From fiber bundle: a differentiable point of view

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1 Introduction

In this paper we will talk about fiber bundle and fibration, and their connections with vector field and de Rham cohomology. It will involve a certain amount of differential geometry, like tangent bundles or differential forms, which has solid applications in dynamical systems and theoretical physics, but all these concepts will be carefully introduced. As the point of view in this paper is very geometrical, we shall not expect heavy algebra but rather some intuition about what we learned in this semester. The goal of this paper is to make connections of concepts we learned and some topic that we don't have time to talk about. So it will cover a wide range of topics but won't go very deep into most of them.

2 Fiber bundle and fibration

First we would like to introduce fiber bundle. A fiber bundle is a structure (E, B, π, F) such that π : E $\rightarrow$ B is a continuous, locally trivial surjection. Here E is called the total space, B is the base space, and F is, of course, fiber. What is important about this structure is the local triviality. It means that, for each x $\in$ B, there is a neighborhood U of x such that $\pi^{-1}(U)$ is homeomorphic to $U \times F$ in such a way that the homeomorphism ϕ satisfies $\phi \circ proj = \pi|_U$, where $proj$ means the natural projection from $U \times F$ to U. Use the language that we are more familiar, we have the commutative diagram in Fig.1. Such *phi* is usually called a fiber preserving homeomorphism. The most common visualization of fiber bundle is something like a brush. Imagine that the handle of the brush (in fact not the whole handle but the front part) as the base space and the hairs are fibers stick to the base space, then the brush makes a total space. If we define π as compressing each hair into its base, we make a fiber bundle. In fact, fiber

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\
U &
\end{tikzcd}$$

Figure 1: Commutative diagram for fiber bundle

bundle doesn't necessarily have to look like a brush. As we will see later, any covering space with the covering map can make a fiber bundle.

Why do we need fiber bundles in algebraic topology? Surely, it doesn't help if we are computing the fundamental groups. However, if we want to go deeper and define homotopy groups, like we what did for homology groups, we face the trouble that we have no long exact sequence, as we lost the short exact sequence $A \to X \to X/A$ from which we constructed our long exact sequence. Here fiber bundle comes in. We do have $F \to E \xrightarrow{\pi} B$ as our new short exact sequence. This short exact sequence has more homogeneity in the way that all $\pi^{-1}(x) \subset E$ (that we call fibers) are homeomophic (in fact, homoemorphic to F as the name suggests). Aren't these definitions familiar? Yes, if we do allow fibers to be discrete (after all why not), we can immediately see that covering space of a connected space is a fiber bundle with the covering map, and the fiber is decided by how many sheets we have for our covering space. We need the condition for connectedness, or otherwise we cannot guarantee that the number of sheets is constant throughout the space. For example, if we take the base space B to be the wedge of two circles and E to be the total space obtained by wedging three circles onto a central circle, shown in Fig.2, we make a fiber bundle with π the covering map here. Since our total space here is a two-fold covering space of base space, it's easy to see that fiber F is just any two element set. From this example we get an idea that fiber bundle is not necessarily obtained by inserting fibers to get something like a hairy toothbrush. It can be a much 'smoother' object.

Now let's give a more interesting example of fiber bundle called tangent bundle, whose fibers are not discrete. First we need to define 'tangent' in our context. Given a n-dimensional differentiable manifold M (which is a space that is locally homeomorphic to n-dimensional Euclidean space), we can define

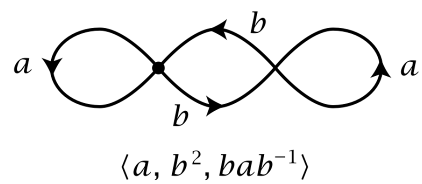

Figure 2: Two-fold covering space of wedge of two circles

the tangent space at each point $x \in M$ a tangent space T_x. As we won't go into much details about differential geometry, we shall use somehow informal definition of tangent space. Intuitively, tangent space can be viewed as a generalization of tangent plane of a surface. It is a real vector space that contains the possible directions in which one can tangentially pass through x (Wikipedia). Going along this direction, we define the tangent space as below. Take a local coordinate (a chart) in the neighborhood U of x $\varphi : U \to \mathbb{R}^n$ and two smooth curves $\gamma_1, \gamma_2 : (-1, 1) \to U$ such that $\gamma_1(0) = \gamma_2(0) = x$. We define the equivalence relation $\sim$, such that $\gamma_1 \sim \gamma_2$ iff the derivative of $\gamma_1 \circ \varphi$ and $\gamma_2 \circ \varphi$ at 0. The equivalence class of $\sim$ is then called a vector at x, and the collection of all vectors at x makes a vector space, which is the tangent space T_x. It's rather easy to show that T_x is a real vector space and such definition of vectors are independent of the choice of chart φ. Now the tangent bundle of M is $\bigcup_{x \in M} \{x\} \times T_x = \{(x, y) | x \in M, y \in T_x\}$. As it is naturally defined in the product form, to give tangent bundle a structure of fiber bundle, we just define π as the natural projection $proj$, and the fiber is just $\mathbb{R}^n$ (here we used the conclusion that the tangent space at each point of a n-dimensional differentiable manifold is homeomorphic to $\mathbb{R}^n$).

In fact, tangent bundle is just a specific example of vector bundles, for which we get a topological space X and attach to each point a vector space in a local trivial way. Vector bundles have a lot of interesting features and is closely related to K-theory, but we shall not go so deep into it now.

Now we have the definition for tangent bundle, and it's easy to define a vector field over a manifold. Let's define a section $\sigma : M \to TM$ such that $\pi \circ \sigma = id_M$. There are many question regarding whether section exists on a given manifold. Though I didn't understand this part very well, I guess it's similar to the kind of question whether there is non-zero vector field over S^n, and whose proof we saw in class.

$$
\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{h}_0"] \arrow[d, "incl"'] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & B
\end{tikzcd}
$$

Figure 3: Commutative diagram for fibration

Here we should talk more about algebra and start our discussion of fibration. The notion of fibration is a generalization of fiber bundle. Formally, fibration is a map $\pi : E \to B$ such that π has homotopy lifting property for any space X. Homotopy lifting property means that for any homotopy $h : X \times [0,1] \to B$, and each $\tilde{h_0} : X \to E$ lifting $h|_{X\times\{0\}}$, there is a unique homotopy $\tilde{h}$ so that $\tilde{h_0} = \tilde{h}|_{X\times\{0\}}$. This relation is shown by the commutative diagram in Fig.3. Thus, the map π in a fiber bundle is just a locally trivial fibration, and it's called trivial because the homeomorphism is induced by the natural projection. We say that fibration is a generalization of fiber bundle because it can have different fibers that are not homeomorphic to each other (a fiber at x is defined as $\pi^{-1}(x) \subset E$), and a fiber bundle does have homotopy lifting property. We shall restate the proof in the next section.

3 Homotopy lifting property

I looked through many different versions of this proof, but they all basically follow the way train of thought. Here we will state the proof for compact base space. The ultimate general version can be found in §2.7 of [Spanier 1966]. The idea is basically to find a finite open cover of B such that each of the open sets lies in the domain of a local trivialization $h_\alpha : p^{-1}U_\alpha \to U_\alpha \times F$.

Theorem: A fiber bundle $\pi : E \to B$ has homotopy lifting property with respect to all CW-pairs (X,A)

Proof: As we are dealing with CW-complexes, we will just prove the claim for a n-dimensional disk. Let $g : I^n \times I \to B$ be the homotopy we want to lift and let g_0 be the starting point. The next step is to subdivide $I^n \times I$ into cells that are mapped by G into the domains of local trivialization U_α and

$$
\begin{array}{l}
\cdots \xrightarrow{\partial_*} \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial_*} \pi_{n-1}(F) \to \\
\cdots \to \pi_1(F) \xrightarrow{i_*} \pi_1(E) \xrightarrow{p_*} \pi_1(B).
\end{array}
$$

Figure 4: Long exact chain form from fibration p at base point e_0 (F is the fiber at e_0)

construct the lift piecewisely. Since the process is the same for each U_α, let's just assume the all cells are mapped into a single U_α. Now, by induction on n, we can assume that $\tilde{f}_t$ is already constructed on $\partial I^n \simeq I^{n-1}$, so we have defined $G(I^n \times \{0\} \cup \partial I^{n-1} \times I) \in \pi^{-1}(U_\alpha)$, and the problem is reduced to define $\tilde{G}$ over $U_\alpha \times F$. The first component is given by definition, and the second component we can construct by the composition of $r : I^n \times I \to I^n \times \{0\} \cup \partial I^{n-1} \times I$, which is the retraction, and $g : I^n \times \{0\} \cup \partial I^{n-1} \times I \to F$ is what we figured out previously.

The proof is not very novice, as we have done this many times when we were studying the homotopy theory for CW-complexes. It almost looks like a standard process for extension. However, the interesting thing here is to consider when such property fails. This is specifically considered in Obstruction theory, which is related to characteristic class, and we will not go into this topic any further.

We take a detour to talk about fibration because it is very important in homotopy theory, because every continuous map $f : X \to Y$ is homotopic to a fibration, and for every fibration we can construct a long exact sequence in homotopy groups, as shown in Fig.4. In fact, this result is closely related to Hopf's calculation of homotopy groups of S^2 and S^3.

4 Forms and de Rham cohomology

With all these differentiable structures in mind, let's finally go back to algebraic topology and look at de Rham cohomology as an example of cohomology we learned in class. But first we need to define differential forms. Given a manifold M, we defined its tangent bundle TM. Remember that differential forms are dual to vector field, so we define the kth-form to be a section on the k-th exterior power of cotangent bundle of M (cotangent bundle is usually denoted by T^*M). Cotangent bundle is a vector bundle, and the vector space we attach at each point is the cotangent space, which is dual to its tangent space, and by kth exterior power, it means taking exterior product with itself k times. Immediately, we notice here that this exterior product makes all the difference. Remember that tangent and cotangent space are dual, and in manifolds they are always homeomorphic. The only difference is that tangent is covariant, and cotangent is contravariant. It is this contravariance that enables us to introduce Grassman algebra and define the natural exterior product on cotangent bundle. Exterior product is defined as $\wedge : \Omega^i(M) \times \Omega^j(M) \to \Omega^{i+j}(M)$ such that $x \wedge y = -y\ wedgex$ ($\Omega^n(M)$ is a collection of n forms over M, which will be explained later). So it's an anticommutative binary operator.

Intuitively, a one-form is just a real valued function over TM, and for higher order we can just see them as obtained by taking exterior product (or wedge product). Since exterior product is very natural, we can define the exterior derivative d such that $d\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \wedge x_\alpha$ (for $\omega = f dx_\alpha$). If ω is a n form, then $d\omega$ is a n+1 form. Usually, it's more convenient to denote by $d_n : \Omega^n(M) \to \Omega^{n+1}(M)$, so that we specify what forms it acts on. And the interesting thing, as we would expect from this reminiscent look, is that $d_n \circ d_{n-1} = 0$ for all $n \in \mathbb{N}$, which means that exterior derivatives are closed. (Proof should be presented in the next version). This means that $ker(d_n) \supset im(d_{n-1})$. So we are just one step from our exact sequence, and it is given by Poincare lemma: if the base space M is contractible, then all closed forms are exact (by exact, we means that it is the exterior derivative of another differential form). Not a big surprise, we will define de Rham cohomology for M (contractible) as $H^k_{dR}(M) = ker(d_{n-1})/im(d_n)$. But for a more generalized base space, we define de Rham cohomology by constructing equivalence classes. We define relation $\sim$ such that $\alpha \sim \beta$ iff $\alpha - \beta$ is exact. It's easy to check that $\sim$ induces an equivalence relation on the closed forms in $\Omega^k(M)$. And we define $H^k_{dR}(M)$ as the equivalence classes of $\sim$ in $\Omega^k(M)$.

By far we have finished our construction of de Rham cohomology. Now it's time to ask why we should care about it, since it is somehow too differential. However, de Rham theorem shows that de Rham cohomology is homomorphic to singular cohomology groups, and the exterior product on de Rham cohomology is analogous to cup product on singular cohomology, which gives us the ring structure that homology groups don't have. Thus, for me, I choose this as my topic for term paper because I want to give some intuition about the abstract structure that we always approach by algebraic methods.

5 Conclusion

In this paper we mainly take the differential point of view. In fact, it is very surprising to think about the fact that we can understand topology from calculus on manifolds. There are many deeper ideas that we cannot go into, like homotopy groups generated from fiber bundle and fibration, but I believe that this unique point of view does, at least, provide some important geometric insight into cohomology that would be helpful when we study the generalized cohomology theory later.

6 Reference

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