536

Xinyang Wang

189001002

1 HW0

1.1

$$\begin{aligned}
MSE(\hat{L}) &= \mathbb{E}[(\hat{L} - L)^2] \\
&= \mathbb{E}[\hat{L}^2 - 2\hat{L}L + L^2] \\
&= \mathbb{E}[\hat{L}^2] - \mathbb{E}[2\hat{L}L] + \mathbb{E}[L^2] \\
&= \mathbb{E}[\hat{L}^2] - 2L\mathbb{E}[\hat{L}] + L^2 \\
&= \mathbb{E}[\hat{L}^2] - 2L\mathbb{E}[\hat{L}] + L^2 + \mathbb{E}[\hat{L}]^2 - \mathbb{E}[\hat{L}]^2 \\
&= (\mathbb{E}[\hat{L}^2] - \mathbb{E}[\hat{L}]^2) + (L - \mathbb{E}[\hat{L}])^2 \\
&= var(\hat{L}) + bias(\hat{L})^2
\end{aligned}$$

1.2

1. $for \quad \hat{L}_{MOM}:$

$$\begin{aligned}
\mathbb{E}[\hat{L}_{MOM}] &= \mathbb{E}[2\overline{X_n}] \\
&= 2\frac{1}{n}\mathbb{E}[\sum_{i=1}^{n} X_i] \\
&= 2\mathbb{E}[X_i] \\
&= L
\end{aligned}$$

$$\begin{aligned}
bias(\hat{L}_{MOM}) &= L - \mathbb{E}[\hat{L}_{MOM}] \\
&= 0
\end{aligned}$$

2. $for \quad \hat{L}_{MLE}:$

$\mathbb{E}[\hat{L}_{MLE}] = \mathbb{E}[\max_{i=1,\ldots,n} X_i]$

According to conclusion of problem 1) in HW0:

$$\begin{aligned}
\mathbb{E}[\max_{i=1,\ldots,n} X_i] &= \int_0^L \frac{n}{L}(\frac{x}{L})^{n-1} x dx \\
&= \frac{n}{n+1} L
\end{aligned}$$

$$
\begin{aligned}
bias(\hat{L}_{MLE}) &= L - \mathbb{E}[\hat{L}_{MLE}] \\
&= \frac{1}{n+1}L
\end{aligned}
$$

1.3

1. $for \quad \hat{L}_{MOM}:$

$$
\begin{aligned}
Var(\hat{L}_{MOM}) &= var(2\overline{X_n}) \\
&= \frac{4}{n^2}var(\sum_{i=1}^{n} X_i) \\
&= \frac{4}{n}var(X_i) \\
&= \frac{4}{n} * \frac{L^2}{12} \\
&= \frac{L^2}{3n}
\end{aligned}
$$

2. $for \quad \hat{L}_{MLE}:$

$Var(\hat{L}_{MLE}) = \mathbb{E}[\hat{L}_{MLE}^2] - \mathbb{E}[\hat{L}_{MLE}]^2$

According to conclusion of problem 1) in HW0:

$$
\begin{aligned}
\mathbb{E}[\max_{i=1,\dots,n} X_i^2] &= \int_0^L \frac{n}{L}(\frac{x}{L})^{n-1}x^2dx \\
&= \frac{n}{n+2}L^2
\end{aligned}
$$

According to 2) in 1.2, $\mathbb{E}[\hat{L}_{MLE}] = \frac{n}{n+1}L$,so

$$
\begin{aligned}
Var(\hat{L}_{MLE}) &= \mathbb{E}[\hat{L}_{MLE}^2] - \mathbb{E}[\hat{L}_{MLE}]^2 \\
&= \frac{n}{n+2}L^2 - \frac{n^2}{(n+1)^2}L^2 \\
&= \frac{n}{(n+1)^2(n+2)}L^2
\end{aligned}
$$

1.4

According to 1.1, we have:
$MSE(\hat{L}) = var(\hat{L}) + bias(\hat{L})^2$

$for \quad \hat{L}_{MOM}:$

$$
\begin{aligned}
MSE(\hat{L}_{MOM}) &= var(\hat{L}_{MOM}) + bias(\hat{L}_{MOM})^2 \\
&= \frac{L^2}{3n}
\end{aligned}
$$

$for \quad \hat{L}_{MLE}:$

$$
\begin{aligned}
MSE(\hat{L}_{MLE}) &= var(\hat{L}_{MLE}) + bias(\hat{L}_{MLE})^2 \\
&= \frac{n}{(n+1)^2(n+2)}L^2 + (L - \frac{n}{n+1}L)^2 \\
&= \frac{2L^2}{(n+1)(n+2)}
\end{aligned}
$$

Based on the conclusion above, we can see that when n is bigger than 2, $MSE(\hat{L}_{MLE})$ is always smaller than $MSE(\hat{L}_{MOM})$, So MLE is the better estimator.

1.5

Below is the code in Python to verify experimentally:

```
import random
import sys
from sys import argv

#script, log = argv
#f = open(log,'a')
#__con__ = sys.stderr
#sys.stderr = f

n=100
L=10
theo_MSE_MOM = L**2/(3*n)
theo_MSE_MLE = 2*L**2/((n+1)*(n+2))
print(f'theoretical MSE of MOM is {theo_MSE_MOM}\n')
print(f'theoretical MSE of MLE is {theo_MSE_MLE}\n')

list_for_MOM = []
list_for_MLE = []
for i in range(1000):
    list_for_sample = []
    for j in range(100):
        sample = random.random()
        sample *= L
        list_for_sample.append(sample)
    L_MOM = sum(list_for_sample)/n*2
    L_MLE = max(list_for_sample)
    list_for_MOM.append(L_MOM)
    list_for_MLE.append(L_MLE)

# compute expectation of MSE_MOM
MSE_for_MOM = 0
for i in range(1000):
    MSE_for_MOM += (list_for_MOM[i]-L)**2
MSE_for_MOM /= 1000
print(f'estimated MSE of MOM is {MSE_for_MOM}\n')

# compute expectation of MSE_MLE
```

```
MSE_for_MLE = 0
for i in range(1000):
    MSE_for_MLE += (list_for_MLE[i]-L)**2
MSE_for_MLE /= 1000
print(f'estimated MSE of MOM is {MSE_for_MLE}\n')
```

Here is the result of the code:

```
theoretical MSE of MOM is 0.3333333333333333
theoretical MSE of MLE is 0.01941370607649
estimated MSE of MOM is 0.33047168040779124
estimated MSE of MOM is 0.01927705870982061
```

As we can see from above, MSEs from the experiments are very close to their theoretical values,

1.6

We take a look at the definition of two estimators:
$\hat{L}_{MOM} = 2\overline{X}_n$, while $\hat{L}_{MLE} = \max\limits_{i=1,\ldots,n} X_i$

We find that $\hat{L}_{MLE}$ tries to reach the real value of L from left,which means that $\hat{L}_{MLE}$ can not exceed L. While $\hat{L}_{MOM}$ can be smaller or bigger than L. So by intuition, $\hat{L}_{MOM}$ can deviate more from the real value of L. This means a bigger variance which leads to a bigger MSE.
Besides, the real value of L should be bigger than $\hat{L}_{MLE}$. So it is reasonable that $\hat{L}_{MOM} > \hat{L}_{MLE}$. While there are situations where $\hat{L}_{MOM} < \hat{L}_{MLE}$ (in this situation,$\hat{L}_{MOM}$ is a bad estimation). So $\hat{L}_{MOM}$ is less reliable.

1.7

According to conclusion of problem 1) in HW0:

$$
\begin{aligned}
P(\hat{L}_{MLE} < L - \epsilon) &= P(\max_{i=1,\ldots,n} X_i < L - \epsilon) \\
&= P(X < L - \epsilon)^n \\
&= (\frac{L-\epsilon}{L})^n
\end{aligned}
$$

According to requirement:
We need: $P(\hat{L}_{MLE} < L - \epsilon) \geq 1 - \delta$
So we need $n \geq \frac{\ln(1-\delta)}{\ln(L-\epsilon) - \ln L}$

1.8

With new estimator $\hat{L}_{MLE-new} = \frac{n+1}{n}\hat{L}_{MLE-old}$ ($\hat{L}_{MLE-old} = \max\limits_{i=1,\ldots,n} X_i$)

$$
\begin{aligned}
bias(\hat{L}_{MLE-new}) &= L - \mathbb{E}[\frac{n+1}{n}\hat{L}_{MLE-old}] \\
&= L - \frac{n+1}{n}\mathbb{E}[\hat{L}_{MLE-old}] \\
&= L - \frac{n+1}{n} * \frac{n}{n+1} L \\
&= 0
\end{aligned}
$$

So $\hat{L}_{MLE-new}$ is an unbiased estimator
Now we recompute the MSE for this new MLE estimator

$$
\begin{aligned}
var(\hat{L}_{MLE-new}) &= var(\frac{n+1}{n}\hat{L}_{MLE-old}) \\
&= (\frac{n+1}{n})^2 var(\hat{L}_{MLE-old}) \\
&= (\frac{n+1}{n})^2 \frac{n}{(n+1)^2(n+2)} L^2 \\
&= \frac{L^2}{n(n+2)}
\end{aligned}
$$

So $MSE(\hat{L}_{MLE-new}) = var(\hat{L}_{MLE-new}) = \frac{L^2}{n(n+2)}$
Compared with $MSE(\hat{L}_{MOM})$,when $n > 1$, $MSE(\hat{L}_{MLE-new})$ is always smaller than $MSE(\hat{L}_{MOM})$.