NUCLEAR ENGINEERING, UC BERKELEY

ME 280A Finite Element Analysis HOMEWORK 4: ERROR ESTIMATION and ADAPTIVE MESHING

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1 INTRODUCTION

The objective of this project is to solve two 1-D differential equation systems using linear elements.

$$
\begin{aligned}
\frac{d}{dx}(A_1(x)\frac{du}{dx}) &= f(x) \\
f(x) &= 90\pi^2 sin(3\pi x) sin(36\pi x^3) - (10 sin(3\pi x) + 5)... \\
(216\pi x cos(36\pi x^3) &- 11664\pi^2 x^4 sin(36\pi x^3)) - 6480\pi^2 x^2 cos(3\pi x) cos(36\pi x^3) \\
A_1(x) &= 1.0 \\
L &= 1 \\
u(0) &= 0 \\
u(L) &= 0 \\
u^{true} &= (10 sin(3\pi x) + 5) sin(36\pi x^3) \\
du/dx^{true} &= 6\pi(5 sin(36\pi x^3) cos(3\pi x) + 90x^2(2 sin(3\pi x) + 1) cos(36\pi x^3))
\end{aligned} \tag{1}
$$

and

$$
\begin{aligned}
\frac{d}{dx}(A_1(x)\frac{du}{dx}) &= f(x) \\
f(x) &= 256sin(\frac{3\pi x}{4})cos(16\pi x) \\
A_1(x) &= \begin{cases} 0.2 & x < 1/3 \\ 2.0 & x \geq 1/3 \end{cases} \\
L &= 1 \\
u(0) &= 0 \\
A1(L)\frac{du}{dx}(L) &= 1 \\
u^{true} &= \frac{512}{4087\pi A_1}(\frac{268sin(\frac{61\pi x}{4})}{61\pi} - \frac{244sin(\frac{67\pi x}{4})}{67\pi}) \\
+ & \begin{cases} (5+\frac{7680\sqrt{2}}{4087\pi})x, x < 1/3 \\ \frac{2304\sqrt{2}(8210-768\sqrt{3}+4087\pi)}{16703569\pi^2} + \frac{3}{2} + (\frac{1}{2} + \frac{768\sqrt{2}}{4087\pi})x, x \geq 1/3 \end{cases} \\
du/dx^{true} &= \frac{512}{4087\pi A_1}(67cos(\frac{61\pi x}{4}) - 61cos(\frac{67\pi x}{4})) \\
+ & \begin{cases} (5+\frac{7680\sqrt{2}}{4087\pi}), x < 1/3 \\ (\frac{1}{2} + \frac{768\sqrt{2}}{4087\pi}), x \geq 1/3 \end{cases}
\end{aligned} \tag{2}
$$

2 FINITE ELEMENT METHOD

2.1 WEAK FORM

3 ADAPTIVE MESHING

3.1 DATA STRUCTURE

The node coordinates are stored in a Nx1 table. And the connectivity talbe contains the node numbers for each element, i.e $ce = connectivity(e) = [ce(1), ce(2)]$ for linear element, where ce(1) and ce(2) are the number of the nodes at the element e.

The stiffness matrix K is no more a band matrix, but instead filled as:

$$\begin{aligned} K[ce(i), ce(j)] &= Ke(i, j) \\ i = 1,2; j &= 1,2 \end{aligned} \tag{3}$$

The load matrix is not filled in order anymore but still a 1xN array:

$$\begin{aligned} R(ce(i)) &= Re(i) \\ i &= 1,2 \end{aligned} \tag{4}$$

3.2 Adaptive meshing process

The process to solve the problem using adaptive meshing is:

- solve with an initial mesh, Ne = 16
- loop through the mesh and calculate the element error
  - fetch the node numbers in this element using connectivity table ce= [2, 5]
  - fetch xe and ae with ce
  - compute E_e with gaussian quadriture
- if the error is larger than the tolerance for an element, i.e. $E_e > tol_e$, then devide the element into 2
  - add the new node nodex(end+1) = (xe(1)+xe(2))/2
  - connectivity(e) = [ce(1), len(nodex)]
  - add a new element: connectivity(end+1) = [len(nodex), ce(2)]
  - changed element = true
- check if any element was modified, if so, resolve with the new mesh, otherwise done.

The error is calculated on every mesh element as:

$$
\begin{aligned}
E_I &= \frac{\frac{1}{h_I}\|u-u^N\|^2_{A_1(\Omega_I)}}{\frac{1}{L}\|u\|^2_{A_1(\Omega)}} \\
&= \frac{\frac{1}{h_I}\int_{\Omega_I} A_1(x)(\frac{du^{True}}{dx}-\frac{du}{dx})^2 dx}{\frac{1}{L}\int_{\Omega} A_1(x)(\frac{du^{True}}{dx})^2 dx}
\end{aligned}
\tag{5}
$$

Etol is defined as the overall tolerance devided by the number of elements.

4 Error Calculations

The error is defined as

$$
\begin{aligned}
e^N &= \frac{\|u-u^N\|_{A_1(\Omega)}}{\|u\|_{A_1(\Omega)}} \\
\|u\|_{A_1(\Omega)} = \sqrt{\int_\Omega \frac{du}{dx}A_1\frac{du}{dx}dx} &= \sqrt{\int_\Omega A_1(\frac{du}{dx})^2 dx} \\
\|u-u^N\|_{A_1(\Omega)} &= \sqrt{\int_\Omega A_1(\frac{d(u-u^N)}{dx})^2 dx}
\end{aligned}
\tag{6}
$$

To compute the error numerically, we calculate the two following quantities element by element, and then assemble them to obtain the overall error:

$$
\begin{aligned}
\|u\|^2_{A_1(x)(\Omega)} &= \int_\Omega A_1(x)(\frac{du}{dx})^2 dx \\
\|u-u^N\|^2_{A_1(x)(\Omega)} &= \int_\Omega A_1(x)(\frac{d(u-u^N)}{dx})^2 dx \\
&= \int_\Omega A_1(x)(\frac{du}{dx}-\frac{du_N}{dx})^2 dx \\
&= \sum_{e=1}^{N_e}\int_{x_e}^{x_{e+1}} A_1(x)(\frac{du}{dx}-\frac{du_N}{dx})^2 dx \\
&= \sum_{e=1}^{N_e}\int_{-1}^{1} A_1(x(\xi))(\frac{du}{d\xi}-\frac{du_N}{d\xi})^2 J d\xi
\end{aligned}
\tag{7}
$$

5 RESULTS

5.1 EQUALLY SPACED LINEAR ELEMENTS

For each problem, the minimum number of equally spaced linear elements needed in order to achieve $e^N \leq 0.05$ are 1466 and 382.

For the second problem, the left side of the domains was divided into 85 elements and the right side into 171 elements.

5.2 ADAPTIVE MESHING

Solving both adaptive meshing was also used to solve the equation.

6 CONCLUSION