NUCLEAR ENGINEERING, UC BERKELEY

ME 280A Finite Element Analysis
HOMEWORK 2: HIGHER ORDER ELEMENTS

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1 Introduction

The objective of this project is to solve a 1-D differential equation using linear, quadratic and cubic elements.

$$
\begin{aligned}
\frac{d}{dx}(A_1(x)\frac{du}{dx}) &= f(x)\\
f(x) &= 256sin(\frac{3\pi kx}{4})cos(16\pi x)\\
A_1(x) &= 0.2\\
L &= 1\\
u(0) &= 0\\
A1(L)\frac{du}{dx}(L) &= 1
\end{aligned}
\tag{1}
$$

2 Analytical solution

This equation can be solved analytically after two integrations:

$$
\begin{aligned}
u(x) &= 5\frac{4087\pi x + 1536\sqrt{2}x + \frac{137216sin(\frac{61\pi x}{4})}{61\pi} - \frac{124928sin(\frac{67\pi x}{4}))}{67\pi}}{4087\pi}\\
\frac{du}{dx} &= 5(1 + \frac{512(67cos(\frac{61\pi x}{4}) - 61cos(\frac{67\pi x}{4}))}{4087\pi} - \frac{512(67cos(\frac{61\pi}{4}) - 61cos(\frac{67\pi}{4}))}{4087\pi}
\end{aligned}
\tag{2}
$$

3 Error

The error is defined as

$$
\begin{aligned}
e^N &= \frac{\|u-u^N\|_{A_1(\Omega)}}{\|u\|_{A_1(\Omega)}}\\
\|u\|_{A_1(\Omega)} &= \sqrt{\int_\Omega \frac{du}{dx}A_1\frac{du}{dx}dx} = \sqrt{\int_\Omega A_1(\frac{du}{dx})^2 dx}\\
\|u-u^N\|_{A_1(\Omega)} &= \sqrt{\int_\Omega A_1(\frac{d(u-u^N)}{dx})^2 dx}
\end{aligned}
\tag{3}
$$

To compute the error numerically, we calculate the two following quantities element by element, and then assemble them to obtain the overall error:

$$
\begin{aligned}
\|u\|^2_{A_1(\Omega)} &= \int_\Omega A_1(\frac{du}{dx})^2 dx \\
\|u-u^N\|^2_{A_1(\Omega)} &= \int_\Omega A_1(\frac{d(u-u^N)}{dx})^2 dx \\
&= A_1 \int_\Omega (\frac{du}{dx} - \frac{du_N}{dx})^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{x_e}^{x_{e+1}} (\frac{du}{dx} - \frac{du_N}{dx})^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{-1}^{1} (\frac{du}{d\xi} - \frac{du_N}{d\xi})^2 J d\xi
\end{aligned} \tag{4}
$$

4 Finite Element Method

The matrix system is computed using the same technics for linear, quadratic or cubic elements:

$$
\begin{aligned}
K_{ij} &= \int_\Omega \frac{d}{dx}\phi_j(x) A_1 \frac{d}{dx}\phi_i(x) dx \\
R_i &= \int_\Omega f\phi_i(x)dx + \phi_i(x)t\,|_{\Gamma t} \\
Ka &= R
\end{aligned} \tag{5}
$$

The numerical computation is done over the corresponding master elements. Shape functions are defined for element with different orders(figure 1).

Linear shape functions are:

$$
\begin{aligned}
\hat{\phi}_1 &= \frac{1-\xi}{2} \\
\hat{\phi}_2 &= \frac{1+\xi}{2}
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
\frac{d\hat{\phi}_1}{d\xi} &= -\frac{1}{2} \\
\frac{d\hat{\phi}_2}{d\xi} &= \frac{1}{2}
\end{aligned} \tag{7}
$$

Quadratic shape functions are:

$$\begin{aligned}\hat{\phi}_1 &= \frac{\xi(\xi-1)}{2}\\ \hat{\phi}_2 &= -(\xi-1)(\xi+1)\\ \hat{\phi}_3 &= \frac{\xi(\xi+1)}{2}\end{aligned}\tag{8}$$

and

$$\begin{aligned}\frac{d\hat{\phi}_1}{d\xi} &= \frac{2\xi-1}{2}\\ \frac{d\hat{\phi}_2}{d\xi} &= -2\xi\\ \frac{d\hat{\phi}_3}{d\xi} &= \frac{2\xi+1}{2}\end{aligned}\tag{9}$$

The cubic shape functions are:

$$\begin{aligned}\hat{\phi}_1 &= \frac{-(\xi-1)(3\xi+1)(3\xi-1)}{16}\\ \hat{\phi}_2 &= \frac{9(\xi-1)(\xi+1)(3\xi-1)}{16}\\ \hat{\phi}_3 &= \frac{-9(\xi-1)(\xi+1)(3\xi+1)}{16}\\ \hat{\phi}_4 &= \frac{(\xi+1)(3\xi+1)(3\xi-1)}{16}\end{aligned}\tag{10}$$

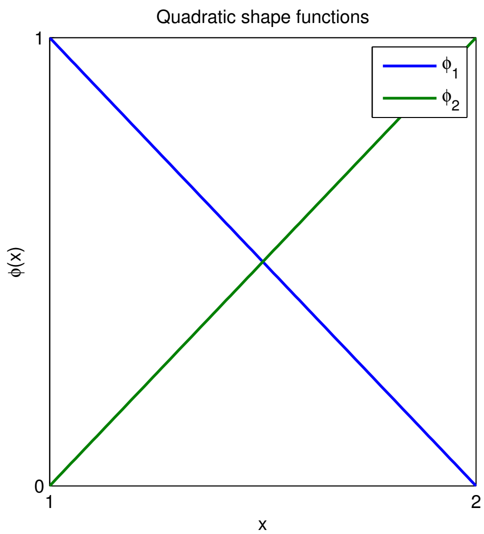
and

$$\begin{aligned}\frac{d\hat{\phi}_1}{d\xi} &= -\frac{1}{16}(27\xi^2-18\xi-1)\\ \frac{d\hat{\phi}_2}{d\xi} &= \frac{9}{16}(9\xi^2-2\xi-3)\\ \frac{d\hat{\phi}_3}{d\xi} &= -\frac{9}{16}(9\xi^2+2\xi-3)\\ \frac{d\hat{\phi}_4}{d\xi} &= \frac{1}{16}(27\xi^2+18\xi-1)\end{aligned}\tag{11}$$

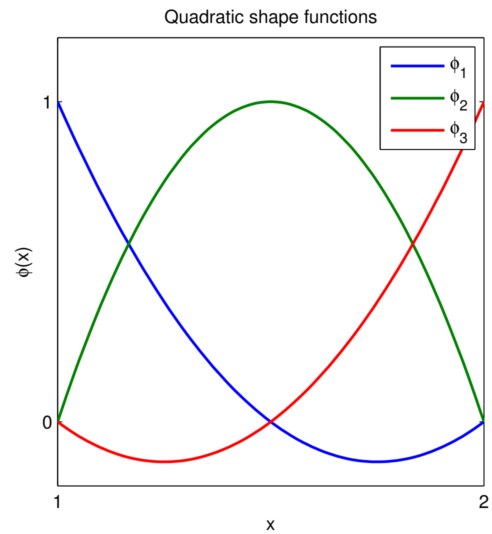
To mapping between master elements and the real spatial coordinates, we calculate x from ξ:

$$x = \sum \chi_i \hat{\phi}_i \tag{12}$$

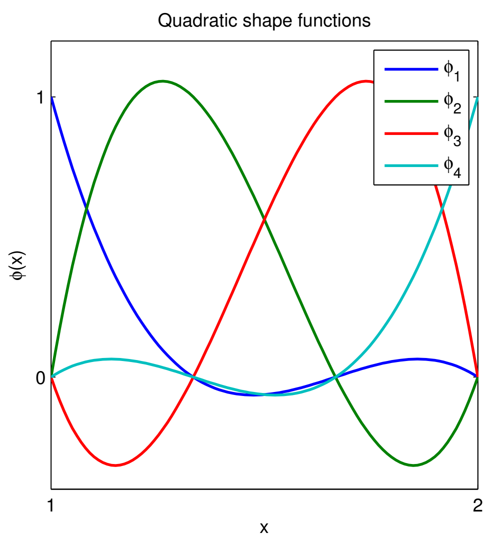
where χ_i are the coordinates of the nodes in the global element.



(a) linear shape function



(b) quadratic shape function



(c) cubic shape function

Figure 1: nodel shape functions on an element [1, 2]

5 POSTPROCESSING

Following the same procedures as in homework 1 would give the matrix A for the system. However, the postprocessing is more complicated for higher order element. In order to plot the result, we calculate the value of the numerical solution for 10 points in each elements. For any given ξ, the corresponding value for the solution is straightforward:

$$\begin{aligned} u(x) &= u(x(\xi)) = \sum_{i=1}^{P+1} a_i^e \hat{\phi}_i \\ x(\xi) &= \sum x^e \hat{\phi}(\xi) \end{aligned} \tag{13}$$

For each element, we can also calculate the derivative of u(x):

$$\frac{du}{dx} = \frac{d}{dx} \sum_{i=1}^{P+1} a_i \phi_i = \left(\frac{d}{d\xi} \sum_{i=1}^{P+1} a_i \hat{\phi}_i\right) \frac{d\xi}{dx} \tag{14}$$

6 RESULTS

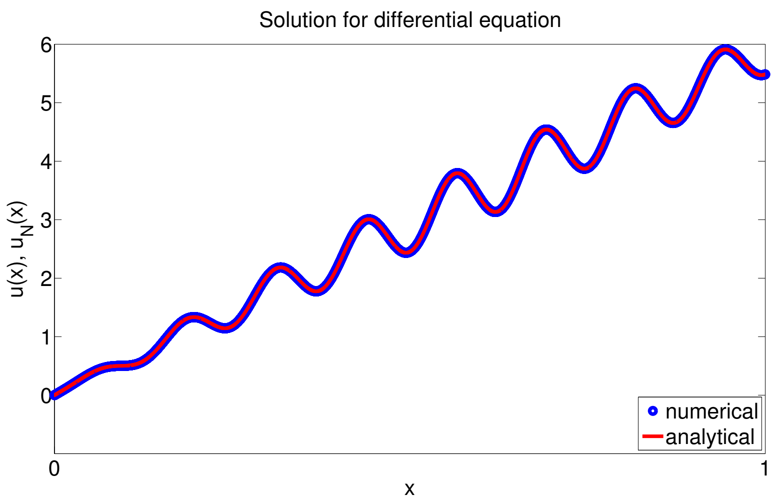
Minimum number of elements and number of nodes to achieve to error criteria for different order of elements are tabulated bellow:

P	Ne_{opt}	$N_{opt} = Ne_{Opt} * P + 1$
1	1348	1349
2	94	189
3	34	103

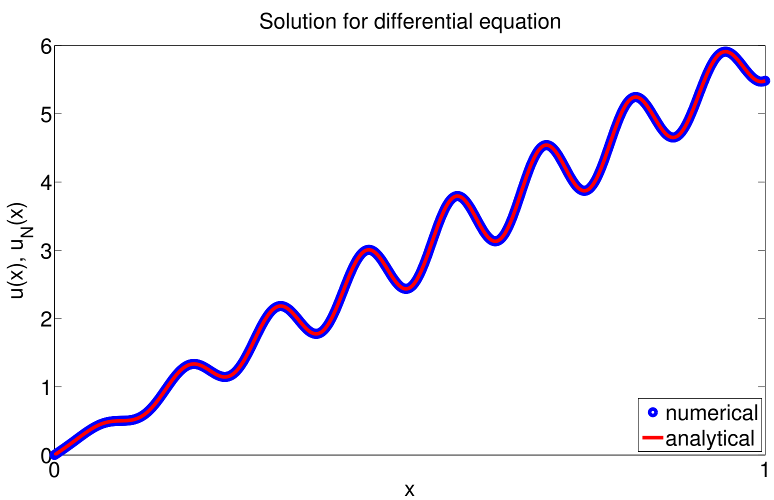
The numerical solution and the analytical solution are compared in figure 2. And the error between the analytical solution and the numerical solution at each elements are shown in figure 3. The error is not uniform throughout the domain of validation.

7 CONCLUSION

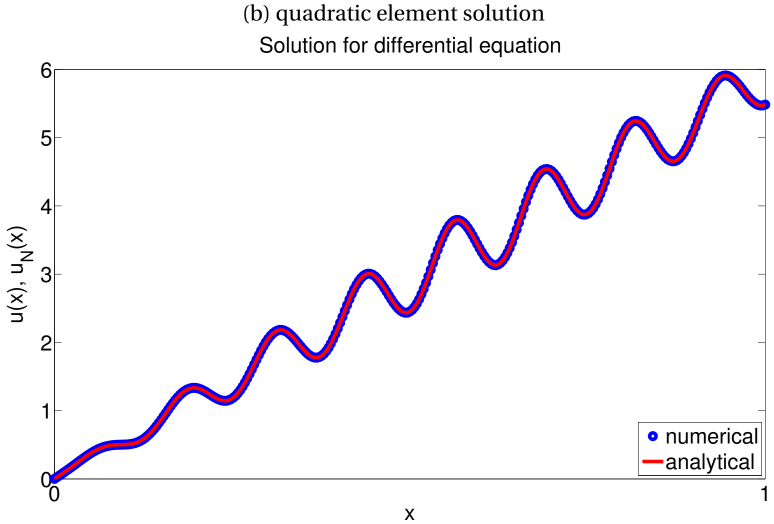
This homework extends the previous one to higher order of element. As the error of ... . Having higher order element shape functions can capture the fluctuations



(a) linear element solution

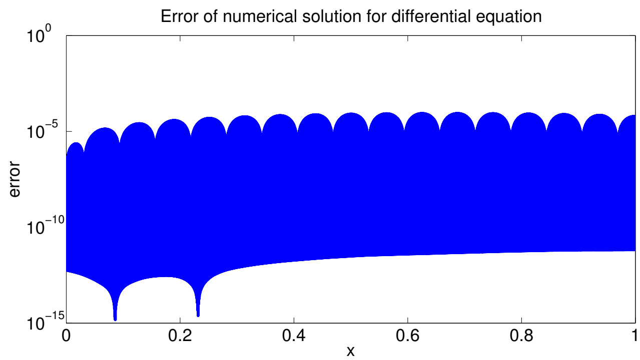


(b) quadratic element solution

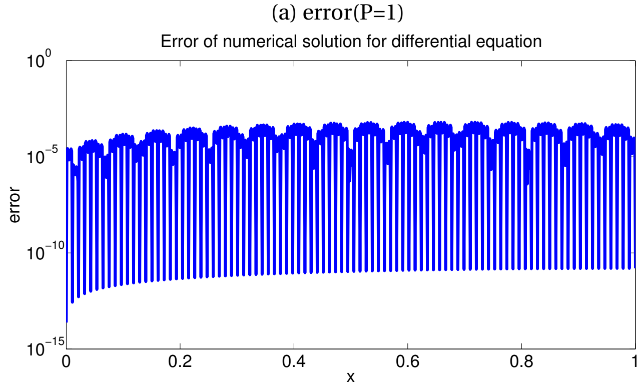


(c) cubic element solution

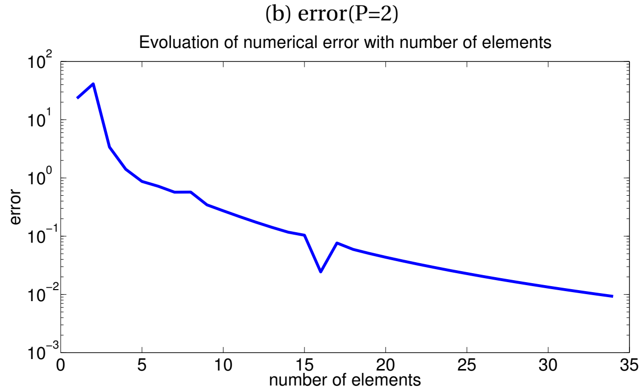
Figure 2: solution u(x) and $u_N(x)$



(a) error(P=1)



(b) error(P=2)



(c) error(P=3)

Figure 3: Evoluation of numerical error with number of elements