Messy Notes for Applied

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1 Circuits

1.1 Null-Spaces

Consider an adjacency matrix A representing a circuit.

Then (if we let Φ be the vector representing the potiential at each node) we have $A\Phi = w$ is the potiential difference across each edge.

With this, we see that there is the trivial right null-vector $\Phi_0 = (\mathbf{1})$, as if each node has the same potiential, the the potiential difference across each edge is zero. In fact, this is the **only** right null-vector of A (given that the circuit is connected) since if there is two nodes that have different potiential, then of course there is going to be some edge with some potiential difference.

On the other hand, for $-A^T w = f$ where w is the potiential difference across each node, we have f being the vector representing the *sum* of the potientials out of each node.

Then the dimension of the left null-space of A is represented by the number of "loops" of the circuit and we can find the basis of the left null-space by the following method:

1. Let $w_i \in \mathbb{R}^n$ be the vector representing the path of a loop where n is the number of edges of the circuit.

2. For each edge the path traverses, put a 1 in that position if the path follows the directed graph and a -1 if it goes against it.
3. Fill in the rest with 0s.

Thus, by rank-nullity, we can easily verify our previous statement that the trivial right-null vector is the only right-null vector.

1.2 Connected Graphs

The question is what happens when we have two disconnected graphs representing a single graph?

Well, with two disconnected graphs, we can see there are two trivial right-null vectors, i.e. $\Phi_0 = (\mathbf{1}, \mathbf{0})$ and $\Phi'_0 = (\mathbf{0}, \mathbf{1})$. We can easily see that our original trivial null-vector is in the span of these two new null-vectors.

1.3 Useful Terms

- A is the *adjacency matrix* of the graph
- $K = A^T A$ is the *Laplacian matrix* of the graph
- $C = \mathrm{diag}(c_1, \ldots c_n)$ the *conductance* matrix of the graph
- $K_c = A^T C A$ is the *weighted Laplacian matrix* of the graph
- $-A^T w$ is called the *divergence* of the currents (at the nodes)
- C_e is the *effective conductance* of a circuit

1.4 KCL Cannot Hold at All nodes!

Consider the result from the *fundamental theorem of linear algebra*:

- The null space of A^T is orthogonal to the row space of A, i.e. $\ker A^T \perp \mathrm{csp}A$.

(This is true because if $v \in \ker A$, then the dot product of v and the rows of A must be zero, so it must be perpendicular to the row vectors of A. But for all $w \in \mathrm{rsp}A$, w is a linear combination of the row vectors of A, so $v \cdot w = 0$ and hence, as $\mathrm{rsp}A = \mathrm{csp}A^T$, we have $\ker A \perp \mathrm{csp}A^T$.)

With that, let use suppose *KCL* holds at all the nodes. Then, we have $\mathbf{0} = A^T w$ for some w representing the potiential difference across each edge. Then, by definition, w is in the null-space of A^T. But this, by our aforementioned lemma must be perpendicular to any vectors in $\mathrm{csp}A = \mathrm{Im}A$ and hence, if $w \neq 0$ (which we are assuming as otherwise we would have a circuit not doing anything!) there does not exist some potiential vector Φ such that $w = A\Phi$.

Physically, this means that under the assumption that KCL holds at all nodes, there will be **no** non-trivial potiential that will produce a current through our circuit (but of course the trivial null-vectors of A satisfy our equation $\mathbf{0} = A^T A\Phi$).

1.5 Method for Finding the Potientials

Just solve

$$f = A^T C A\Phi = K_c \Phi$$

and we are done!

Okay, perhaps this is not very helpful so to expand on what I've said above, let us consider two types of questions.

1.5.1 Specified Conductance

First, perhaps we are given the net currents at the grounded nodes, i.e. f is known and we are asked to find Φ such that $f = K_c\Phi$. Then, you might just think to do something like $\Phi = K_c^{-1} f$ to determine Φ. But this is no good as K_c is singular and therefore not invertible (recall that we have the trivial null-vector for A, Φ_0 so $K_c\Phi_0 = A^T C(A\Phi_0) = A^T C\mathbf{0} = \mathbf{0}$). So what we do is we set one of the nodes to have potential zero, i.e. we will have that node as our reference potential (if you think about this physically what I'm saying is that the potential at one node doesn't really mean much - its the potential difference that actually does the work so fixing one node potential doens't effect what we are really after - the potential difference).

So what we do have now is some mathematical trickery. If we fix the i-th nodes to have potential zero, then when we multiply K_c with this potential vector, you can see that the i-th row and collumn of K_c nolonger contribute to the multiplication so by removing them, we receive a new linear system $K_c'\Phi' = f'$. Now, by repeating this process until its solvable, we should receive some vector $\bar{x}$ that is satisfied by our original system and hence we have a solution space

$$S = \bar{x} + \ker A := \{\bar{x} + x_0 \mid x_0 \in \ker A\}.$$

1.5.2 Specified Potientials

So now what if we are given the potientials at the grounded nodes and are asked to find the potientials at the non-grounded nodes while also finding the net-currents out of the grounded nodes?

Well, thats a lot asked from us, but I think we can manage. So in this case we have $K_c\Phi = f$ where $\Phi = (x_1, x_2, x_3, \dots)^T$ and $f = (f_1, -f_1, 0, \dots)^T$ where x_1, x_2 are given and we are asked to find the rest (if it's not in this form then just rewrite it in this form). And then all we have to do is to use the Schur complement and Wikipedia can explain the rest.

One more comment. The effective conductance of the system $C_e = f_1$.

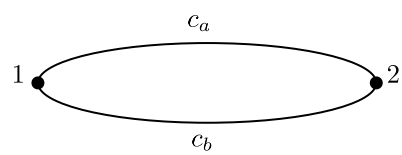
1.6 Calculating the Effective Conductance

We will now present some trickery in calculating the effective conductance of a circuit by simplifying the circuit whenever we see edges in series or parallel.

1.6.1 Edges in Parallel

I'm going to talk about parallel first because it is nicer to work out mathematically.

Suppose we have the following graph represting a particular section of another graph with parallel edges.



Let us write out the incidence matrix and the conductance matrix,

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} c_a & 0 \\ 0 & c_b \end{pmatrix}.$$

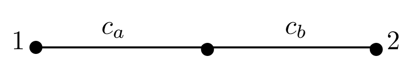
Then, if let the system have unit potiential at node 1 and zero potiential at node 2, we have

$$f = A^T C A \Phi = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_a & 0 \\ 0 & c_b \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_a + c_b \\ -(c_a + c_b) \end{pmatrix}$$

implying that for edges in parallel, we shall add the conductance.

1.6.2 Edges in Series

It's a very similar story for edges in series but the matrices doesn't multiply as nicely so we will omit it the proof but the result is that given the below graph,



we can treat the system as one edge with conductance $c_a c_b/(c_a + c_b)$.