

NUCLEAR ENGINEERING, UC BERKELEY

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# ME 280A Finite Element Analysis HOMEWORK 2: HIGHER ORDER ELEMENTS

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## 1 INTRODUCTION

The objective of this project is to solve a 1-D differential equation using linear, quadrilateral and cubic elements.

$$\begin{aligned}
 \frac{d}{dx}(A_1(x)\frac{du}{dx}) &= f(x) \\
 f(x) &= 256\sin(\frac{3\pi kx}{4})\cos(16\pi x) \\
 A_1(x) &= 0.2 \\
 L &= 1 \\
 u(0) &= 0 \\
 A_1(L)\frac{du}{dx}(L) &= 1
 \end{aligned} \tag{1}$$

## 2 ANALYTICAL SOLUTION

The analytical solution of this equation is easily found after two integrations:

$$\begin{aligned}
 u(x) &= 5 \frac{4087\pi x + 1536\sqrt{2}x + \frac{137216\sin(\frac{61\pi x}{4})}{61\pi} - \frac{124928\sin(\frac{67\pi x}{4})}{67\pi}}{4087\pi} \\
 \frac{du}{dx} &= 5(1 + \frac{512(67\cos(\frac{61\pi x}{4}) - 61\cos(\frac{67\pi x}{4}))}{4087\pi} - \frac{512(67\cos(\frac{61\pi}{4}) - 61\cos(\frac{67\pi}{4}))}{4087\pi}) \quad (2)
 \end{aligned}$$

## 3 ERROR

The error is defined as

$$\begin{aligned}
 e^N &= \frac{\|u - u^N\|_{A_1(\Omega)}}{\|u\|_{A_1(\Omega)}} \\
 \|u\|_{A_1(\Omega)} &= \sqrt{\int_{\Omega} \frac{du}{dx} A_1 \frac{du}{dx} dx} = \sqrt{\int_{\Omega} A_1 (\frac{du}{dx})^2 dx} \\
 \|u - u^N\|_{A_1(\Omega)} &= \sqrt{\int_{\Omega} A_1 (\frac{d(u - u^N)}{dx})^2 dx}
 \end{aligned} \tag{3}$$

To compute the error numerically, we calculate the two following quantities element by element, and then assemble them to obtain the overall error:

$$\begin{aligned}
\|u\|_{A_1(\Omega)}^2 &= \int_{\Omega} A_1 \left( \frac{du}{dx} \right)^2 dx \\
\|u - u^N\|_{A_1(\Omega)}^2 &= \int_{\Omega} A_1 \left( \frac{d(u - u^N)}{dx} \right)^2 dx \\
&= A_1 \int_{\Omega} \left( \frac{du}{dx} - \frac{du_N}{dx} \right)^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{x_e}^{x_{e+1}} \left( \frac{du}{dx} - \frac{du_N}{dx} \right)^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{x_e}^{x_{e+1}} \left( \frac{du}{dx} - \frac{a_{i+1} - a_i}{h_i} \right)^2 dx
\end{aligned} \tag{4}$$

## 4 FINITE ELEMENT METHOD

The matrix system to solve stays the same for elements with different orders:

$$\begin{aligned}
K_{ij} &= \int_{\Omega} \frac{d}{dx} \phi_j(x) A_1 \frac{d}{dx} \phi_i(x) dx \\
R_i &= \int_{\Omega} f \phi_i(x) dx + \phi_i(x) t|_{\Gamma_t} \\
Ka &= R
\end{aligned} \tag{5}$$

Linear shape functions are:

$$\begin{aligned}
\hat{\phi}_1 &= \frac{1 - \xi}{2} \\
\hat{\phi}_2 &= \frac{1 + \xi}{2}
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\frac{d\hat{\phi}_1}{d\xi} &= -\frac{1}{2} \\
\frac{d\hat{\phi}_2}{d\xi} &= \frac{1}{2}
\end{aligned} \tag{7}$$

Quadratic shape functions are:

$$\begin{aligned}\hat{\phi}_1 &= \frac{\xi(\xi-1)}{2} \\ \hat{\phi}_2 &= -(\xi-1)(\xi+1) \\ \hat{\phi}_3 &= \frac{\xi(\xi+1)}{2}\end{aligned}\tag{8}$$

and

$$\begin{aligned}\frac{d\hat{\phi}_1}{d\xi} &= \frac{2\xi-1}{2} \\ \frac{d\hat{\phi}_2}{d\xi} &= -2\xi \\ \frac{d\hat{\phi}_3}{d\xi} &= \frac{2\xi+1}{2}\end{aligned}\tag{9}$$

The cubic shape functions are:

$$\begin{aligned}\hat{\phi}_1 &= \frac{-(\xi-1)(3\xi+1)(3\xi-1)}{16} \\ \hat{\phi}_2 &= \frac{9(\xi-1)(\xi+1)(3\xi-1)}{16} \\ \hat{\phi}_3 &= \frac{-9(\xi-1)(\xi+1)(3\xi+1)}{16} \\ \hat{\phi}_4 &= \frac{(\xi+1)(3\xi+1)(3\xi-1)}{16}\end{aligned}\tag{10}$$

and

$$\begin{aligned}\frac{d\hat{\phi}_1}{d\xi} &= -\frac{1}{16}(27\xi^2-18\xi-1) \\ \frac{d\hat{\phi}_2}{d\xi} &= \frac{9}{16}(9\xi^2-2\xi-3) \\ \frac{d\hat{\phi}_3}{d\xi} &= -\frac{9}{16}(9\xi^2+2\xi-3) \\ \frac{d\hat{\phi}_4}{d\xi} &= \frac{1}{16}(27\xi^2+18\xi-1)\end{aligned}\tag{11}$$

The mapping between master elements and the real spatial coordinates, we calculate  $x$  from  $\xi$ :

$$x = \sum \chi_i \hat{\phi}_i\tag{12}$$

where  $\chi_i$  are the coordinates of the nodes in the global element.

## 5 PROCEDURES

## 6 RESULTS

Minimum number of element to achieve to error criteria for different order of elements are tabulated bellow:

| P | $Ne_{opt}$ |
|---|------------|
| 1 | 1347       |
| 2 |            |
| 3 |            |

The numerical solution and the analytical solution are compared in figure ???. And the error between the analytical solution and the numerical solution at each elements are shown in figure ??. The error is not uniform throughout the domain of validation.

## 7 CONCLUSION

With the constant  $k$  in the right side of the equation getting larger, the frequency of  $\sin(\pi kx/L)$  increases, so we need more nodes to capture the fluctuation of the solution. The number of elements needed for the same error criteria is almost proportional to  $k$ .

Three points gaussian quadrature is used in the project, giving exact integral values for polynomials with order 5 or less.

Keeping in mind the computation cost is important for engineers. Several technics were used in this project to save the memory and computation cost, such as binary search and sparse matrix.