NUCLEAR ENGINEERING, UC BERKELEY

ME 280A Finite Element Analysis HOMEWORK 2: HIGHER ORDER ELEMENTS

Xin Wang

October 19, 2014

1 INTRODUCTION

The objective of this project is to solve a 1-D differential equation using linear, quadratic and cubic elements.

$$\begin{aligned}
\frac{d}{dx}(A_1(x)\frac{du}{dx}) &= f(x) \\
f(x) &= 256sin(\frac{3\pi kx}{4})cos(16\pi x) \\
A_1(x) &= 0.2 \\
L &= 1 \\
u(0) &= 0 \\
A1(L)\frac{du}{dx}(L) &= 1
\end{aligned} \tag{1}$$

2 ANALYTICAL SOLUTION

This equation can be solved analytically after two integrations:

$$\begin{aligned}
\frac{du}{dx} &= 5\int 256sin(\frac{3\pi kx}{4})cos(16\pi x) \\
&= 5\int 128[sin(\frac{67\pi x}{4}) - sin(\frac{61\pi x}{4})] \\
&= 5\frac{512(67cos(\frac{61\pi x}{4}) - 61cos(\frac{67\pi x}{4}))}{4087\pi} + constant
\end{aligned} \tag{2}$$

Apply the bondary condition on x=L, we obtain the constant. The equation 2 becomes:

$$\begin{aligned}
\frac{du}{dx} &= 5(1 + \frac{512(67cos(\frac{61\pi x}{4}) - 61cos(\frac{67\pi x}{4}))}{4087\pi} - \frac{512(67cos(\frac{61\pi}{4}) - 61cos(\frac{67\pi}{4}))}{4087\pi}) \\
&= 5(\frac{512(67cos(\frac{61\pi x}{4}) - 61cos(\frac{67\pi x}{4}))}{4087\pi} + 1 - \frac{1536\sqrt{2}}{4087\pi})
\end{aligned} \tag{3}$$

Integrating $\frac{du}{dx}$ and applying the boundary condition at x=0, we obtain the analytical solution for the differencial equation:

$$u(x) = 5\frac{4087\pi x + 1536\sqrt{2}x + \frac{137216 sin(\frac{61\pi x}{4})}{61\pi} - \frac{124928 sin(\frac{67\pi x}{4}))}{67\pi}}{4087\pi} \tag{4}$$

3 Finite Element Method

The first step of FEM is to derive the weak form of the differential equation:

Find u, u|Γu =d, such that ∀ v, v|Γu =0

$$\int_\Omega \frac{dv}{dx} A_1 \frac{du}{dx} dx = \int_\Omega f v dx + t v \mid_{\Gamma t} \tag{5}$$

We approximate the real solution u by

$$u(x) = \sum_{j=1}^{N} a_j \phi_j(x) \tag{6}$$

and we choose the test function v with the same approximation functions

$$v(x) = \sum_{i=1}^{N} b_i \phi_i(x) \tag{7}$$

where N is the number of degree of freedom(number of nodes).

Then the equation becomes:

$$\int_\Omega \frac{d}{dx}(\sum_{j=1}^{N+1} a_j \phi_j(x)) A_1 \frac{d}{dx}(\sum_{i=1}^{N+1} b_i \phi_i(x)) dx = \int_\Omega f(\sum_{i=1}^{N+1} b_i \phi_i(x)) dx + (\sum_{i=1}^{N+1} b_i \phi_i(x) t) \mid_{\Gamma t}, \forall b_i \tag{8}$$

We can regroup the terms into:

$$\sum_{i=1}^{N+1} b_i \int_\Omega (\sum_{j=1}^{N+1} a_j \frac{d}{dx}\phi_j(x) A_1 \frac{d}{dx}\phi_i(x)) dx = \sum_{i=1}^{N+1} b_i \int_\Omega f \phi_i(x) dx + \sum_{i=1}^{N+1} b_i (\phi_i(x) t) \mid_{\Gamma t} \tag{9}$$

As the equation should be valid for any b_i, we obtain the matrix system to solve:

$$
\begin{aligned}
K_{ij} &= \int_\Omega \frac{d}{dx}\phi_j(x) A_1 \frac{d}{dx}\phi_i(x) dx \\
R_i &= \int_\Omega f\phi_i(x)dx + \phi_i(x)t\,|_{\Gamma t} \\
Ka &= R
\end{aligned} \tag{10}
$$

In this homework, piece-wise linear, quadratic or cubic basis functions are used. The numerical computation is carried over the corresponding master elements and mapped to the global elements.

Shape functions are defined for elements with different polynomial orders(figure 1).

Linear shape functions are:

$$
\begin{aligned}
\hat{\phi}_1 &= \frac{1-\xi}{2} \\
\hat{\phi}_2 &= \frac{1+\xi}{2}
\end{aligned} \tag{11}
$$

and

$$
\begin{aligned}
\frac{d\hat{\phi}_1}{d\xi} &= -\frac{1}{2} \\
\frac{d\hat{\phi}_2}{d\xi} &= \frac{1}{2}
\end{aligned} \tag{12}
$$

Quadratic shape functions are:

$$
\begin{aligned}
\hat{\phi}_1 &= \frac{\xi(\xi-1)}{2} \\
\hat{\phi}_2 &= -(\xi-1)(\xi+1) \\
\hat{\phi}_3 &= \frac{\xi(\xi+1)}{2}
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
\frac{d\hat{\phi}_1}{d\xi} &= \frac{2\xi-1}{2} \\
\frac{d\hat{\phi}_2}{d\xi} &= -2\xi \\
\frac{d\hat{\phi}_3}{d\xi} &= \frac{2\xi+1}{2}
\end{aligned} \tag{14}
$$

The cubic shape functions are:

$$
\begin{aligned}
\hat{\phi}_1 &= \frac{-(\xi-1)(3\xi+1)(3\xi-1)}{16} \\
\hat{\phi}_2 &= \frac{9(\xi-1)(\xi+1)(3\xi-1)}{16} \\
\hat{\phi}_3 &= \frac{-9(\xi-1)(\xi+1)(3\xi+1)}{16} \\
\hat{\phi}_4 &= \frac{(\xi+1)(3\xi+1)(3\xi-1)}{16}
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
\frac{d\hat{\phi}_1}{d\xi} &= -\frac{1}{16}(27\xi^2-18\xi-1) \\
\frac{d\hat{\phi}_2}{d\xi} &= \frac{9}{16}(9\xi^2-2\xi-3) \\
\frac{d\hat{\phi}_3}{d\xi} &= -\frac{9}{16}(9\xi^2+2\xi-3) \\
\frac{d\hat{\phi}_4}{d\xi} &= \frac{1}{16}(27\xi^2+18\xi-1)
\end{aligned} \tag{16}
$$

Corresponding global coordinate x can be calculate x from ξ:

$$
x = \sum \chi_i \hat{\phi}_i \tag{17}
$$

where χ_i are the coordinates of the nodes in the global element.

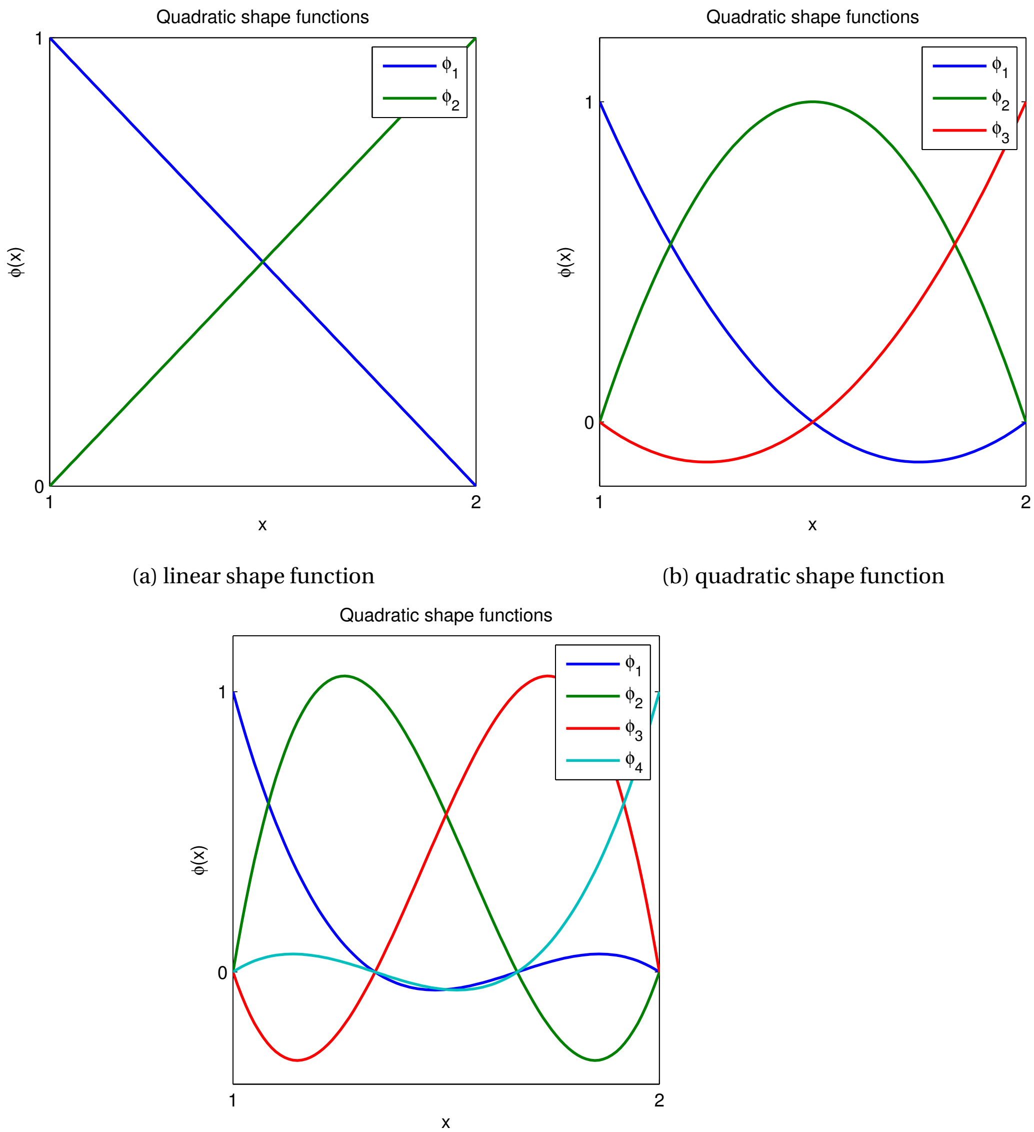
The method to compute the matrices K and R element by element and assembling procedures are the same as in homework 1. The difference for higher order elements is the size of Ke and Re.

3.1 Apply the boundary conditions

After inital implementation of K and R matrices, one needs to modify them in order to incorporate the boundary conditions. In this homework, we deal with the two types of boundary conditions discussed in the following section.

3.1.1 Dirichlet boundary condition

Dirichlet boundary condition gives the value of u at an end point(x=0 or x=L), i.e the value in solution vector a(0 or N). We force the test function v(0) =0 at these boundaries. The implementation of Dirichlet boundary condition in 1D is to modify the first or the last line of K and R for boundary condition at the left or right end of the domain.



(a) linear shape function

(b) quadratic shape function

(c) cubic shape function

Figure 1: nodel shape functions on an element [1, 2]

For Dirichlet BC on u(0):

K(1,1) =1 and K(1, 2:N) =0

R(1) = u(0)

For Dirichlet BC on u(L):

K(N,N) =1 and K(N, 1:N-1) =0

R(N) = u(L)

After the modification, a = K/R results in for the line corresponds to Dirichlet boundary condition:

a(1) = R(1)/K[1,1] = R(1)

or

a(N) = R(N)/K[N,N] = R(N)

3.1.2 Neumann boundary condition

Neumann boundary condition gives the value of traction $A1(x)\frac{du}{dx}$ on an end point.

For Neumann BC on x=0:

$R(1) = R(1) + A_1\frac{du}{dx}[x = 0]$

For Neumann BC on x=L:

$R(N) = R(N) + A_1\frac{du}{dx}[x = L]$

4 Gaussian Quadrature Integration

The integration of a function f over the domain [-1 ,1] can be computed numerically as:

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{N} w_i f(\xi_i) \tag{18}$$

Increasing the number of Gaussian quadrature points increase the precision of the integration result, but only up to a certain point. For a N order polynomial integration, (N+1)/2 points is enough for an exact result. More Gaussian point would cost unnecessary computation time. So it's important to optimize the number of GQ points used for each integration.

5 points GQ integration is used for quadratic and cubic element calculation. 3 points GQ integration is used for linear element calculation.

5 Error Calculations

The error is defined as

$$
\begin{aligned}
e^N &= \frac{\|u-u^N\|_{A_1(\Omega)}}{\|u\|_{A_1(\Omega)}} \\
\|u\|_{A_1(\Omega)} &= \sqrt{\int_\Omega \frac{du}{dx} A_1 \frac{du}{dx} dx} = \sqrt{\int_\Omega A_1 (\frac{du}{dx})^2 dx} \\
\|u-u^N\|_{A_1(\Omega)} &= \sqrt{\int_\Omega A_1 (\frac{d(u-u^N)}{dx})^2 dx}
\end{aligned}
\tag{19}
$$

To compute the error numerically, we calculate the two following quantities element by element, and then assemble them to obtain the overall error:

$$
\begin{aligned}
\|u\|^2_{A_1(\Omega)} &= \int_\Omega A_1 (\frac{du}{dx})^2 dx \\
\|u-u^N\|^2_{A_1(\Omega)} &= \int_\Omega A_1 (\frac{d(u-u^N)}{dx})^2 dx \\
&= A_1 \int_\Omega (\frac{du}{dx} - \frac{du_N}{dx})^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{x_e}^{x_{e+1}} (\frac{du}{dx} - \frac{du_N}{dx})^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{-1}^{1} (\frac{du}{d\xi} - \frac{du_N}{d\xi})^2 J d\xi
\end{aligned}
\tag{20}
$$

6 Postprocessing

The postprocessing is more complicated for higher order element. In order to plot the result, we calculate the value of the numerical solution for 10 points in each elements. For any given ξ, the corresponding value for the solution is calculated, as well as the global coordinate x for the given ξ:

$$
\begin{aligned}
u(x) &= u(x(\xi)) = \sum_{i=1}^{P+1} a_i^e \hat{\phi}_i \\
x(\xi) &= \sum x^e \hat{\phi}(\xi)
\end{aligned}
\tag{21}
$$

For each element, we can also calculate the derivative of u(x):

P	Ne_{opt}	$N_{opt} = Ne_{Opt} * P + 1$
1	1348	1349
2	94	189
3	34	103

Table 1: Minimum number of nodes needed for the error criteria

$$\frac{du}{dx} = \frac{d}{dx}\sum_{i=1}^{P+1} a_i \phi_i = (\frac{d}{d\xi}\sum_{i=1}^{P+1} a_i \hat{\phi}_i)\frac{d\xi}{dx} \tag{22}$$

7 RESULTS

Minimum number of elements and number of nodes to achieve to error criteria for different order of elements are tabulated bellow. N_{Opt} drops dramatically with the order of the elements.

The numerical solution and the analytical solution are compared in figure 2.

7.1 RELATIONSHIP BETWEEN THE ERROR AND THE ELEMENT SIZE

Error estimate for FEM is:

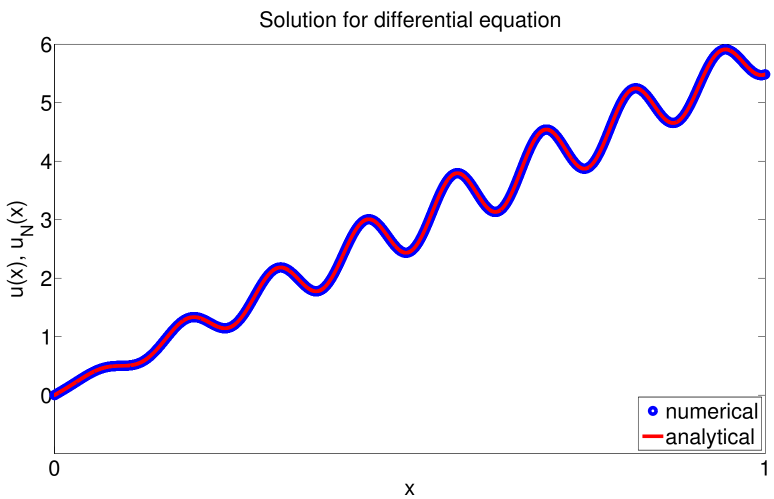
$$e^N \leq Ch^{\gamma = min(r-1,P)} \tag{23}$$

r is smoothness, and P is the polynomial order.

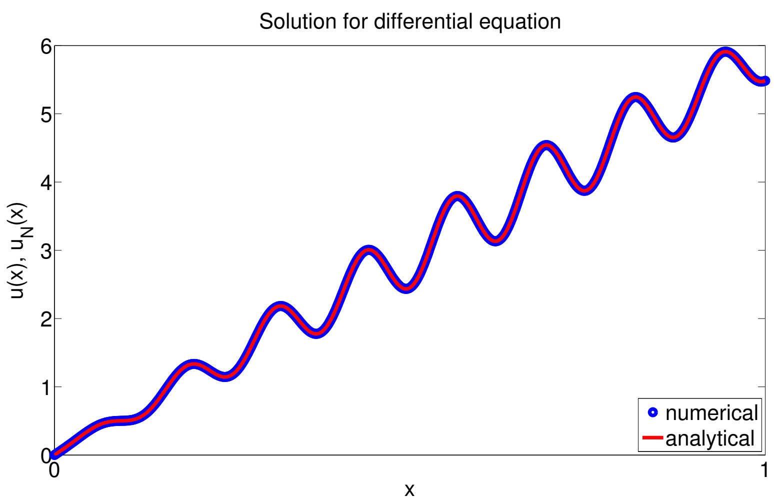
From the error plotting, we can observe the linear relation between $log(e^N)$ and $log(h)$,

$$\begin{aligned} \gamma &= 0.98, P = 1 \\ \gamma &= 1.98, P = 2 \\ \gamma &= 2.93, P = 3 \end{aligned} \tag{24}$$

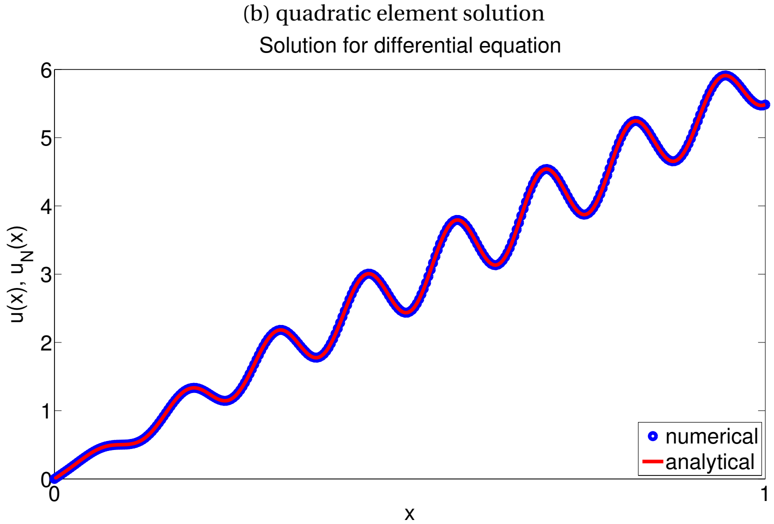
Overall, γ is similar to P.



(a) linear element solution



(b) quadratic element solution



(c) cubic element solution

Figure 2: Comparison of analytical solution and numerical solution at optimum number of element

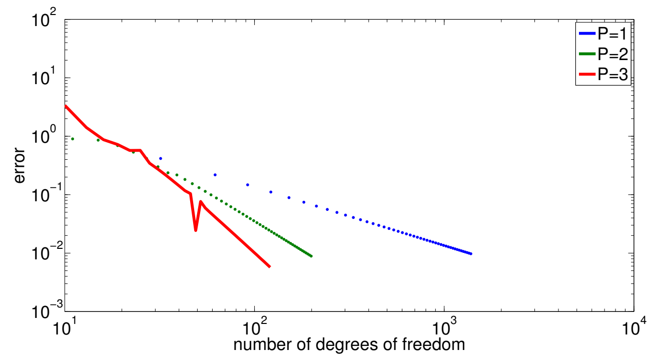


Figure 3: Evoluation of numerical error with number of elements

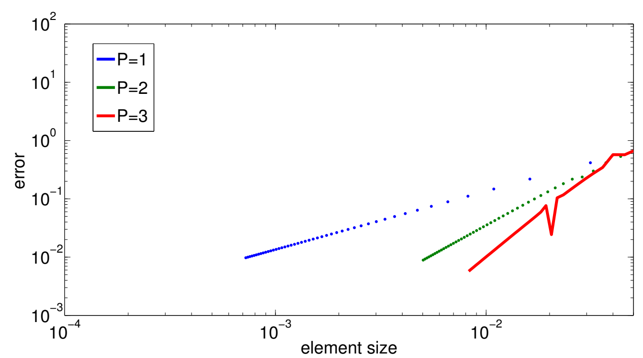


Figure 4: Evoluation of numerical error with number of elements

8 Conclusion

This homework extends the previous one to higher order of element. Having higher order element shape functions can reduce the number of degree of freedom needed for solving the problem by better capturing the fluctuations in the solution.