NUCLEAR ENGINEERING, UC BERKELEY

# ME 280A Finite Element Analysis
# HOMEWORK 1: THE BASICS

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# 1 INTRODUCTION

The objective of this project is to solve a 1-D differential equation using finite element method.

$$\begin{aligned}
\frac{d}{dx}(A_1(x)\frac{du}{dx}) &= f(x) \\
f(x) &= k^2 sin(\frac{\pi kx}{L}) + 2x \\
A_1(x) &= 0.2 \\
k &= givenconstant \\
L &= 1 \\
u(0) &= \Delta_1 = givenconstant = 0 \\
u(L) &= \Delta_2 = givenconstant = 1
\end{aligned} \tag{1}$$

# 2 ANALYTICAL SOLUTION

The analytical solution of this equation is easily found after two integrations:

$$\begin{aligned}
A_1\frac{du}{dx} &= \int (k^2 sin(\frac{\pi kx}{L}) + 2x)dx + c_1 \\
&= \frac{-Lkcos(\frac{\pi kx}{L})}{\pi} + x^2 + c_1 \\
\frac{du}{dx} &= \frac{1}{A_1}(\frac{-Lkcos(\frac{\pi kx}{L})}{\pi} + x^2 + c_1) \\
u(x) &= \frac{1}{A_1}\int (\frac{-Lkcos(\frac{\pi kx}{L})}{\pi} + x^2 + c_1)dx \\
&= -\frac{1}{A_1}\frac{L^2}{\pi^2} sin(\frac{\pi kx}{L}) + \frac{1}{A_1}\frac{x^3}{3} + c_2 x + c_3
\end{aligned} \tag{2}$$

where $c_1$, $c_2$ and $c_3$ are constants to be determined using boundary conditions.

For L=1, $A_1$=0.2, the general solution for this equation becomes:

$$u(x) = \frac{-5}{\pi^2} sin(\pi kx) + \frac{5x^3}{3} + c_2 x + c_3 \tag{3}$$

The boundary conditions at 0 and L requires:

$$
\begin{aligned}
u(0) = c_3 = 0 \\
u(L) = u(1) = \frac{-5}{\pi^2} sin(\pi k) + \frac{5}{3} + c_2 = 1 \\
c_2 = -\frac{2}{3}
\end{aligned} \tag{4}
$$

So the solution for the differential equation is:

$$
\begin{aligned}
u(x) &= \frac{-5}{\pi^2} sin(\pi k x) + \frac{5x^3}{3} - \frac{2}{3}x \\
\frac{du}{dx} &= \frac{-5k}{\pi} cos(\pi k x) + 5x^2 - \frac{2}{3}
\end{aligned} \tag{5}
$$

# 3 Error

The error is defined as

$$
\begin{aligned}
e^N = \frac{\|u - u^N\|_{A_1(\Omega)}}{\|u\|_{A_1(\Omega)}} \\
\|u\|_{A_1(\Omega)} = \sqrt{\int_\Omega \frac{du}{dx} A_1 \frac{du}{dx} dx} = \sqrt{\int_\Omega A_1 (\frac{du}{dx})^2 dx} \\
\|u - u^N\|_{A_1(\Omega)} = \sqrt{\int_\Omega A_1 (\frac{d(u-u^N)}{dx})^2 dx}
\end{aligned} \tag{6}
$$

To compute the error numerically, we calculate the two following quantities element by element, and then assemble them to obtain the overall error:

$$
\begin{aligned}
\|u\|^2_{A_1(\Omega)} &= \int_\Omega A_1 (\frac{du}{dx})^2 dx \\
\|u - u^N\|^2_{A_1(\Omega)} &= \int_\Omega A_1 (\frac{d(u-u^N)}{dx})^2 dx \\
&= A_1 \int_\Omega (\frac{du}{dx} - \frac{du_N}{dx})^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{x_e}^{x_{e+1}} (\frac{du}{dx} - \frac{du_N}{dx})^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{x_e}^{x_{e+1}} (\frac{du}{dx} - \frac{a_{i+1} - a_i}{h_i})^2 dx
\end{aligned} \tag{7}
$$

To find the optimal number of elements needed to solve the equation with the acceptable tolerance, the binary search approach is adopted. Starting from as few as two elements, connected at the middle point of the domain, the equation is solved using Finite Element Method and the error computed. If the error doesn't meet the tolerance criteria, then the number of elements is increased by a power of 2. When the error is satisfactory after n iteration, the optimal number of element should be in the range between $2^{n-1}$ and $2^n$. Based on the result of n, test the error at $(2^{n-1}+2^n)/2$, if the error is larger than the tolerance, then try the middle point of the right half, and vice versa.

# 4 FINITE ELEMENT METHOD

The first step of FEM is to derive the weak form of the differential equation:

$$\int_\Omega \frac{dv}{dx} A_1 \frac{du}{dx} dx = \int_\Omega f v dx + A_1 \frac{du}{dx} v \mid_{\partial\Omega}, \forall v \tag{8}$$

We approximate the real solution u by

$$u(x) = \sum_{j=1}^{N+1} a_j \phi_j(x) \tag{9}$$

and we choose the test function v with the same approximation functions

$$v(x) = \sum_{i=1}^{N+1} b_i \phi_i(x) \tag{10}$$

where N is the number of elements and N+1 the number of nodes.

Then the equation becomes

$$\int_\Omega \frac{d}{dx}(\sum_{j=1}^{N+1} a_j \phi_j(x)) A_1 \frac{d}{dx}(\sum_{i=1}^{N+1} b_i \phi_i(x)) dx = \int_\Omega f(\sum_{i=1}^{N+1} b_i \phi_i(x)) dx + (\sum_{i=1}^{N+1} b_i \phi_i(x)\, t) \mid_{\Gamma t}, \forall b_i \tag{11}$$

As the equation should be valid for any $b_i$, we can regroup the terms into:

$$\sum_{i=1}^{N+1} b_i \int_\Omega (\sum_{j=1}^{N+1} a_j \frac{d}{dx}\phi_j(x) A_1 \frac{d}{dx}\phi_i(x)) dx = \sum_{i=1}^{N+1} b_i \int_\Omega f \phi_i(x) dx + \sum_{i=1}^{N+1} b_i (\phi_i(x)\, t) \mid_{\Gamma t} \tag{12}$$

We obtain the matrix system to solve:

$$\begin{aligned} K_{ij} &= \int_\Omega \frac{d}{dx}\phi_j(x) A_1 \frac{d}{dx}\phi_i(x)\,dx \\ R_i &= \int_\Omega f\phi_i(x)\,dx + \phi_i(x)\,t\,|_{\Gamma t} \\ Ka &= R \end{aligned} \tag{13}$$

A set of basis functions are defined by

$$\begin{aligned} \phi_i(x) = \frac{x_i - x_{i-1}}{h_i}, \qquad & for \qquad x_{i-1} \le x \le x_i \\ \phi_i(x) = 1 - \frac{x - x_i}{h_{i+1}}, \qquad & for \qquad x_i \le x \le x_{i+1} \\ \phi_i(x) = 0 \quad & otherwise \end{aligned} \tag{14}$$

The derivative of the function is

$$\begin{aligned} \frac{\phi_i}{dx} = \frac{1}{h_i}, \qquad & for \qquad x_{i-1} \le x \le x_i \\ \frac{\phi_i}{dx} = \frac{-1}{h_{i+1}}, \qquad & for \qquad x_i \le x \le x_{i+1} \\ \frac{\phi_i}{dx} = 0 \quad & otherwise \end{aligned} \tag{15}$$

To compute the elements in K and R numerically, we partition the domain into N elements and map the mesh elements to a master element with two shape functions:

$$\begin{aligned} \hat{\phi_1} &= \frac{1}{2}(1-\zeta) \\ \hat{\phi_2} &= \frac{1}{2}(1+\zeta) \end{aligned} \tag{16}$$

The stiffness matrix and the load matrix for element number e are

$$\begin{aligned} K^e_{ij} &= \sum_{q=1}^{g} w_q \left(\frac{d}{d\zeta}\phi_i(M(\zeta))\right) \frac{d\zeta}{dx} A_1 \left(\frac{d}{d\zeta}(\phi_j(M(\zeta))\right) \frac{d\zeta}{dx} J \\ R^e_i &= \sum_{q=1}^{g} w_q \phi_i(M(\zeta))\, fJ + \phi_i(M(\zeta))\, t\,|_{\Gamma t} \end{aligned} \tag{17}$$

where $w_q$ are Gauss weights, and g is the number of points used in Gauss Integration.

## 5 PROCEDURES

The procedure for solving the differential equation using Finite Element Method is:

1. Define the elements connected at nodes
2. For each elements, compute the stiffness matrix K and the load matrix R
3. Assemble the contribution of all elements into the global system KA=R
4. Modify the system taking into account of boundary conditions
5. Solve the system for unknown A, using 'backslash' in Matlab: A= K/R
6. Evaluate the overall error, and compare to the tolerance value
7. Connect the numerical solution at each nodes by straight lines to construct the final solution. This simple method is sufficient in 1D linear system.

## 6 RESULTS

Minimum number of element to achieve to error criteria for different k values are tabulated bellow:

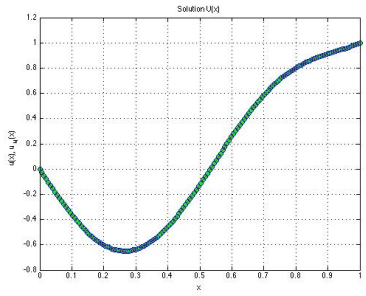
| k | $Ne_{opt}$ |
|---|---|
| 1 | 92 |
| 2 | 147 |
| 4 | 337 |
| 8 | 712 |
| 16 | 1443 |
| 32 | 2898 |

The numerical solution and the analytical solution are compared in figure 1. And the error between the analytical solution and the numerical solution at each elements are shown in figure 2. The error is not uniform throughout the domain of validation.

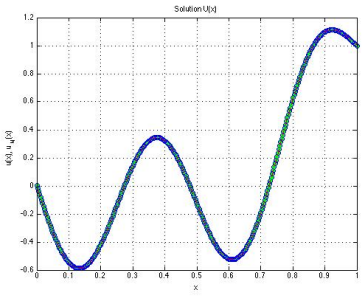
## 7 CONCLUSION

With the constant k in the right side of the equation getting larger, the frequency of $sin(\pi kx/L)$ increases, so we need more nodes to capture the fluctuation of the solution. The number of elements needed for the same error criteria is almost proportional to k.

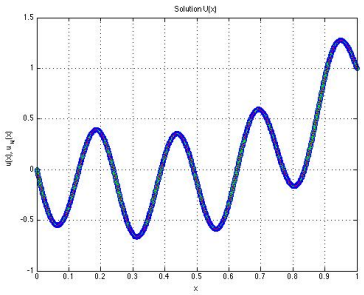
Three points gaussian quadriture is used in the project, giving exact integral values for polynomials with order 5 or less.



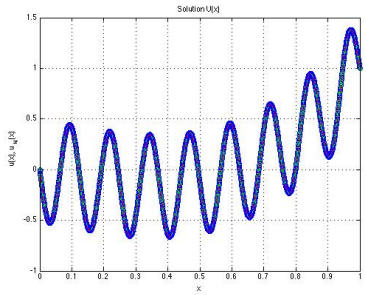

(a) solution(k=2)




(b) solution(k=4)

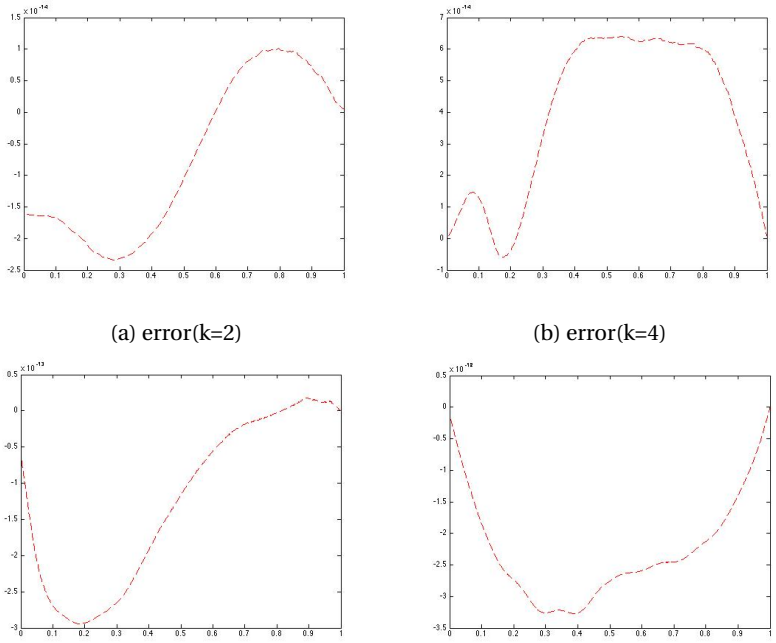



(c) solution(k=8)




(d) solution(k=16)

Figure 1: solution u(x) and $u_N(x)$



(a) error(k=2)

(b) error(k=4)

(c) error(k=8)

(d) error(k=16)

Figure 2: error between u(x) and $u_N(x)$

Keeping in mind the computation cost is important for engineers. Several technics were used in this project to save the memory and computation cost, such as binary search and sparse matrix.