EE2-08 Mathematics

Solutions to Example Sheet 6: Complex Integration

1. By taking the contour C as the unit circle $|z| = 1$ (positive is anti-clockwise), evaluate the following contour integrals $\oint_C F(z)dz$:

   (a) $F(z) = (z^2 - 2z)^{-1}$,

   (b) $F(z) = (z+1)(4z^3 - z)^{-1}$,

   (c) $F(z) = z(1 + 9z^2)^{-1}$.

   Solution:(a) $F(z) = (z^2 - 2z)^{-1} = \frac{1}{z(z-2)}$ has simple poles at $z = 0$ and $z = 2$ Only $z = 0$ lies in the unit circle $|z| = 1$. The residue is

$$\lim_{z\to 0}\left[\frac{z}{z(z-2)}\right] = -\tfrac{1}{2}$$

   Using the residue theorem, $\oint_C F(z)dz = 2\pi i \times -\frac{1}{2} = -\pi i.$

   (b) $F(z) = \frac{z+1}{4z^3 - z} = \frac{z+1}{z(2z+1)(2z-1)}$ has simple poles at $z = 0, \ \pm\frac{1}{2}$. All of these count as they lie inside $|z| = 1$.

   i) Residue at $z = 0$ is $\lim_{z\to 0}\left[\frac{z(z+1)}{z(2z+1)(2z-1)}\right] = -1$

   ii) Residue at $z = -\frac{1}{2}$ is $\lim_{z\to -\frac{1}{2}}\left[\frac{(z+\frac{1}{2})(z+1)}{z(2z+1)(2z-1)}\right] = \frac{1}{4}$

   iii) Residue at $z = \frac{1}{2}$ is $\lim_{z\to \frac{1}{2}}\left[\frac{(z-\frac{1}{2})(z+1)}{z(2z+1)(2z-1)}\right] = \frac{3}{4}$

   The sum of the residues is $-1 + \frac{1}{4} + \frac{3}{4} = 0$. Hence the value of the integral is $2\pi i \times 0 = 0$.

   ((c) $F(z) = \frac{z}{1+9z^2} = \frac{z}{(3z+i)(3z-i)}$ has simple poles at $\pm i/3$. Both count as they lie inside $|z| = 1$.

   i) Residue at $z = i/3$ is $\lim_{z\to i/3}\left[\frac{(z-i/3)z}{9(z-i/3)(z+i/3)}\right] = 1/18$

   ii) Residue at $z = -i/3$ is $\lim_{z\to -i/3}\left[\frac{(z+i/3)z}{9(z-i/3)(z+i/3)}\right] = 1/18$

   The sum of the residues is $1/18 + 1/18 = 1/9$. Hence the value of the integral is $2\pi i \times 1/9 = 2\pi i/9$.

2. Use the Residue Theorem to show that

$$\oint_C \frac{z\,dz}{(z-i)^2} = 2\pi i\,.$$

   where the contour C is the rectangle with vertices at $\pm\frac{1}{2} + 2i$ and $\pm\frac{1}{2} - 2i$.

   Solution:$F(z) = \frac{z}{(z-i)^2}$ has a double pole at $z = i$ lying inside the contour C, which is the rectangle with vertices at $\pm\frac{1}{2}+2i$ and $\pm\frac{1}{2}-2i$.

$$\text{Residue at the double pole } z=i \text{ is:} \quad \lim_{z\to i}\left[\frac{d}{dz}\left\{\frac{(z-i)^2 z}{(z-i)^2}\right\}\right]=1$$

Hence the integral takes the value $2\pi i$.

3. Show that

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2}=\tfrac{1}{2}\pi\,.$$

Solution: From the lectures we know that

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^2}=2\pi i\times\left\{\text{Sum of residues in upper } \tfrac{1}{2}\text{-plane of } \; F(z)=\frac{1}{(1+z^2)^2}\right\}$$

$F(z)=\frac{1}{(1+z^2)^2}$ has double poles at $z=i$ and at $z=-i$: count only the double pole at $z=i$.

$$\text{Residue at the pole } z=i \text{ is:} \quad \lim_{z\to i}\left[\frac{d}{dz}\left\{\frac{(z-i)^2}{(1+z^2)^2}\right\}\right]=-\frac{2}{(2i)^3}=-\tfrac{1}{4}i$$

The Residue Theorem then gives $2\pi i\times(-\frac{1}{4}i)=\frac{1}{2}\pi$ as the answer.

4. Given the real integral

$$I=\int_0^{2\pi}\frac{d\theta}{1-2p\cos\theta+p^2} \qquad\qquad (|p|\neq 1)$$

show that the substitution $z=e^{i\theta}$ converts it into

$$I=\frac{i}{p}\oint_C\frac{dz}{(z-p)(z-p^{-1})},$$

where C is the unit circle $|z|=1$. Evaluate the residues at the poles and hence show that

(i) $I=-2\pi\left(p^2-1\right)^{-1}$ when $|p|<1\,,$

(ii) $I=+2\pi\left(p^2-1\right)^{-1}$ when $|p|>1\,.$

Solution: With $z=e^{i\theta}$ we use the fact that $\cos\theta=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)=\frac{1}{2}(z+z^{-1})$ and $dz=iz\,d\theta$. Take C as the unit circle $|z|=1$ with $\theta:\ 0\to 2\pi$. Then

$$I=\int_0^{2\pi}\frac{d\theta}{1-2p\cos\theta+p^2}=\frac{1}{i}\oint_C\frac{dz}{z\,(1-p(z+z^{-1})+p^2)}=\frac{i}{p}\oint_C\frac{dz}{(z-p)(z-p^{-1})}$$

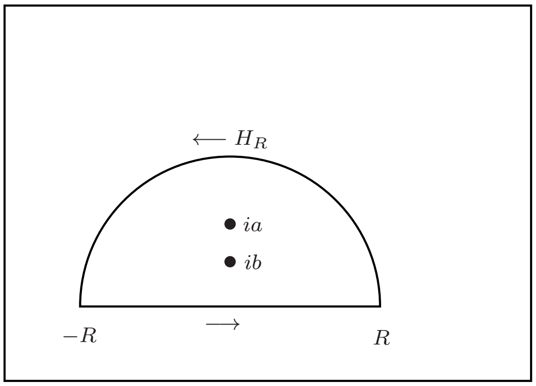
This has simple poles at $z=p$ and $z=p^{-1}$. When $|p|<1$ the pole at $z=p$ lies inside C while $z=p^{-1}$ lies outside and doesn't count. The reverse is true when $|p|>1$.

(i) When $|p|<1$ the residue of the last integral at $z=p$ is $\frac{p}{p^2-1}$. Thus $I=2\pi i\times\frac{i}{p^2-1}=-\frac{2\pi}{p^2-1}$. (ii) When $|p|>1$ the residue of the last integral at $z=p^{-1}$ is $\frac{p}{1-p^2}$, so $I=2\pi i\times\frac{i}{1-p^2}=\frac{2\pi}{p^2-1}$.

5. By choosing a suitable contour in the upper half of the complex plane, use the Residue Theorem & Jordan's Lemma to show that for $a > b > 0$

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

Solution:



The closed contour C is comprised of the semicircular contour $H_R: \quad z = Re^{i\theta}$ for $0 \leq \theta \leq \pi$ in the upper $\frac{1}{2}$-plane plus that part of the real axis from $x = -R$ to $x = R$.

$$F(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$$

Now we know that

$$\oint_C e^{iz} F(z) dz = \int_{-R}^{R} e^{ix} F(x) dx + \int_{H_R} e^{iz} F(z) dz$$

where H_R is the semi-circle. We know that $(z^2+a^2)^{-1}(z^2+b^2)^{-1}$ decays as $R \to \infty$ in such a way that Jordan's lemma is satisfied; thus

$$\lim_{R\to\infty} \int_{H_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} = 0$$

Now consider the full closed contour integral $\oint_C e^{iz} F(z) dz$:

$$\text{Residue at the simple pole at} \quad z = ia \quad \text{is} \quad \frac{e^{-a}}{2ia(b^2-a^2)}$$

$$\text{Residue at the simple pole at} \quad z = ib \quad \text{is} \quad \frac{e^{-b}}{2ib(a^2-b^2)}.$$

Hence

$$\oint_C e^{iz} F(z) dz = 2\pi i \times \frac{1}{2i(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

and

$$\int_{-\infty}^{\infty} e^{ix} F(x) dx = \int_{-\infty}^{\infty} \frac{e^{ix} dx}{(x^2+a^2)(x^2+b^2)} = \int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2+a^2)(x^2+b^2)}$$

The imaginary part $i \sin x$ of e^{ix} within the integral is not present because this has cancelled over the two halves of the domain $(-\infty, \infty)$. Thus we have the answer

$$\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{(a^2-b^2)}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$