EE2 Mathematics – Probability & Statistics

Exercise 4

1. The number of α-particles emitted in a one second interval by 1 gram of a radioactive substance follows a Poisson distribution with a rate of 3.2 particles per second. What is the probability that no more than two particles will be emitted in a half-second interval?

2. The time, T months, required to build an aircraft component has an exponential distribution with parameter $\lambda = 2$. Calculate the probability that
   (a) it takes less than 1 month to build the component.
   (b) it takes longer than 3 months to build the component.
   (c) it takes longer than 3 months to build the component, given that the work has already lasted 2 months

3. Consider the random variable X with probability density function
$$f_X(x) = k \cos x \,, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
   and zero otherwise.
   (a) What is the value of k that makes this a valid density?
   (b) Compute the mean and variance of X.
   (c) The interquartile range (IQR) is defined as
$$IQR = q_u - q_l$$
   where $F_X(q_u) = 0.75$ and $F_X(q_l) = 0.25$. Compute the IQR for the random variable X.

4. Consider the function
$$f(x) = K\sqrt{1 - x^2}\,, \quad -1 \leq x \leq 1.$$
   Find the value K that makes the following a valid probability density function. Sketch this probability density function.

5. If X is a positive continuous random variable with density function $f_X(x)$ and mean μ, show that
$$f(y) = \begin{cases} 0 & y < 0 \\ y f_X(y)/\mu & y \geq 0 \end{cases}$$
   is a density function, and hence show that
$$\mathrm{E}(X^3)\mathrm{E}(X) \geq \{\mathrm{E}(X^2)\}^2.$$

6. Find the cumulative distribution functions corresponding to the following probability density functions:

(a)	$f_X(x) = 1/[\pi(1+x^2)]$	$-\infty < x < \infty$	(Cauchy)
(b)	$f_X(x) = e^{-x}/(1+e^{-x})^2$	$-\infty < x < \infty$	(Logistic)
(c)	$f_X(x) = (a-1)/(1+x)^a$	$0 < x < \infty$, $a > 1$	(Pareto)
(d)	$f_X(x) = c\tau x^{\tau-1}e^{-cx^\tau}$	$0 < x < \infty$, $\tau > 0$, $c > 0$	(Weibull)

7. Find the mean and the variance for the following distributions:

(a)	$f_X(x) = e^{-kx}x^{r-1}k^r/(r-1)!$	$x > 0$, $r \in \mathbb{N}^*$, $k > 0$	(Gamma)
(b)	$f_X(x) = \binom{x+a-1}{x}p^x(1-p)^a$	$x \in \mathbb{N}_0$, $a \in \mathbb{N}^*$, $p \in (0,1)$	(Neg. Binomial)

8. If X is a non-negative continuous random variable, then

$$\bar{F}(x) = P(X > x)$$

is called the *survival function*, and

$$\lambda(x) = f(x)/\bar{F}(x)$$

is called the *hazard rate* at x.

(a) Find the hazard rate for the Exponential, Weibull and Pareto distributions.

(b) Show that in general the hazard rate does not decrease as x increases if

$$\bar{F}(x+y)/\bar{F}(x)$$

does not increase as x increases, for all $y \geq 0$.

Hint: Differentiate the logarithm of the given expression with respect to x.