

Messy Notes for Applied

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Contents

1	Circuits	2
1.1	Null-Spaces	2
1.2	Connected Graphs	2
1.3	Useful Terms	2
1.4	KCL Cannot Hold at All nodes!	3
1.5	Method for Finding the Potentials	3
1.6	Calculating the Effective Conductance	4
2	Random Walks	5
2.1	A Stroll Through Central London	5
2.2	An Algebraic Method	6
2.3	Connection Between the Two Systems	7
2.4	Method of Relaxation	8
3	Harmonic Graphs	10
3.1	Maximum Principle	10
3.2	Uniqueness Principle	10
3.3	Dirichlet's Principle	10
3.4	Thomson's Principle	11
3.5	Tellegen's Theorem	11
4	Spring-mass Systems	13
4.1	Hooke's Law	13
4.2	Boundary Condition	13
4.3	Spring-mass System in Motion	14
5	As $n \rightarrow \infty$ and Beyond!	16
5.1	Conductances in \mathbb{R} and \mathbb{R}^2	16
5.2	1-D Continuum Limit	16
5.3	2-D Continuum Limit	17
5.4	Introduction to Harmonic Functions	18
5.5	Boundary Conditions on a Domain	19

1 Circuits

In this section we will discuss the method of using matrices in solving the potential and conductance problem regarding circuits.

I will assume you can manipulate matrices and use basic linear algebra, I will also assume you can construct the incidence matrix of a graph.

1.1 Null-Spaces

Consider an incidence matrix A representing a circuit.

Then (if we let Φ be the vector representing the potential at each node) we have $A\Phi = w$ is the potential difference across each edge.

With this, we see that there is the trivial right null-vector $\Phi_0 = (\mathbf{1})$, as if each node has the same potential, the the potential difference across each edge is zero. In fact, this is the **only** linearly independent right null-vector of A (given that the circuit is connected) since if there is two nodes that have different potential, then of course there is going to be some edge with some potential difference.

On the other hand, for $-A^T w = f$ where w is the potential difference across each node, we have f being the vector representing the *sum* of the potentials out of each node.

Then the dimension of the left null-space of A is represented by the number of “loops” of the circuit and we can find the basis of the left null-space by the following method:

1. Let $w_i \in \mathbb{R}^n$ be the vector representing the path of a loop where n is the number of edges of the circuit.
2. For each edge the path traverses, put a 1 in that position if the path follows the directed graph and a -1 if it goes against it.
3. Fill in the rest with 0s.

Thus, by rank-nullity, we can easily verify our previous statement that the trivial right-null vector is the only right-null vector.

1.2 Connected Graphs

The question is what happens when we have two disconnected graphs representing a single graph?

Well, with two disconnected graphs, we can see there are two trivial right-null vectors, i.e. $\Phi_0 = (\mathbf{1}, \mathbf{0})$ and $\Phi'_0 = (\mathbf{0}, \mathbf{1})$. We can easily see that our original trivial null-vector is in the span of these two new null-vectors.

1.3 Useful Terms

- A is the *incidence matrix* of the graph
- $K = A^T A$ is the *Laplacian matrix* of the graph
- $C = \text{diag}(c_1, \dots, c_n)$ the *conductance* matrix of the graph

- $K_c = A^T C A$ is the *weighted Laplacian matrix* of the graph
- $-A^T w$ is called the *divergence* of the currents (at the nodes)
- C_e is the *effective conductance* of a circuit

1.4 KCL Cannot Hold at All nodes!

Consider the result from the *fundamental theorem of linear algebra*:

- The null space of A^T is orthogonal to the row space of A , i.e. $\ker A^T \perp \text{csp} A$.

(This is true because if $v \in \ker A$, then the dot product of v and the rows of A must be zero, so it must be perpendicular to the row vectors of A . But for all $w \in \text{rsp} A$, w is a linear combination of the row vectors of A , so $v \cdot w = 0$ and hence, as $\text{rsp} A = \text{csp} A^T$, we have $\ker A \perp \text{csp} A^T$.)

With that, let us suppose *KCL* holds at all the nodes. Then, we have $\mathbf{0} = A^T w$ for some w representing the potential difference across each edge. Then, by definition, w is in the null-space of A^T . But this, by our aforementioned lemma must be perpendicular to any vectors in $\text{csp} A = \text{Im} A$ and hence, if $w \neq 0$ (which we are assuming as otherwise we would have a circuit not doing anything!) there does not exist some potential vector Φ such that $w = A\Phi$.

Physically, this means that under the assumption that *KCL* holds at all nodes, there will be **no** non-trivial potential that will produce a current through our circuit (but of course the trivial null-vectors of A satisfy our equation $\mathbf{0} = A^T A\Phi$).

1.5 Method for Finding the Potentials

Just solve

$$f = A^T C A \Phi = K_c \Phi$$

and we are done!

Okay, perhaps this is not very helpful so to expand on what I've said above, let us consider two types of questions.

1.5.1 Specified Conductance

First, perhaps we are given the net currents at the boundary nodes, i.e. f is known and we are asked to find Φ such that $f = K_c \Phi$. Then, you might just think to do something like $\Phi = K_c^{-1} f$ to determine Φ . But this is no good as K_c is singular and therefore not invertible (recall that we have the trivial null-vector for A , Φ_0 so $K_c \Phi_0 = A^T C (A\Phi_0) = A^T C \mathbf{0} = \mathbf{0}$). So what we do is we set one of the nodes to have potential zero, i.e. we will have that node as our reference potential (if you think about this physically what I'm saying is that the potential at one node doesn't really mean much - it's the potential difference that actually does the work so fixing one node potential doesn't effect what we are really after - the potential difference).

So what we do have now is some mathematical trickery. If we fix the i -th nodes to have potential zero, then when we multiply K_c with this potential vector, you can see that the i -th row and column of K_c no longer contribute to the multiplication so by removing them,

we receive a new linear system $K'_c \Phi' = f'$. Now, by repeating this process until its solvable, we should receive some vector \bar{x} that is satisfied by our original system and hence we have a solution space

$$S = \bar{x} + \ker A := \{\bar{x} + x_0 \mid x_0 \in \ker A\}.$$

1.5.2 Specified Potentials

So now what if we are given the potentials at the boundary nodes and are asked to find the potentials at the non-boundary nodes while also finding the net-currents out of the boundary nodes?

Well, that's a lot asked from us, but I think we can manage. So in this case we have $K_c \Phi = f$ where $\Phi = (x_1, x_2, x_3, \dots)^T$ and $f = (f_1, -f_1, 0, \dots)^T$ where x_1, x_2 are given and we are asked to find the rest (if it's not in this form then just rewrite it in this form). And then all we have to do is to use the **Schur complement** and Wikipedia can explain the rest.

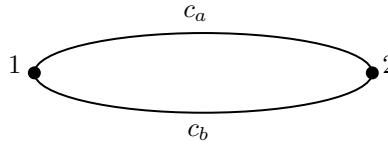
One more comment. The effective conductance of the system $C_e = f_1$.

1.6 Calculating the Effective Conductance

We will now present some trickery in calculating the effective conductance of a circuit by simplifying the circuit whenever we see edges in series or parallel.

1.6.1 Edges in Parallel

Suppose we have the following graph representing a particular section of another graph with parallel edges.



Let us write out the incidence matrix and the conductance matrix,

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} c_a & 0 \\ 0 & c_b \end{pmatrix}.$$

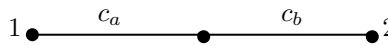
Then, if let the system have unit potential at node 1 and zero potential at node 2, we have

$$f = A^T C A \Phi = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_a & 0 \\ 0 & c_b \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_a + c_b \\ -(c_a + c_b) \end{pmatrix}$$

implying that for edges in parallel, we shall add the conductance.

1.6.2 Edges in Series

It's a very similar story for edges in series but the matrices doesn't multiply as nicely so we will omit it the proof but the result is that given the below graph,



we can treat the system as one edge with conductance $c_a c_b / (c_a + c_b)$.

2 Random Walks

In this section, we shall examine another system - random walks on a graph and explore its similarity to a circuit system. With this, we will conclude that in fact the two systems are the same and are said to be *harmonic* and for each method of solving the specific problem, an equivalent method can be used for the other.

We shall also consider two additional methods for solving problems involving harmonic graphs - *algebraic* and the *method of relaxation*.

2.1 A Stroll Through Central London

Consider the much reduced (an worse) tubemap of central London,

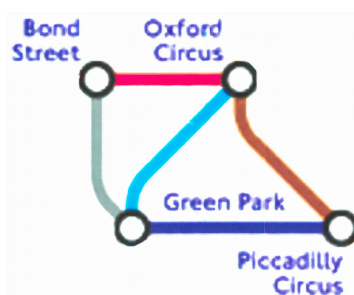


Figure 1: A fictional tubemap of central London.

and suppose, your very drunk friend, Xin, gotten himself wasted at *Station A*. Wanting to get home, Xin takes a random tube at each station and gets off at the next rinse and repeat until he reaches some station we will specify.

We would now like to pose three questions:

1. What is the probability that, if Xin starts at *Oxford Circus*, he reaches *Bond Street* before reaching *Green Park*?
2. What is the probability if Xin starts at *Piccadilly Circus* instead?
3. What is the probability that if Xin starts as *Bond Street* and leaves, he reaches *Green Park* rather than returning to *Bond Street*?

2.1.1 An Recursive Algorithm for Question 1

Often times, it is helpful to first simulate our problem with a computer programme so we can see perhaps there is a pattern to our answer. To achieve this however, there are many different methods. Here I have prepared a pseudocode representation for a recursive implementation of simulating one random walk however there are different ways of achieving

the same without recursions.

Algorithm 1: An recursive method of simulating random walks

```

Class stations(name, connected):
    | let self.name = name;
    | let self.connected = connected;
Function walkNext(currentStation):
    | if currentStation  $\neq$  bondStreet  $\vee$  greenPark then
    | | walkNext(choose from currentStation.connected);
    | else
    | | return currentStation
    | end

```

By making each station a data structure containing its name and its adjacent stations, we can recursively choose the next station until we reach Bond Street or Green Park.

Algorithm 2: Finding the probability given our method of simulating one random walk

```

let reachedBondStreet = 0;
while runs  $\leq$  n do
    | if walkNext(oxfordCircus) = bondStreet then
    | | bondStreet += 1;
    | else
    | | Pass
    | end
    | runs += 1;
end
let probability = reachedBondStreet/n

```

Thus, by doing sufficiently large number of such random walks, we can compare the ratio of the number times *X* reached Bond Street against the total number of runs.

By implementing this algorithm in an actual programming language, it seems that this probability appears to be around 2/5 which is exactly the same value if we attempted to compute this as if this was a circuit with unit conductances on each edge.

This is surprising as on the surface, neither systems seemed to have anything to do with each other, but however, later we will find out that both systems are *harmonic* and therefore are essentially the same.

2.2 An Algebraic Method

As promised, I will now present an algebraic method in solving the random walker problem.

Let use first rephrase the question so we can refer to it easier. Consider the graph drawn below. We can easily see that the graph and the tubemap are infact the same with Bond Street being node 1, Green Park as node 2 and so on.

With that, we can rephrase the question such that we are asked to find p_3 where p_i represents the probability of reaching node 1 before reaching node 4.

Consider the *law of total probability*:

- Given a event $E \in \mathcal{F}$ and $B = \{B_i | i \in \mathcal{I}\} \subseteq \mathcal{F}$ for some index set \mathcal{I} , if B forms a partition of the sample space Ω , then

$$P(E) = \sum_{i \in \mathcal{I}} P(E|B_i)P(B_i).$$

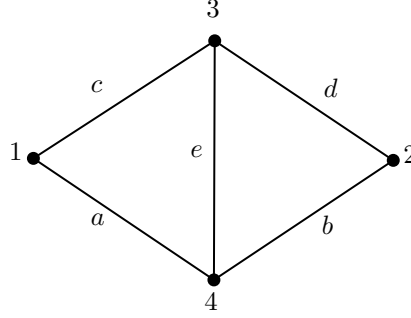


Figure 2: A graph representation of the tubemap.

Then, starting at node 3, we see that Xin can either go to node 1, 2 or 4, so the events of Xin going to node 1, node 2 and goint to node 4 partitions the sample space. Therefore by the law of total probability, we have

$$p_3 = \frac{1}{3} \times p_1 + \frac{1}{3} \times p_2 + \frac{1}{3} \times p_4.$$

Similarly, we see that going to node 3 or node 4 from node 2 partitions the sample space, so

$$p_2 = \frac{1}{2} \times p_3 + \frac{1}{2} \times p_4.$$

Now, if we started from node 4, then there is of course no change of reaching node 1 before node 4 (as you are already at node 4!) so $p_4 = 0$. Similarly if we started from node 1, then the probability of reaching 1 before 4 is 1, so $p_1 = 1$.

Thus, by plugging in $p_1 = 1$ and $p_4 = 0$, we have a system of equations with two unknowns,

$$\begin{aligned} p_3 &= \frac{1}{3} + \frac{1}{3} \times p_2 \\ p_2 &= \frac{1}{2} \times p_3 \end{aligned}$$

which is easily solvable resulting in $p_3 = 2/5$ and $p_2 = 1/5$ would you imagine it! It's exactly what we expected from the simulated solution (and the same solution as the matrix method).

2.3 Connection Between the Two Systems

Recall our equation for circuits, $f = K\Phi$ where in which at the non-boundary nodes, we had the enforcement of *KCL*. So suppose ϕ_i is the potential at a non-boundary node. As *KCL* is upheld at this node, we have the i -th row of $K\Phi$ being zero. So by denoting $\deg(i)$ by the number of edges connected to node i , we have

$$\deg(i)x_i - \sum_{\substack{j \in \text{connected} \\ \text{nodes}}} x_j = 0,$$

so by rearranging, we have,

$$x_i = \sum_{\substack{j \in \text{connected} \\ \text{nodes}}} x_j / \deg(i), \quad (1)$$

which is exactly what we have done for the algebraic method for random walks.

2.3.1 Intuitive Interpretation

Intuitively we can interpret both system as graphs with some variables associated with the nodes. Furthermore, as demonstrated by the above equation, we see that in both system, the node variable of each node is the **average** of the its neighbouring node variables. In the case of circuits, the potentials (the node variable) of each node is the average of its neighbours (assuming uniform conductance) and similarly in the case of the random walk, the probability (the node variable) is the average of its neighbours (assuming each path is equally likely).

With that, we can abstract graph systems such as these in which each nodes of the graph is associated with a node variable that is the average of its neighbouring nodes and we will call such systems *harmonic*.

And thus, as these systems are essentially the same in our framework, we can utilize a method of one system interchangeably with another.

2.3.2 Weighted Harmonic Graphs

However, what if our conductances/ probabilities of going to each neighbouring stations are not the same? Then we shall call the harmonic system *weighted*.

As an extension of the regular harmonic system, we see that we do not need to change our original framework much but to weight each node variable according to the conductances/ probabilities or whatever other system we are using resulting in

$$x_i = \sum_{\substack{j \in \text{connected} \\ \text{nodes}}} (x_j / c_j), \quad (2)$$

where c_j denotes the weight we have assigned to the edge connecting the i -th node and the j -th node.

2.4 Method of Relaxation

With our interpretation in mind, we can now construct another algorithm that converges to our desired node variables by averaging the each node variables with the adjacent ones (with weighing dependent on the edge variables if the edge variables are not uniform) over and over agains for sufficiently large number of times.

Below are some pseudocode describing this method.

We first create a object called **Node** containing information about its neighbours and it's initial potential and we will let a **graph** to be a collection of **node : Node**.

Algorithm 3: Constructing the object of nodes and creating a graph as a set of nodes

Class Node(*name, nodeVariable, connected, weighing*):

```

    let self.name = name;
    let self.connected = connected;
    let self.weighingi = weighingi;
    if declared nodeVariable then
        let self.nodeVariable = nodeVariable;
        let self.boundary = True
    else
        let self.nodeVariable = 0;
        let self.boundary = False
    end
end

```

let *graph* = {*node*₁, *node*₂, ..., *node*_{*k*}};

Then by looping over all the non-boundary nodes for sufficient amount of times the node variables of each node converges to the true value.

Algorithm 4: Constructing the object of nodes and creating a graph as a set of nodes

while *runs* ≤ *n* **do**

```

    for node ∈ graph ∧ node.boundary = False do
        node.nodeVariable =  $\sum_{i \in \text{node.connected}} \text{weighing}_i \times \text{node}_i.\text{nodeVariable}$ 
    end
    runs += 1
end

```

3 Harmonic Graphs

In this section we shall examine some properties of harmonic graphs.

3.1 Maximum Principle

Given a harmonic graph, it is rather natural to separate its nodes into *boundary nodes* and *internal nodes* (all other nodes). Then the *maximum principle* states the following:

- For a harmonic graph, the node variables attains maximum and minimum at the boundary nodes.

To see why this is true recall that for a (weighted) harmonic graph, each internal node has node variable that is a weighted average of its neighbouring node variables. So if a harmonic graph attains its maximum node variable at a internal node, then all nodes adjacent must be less equal to it. However, as the node variable is the weighted average of the adjacent node variables, if there is one adjacent node variable strictly less than it, the weighted average will be smaller resulting in a contradiction. Therefore, all adjacent nodes must have the same node variable as the maximum. Now repeat this until it reaches a boundary node, resulting in our original statement.

The same argument works for the minimum.

3.2 Uniqueness Principle

Given two harmonic graphs, the uniqueness principle states that if the graphs have the same boundary conditions (and the same edge variables if it is not uniform), then they also have the same node variables everywhere.

This is true because if we let x_i and p_i denote the i -th node variable of the two graphs, then the graph with node variables defined by $d_i = x_i - p_i$ attains maximum and minimum at one of its boundary nodes. But at the boundary nodes, $x_i = p_i$ implying $\max_j d_j = \min_j d_j = 0$ and hence has node variable zero everywhere.

This is a very important principle as this ensures that our solution for the random walk problem is the same as the circuit and is the same for any other harmonic graphs for that matter!

3.3 Dirichlet's Principle

The energy dissipation associated with any circuit with two boundary nodes with potential 1 and 0 is defined to be

$$\xi(\Phi) = \Phi^T K \Phi$$

where K is the (weighted) Laplacian and Φ is any potential vector.

Dirichlet's principle states that the potential vector Φ_* satisfying *KCL* at all internal nodes minimizes the energy dissipation, i.e.

$$\xi(\Phi_*) = \min_{\Phi} \xi(\Phi).$$

3.4 Thomson's Principle

By mathematical manipulation the energy dissipation from a circuit can also be written as

$$\begin{aligned}\xi(\Phi) &= \Phi^T K \Phi = \Phi^T A^T C A \Phi \\ &= (-A\Phi)^T (-CA\Phi) \\ &= \sum_{k \in \text{edges}} w_k \left(\frac{w_k}{c_k} \right) \\ &= \hat{\xi}(w),\end{aligned}$$

where w_k is the current in the k -th edge and c_k is the conductance in the k -th edge.

Notice that this new expression of energy dissipation does not have potential as it's variable but the edge currents. So as a corollary of Dirichlet's principle, the current vector such that all the internal nodes satisfies *KCL* also minimizes energy dissipation.

3.5 Tellegen's Theorem

Given a circuit with incidence matrix A and any vector w satisfying *KCL* at all nodes (w does not need to equal some $A\Phi$ for some potential vector Φ), then for any e a potential difference vector, Tellegen's theorem states,

$$e^T w = \mathbf{0}.$$

Proof. Since e is some vector of potential difference, then there exist some $\Phi \in \mathbb{R}^n$, a potential vector, such that $e = -A\Phi$. Thus

$$\begin{aligned}e^T w &= (-A\Phi)^T w = -\Phi^T A^T w \\ &= -\Phi^T (w^T A)^T \\ &= -\Phi^T \mathbf{0} = \mathbf{0}\end{aligned}$$

as required. □

Now suppose that w is some vector that satisfies $-A^T w = f$ then for all e a potential difference vector such that $e = -A\Phi$, we have

$$e^T w = x^T f.$$

Proof.

$$\begin{aligned}e^T w &= (-A\Phi)^T w = -\Phi^T A^T w \\ &= -\Phi^T (-(-A^T w)) \\ &= \Phi^T f.\end{aligned}$$

□

Corollary 0.1. *Suppose we have two boundary nodes with potential 1 and 0. Let w be the current in the circuit and let e be the potential difference of the circuit, i.e. $e = -A\Phi$ where Φ is the potential vector of the circuit. Then the effective conductance is the energy dissipation.*

Proof. By considering Tellegen's theorem and the fact that other than the two boundary nodes, all other nodes satisfy *KCL*, we have

$$e^T w = x^T f = x_1 C_e + x_0 (-C_e) = C_e.$$

But we also have,

$$\begin{aligned} e^T w &= (-A\Phi)^T (-CA\Phi) \\ &= \Phi^T A^T C A \Phi \\ &= \Phi^T K \Phi = \xi(\Phi). \end{aligned}$$

So therefore $C_e = \xi(\Phi)$ as required. □

4 Spring-mass Systems

In this section we will consider a spring-mass system as a harmonic graph and through the usage of physical laws such as *Hooke's* and *Newton's second law* we will find a method for finding the position of the individual masses as a function of time.

4.1 Hooke's Law

We will use two laws from physics to solve our problem so I shall briefly introduce them here.

Consider the spring-mass system described in the figure below, we can easily imagine the system as a harmonic graph with the node variables being the distance of each mass to their equilibrium positions (i.e. the displacements) $\phi_i = \hat{x}_i - x_i$ and the edge variables being the spring constants c_i .

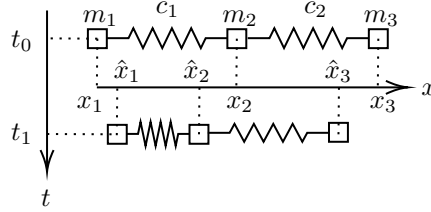


Figure 3: A spring-mass system with three masses in two position in time.

Then *Hooke's law* states that the tension in spring i is given by

$$T_i = c_i(\phi_{i+1} - \phi_i).$$

Thus, by writing in matrix form, we have,

$$\mathbf{T} = C A \Phi,$$

where \mathbf{T} is the tension vector, C the spring constant vector, A the incidence matrix and Φ the displacement vector.

Now by considering the sum of the tension connected to the i -th mass is the force experienced by that mass, we have,

$$-A^T \mathbf{T} = f,$$

where f is the force vector denoting the force experienced by each mass.

4.2 Boundary Condition

With this equation in mind, let us consider the boundary condition where the system is in equilibrium.

At equilibrium, the net force at each mass other than the boundary masses is zero. Suppose there is some external force acting on the system $f_{\text{ext}} = (f_i)^T$, then we have

$$f_{\text{ext}} + f = 0,$$

where $f = -A^T C A \Phi$, thus,

$$f_{\text{ext}} = A^T C A \Phi = K \Phi.$$

Remark. Note the similarity of this with the equation derived from circuits.

4.3 Spring-mass System in Motion

Now suppose that our system is no longer in equilibrium, then we can no longer assume the net force at each mass is zero; but we can instead utilize Newton's second law ($F = ma$),

$$f_{\text{ext}} + f = M \frac{d^2 \mathbf{x}}{dt^2} = M \frac{d^2}{dt^2} (\mathbf{x} - \mathbf{x}_0) = M \frac{d^2 \Phi}{dt^2},$$

where $M := \text{diag}(m_1, \dots, m_n)$ is the mass matrix, \mathbf{x} is the position vector of the masses and \mathbf{x}_0 is the initial position of the masses.

Remark. Note that the second equality is true since \mathbf{x}_0 is a constant.

Thus, by rewriting the above equation, we have

$$f_{\text{ext}} - K\Phi = M \frac{d^2 \Phi}{dt^2}.$$

But now the question becomes: how do we solve such a system?

I will outline the main steps below and then elaborate on how to solve the differential equation as solving the boundary condition equation is exactly the same as what we have done for circuits.

- We will first find the general solution to the complementary function of the differential equation, i.e. the solution to the equation

$$M \frac{d^2 \Phi}{dt^2} + K\Phi_{\text{CF}} = 0.$$

- Then I will find a particular solution to the boundary condition equation,

$$f_{\text{ext}} = K\Phi.$$

- Then general solution will simply be the sum of the complementary function with the particular solution.

To solve $M \frac{d^2 \Phi}{dt^2} + K\Phi_{\text{CF}} = 0$, we will use the eigenvalue method. Suppose our solution is in the form $\Phi(t) = v \exp(i\omega t)$ for some $\omega \in \mathbb{R}$ and $v \in \mathbb{C}^n$. Then by substituting Φ into our differential equation, we have

$$\begin{aligned} -K(v e^{i\omega t}) &= -\omega^2 v e^{i\omega t}, \\ \implies K v &= \omega^2 v, \end{aligned}$$

thus v is an eigenvector of K (whose existence is guaranteed by the fact that K is symmetric) so we have found $\Phi(t) = v \exp(i\omega t)$ satisfying the differential equation.

But now we've encountered the problem that our solution is not necessarily real as $v \in \mathbb{C}^n$. Thus, by considering that, as both K and ω^2 are real, by taking the complex conjugate on both sides of the equation

$$K v = \omega^2 v,$$

we have,

$$K\bar{v} = \omega^2\bar{v},$$

implying the conjugate of v is also an eigenvector of K with eigenvalue ω^2 . Thus, with this, we can find the real-valued solutions of our differential equation by considering

$$\begin{aligned}\frac{1}{2}ve^{i\omega t} + \frac{1}{2}\bar{v}e^{-i\omega t} &= \Re\{ve^{i\omega t}\} \in \mathbb{R} \\ \frac{1}{2i}ve^{i\omega t} - \frac{1}{2i}\bar{v}e^{-i\omega t} &= \Im\{ve^{i\omega t}\} \in \mathbb{R}\end{aligned}$$

we have found real-valued solutions to our differential equation and the complementary function would be the linear combination of the solutions. See [here](#) for an example.

5 As $n \rightarrow \infty$ and Beyond!

In this section we will consider a harmonic graph with n nodes as $n \rightarrow \infty$. Physically, we can interpret this as the problem of finding the electric potential at some position in a metal rod among other things.

We will first examine what happens to our model in a one dimensional case - as in the rod perspective and then we will also expand the system to a two dimensional case such as the problem of finding the electric potential on a metal plate.

5.1 Conductances in \mathbb{R} and \mathbb{R}^2

Up to now, we have simply assigned a number to an edge saying that corresponds to its conductance without regard to the physical length of the wire or any other physical attributes. This was perfectly fine since we only cared for the ‘topology’ of the graph but it’s less than ideal when working with one dimensional rods that conduct electricity. This is because as the rod increases in length, it is natural to expect it will be increasingly ‘harder’ for electrons to traverse the entire distance so the conductance should be negatively proportional to the length. With that in mind, we will define the conductance of a rod to be

$$c = \frac{\hat{c}}{L},$$

where \hat{c} is the *conductivity* of the material and L the length of the rod.

Now, suppose that our wire is no longer one dimensional and has now a mysterious property called “width”. As the width of the rod increases, physically we can imagine there being more space for the electrons so we would expect the conductance to increase as the width increases. Hence, in a two dimensional case, we define the conductance as

$$c = \frac{H}{L} \hat{c},$$

where H denotes the width increases.

5.2 1-D Continuum Limit

Suppose we have a wire along the real interval $[0, 1]$ and we place $N + 1$ nodes equidistant apart, i.e. for all $i \in 0, 1, \dots, N$ there is a node at the position i/N . Now let us denote the position of the i -th node on the interval by x_i , and the potential at this position by $\phi(x_i)$ thus $x_i = i/N = i\Delta x$ where $\Delta x = 1/N$.

Then by considering our incidence matrix A_N for all N , we have $A_N \Phi$ being the potential difference vector at each node i.e. $(A_N \Phi)_i = \phi(x_{i+1}) - \phi(x_i) = \Delta \phi_i$. Now, by defining the conductance between the $i + 1$ -th node and the i -node as $c_i/\Delta x$ (recall the reasoning above), we have $(-C A_N \Phi)_i = -c_i \frac{\Delta \phi_i}{\Delta x} = w_i$, where w_i represent the current on the edge between the $i + 1$ -th and the i -th node.

Now, by considering $-A^T w = f$ where f is the vector representing the sum of currents out of the nodes, let’s define $\hat{f}_i \Delta x = (f)_i$, where \hat{f}_i represents the source current density at the i -th node. The reason why we are doing this is because the net current out of each node is dependent on the conductance of its neighbouring edges; but as $\Delta x \rightarrow 0$ the conductance

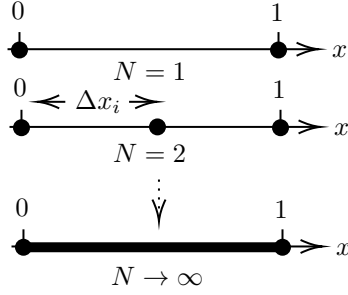


Figure 4: Bringing the number of nodes to infinity.

tends to infinity which doesn't have any physical representations. Thus, by dividing f by Δx we can receive a notion of the density of the net current out of each position (also known as *electric flux*). Then, by rewriting, we have $(-A^T w)_i = w_{i+1} - w_i = \Delta w_i = \hat{f}_i \Delta x$.

Finally, by plugging in our equation for w_i , we have

$$\hat{f}_i = \frac{\Delta w_i}{\Delta x} = \frac{1}{\Delta x} \left(-c_i \frac{\Delta \phi_i}{\Delta x} \right),$$

and as $N \rightarrow \infty$, we have $\Delta x \rightarrow 0$ and

$$\hat{f}(x) = \frac{dw}{dx} = \frac{d}{dx} \left(-c(x) \frac{d\phi}{dx} \right).$$

(Oftentimes, we have $c(x)$ being a constant as the material throughout the rod is the same, then we can bring out c from the derivative resulting in the differential equation $\hat{f}(x) = -c \frac{d^2 \phi}{dx^2}$.)

5.3 2-D Continuum Limit

Let's now consider the two dimensional limit. This time, rather than having each node representing some length Δx that tends to 0, we will let each node represent some area $\Delta x \Delta y \rightarrow 0$. As we are working with a two dimensional plane like the Cartesian coordinates, we will denote each node by a tuple of two numbers i, j . With that, we shall again define the source density by $f_{i,j} = \hat{f}_{i,j} \Delta x \Delta y$. As argued before, we will also define the conductance at each position by $c = \hat{c} \frac{\Delta y}{\Delta x}$.

Then, by decomposing the contributing currents at each node into the x and y directions denoted by u_i and v_j we have,

$$\begin{aligned} u_{i+1} &= -c(\phi_{i+1,j} - \phi_{i,j}) = -\hat{c} \frac{\Delta y}{\Delta x} (\phi_{i+1,j} - \phi_{i,j}) \\ u_i &= -c(\phi_{i,j} - \phi_{i-1,j}) = -\hat{c} \frac{\Delta y}{\Delta x} (\phi_{i,j} - \phi_{i-1,j}) \\ v_{j+1} &= -c(\phi_{i,j+1} - \phi_{i,j}) = -\hat{c} \frac{\Delta x}{\Delta y} (\phi_{i,j+1} - \phi_{i,j}) \\ v_j &= -c(\phi_{i,j} - \phi_{i,j-1}) = -\hat{c} \frac{\Delta x}{\Delta y} (\phi_{i,j} - \phi_{i,j-1}) \end{aligned}$$

where $\phi_{i,j}$ is the potential at the i, j -th node.

Then as $u_{i+1} - u_i + v_{j+1} - v_j = \hat{f}_{i,j}\Delta x\Delta y$, we have

$$\begin{aligned}\hat{f}_{i,j} &= \frac{1}{\Delta x\Delta y}(u_{i+1} - u_i + v_{j+1} - v_j) \\ &= -\hat{c}\left(\frac{1}{(\Delta x)^2}(\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j})\right) - \hat{c}\left(\frac{1}{(\Delta y)^2}(\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1})\right).\end{aligned}$$

Then by letting $\Delta x \rightarrow 0$ and $\Delta y \rightarrow 0$, we have,

$$\hat{f}(x, y) = -\hat{c}\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) = \nabla^2 \phi$$

5.3.1 Current Density

We will now define the notion of *current densities* for later use.

Recall our decomposition of the source density in the x direction,

$$\begin{aligned}u_i &= -\hat{c}\frac{\Delta y}{\Delta x}(\phi_{i,j} - \phi_{i-1,j}) \\ &= -\hat{c}\Delta y\frac{(\phi(x, y) - \phi(x - \Delta x, y))}{\Delta x},\end{aligned}$$

and thus as $\Delta x \rightarrow 0$, we define,

$$\lim_{\Delta x \rightarrow 0} u_i = \Delta y j_x(x) = -\Delta y \hat{c} \frac{\partial \phi}{\partial x},$$

where j_x is the current density in the x direction and $j_x = -\hat{c} \frac{\partial \phi}{\partial x}$.

Similarly, we define the current density in the y direction as $j_y = -\hat{c} \frac{\partial \phi}{\partial y}$. Then, by using this notation, we can write the flux as,

$$\hat{f} = \nabla^2 \phi = \frac{\partial}{\partial x} \left(-\hat{c} \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left(-\hat{c} \frac{\partial \phi}{\partial y} \right) = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = \nabla \cdot \mathbf{J},$$

where $\mathbf{J} = (j_x, j_y)^T$.

5.4 Introduction to Harmonic Functions

Harmonic functions are functions with the complex variable $z = x + iy$, that is, h is harmonic if, $h = h(z)$. It turns out, we can treat harmonic function in much of the same way as real function when doing calculus.

This is useful to us as we would like to find some *real*-valued function ϕ satisfying the continuous version of the *KCL* equation, $\nabla^2 \phi = 0$. What we can do is find some harmonic function h satisfying the equation instead, i.e. $\nabla^2 h = 0$, then the conjugate of h must also satisfy the equation also resulting $\nabla^2(h + \bar{h}) = 0$ which is a real valued function.

While this is all well and good, the question of course is now why bother with the complex? Well, the answer is simple, it turns out **all** harmonic functions satisfy our *KCL* equation!

Proof. Let h be a harmonic function $h = h(z) = h(x + iy)$. Then by chain rule,

$$\frac{\partial h}{\partial x} = h'(z) \frac{\partial}{\partial x}(x + iy) = h'(z) \implies \frac{\partial^2 h}{\partial x^2} = \frac{\partial h'}{\partial x} = h''(z)$$

and similarly,

$$\frac{\partial^2 h}{\partial y^2} = -h''(z).$$

Thus,

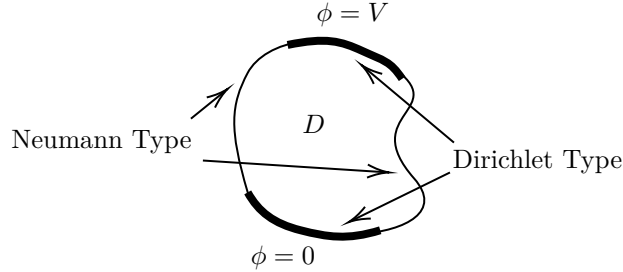
$$\nabla^2 h = h''(z) - h''(z) = 0.$$

□

Furthermore, consider the concept of *current density* introduced earlier. By algebraic manipulation, we can easily show that if $\phi = \Re\{h(z)\}$ then $j_x - ij_y = -\hat{c}h'(z)$.

5.5 Boundary Conditions on a Domain

Just like the discrete harmonic graphs, the nodes within a domain either satisfy *KCL* or has potential specified. If, at a position, we specify the potential, then we call this a boundary condition of *Dirichlet type*. On the other hand, if the position satisfies *KCL*, then we call the boundary condition of *Neumann type*.



With this in mind, we can then ask the usual questions:

- What is the potential function $\phi(x, y)$ inside the domain D .
- What is the effective conductance of the whole system.
- If a Dirichlet type boundary condition holds at the same boundary point, what is the current there. Or similarly, if the Neumann type condition holds, what is the potential there (Dirichlet-to-Neumann map)?

5.5.1 Example 1. Rectangular Conductor

Suppose we have a rectangular sheet of metal with uniform conductivity $\hat{c} = 1$. Suppose that this rectangular sheet had height H and width L with the left side attached to an electrode rod held at potential $\phi = 1$ and the right side grounded. Thus we have the left and right boundary of the metal sheet have boundary condition of Dirichlet type while the top and bottom with Neumann type. We can therefore immediately deduce that $j_y(\pm H/2) = 0$.

With this in consideration, we are now asked to find some harmonic function h such that $\Re h$ satisfy the boundary conditions.

Let's try $h(z) = -z/L$. Then $\phi(x, y) = \Re h(z) = -x/L$ which satisfies the boundary conditions of Dirichlet type as $\phi(-L, y) = 1$ and $\phi(0, y) = 0$ as required. Now, by considering $j_x - ij_y = -h'(z) = 1/L \in \mathbb{R}$ so $j_y = 0$ everywhere, and certainly also zero at the boundary conditions of Neumann type.

With that, by the uniqueness principle, we can now concluded $\phi(x, y) = -x/L$ is indeed the require function of potentials.

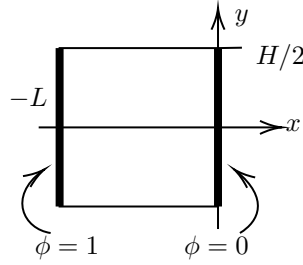


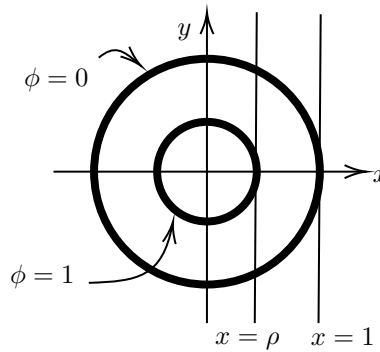
Figure 5: A rectangular sheet of metal with two electrodes connected.

Let's now find the effective conductance of this system. As in the discrete case the effective conductance of this system is defined to be the net current entering the circuit via the electrodes. Thus, by integrating j_x with respect to y would do the trick in this case,

$$C_e = \int_{-H/2}^{H/2} j_x dy = \frac{H}{L}.$$

5.5.2 Example 2. Circular Conductor

Now suppose that we have a circular sheet with uniform conductivity such that the inner electrode is held at unit potential while the outer boundary is grounded.



Let's try $h(z) = \log z / \log \rho$. As then $\phi(x, y) = \Re h(z) = \log |z| / \log \rho$ which satisfies the boundary conditions.

But what about the effective conductance?

Again, as the effective conductance is defined to be $\int_{s_0}^{s_1} \mathbf{J} \cdot n ds$, we need to find n - the normal vector from the inner circle. But this is simply $n = \exp(i\theta)$, so

$$\mathbf{J} \cdot n = \Re\{(j_x - ij_y)e^{i\theta}\} = \Re\{-\hat{c}h'(z)e^{i\theta}\} = -\frac{1}{\log \rho}.$$

And thus, by multiply this density by the arc-length of the circle, we have $C_e = -\frac{2\pi}{\log \rho}$.