EE2-08C Mathematics

Solutions to Sheet 2

LINE INTEGRALS & INDEPENDENCE OF PATH

1. Find $\int_C (x^2+y^2+z^2)\,ds$ where C is the helix $\boldsymbol{r} = \hat{\boldsymbol{\imath}}\cos t + \hat{\boldsymbol{\jmath}}\sin t + \hat{\boldsymbol{k}}t$ from $(1,0,0)$ to $(1,0,2\pi)$ and where ds is an element of arc length on C.

SolutionThe equation of the helix is $\boldsymbol{r} = \hat{\boldsymbol{\imath}}\cos t + \hat{\boldsymbol{\jmath}}\sin t + \hat{\boldsymbol{k}}t$ So $x = \cos\theta$, $y = \sin\theta$ and $z = \theta$, already in parametrized form.

With C going from $(1,0,0)$ to $(1,0,2\pi)$, we see that this is exactly one turn of the helix and we have the interval of the parameter θ running from $0 \to 2\pi$.

Therefore $dx/d\theta = -\sin\theta$, $dy/d\theta = \cos\theta$ and $dz/d\theta = 1$. Thus

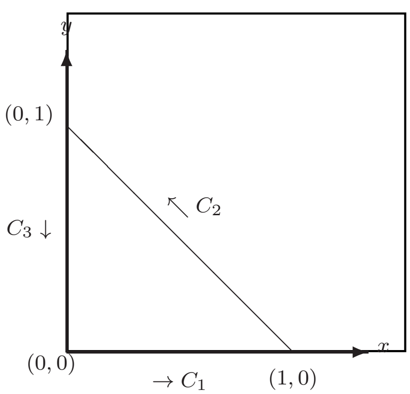
$$ds = \sqrt{\sin^2\theta + \cos^2\theta + 1}\; d\theta = \sqrt{2}\; d\theta\,. \tag{1}$$

Therefore we can write the integral as

$$\int_C (x^2+y^2+z^2)\,ds = \sqrt{2}\int_0^{2\pi}\left(1+\theta^2\right)\,d\theta = 2\pi\sqrt{2}\left(1+4\pi^2/3\right)\,. \tag{2}$$

2. Find $\oint_C xy\,ds$ where C is the closed path of straight lines from $(0,0)$ to $(1,0)$ to $(0,1)$ and then back to $(0,0)$.

Solution



C_1: $y = 0$ so $ds = dx$ and $\int_{C_1} xy\,ds = 0$.

C_3: $x = 0$ so $ds = dy$ and $\int_{C_3} xy\,ds = 0$.

C_2: $y = 1-x$ so parametrize $y = t$ and $x = 1-t$ with $0 \le t \le 1$ giving C. Hence $\dot{y} = 1\,, \dot{x} = -1$ and

$$ds = \sqrt{\dot{x}^2+\dot{y}^2}dt = \sqrt{2}dt\,.$$

Therefore

$$\int_{C_2} xy\,ds = \sqrt{2}\int_0^1 (1-t)t\;dt = \sqrt{2}/6$$

Hence $\oint xy\,ds = \sqrt{2}/6$.

3. Evaluate $\int_C \left[(x^2+y^2)\,dx - 2xy\,dy\right]$ where C is a path in the (x, y) plane from the point $(0, 0)$ to the point $(1, 1)$ along the curves:

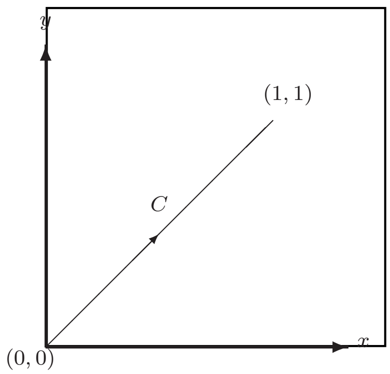
a) $y = x$

b) $y = \sqrt{x}$

c) $y = x^2$

Solution

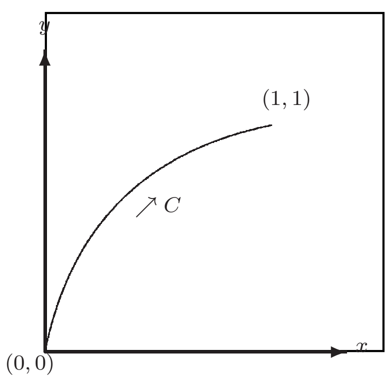
a) To evaluate $\int_C \left[(x^2+y^2)\,dx - 2xy\,dy\right]$ where C is the straight line $y = x$ from $(0, 0)$ to $(1, 1)$:



On C, which is the line $y = x$, we have $dy = dx$. Hence the integral becomes

$$I = \int_C \left(2x^2\,dx - 2x^2\,dx\right) = 0$$

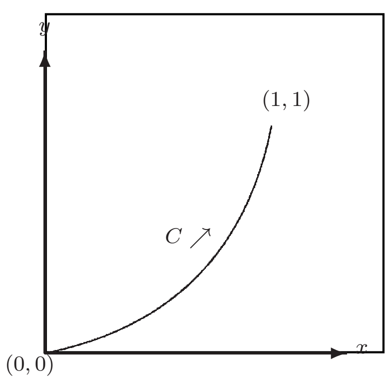
b) To evaluate $\int_C \left[(x^2+y^2)\,dx - 2xy\,dy\right]$ where C is the curve $y = x^{1/2}$ from $(0, 0)$ to $(1, 1)$:



On C, which is the curve $y = x^{1/2}$, we have $dy = \frac{1}{2}x^{-1/2}dx$. Hence the integral becomes

$$\begin{aligned} I &= \int_0^1 \left\{\left(x^2+x\right)\,dx - 2x\,x^{1/2}\left(\tfrac{1}{2}x^{-1/2}dx\right)\right\} \\ &= \int_0^1 x^2\,dx = 1/3 \end{aligned}$$

c) To evaluate $\int_C \left[(x^2+y^2)\,dx - 2xy\,dy\right]$ where C is the curve $y = x^2$ from $(0, 0)$ to $(1, 1)$:



On C, which is the curve $y = x^2$, we have $dy = 2x\,dx$. Hence the integral becomes

$$\begin{aligned} I &= \int_0^1 \left\{\left(x^2+x^4\right)\,dx - 2x\,x^2\,(2x\,dx)\right\} \\ &= \int_0^1 \left(x^2 - 3x^4\right)\,dx = -4/15 \end{aligned}$$

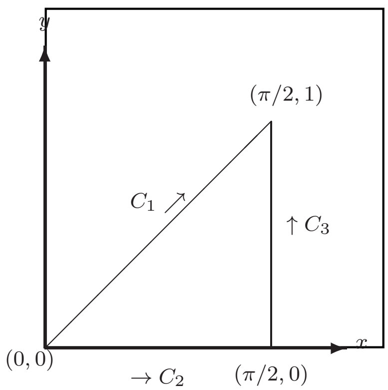
4. Evaluate the line integrals:

$$I_1 = \int_C [y^2 \cos x \, dx + 2y \sin x \, dy] \qquad\qquad I_2 = \int_C [2y^2 \, dx - x \, dy]$$

a) Where C is the straight line between $(0,0)$ and $(\pi/2, 1)$.

b) Where C is the line from $(0,0)$ to $(\pi/2, 0)$ and then a line from $(\pi/2, 0)$ to $(\pi/2, 1)$.

Solution To evaluate over a path C_1 which is the straight line between $(0,0)$ and $(\pi/2, 1)$: that is the line $y = 2x/\pi$. For I_1 we have $F_1 = y^2 \cos x$; $F_2 = 2y \sin x$ so $F_{1,y} = F_{2,x} = 2y \cos x$. Hence the integral is independent of path and so I_1 over C_1 must be the same as I_1 over $C_2 + C_3$. We calculate I_1 over C_1 only.



The integral I_1 over C_1 can be written as

$$\begin{aligned} I_1 &= \int_{C_3} [y^2 \cos x \, dx + 2y \sin x \, dy] \\ &= \int_{C_3} d\left(y^2 \sin x\right) = \left[y^2 \sin x\right]_{(0,0)}^{(\frac{\pi}{2},1)} \\ &= 1 \end{aligned}$$

To evaluate I_2, which is not independent of path ($F_{1,y} \neq F_{2,x}$), we integrate first over C_1; that is, along the line $y = \frac{2}{\pi}x$

$$\int_{C_1} \left(2y^2 dx - x dy\right) = \int_0^{\frac{\pi}{2}} \left\{2\left(\frac{2}{\pi}\right)^2 x^2 - \frac{2}{\pi}x\right\} dx = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

Now we find $\int_{C_2+C_3} = \int_{C_2} + \int_{C_3}$ by evaluating first over C_2 (which is the line $y = 0$) and then over C_3 (which is the line $x = \pi/2$). On C_2 we have $y = 0$ and so $dy = 0$ whereas on C_3 we have $x = \pi/2$ so $dx = 0$. Hence

$$\int_{C_2} \left(2y^2 dx - x dy\right) = 0 \qquad\qquad \int_{C_3} \left(2y^2 dx - x dy\right) = -\frac{\pi}{2} \int_0^1 dy = -\frac{\pi}{2}$$

Hence $\int_{C_2+C_3} = -\frac{\pi}{2}$. Note that $\int_{C_1} \neq \int_{C_2+C_3}$ because I_2 is not independent of path.

5. We want to evaluate $\oint_C (x \, dy - y \, dx)$, where C is the unit circle $x = \cos t, \quad y = \sin t$.

Solution: For C to be closed we need $t: \quad 0 \to 2\pi$. We have $dx = -\sin t \, dt, \quad dy = \cos t \, dt$ so the integral is

$$\oint_C (x \, dy - y \, dx) = \int_0^{2\pi} (\sin^2 t + \cos^2 t) \, dt = \int_0^{2\pi} dt = 2\pi$$

6. If $\boldsymbol{E} = (3x^2 + 6y)\hat{\boldsymbol{\imath}} - 14yz\hat{\boldsymbol{\jmath}} + 20xz^2\hat{\boldsymbol{k}}$, evaluate $\int_C \boldsymbol{E} \cdot d\boldsymbol{r}$ from $(0,0,0)$ to $(1,1,1)$ along the path $x = t, \quad y = t^2, \quad z = t^3$.

Solution: The integral can be written as

$$\int_C \boldsymbol{E} \cdot d\boldsymbol{r} = \int \left[\left(3x^2 + 6y\right)\, dx - 14yz\, dy + 20xz^2\, dz\right]$$

Along the path $x = t, \quad y = t^2, \quad z = t^3$ we have $dx = dt, \quad dy = 2t\, dt$ and $dz = 3t^2\, dt$, in which case the integral becomes

$$\begin{aligned}
\int_C \boldsymbol{E} \cdot d\boldsymbol{r} &= \int_0^1 \left[\left(3t^2 + 6t^2\right)\, dt - 14t^2 t^3 (2t\, dt) + 20tt^6 (3t^2\, dt)\right] \\
&= \int_0^1 \left[9t^2 - 28t^6 + 60t^9\right]\, dt = 3 - 4 + 6 = 5
\end{aligned}$$

7. If $\boldsymbol{E} = \left(2xy + z^3\right)\hat{\boldsymbol{\imath}} + x^2\hat{\boldsymbol{\jmath}} + 3xz^2\hat{\boldsymbol{k}}$, show that $\boldsymbol{E}$ is a conservative field i.e. find the scalar potential ϕ where $\boldsymbol{E} = -\nabla\phi$, and then find the value of $\int_C \boldsymbol{E} \cdot d\boldsymbol{r}$ in moving from $(1,-2,1)$ to $(3,1,4)$.

Solution: If $\boldsymbol{E} = \left(2xy + z^3\right)\hat{\boldsymbol{\imath}} + x^2\hat{\boldsymbol{\jmath}} + 3xz^2\hat{\boldsymbol{k}} = -\nabla\phi$, then $\phi_x = -2xy - z^3; \quad \phi_y = -x^2$ and $\phi_z = -3xz^2$. We can check that $\phi_{xy} = \phi_{yx}$, $\phi_{xz} = \phi_{zx}$ and $\phi_{yz} = \phi_{zy}$. Integrating the three equations for ϕ we find that (c is an arbitrary constant)

$$\phi = -\left(x^2 y + xz^3\right) + c$$

Therefore, because $\boldsymbol{E} = -\nabla\phi$

$$\int_C \boldsymbol{E} \cdot d\boldsymbol{r} = -\int_C \nabla\phi \cdot d\boldsymbol{r} = -\int_C d\phi = \left[x^2 y + xz^3\right]_{(1,-2,1)}^{(3,1,4)} = 202$$