*t} * \delta(t - t_0) dt = \begin{cases} e^{-s*t_0} \; t_0 \geq 0 \\ 0 \; t_0 < 0 \end{cases}$$

2.2 Theorems

1. Shift theorem:

$$\mathcal{L}[e^{a*t} * f(t)] \Leftrightarrow \overline{f}(s-a)$$

where $\Re(s-a) > 0$

Proof:

$$\begin{aligned}\mathcal{L}[e^{a*t} * f(t)] &= \int_0^\infty e^{-(s-a)t} * f(t)dt \\ &= \overline{f}(s-a)\end{aligned}$$

2. Second shift theorem:

$$\mathcal{L}[H(t-a) * f(t-a)] = e^{-sa} * \overline{f}(s)$$

Proof: Let $\tau = t - a$

$$\begin{aligned}\mathcal{L}[H(t-a) * f(t-a)] &= \int_0^\infty e^{-s*t} * H(t-a) * f(t-a)dt \\ &= e^{-s*a} \int_{-a}^\infty e^{-s*\tau} * H(\tau) * f(\tau)d\tau \\ &= e^{-s*a} \int_0^\infty e^{-s*\tau} * f(\tau)d\tau \\ &= e^{-s*a} * \overline{f}(s)\end{aligned}$$

3. Convolution theorem:

$$\mathcal{L}\{f \star g\} = \overline{f}(s) * \overline{g}(s)$$

where the convolution between two functions $f(t)$ and $g(t)$ is defined as:

$$f \star g = \int_0^t f(u) * g(t-u)du$$

Proof:

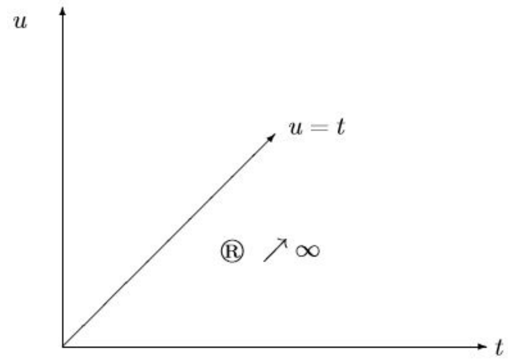

Figure 1: Region of integration obtained from original integral

$$\begin{aligned}\mathcal{L}\{f \star g\} &= \int_0^\infty e^{-s*t}\left(\int_0^t f(u) * g(u-t)du\right)dt\\ &= \int_0^\infty \left(\int_u^\infty e^{-s*t} * g(t-u)dt\right) * f(u)du\end{aligned}$$

Change of variables $\tau = t - u$

$$\begin{aligned}&= \int_0^\infty e^{-s*u}\left(\int_{\tau=0}^{\tau=\infty} e^{-s*\tau} * g(\tau)d\tau\right) f(u)du\\ &= \overline{f}(s) * \overline{g}(s)\end{aligned}$$

4. Integral theorem:

$$\mathcal{L}\left(\int_0^t f(u)du\right) = \frac{\overline{f}(s)}{s}$$

5. Derivative theorem:

$$\mathcal{L}[f'(t)] = s\overline{f}(s) - f(0)$$

Proof:

$$\begin{aligned}
\mathcal{L}[\dot{f}] &= \int_0^\infty e^{-s*t} * \dot{f} dt \\
&= \int_0^\infty e^{-s*t} df \\
&= \left[e^{-s*t} * f(t)\right]_0^\infty + s\int_0^\infty e^{-s*t} * f dt \\
&= s\overline{f}(s) - f(0)
\end{aligned}$$

6. Second derivative theorem:

$$\boxed{\mathcal{L}[f''(t)] = s^2\overline{f}(s) - s\overline{f}(0) - f'(0)}$$

Proof:

$$\begin{aligned}
\mathcal{L}[f''(t)] &= \int_0^\infty e^{-s*t} * \dot{f} dt \\
&= \int_0^\infty e^{-s*t} df' \\
&= \left[e^{-s*t} * f'(t)\right]_0^\infty + s\int_0^\infty e^{-s*t} * f' dt \\
&= s\mathcal{L}[f'] - f'(0) \\
&= s^2\overline{f}(s) - s\overline{f}(0) - f'(0)
\end{aligned}$$