NUCLEAR ENGINEERING, UC BERKELEY

ME 280A Finite Element Analysis HOMEWORK 2: HIGHER ORDER ELEMENTS

Xin Wang

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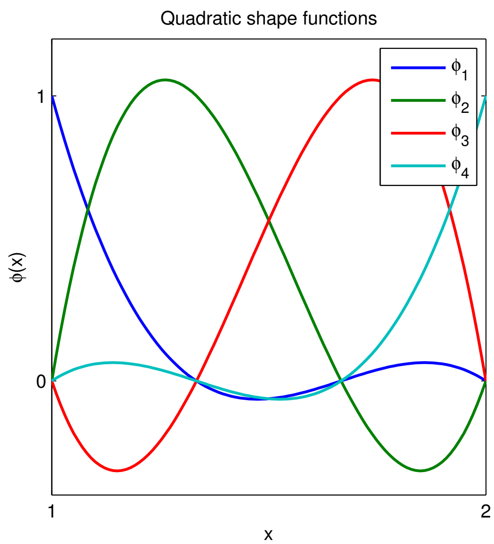


Figure 1: cubic shape function

1 Introduction

The objective of this project is to solve a 1-D differential equation using linear, quadrical and cubic elements.

$$\begin{aligned}
\frac{d}{dx}(A_1(x)\frac{du}{dx}) &= f(x) \\
f(x) &= 256sin(\frac{3\pi kx}{4})cos(16\pi x) \\
A_1(x) &= 0.2 \\
L &= 1 \\
u(0) &= =0 \\
A1(L)\frac{du}{dx}(L) &= 1
\end{aligned} \tag{1}$$

2 Analytical solution

The analytical solution of this equation is easily found after two integrations:

$$
\begin{aligned}
u(x) &= 5\frac{4087\pi x + 1536\sqrt{2}x + \frac{137216 sin(\frac{61\pi x}{4})}{61\pi} - \frac{124928 sin(\frac{67\pi x}{4}))}{67\pi}}{4087\pi} \\
\frac{du}{dx} &= 5(1 + \frac{512(67cos(\frac{61\pi x}{4}) - 61cos(\frac{67\pi x}{4}))}{4087\pi} - \frac{512(67cos(\frac{61\pi}{4}) - 61cos(\frac{67\pi}{4}))}{4087\pi} \qquad (2)
\end{aligned}
$$

3 Error

The error is defined as

$$
\begin{aligned}
e^N = \frac{\|u - u^N\|_{A_1(\Omega)}}{\|u\|_{A_1(\Omega)}} \\
\|u\|_{A_1(\Omega)} = \sqrt{\int_\Omega \frac{du}{dx} A_1 \frac{du}{dx} dx} = \sqrt{\int_\Omega A_1 (\frac{du}{dx})^2 dx} \\
\|u - u^N\|_{A_1(\Omega)} = \sqrt{\int_\Omega A_1 (\frac{d(u - u^N)}{dx})^2 dx}
\end{aligned} \tag{3}
$$

To compute the error numerically, we calculate the two following quantities element by element, and then assemble them to obtain the overall error:

$$
\begin{aligned}
\|u\|^2_{A_1(\Omega)} &= \int_\Omega A_1 (\frac{du}{dx})^2 dx \\
\|u - u^N\|^2_{A_1(\Omega)} &= \int_\Omega A_1 (\frac{d(u - u^N)}{dx})^2 dx \\
&= A_1 \int_\Omega (\frac{du}{dx} - \frac{du_N}{dx})^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{x_e}^{x_{e+1}} (\frac{du}{dx} - \frac{du_N}{dx})^2 dx \\
&= A_1 \sum_{e=1}^{N_e} \int_{x_e}^{x_{e+1}} (\frac{du}{dx} - \frac{a_{i+1} - a_i}{h_i})^2 dx
\end{aligned} \tag{4}
$$

4 Finite Element Method

The matrix system to solve stays the same for elements with different orders:

$$
\begin{aligned}
K_{ij} &= \int_{\Omega} \frac{d}{dx}\phi_j(x) A_1 \frac{d}{dx}\phi_i(x)\,dx \\
R_i &= \int_{\Omega} f\phi_i(x)\,dx + \phi_i(x)\,t\,|_{\Gamma t} \\
Ka &= R
\end{aligned} \tag{5}
$$

Linear shape functions are:

$$
\begin{aligned}
\hat{\phi}_1 &= \frac{1-\xi}{2} \\
\hat{\phi}_2 &= \frac{1+\xi}{2}
\end{aligned} \tag{6}
$$

and

$$
\begin{aligned}
\frac{d\hat{\phi}_1}{d\xi} &= -\frac{1}{2} \\
\frac{d\hat{\phi}_2}{d\xi} &= \frac{1}{2}
\end{aligned} \tag{7}
$$

Quadratic shape functions are:

$$
\begin{aligned}
\hat{\phi}_1 &= \frac{\xi(\xi-1)}{2} \\
\hat{\phi}_2 &= -(\xi-1)(\xi+1) \\
\hat{\phi}_3 &= \frac{\xi(\xi+1)}{2}
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
\frac{d\hat{\phi}_1}{d\xi} &= \frac{2\xi-1}{2} \\
\frac{d\hat{\phi}_2}{d\xi} &= -2\xi \\
\frac{d\hat{\phi}_3}{d\xi} &= \frac{2\xi+1}{2}
\end{aligned} \tag{9}
$$

The cubic shape functions are:

$$
\begin{aligned}
\hat{\phi}_1 &= \frac{-(\xi-1)(3\xi+1)(3\xi-1)}{16} \\
\hat{\phi}_2 &= \frac{9(\xi-1)(\xi+1)(3\xi-1)}{16} \\
\hat{\phi}_3 &= \frac{-9(\xi-1)(\xi+1)(3\xi+1)}{16} \\
\hat{\phi}_4 &= \frac{(\xi+1)(3\xi+1)(3\xi-1)}{16}
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
\frac{d\hat{\phi}_1}{d\xi} &= -\frac{1}{16}(27\xi^2-18\xi-1) \\
\frac{d\hat{\phi}_2}{d\xi} &= \frac{9}{16}(9\xi^2-2\xi-3) \\
\frac{d\hat{\phi}_3}{d\xi} &= -\frac{9}{16}(9\xi^2+2\xi-3) \\
\frac{d\hat{\phi}_4}{d\xi} &= \frac{1}{16}(27\xi^2+18\xi-1)
\end{aligned} \tag{11}
$$

The mapping between master elements and the real spatial coordinates, we calculate x from ξ:

$$
x = \sum \chi_i \hat{\phi}_i \tag{12}
$$

where χ_i are the coordinates of the nodes in the global element.

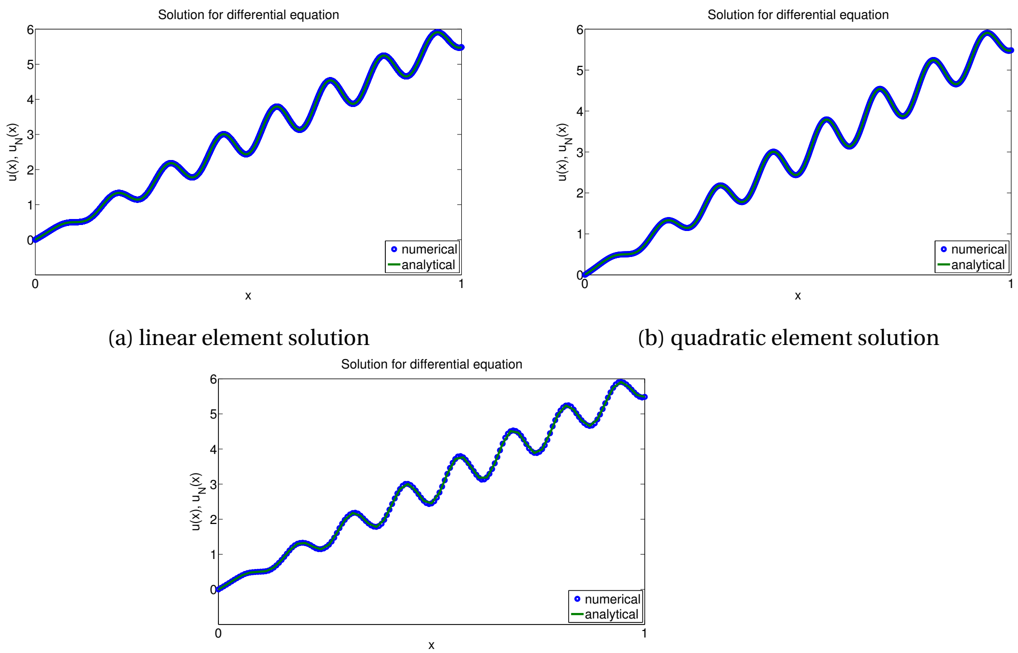
5 PROCEDURES

6 POSTPROCESSING

In order to plot the result, we calculate the value of the numerical solution for 10 points in each elements. For any given ξ, the corresponding value for the solution is straightforward:

$$
\begin{aligned}
u(x) &= u(x(\xi)) = \sum_{i=1}^{P+1} a_i^e \hat{\phi}_i \\
x(\xi) &= \sum x^e \hat{\phi}(\xi)
\end{aligned} \tag{13}
$$

For each element, we can calculate the derivative of u(x):



(a) linear element solution

(b) quadratic element solution

(c) cubic element solution

Figure 2: solution u(x) and $u_N(x)$

$$\frac{du}{dx} = \frac{d}{dx}\sum_{i=1}^{P+1} a_i\phi_i = (\frac{d}{d\xi}\sum_{i=1}^{P+1} a_i\hat{\phi}_i)\frac{d\xi}{dx} \tag{14}$$

7 RESULTS

Minimum number of element to achieve to error criteria for different order of elements are tabulated bellow:

P	Ne_{opt}
1	1347
2	93
3	18

The numerical solution and the analytical solution are compared in figure 2. And the error between the analytical solution and the numerical solution at each elements are shown in figure **??**. The error is not uniform throughout the domain of validation.

8 Conclusion

With the constant k in the right side of the equation getting larger, the frequency of $sin(\pi kx/L)$ increases, so we need more nodes to capture the fluctuation of the solution. The number of elements needed for the same error criteria is almost proportional to k.

Three points gaussian quadriture is used in the project, giving exact integral values for polynomials with order 5 or less.

Keeping in mind the computation cost is important for engineers. Several technics were used in this project to save the memory and computation cost, such as binary search and sparse matrix.