

NUCLEAR ENGINEERING, UC BERKELEY

ME 280A Finite Element Analysis HOMEWORK 2: HIGHER ORDER ELEMENTS

Xin Wang

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1 INTRODUCTION

The objective of this project is to solve a 1-D differential equation using linear, quadratic and cubic elements.

$$\begin{aligned}
 \frac{d}{dx}(A_1(x)\frac{du}{dx}) &= f(x) \\
 f(x) &= 256\sin(\frac{3\pi kx}{4})\cos(16\pi x) \\
 A_1(x) &= 0.2 \\
 L &= 1 \\
 u(0) &= 0 \\
 A_1(L)\frac{du}{dx}(L) &= 1
 \end{aligned} \tag{1}$$

2 ANALYTICAL SOLUTION

This equation can be solved analytically after two integrations:

$$\begin{aligned}
 u(x) &= 5 \frac{4087\pi x + 1536\sqrt{2}x + \frac{137216\sin(\frac{61\pi x}{4})}{61\pi} - \frac{124928\sin(\frac{67\pi x}{4})}{67\pi}}{4087\pi} \\
 \frac{du}{dx} &= 5(1 + \frac{512(67\cos(\frac{61\pi x}{4}) - 61\cos(\frac{67\pi x}{4}))}{4087\pi} - \frac{512(67\cos(\frac{61\pi}{4}) - 61\cos(\frac{67\pi}{4}))}{4087\pi}) \tag{2}
 \end{aligned}$$

3 ERROR

The error is defined as

$$\begin{aligned}
 e^N &= \frac{\|u - u^N\|_{A_1(\Omega)}}{\|u\|_{A_1(\Omega)}} \\
 \|u\|_{A_1(\Omega)} &= \sqrt{\int_{\Omega} \frac{du}{dx} A_1 \frac{du}{dx} dx} = \sqrt{\int_{\Omega} A_1 (\frac{du}{dx})^2 dx} \\
 \|u - u^N\|_{A_1(\Omega)} &= \sqrt{\int_{\Omega} A_1 (\frac{d(u - u^N)}{dx})^2 dx}
 \end{aligned} \tag{3}$$

To compute the error numerically, we calculate the two following quantities element by element, and then assemble them to obtain the overall error:

$$\begin{aligned}
\|u\|_{A_1(\Omega)}^2 &= \int_{\Omega} A_1 \left(\frac{du}{dx} \right)^2 dx \\
\|u - u^N\|_{A_1(\Omega)}^2 &= \int_{\Omega} A_1 \left(\frac{d(u - u^N)}{dx} \right)^2 dx \\
&= A_1 \int_{\Omega} \left(\frac{du}{dx} - \frac{du_N}{dx} \right)^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{x_e}^{x_{e+1}} \left(\frac{du}{dx} - \frac{du_N}{dx} \right)^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{-1}^1 \left(\frac{du}{d\xi} - \frac{du_N}{d\xi} \right)^2 J d\xi
\end{aligned} \tag{4}$$

4 FINITE ELEMENT METHOD

The matrix system is computed using the same technics for linear, quadratic or cubic elements:

$$\begin{aligned}
K_{ij} &= \int_{\Omega} \frac{d}{dx} \phi_j(x) A_1 \frac{d}{dx} \phi_i(x) dx \\
R_i &= \int_{\Omega} f \phi_i(x) dx + \phi_i(x) t|_{\Gamma_t} \\
Ka &= R
\end{aligned} \tag{5}$$

The numerical computation is done over the corresponding master elements. Shape functions are defined for element with different orders (figure 1).

Linear shape functions are:

$$\begin{aligned}
\hat{\phi}_1 &= \frac{1 - \xi}{2} \\
\hat{\phi}_2 &= \frac{1 + \xi}{2}
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\frac{d\hat{\phi}_1}{d\xi} &= -\frac{1}{2} \\
\frac{d\hat{\phi}_2}{d\xi} &= \frac{1}{2}
\end{aligned} \tag{7}$$

Quadratic shape functions are:

$$\begin{aligned}\hat{\phi}_1 &= \frac{\xi(\xi-1)}{2} \\ \hat{\phi}_2 &= -(\xi-1)(\xi+1) \\ \hat{\phi}_3 &= \frac{\xi(\xi+1)}{2}\end{aligned}\tag{8}$$

and

$$\begin{aligned}\frac{d\hat{\phi}_1}{d\xi} &= \frac{2\xi-1}{2} \\ \frac{d\hat{\phi}_2}{d\xi} &= -2\xi \\ \frac{d\hat{\phi}_3}{d\xi} &= \frac{2\xi+1}{2}\end{aligned}\tag{9}$$

The cubic shape functions are:

$$\begin{aligned}\hat{\phi}_1 &= \frac{-(\xi-1)(3\xi+1)(3\xi-1)}{16} \\ \hat{\phi}_2 &= \frac{9(\xi-1)(\xi+1)(3\xi-1)}{16} \\ \hat{\phi}_3 &= \frac{-9(\xi-1)(\xi+1)(3\xi+1)}{16} \\ \hat{\phi}_4 &= \frac{(\xi+1)(3\xi+1)(3\xi-1)}{16}\end{aligned}\tag{10}$$

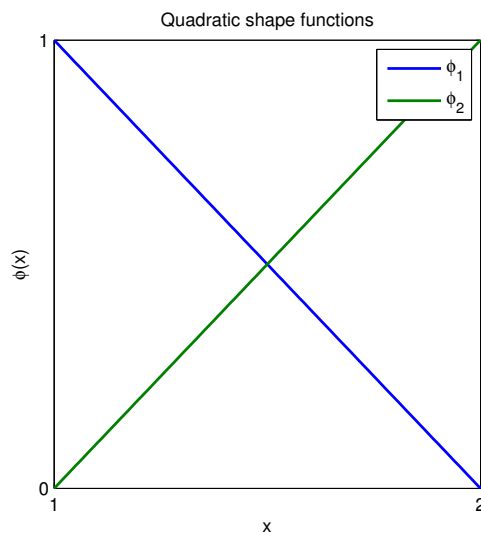
and

$$\begin{aligned}\frac{d\hat{\phi}_1}{d\xi} &= -\frac{1}{16}(27\xi^2-18\xi-1) \\ \frac{d\hat{\phi}_2}{d\xi} &= \frac{9}{16}(9\xi^2-2\xi-3) \\ \frac{d\hat{\phi}_3}{d\xi} &= -\frac{9}{16}(9\xi^2+2\xi-3) \\ \frac{d\hat{\phi}_4}{d\xi} &= \frac{1}{16}(27\xi^2+18\xi-1)\end{aligned}\tag{11}$$

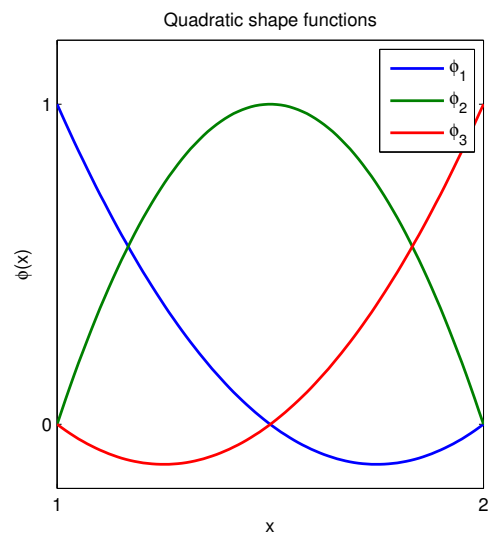
To mapping between master elements and the real spatial coordinates, we calculate x from ξ :

$$x = \sum \chi_i \hat{\phi}_i\tag{12}$$

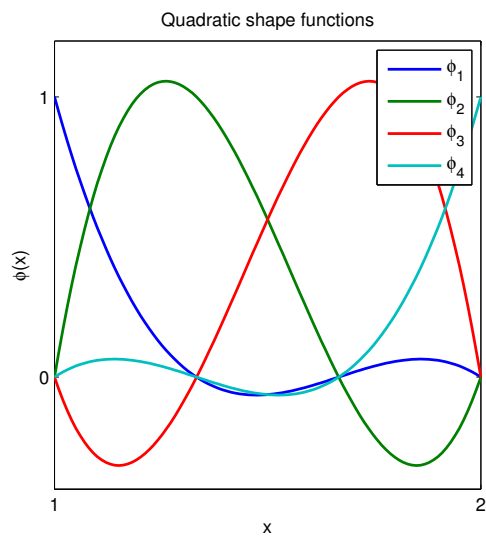
where χ_i are the coordinates of the nodes in the global element.



(a) linear shape function



(b) quadratic shape function



(c) cubic shape function

Figure 1: nodal shape functions on an element $[1, 2]$

5 POSTPROCESSING

Following the same procedures as in homework 1 would give the matrix A for the system. However, the postprocessing is more complicated for higher order element. In order to plot the result, we calculate the value of the numerical solution for 10 points in each elements. For any given ξ , the corresponding value for the solution is straightforward:

$$\begin{aligned} u(x) &= u(x(\xi)) = \sum_{i=1}^{P+1} a_i^e \hat{\phi}_i \\ x(\xi) &= \sum x^e \hat{\phi}(\xi) \end{aligned} \quad (13)$$

For each element, we can also calculate the derivative of u(x):

$$\frac{du}{dx} = \frac{d}{dx} \sum_{i=1}^{P+1} a_i \phi_i = \left(\frac{d}{d\xi} \sum_{i=1}^{P+1} a_i \hat{\phi}_i \right) \frac{d\xi}{dx} \quad (14)$$

6 RESULTS

Minimum number of elements and number of nodes to achieve to error criteria for different order of elements are tabulated bellow:

P	Ne_{opt}	N_{opt}
1	1347	1348
2	93	187
3	18	65

The numerical solution and the analytical solution are compared in figure 2. And the error between the analytical solution and the numerical solution at each elements are shown in figure 3. The error is not uniform throughout the domain of validation.

7 CONCLUSION

With the constant k in the right side of the equation getting larger, the frequency of $\sin(\pi kx/L)$ increases, so we need more nodes to capture the fluctuation of the solution. The number of elements needed for the same error criteria is almost proportional to k.

Three points gaussian quadrature is used in the project, giving exact integral values for polynomials with order 5 or less.

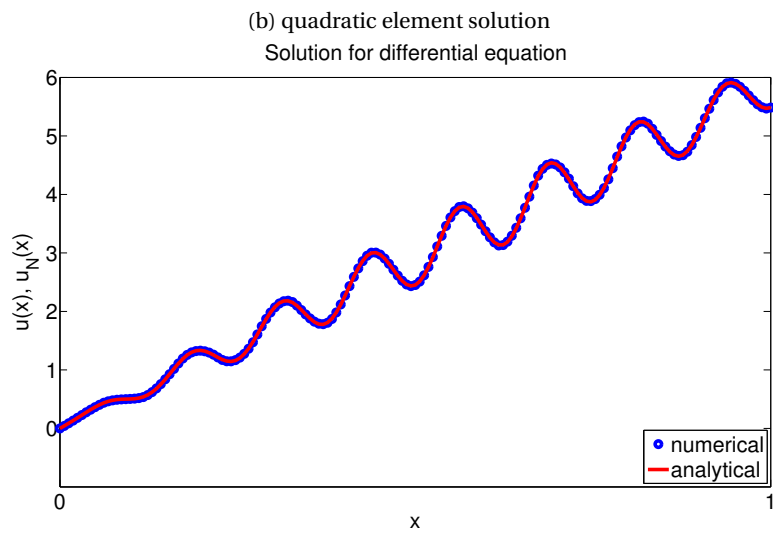
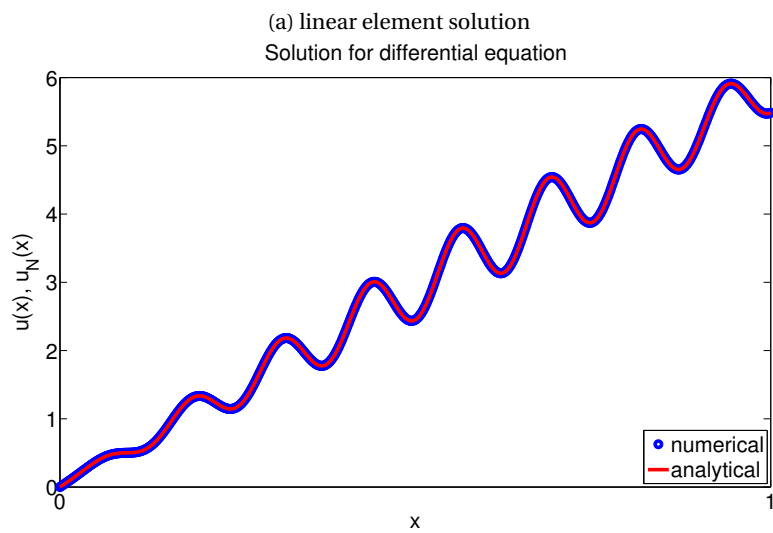
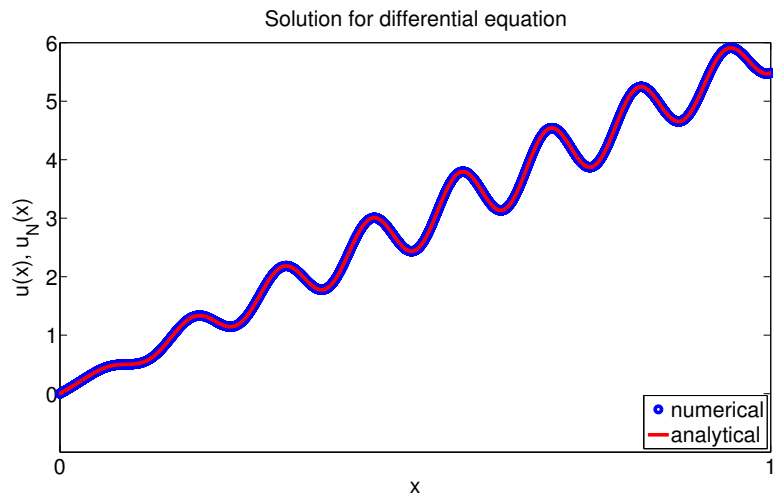
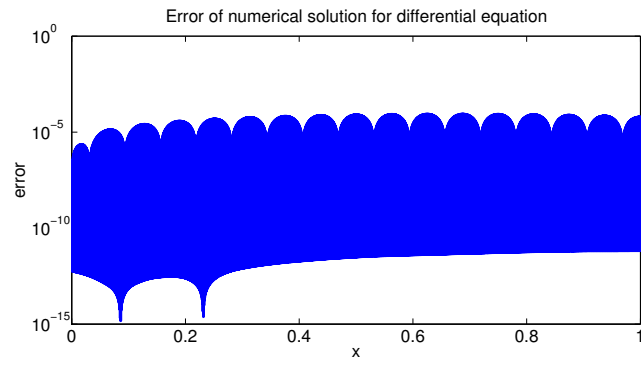
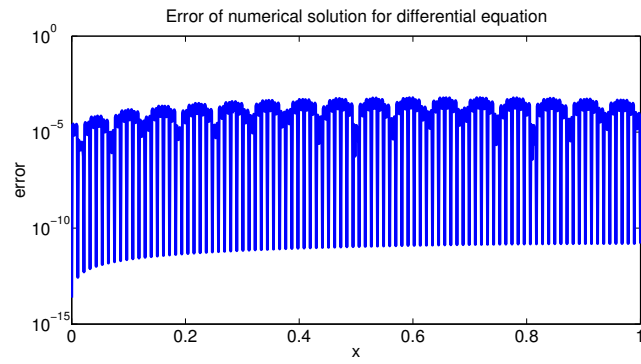


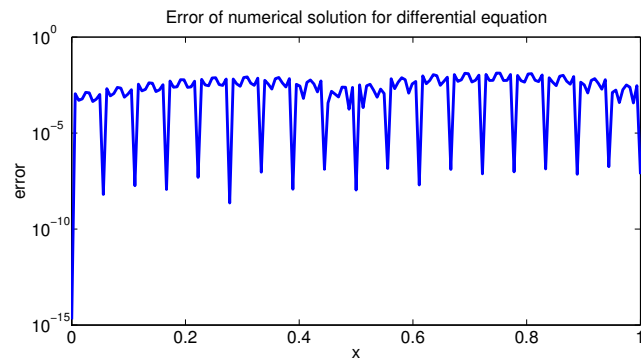
Figure 2: solution $u(x)$ and $u_N(x)$



(a) error(P=1)



(b) error(P=2)



(c) error(P=3)

Figure 3: error between $u(x)$ and $u_N(x)$

Keeping in mind the computation cost is important for engineers. Several technics were used in this project to save the memory and computation cost, such as binary search and sparse matrix.