Spectrum of orbit Jacobian operator $\mathcal{J}$ on Bloch states, with examples in Cat map

Xuanqi Wang

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Here I summarize my understanding of spectrum of $\mathcal{J}$, the orbit Jacobian operator, in detail, which would be useful to fully understand this result and see what can be done further.

This result mainly uses the notion of anti-symmetrized trace in Predrag's group theory. For a temporal periodic state of length n, its $\mathcal{J}$ is reduced to a $n \times n$ matrix $\mathcal{J}$, and the determinant of this matrix can be given by

$$\mathrm{Det}(\mathcal{J}) = \mathrm{Tr}_p A\mathcal{J} = \frac{1}{n}\sum_{m=1}^{n}(-1)^{m-1}(\mathrm{Tr}_{n-m}A\mathcal{J})\mathrm{Tr}\mathcal{J}^m \tag{1}$$

where A denote the anti-symmetrized tensor with $\mathrm{Tr}_0 AM = 1$. Then, we can recursively decompose $\mathrm{Tr}_i AM$ into sum of products of $\mathrm{tr}(M^n)$ for $n \leq i$. If we request $M = \mathrm{diag}(x_1, x_2, ..., x_n)$ is a diagonal matrix, then () exploits all possible terms in a homogenous polynomial $p(x_1, x_2, ..., x_n)$ of degree n. But we know that an extricate leaves only one last term. This expression of $\mathrm{Det}\mathcal{J}$ in a finite sum of traces is especially useful in the calculation of spectrum of $\mathcal{J}$.

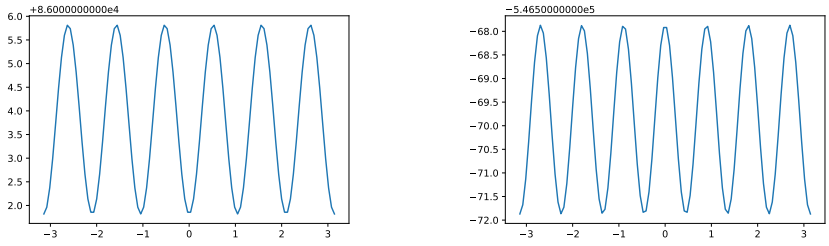
One-dimensional systems

For one-dimensional systems, we realize through Bloch theorem that since $r^n\mathcal{J} = \mathcal{J}$, we have $\mathcal{J}(e^{ikr}\psi_k^\alpha(r)) = \Lambda(k)^\alpha e^{ikr}\psi_k^\alpha(r)$ and the wave function $\psi_k^\alpha(r) = \psi_k^\alpha(r+n)$ has the same periodicity as underlying periodic state. In this case, one should aware that it is a wise choice to absorb all the phase into an operator $\Psi : \mathbb{R}^n \to \mathbb{R}^{\mathbb{Z}}$ such that $\Psi(\psi_k^\alpha(r)) = e^{ikr}\psi_k^\alpha(r)$ is extended from the primitive cell to all space with proper phase factor.

Now, to work our the spectrum of J, we still need to figure out how to put it into primitive cell so that it can be reduced to a matrix whose determinant is well-defined. To do so, we need to figure how to change order of $\mathcal{J}$ and Ψ on lattice. Write $\mathcal{J} = -r - r^{-1} + \mathbf{S}$, where $\mathbf{S}$ is a diagonal operator that commutes with Ψ, we get

$$\mathcal{J}\Psi = (-r - r^{-1} + \mathbf{S})\Psi = -\Psi r e^{ika} - \Psi r^{-1}e^{-ika} + \Phi\mathbf{S} = \Psi(k) \tag{2}$$

where $a = 1$ should be the lattice constant and (k) is the in primitive cell (i.e. matrix), with the shift operator r replaced by re^{ika}.



(a) (b)

Figure 1: Spectrum of $\mathcal{J}(k)$ plotted over $(-\pi, \pi)$ for an random period-6 (left) and period-7 orbit (right) for ϕ^4 theory. We can see all expected features from these figures, includes the number of peaks equal to period, even function, oscillation with amplitude = 2

The spectrum of is $\prod_\alpha \Lambda^\alpha(k) = (k)$ is reduced to the spectrum of a $n \times n$ matrix that suits the setup of trace calculation. As we should find immediately that $(r^m \mathrm{S}) \neq 0$ only for $m|n$, which means that diagonal matrix S has to be put into its original position. Since we have all terms homogenous degree n, it is impossible to shift S exactly n times, so we have to make sure that shift cancels in order to contribute anything in trace. This important observation then asserts that the only term in (k) that depends on k should come from $(re^{ika})^n = ne^{inka}$ and $(re^{-ika})^n = ne^{-inka}$. Then trace calculation gives

$$\mathrm{Det}\mathcal{J}(k) = \mathrm{Det}\mathcal{J}(0) - 4\mathrm{sin}^2(nk/2) \tag{3}$$

which is strikingly simple. Here we have to be aware that $\mathrm{Det}\mathcal{J}(0)$ can be negative, so upon taking the absolute value moment $sin^2(nk/2)$ term can change sign according to the sign of $\mathrm{Det}\mathcal{J}(0)$. In fig.1 I showed two spectra numerically calculated, and we can see that the prediction matches these calculations perfectly.

Higher dimensional theories, still to be worked out better

The natural next step is to generalize this result to higher dimension, and we will first try 2-dimensional square lattice. Similar to the case in one dimension, we can absorb the phase into some operator Φ and multiply $\psi_k^\alpha(r)$ by Φ to extend primitive cell to all space. I found it convenient to define Φ_1 and Φ_2 for two dimensions separately, and clearly $r_i\Phi_j$ commutes when $i \neq j$. Thus, for a $m \times n$ 2-dimensional periodic state, the spectrum of $\mathcal{J}$ can be found by push it into the flattened $mn \times mn$ primitive cell $\mathcal{J}$, where $r_1 = r \otimes \mathbf{1}$ and $r_2 = \mathbf{1} \otimes r$. We define the primitive cell $\mathcal{J}$ to be

$$\mathcal{J}(k) = -(r_1 e^{ik_1 a_1} + r_1^{-1} e^{-ik_1 a_1} + r_2 e^{ik_2 a_2} + r_2^{-1} e^{-ik_2 a_2}) + \mathrm{S} \tag{4}$$

We can still do the trace calculation, but this time it takes up power $m \times n$, and it is possible to complete multiple cycles in both direction. Then, the spectrum is now a trigonometric polynomial

$$\sum_{pm+nq\leq mn}[\sum_{\vec{n}\cdot\vec{\mu}=p}\sum_{\vec{m}\cdot\vec{\nu}=q}C(\mu,\nu)\times\prod_{i=1}^{n}\cos^{\mu_i}(imk_1)\prod_{j=1}^{m}\cos^{\nu_j}(jnk_2)] \tag{5}$$

where we define $\vec{m}=(1,2,..,m)$ and similarly $\vec{n}=(1,2,..,n)$, and we request all p,q,μ_i,ν_j to be non-negative. To reduce notation, it is more convenient to define $\langle\mu\rangle=\vec{n}\cdot\vec{\mu}$ and $\langle\nu\rangle=\vec{m}\cdot\vec{\nu}$.

To understand this formula, we have to recall the explanation of trace formula as a homogenous polynomial. Here, the independent variables are e^{imk_1}, e^{-imk_1}, e^{ink_2}, e^{-ink_2}, which form pair-wise cancellations. Due to symmetry, all of the k-dependent terms appear as $\cos(n_ik_i)$ for some direction i. And to summarize the condition for homogeneity in degrees, we separate the degree for each direction, as different directions can only possibly couple through multiplication, and their sum cannot exceed the volume of the primitive cell $m\times n$. The degree of each term $\cos(n_ik_i)$ is defined to be n_i, and in multiplication (either in the same direction or different directions), degrees add up. In **??** we label by $mp=m\langle\mu\rangle$ the degree of k_1 terms and $nq=n\langle\nu\rangle$ the degree of k_2 terms, and each is composed by product of different modes specified by vector μ,ν. This exponential decay in **??** is an exponential in the total degree of each term, and clearly the constant term, with degree zero, is the most dominant. I took the product to be upper-bounded by n and m respectively because the superposition of the fundamental modes up to these numbers (which is essentially V/n_i) would have reached the volume of the primitive cell.

Although this spectrum is much more complicated than that for one dimension, it still gives us good intuition. First of all, we can see that this spectrum is dominated by the constant part. Then, if we look at the next level of contribution in k_1 and k_2, we can see that the direction with the shorter period dominates its longer counterpart. Finally, this spectrum is an even function for both wave numbers, and the crossing term of k_1 and k_2 comes first with $k_1^2k_2^2$, the fourth order term. This means that in continuous limit, we should be able to decorelate perturbation in different directions. For even higher dimensions, we only need to introduce more sets of vector subscripts, and clearly all of the observations made above generalize to arbitrarily high dimensions.

To get a feeling for this 2D $\mathcal{J}$ spectrum, it is better to work out an example. We first compare the $[2\times2]_0$ state and $[2\times3]_0$ states of cat map with constant stretching. First, we list all possible modes for each case, which is not very hard (7 terms for $[2\times2]_0$ and 10 terms for $[2\times3]_0$)

$$\begin{aligned}&[2\times2]_0:\\&\qquad\cos(2k_1),\cos(2k_2),\\&\qquad\cos(2k_1)^2,\cos(2k_2)^2,\cos(4k_1),\cos(4k_2),\cos(2k_1)\cos(2k_2)\end{aligned} \tag{6}$$

$$
\begin{aligned}
&[2\times 3]_0: \\
&\qquad \cos(2k_1), \\
&\qquad \cos(3k_2), \\
&\qquad \cos(2k_1)^2, \cos(4k_1), \\
&\qquad \cos(2k_1)\cos(3k_2), \\
&\qquad \cos(6k_1), \cos(2k_1)\cos(4k_1), \cos(2k_1)^3, \cos(6k_2), \cos(3k_1)^2
\end{aligned}
\tag{7}
$$

By these two examples, we find that the structure of two-dimensional spectrum is given by high-frequency modes and their products, compared with the single fundamental frequency expression for one-dimensional spectrum. However, we have to notice that through Chebyshev polynomials (first kind), we can reduce all the high-frequency modes to a polynomial of fundamental modes. Therefore, we reduce Eq. to a polynomial of momentum in each direction $p_i = 2\sin^2(T_i k_i/2)$ as

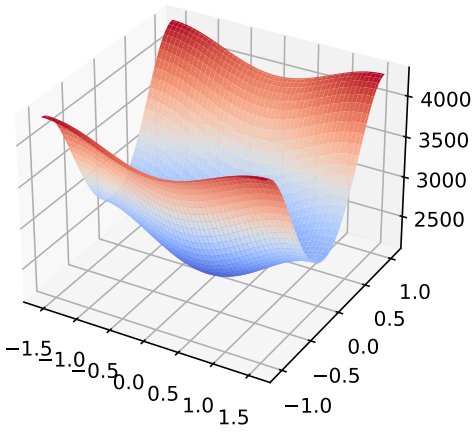
$$
\mathrm{Det}\mathcal{J}(k) = \mathrm{Det}\mathcal{J}(0) + \sum_{n_1 i + n_2 j \leq n_1 n_2} C_{i,j} p_1^i p_2^j \tag{8}
$$

This way it is easier for numerical calculation of coefficients, and I think this "momentum" form is more illustrative to compare to harmonics.

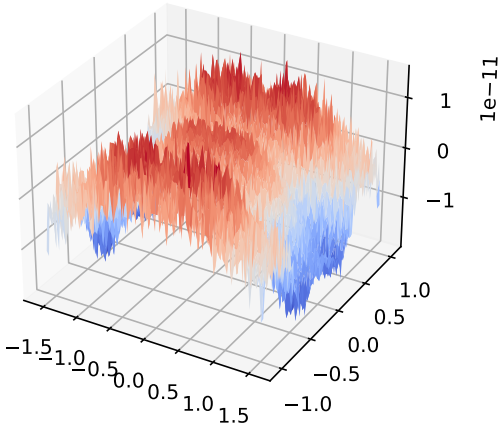
Next, we evaluate $C_{i,j}$ by undetermined coefficients and compare with numerical result. For simplicity we still evaluate $\mathcal{J}$ by Cat map, taking primitive cell to be $[2\times 3]_0$ and $s = 4.1$. Result is shown in fig.2. We can see that fig.2a shows expected trignometric dependence on $\vec{k}$, and fig.2b shows that the error is within range of numerical error (of order 10^{-11}).

Eq. has a very delicate structure due to homogeneity of Eq., which is displayed by the dependence of coefficients $C_{i,j}$ on s, the stretching parameter. Because only Cat map admits a constant stretching, we still take Cat map as an example to illustrate this result. The analytical expression of $C_{i,j}$, with primitive cell $[2\times 3]_0$, is listed below (also plotted in fig.3), when we define $p_i = 2\sin(n_i k_i/2)$

$$
\begin{aligned}
&[2\times 3]_0: \\
&\qquad C_{0,0} = s^6 - 9s^4 - 4s^3 + 12s^2, \\
&\qquad C_{1,0} = \frac{3}{2}s^4 - 6s^2 + 6s - 2, \\
&\qquad C_{0,1} = 2s^3 - 4, \\
&\qquad C_{2,0} = \frac{3}{2}s^2, \\
&\qquad C_{1,1} = -3s, \\
&\qquad C_{3,0} = \frac{1}{2}, C_{0,2} = 1
\end{aligned}
\tag{9}
$$



(a) Spectrum of $\mathcal{J}(k)$ evaluated by the method of undetermined coefficients



(b) Plotted difference of result in fig.2a with numerical

Figure 2: Spectrum calculation of $\mathcal{J}(k)$ for Cat map $[2 \times 3]_0$ $s = 4.1$ plotted over primitive cell

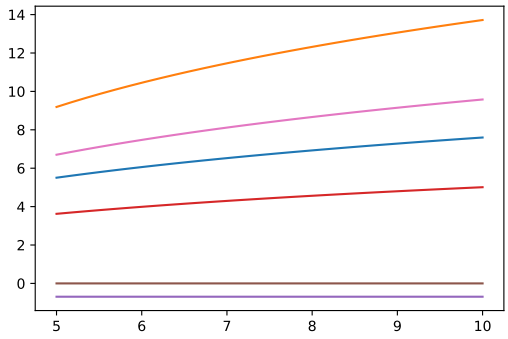


Figure 3: Plotted the dependence of $C_{i,j}$ (in log scale) on s over range $s \in [5, 10]$ for Cat map $[2 \times 3]_0$, with 7 coefficients in total

There are a few interesting observations we can make from this result. First, the highest-order term (i.e. the term with highest power of s) is exactly volume of primitive cell minus total degree on the term of all $C_{i,j}$, and this is due to homogeneity. In principle, we should only lower order terms that has the same parity as the highest-order term, but due to the recursive simplification we introduce some odd powers in Chebyshev polynomials, but this is just a minor detail. The important thing is that the spectrum of $\mathcal{J}$ is exponentially dominated by lower order terms in the anti-integrable limit, and is important when we evaluate the zeta function convergence in two dimensions.

Another interesting observation can be made from half-integer terms, as most of the terms in each $C_{i,j}$ admit integer coefficients. After calculating the coefficients for different primitive cells we found that all half-integer terms are related to the direction where periodicity is 2, and most coefficients in these directions (not all) have a half-integer term in its highest order. Currently we have not yet come to a conclusion to this question, as Chebyshev polynomials should be unaware of the change in independent variable.

To draw a conclusion to this, we found that Eq. is very efficient in evaluating spectrum of $\mathcal{J}$ by providing the structure based on $\vec{k}$ dependence in higher dimensions. However, due to some detours in recursive simplification, in most case the coefficients have to be numerically evaluated, and thus this result is semi-analytical or quasi-analytical in more than one dimension. The insight provided by Eq. enables us to write the spectrum of $\mathcal{J}$ into Eq., which is previously unthinkable. This result already demonstrated its power in evaluation of one-dimensional zeta function, but we believe that it will add more to two-dimensional case.