EE2 Mathematics – Probability & Statistics

Solution 3

1. The number of successes is $X \sim \text{Bin}(14, 0.8)$ and the event of interest is $X = 1$. We have

$$f_X(1) = \binom{14}{1} \cdot 0.8 \cdot 0.2^{13} = 14 \cdot 0.8 \cdot 0.2^{13} \approx 9.2 \cdot 10^{-9}$$

This probability of the observed data is so small that we might suspect the Pentagons claim.

2. The number of possible moves for each square on the board is

2	3	4	4	4	4	3	2
3	4	6	6	6	6	4	3
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
4	6	8	8	8	8	6	4
3	4	6	6	6	6	4	3
2	3	4	4	4	4	3	2

Let the random variable X be the number of possible moves; it has PMF

x	2	3	4	6	8
$f_X(x)$	4/64	8/64	20/64	16/64	16/64

from which we obtain $\mathrm{E}(X) = 21/4$ and $\mathrm{Var}(X) = 61/16$.

3. (a) Just multiply it out, e.g.

$$\begin{aligned} &\mathrm{E}[X(X+1)] - \mathrm{E}(X)[\mathrm{E}(X)+1] \\ &\qquad = \mathrm{E}\left(X^2\right) + \mathrm{E}(X) - \mathrm{E}(X)^2 - \mathrm{E}(X) = \mathrm{Var}(X)\,. \end{aligned}$$

(b) Using the hint, we begin by evaluating the expression

$$\begin{aligned}
&\mathrm{E}\left[X\left(X-1\right)\right] \\
&\quad= \sum_x x\left(x-1\right) f_X(x) \\
&\quad= \sum_x x\left(x-1\right)\binom{n}{x} p^x \left(1-p\right)^{n-x} \\
&\quad= 0+0+\sum_{x=2}^{n} x\left(x-1\right)\frac{n!}{x!\left(n-x\right)!} p^x \left(1-p\right)^{n-x} \\
&\quad= \sum_{x=2}^{n} \frac{n!}{\left(x-2\right)!\left(n-x\right)!} p^x \left(1-p\right)^{n-x} \\
&\quad= n\left(n-1\right)p^2 \sum_{x-2=0}^{n-2}\binom{n-2}{x-2} p^{x-2}\left(1-p\right)^{(n-2)-(x-2)} \\
&\quad= n\left(n-1\right)p^2.
\end{aligned}$$

In the above expression, the sum is equal to 1 because the function we asmming is the PMF of a $\mathrm{Bin}(n-2,p)$ random variable over its support. The variance is thus

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}\left[X\left(X-1\right)\right] - \mathrm{E}\left(X\right)\left[\mathrm{E}\left(X\right)-1\right] \\
&= n(n-1)p^2 - np(np-1) = np(1-p).
\end{aligned}$$

4. (a) Set $G(q) = 1+q+q^2+...$, an infinite geometric series. We know that

$$G(q) = \sum_{k=0}^{\infty} q^k = \left(1-q\right)^{-1},$$

so we can deduce that

$$H(q) = \frac{d}{dq}\left(1-q\right)^{-1} = \left(1-q\right)^{-2}.$$

(b) First part is trivial

$$I(q) = \frac{d}{dq}H(q) = \sum_{k=1}^{\infty} k(k-1)q^{k-2} = \sum_{k=2}^{\infty} k(k-1)q^{k-2}.$$

For the second part, we have

$$I(q) = \frac{d}{dq}\left(1-q\right)^{-2} = 2\left(1-q\right)^{-3}.$$

(c) Recall $X \sim \text{Geo}(p)$ has a PMF

$$f_X(x;p) = (1-p)^{x-1}p, \qquad x = 1, 2, \ldots.$$

Setting $q = 1-p$, the mean is

$$\begin{aligned} \text{E}(X) &= \sum_{x=1}^{\infty} x q^{x-1} p \\ &= p \sum_{x=1}^{\infty} x q^{x-1} \\ &= pH(q) \\ &= p(1-q)^{-2} = p^{-1}. \end{aligned}$$

For the variance, we start with

$$\begin{aligned} \text{E}\left[X(X+1)\right] &= \sum_{x=1}^{\infty} x(x+1) q^{x-1} p \\ &= p \sum_{x=1}^{\infty} x(x+1) q^{x-1} \\ &= \sum_{y=2}^{\infty} y(y-1) q^{y-2} \qquad \text{(using } y = x+1\text{)} \\ &= pI(q) \\ &= p2(1-q)^{-3} = 2p^{-2}, \end{aligned}$$

which gives

$$\begin{aligned} \text{Var}(X) &= \text{E}\left[X(X+1)\right] - \text{E}(X)\left[\text{E}(X) + 1\right] \\ &= 2p^{-2} - p^{-1}(1 + p^{-1}) = p^{-2} - p^{-1} = \frac{1-p}{p^2}. \end{aligned}$$