$\uparrow\uparrow + \downarrow\downarrow$: $\frac{1}{2}(C_A^\dagger C_A C_B^\dagger C_B - C_A C_B + C_A^\dagger C_B^\dagger + C_A C_A^\dagger C_B C_B^\dagger) = P_1$

$\uparrow\uparrow - \downarrow\downarrow$: $\frac{1}{2}(C_A^\dagger C_A C_B^\dagger C_B + C_A C_B - C_A^\dagger C_B^\dagger + C_A C_A^\dagger C_B C_B^\dagger) = P_2$

$\uparrow\downarrow + \downarrow\uparrow$: $\frac{1}{2}(C_A^\dagger C_A C_B C_B^\dagger - C_A C_B^\dagger + C_A^\dagger C_B + C_A C_A^\dagger C_B^\dagger C_B) = P_3$

$\uparrow\downarrow - \downarrow\uparrow$: $\frac{1}{2}(C_A^\dagger C_A C_B C_B^\dagger + C_A C_B^\dagger - C_A^\dagger C_B + C_A C_A^\dagger C_B^\dagger C_B) = P_4$

It's easy to verify that $P_1 + P_2 + P_3 + P_4 = \{C_A, C_A^\dagger\} \cdot \{C_B, C_B^\dagger\} = 1$ and $P_i^2 = 1$ $(i = 1,2,3,4)$

Define
$$\begin{cases} |\psi_1\rangle = \frac{1}{\sqrt{2}}(C_A^\dagger C_B^\dagger + 1)|vac\rangle \\ |\psi_2\rangle = \frac{1}{\sqrt{2}}(C_A^\dagger C_B^\dagger - 1)|vac\rangle \\ |\psi_3\rangle = \frac{1}{\sqrt{2}}(C_A^\dagger + C_B^\dagger)|vac\rangle \\ |\psi_4\rangle = \frac{1}{\sqrt{2}}(C_A^\dagger - C_B^\dagger)|vac\rangle \end{cases}$$

We want to verify that $P_i|\psi_j\rangle = \delta_{ij}|\psi_j\rangle$. In fact we will just do it for P_1 and the rest is similar.

$$\begin{aligned} P_1|\psi_1\rangle &= \frac{1}{2\sqrt{2}}(C_A^\dagger C_A C_B^\dagger C_B - C_A C_B + C_A^\dagger C_B^\dagger + C_A C_A^\dagger C_B C_B^\dagger)(C_A^\dagger C_B^\dagger + 1)|vac\rangle \\ &= \frac{1}{2\sqrt{2}}[(C_A^\dagger C_A C_B^\dagger C_B C_A^\dagger C_B^\dagger - C_A C_B C_A^\dagger C_B^\dagger + C_A^\dagger C_B^\dagger C_A^\dagger C_B^\dagger + C_A C_A^\dagger C_B C_B^\dagger C_A^\dagger C_B^\dagger)|vac\rangle \\ &\quad + (C_A^\dagger C_A C_B^\dagger C_B - C_A C_B + C_A^\dagger C_B^\dagger + C_A C_A^\dagger C_B C_B^\dagger)|vac\rangle)] \\ &= \frac{1}{2\sqrt{2}}(C_A^\dagger C_B^\dagger C_A C_A^\dagger C_B C_B^\dagger|vac\rangle + C_A C_A^\dagger C_B C_B^\dagger|vac\rangle + C_A^\dagger C_B^\dagger|vac\rangle + C_A C_A^\dagger C_B C_B^\dagger|vac\rangle) \\ &= \frac{1}{2\sqrt{2}}(2C_A^\dagger C_B^\dagger|vac\rangle + 2|vac\rangle) \\ &= |\psi_1\rangle \end{aligned}$$

where in the second step we used $C_A^\dagger C_B^\dagger = -C_B^\dagger C_A^\dagger$, $C_B|vac\rangle = 0$, $C_A^\dagger C_A^\dagger = C_B^\dagger C_B^\dagger = 0$, and in the third step we used $C_A C_A^\dagger|vac\rangle = C_B C_B^\dagger|vac\rangle = |vac\rangle$

Similarly, we calculate $P_1|\psi_2\rangle$

$$
\begin{aligned}
P_1|\Psi_2\rangle &= \frac{1}{2\sqrt{2}}\left(C_A^\dagger C_A C_B^\dagger C_B - C_A C_B + C_A^\dagger C_B^\dagger + C_A C_A^\dagger C_B C_B^\dagger\right)\left(C_A^\dagger C_B^\dagger - 1\right)|vac\rangle \\
&= \frac{1}{2\sqrt{2}}\left(C_A^\dagger C_A C_B^\dagger C_B C_A^\dagger C_B^\dagger - C_A C_B C_A^\dagger C_B^\dagger - C_A^\dagger C_B^\dagger - C_A C_A^\dagger C_B C_B^\dagger\right)|vac\rangle \\
&= \frac{1}{2\sqrt{2}}\left(C_A^\dagger C_B^\dagger |vac\rangle + |vac\rangle - C_A^\dagger C_B^\dagger |vac\rangle - |vac\rangle\right) \\
&= 0
\end{aligned}
$$

For $|\Psi_3\rangle$ and $|\Psi_4\rangle$, we have all terms in P_1 identically zero, so $P_1|\Psi_3\rangle = P_1|\Psi_4\rangle = 0$.

The verification for P_i $(i=2,3,4)$ is similar, we just apply the anticommutation relation for fermions to reduce results.

What we need to calculate with these projection operators are

$\langle\Psi_0|P_1^\dagger C_m^\dagger C_n P_1|\Psi_0\rangle$ and $\langle\Psi_0|P_1^\dagger P_1|\Psi_0\rangle$ $(m,n \neq A,B)$, and

using anti-commutation relation we find that $P_1^\dagger C_m^\dagger C_n P_1 = C_m^\dagger C_n P_1^\dagger P_1$, thus a simplification of $P_1^\dagger P_1$ is necessary.

$$
\begin{aligned}
P_1^\dagger P_1 &= \frac{1}{4}\left(C_B^\dagger C_B C_A^\dagger C_A - C_B^\dagger C_A^\dagger + C_B C_A + C_B C_B^\dagger C_A C_A^\dagger\right)\left(C_A^\dagger C_A C_B^\dagger C_B - C_A C_B + C_A^\dagger C_B^\dagger + C_A C_A^\dagger C_B C_B^\dagger\right) \\
&= \frac{1}{4}\Big[\left(C_A^\dagger C_A C_B^\dagger C_B - 0 - C_B^\dagger C_A^\dagger + 0\right) - \left(0 - C_B^\dagger C_B C_A^\dagger C_A + 0 - C_B^\dagger C_A^\dagger\right) \\
&\qquad + \left(-C_B C_A - 0 + C_B C_B^\dagger C_A C_A^\dagger + 0\right) + \left(0 + C_B C_A + 0 + C_A C_A^\dagger C_B C_B^\dagger\right)\Big] \\
&= \frac{1}{2}\left(C_A^\dagger C_A C_B^\dagger C_B - C_A C_B + C_A^\dagger C_B^\dagger + C_A C_A^\dagger C_B C_B^\dagger\right) = P_1
\end{aligned}
$$

Similarly $P_i^\dagger P_i = P_i$ for all i (as this is an orthonormal basis)

All of these results are highly symmetric, and the main challenge is to evalute $\langle c_A^\dagger c_A c_B^\dagger c_B c_m^\dagger c_n \rangle$, which is essentially high-order correlation functions.

$$\langle c_A^\dagger c_A c_B^\dagger c_B \rangle = \langle vac | \prod_{i\in occ} \gamma_i (U_{iA}^\dagger \gamma_i^\dagger)(U_{Aj}\gamma_j)(U_{kB}^\dagger \gamma_k^\dagger)(U_{Bl}\gamma_l) \prod_{i\in occ} \gamma_i^\dagger | vac \rangle$$

Clearly for any term to be non-zero, we need $\{i,k\} = \{j,l\}$

$\Rightarrow \langle c_A^\dagger c_A c_B^\dagger c_B \rangle$

$$= \langle vac | \prod_{i\in occ} \gamma_i \left(\sum_{j,k} U_{jA}^\dagger \gamma_j^\dagger U_{Aj}\gamma_j U_{kB}^\dagger \gamma_k^\dagger U_{Bk}\gamma_k + \sum_{j,k} U_{jA}^\dagger \gamma_j^\dagger U_{Ak}\gamma_k U_{kB}^\dagger \gamma_k U_{Bj}\gamma_j \right) \prod_{i\in occ} \gamma_i^\dagger | vac \rangle$$

To evaluate this, we need to understand $\langle G | \sum_{ij} \gamma_i^\dagger \gamma_j | G \rangle = \sum_{i\in occ} \langle G | \delta_{ij} | G \rangle$ and $\langle G | \sum_{ij} \gamma_i \gamma_j^\dagger | G \rangle = \sum_{i\notin occ} \langle G | \delta_{ij} | G \rangle$, which is the differerence between create-annihilate and annihilate-create.

Define $W_{\alpha\beta} = \sum_{i\in occ} U_{j\alpha}^\dagger U_{\beta j}$, $W'_{\alpha\beta} = \sum_{i\notin occ} U_{j\alpha}^\dagger U_{\beta j}$, then $W + W' = U^\dagger U = I$

and $\langle c_A^\dagger c_A c_B^\dagger c_B \rangle = W_{AA} \cdot W_{BB} + W_{AB} \cdot W'_{BA} = \langle c_A^\dagger c_A \rangle \langle c_B^\dagger c_B \rangle - |\langle c_A^\dagger c_B \rangle|^2$

Similarly, for $\langle c_A^\dagger c_A c_B^\dagger c_B c_m^\dagger c_n \rangle$, we need $\{i_1, i_2, i_3\} = \{j_1, j_2, j_3\}$, so there will be $3! = 6$ terms. This would be hard to calculate by hand, but easy with any script.

$$\begin{aligned}\langle G | P_1^\dagger c_m^\dagger c_n P_1 | G \rangle = \tfrac{1}{2} [& (W_{AA} W_{BB} W_{mn} + W_{AB} W'_{BA} W_{mn} + W_{AA} W_{Bn} W'_{mB} \\ & + W_{An} W'_{BA} W'_{mB} + W_{An} W_{BB} W'_{mA} + W_{AB} W_{Bn} W'_{AA}) \\ & + (W'_{AA} W'_{BB} W_{mn} + W_{AB} W'_{BA} W_{mn} + W'_{AA} W_{Bn} W'_{mB} \\ & + W_{An} W'_{BA} W'_{mB} + W_{An} W'_{BB} W'_{mA} + W_{AB} W_{Bn} W'_{mA})]\end{aligned}$$

If we prepare states with $|G\rangle = \frac{1}{2}(c_1^\dagger - c_2^\dagger)(c_3^\dagger - c_4^\dagger)|vac\rangle$

Before measurement, correlation matrix is
$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & & 0 \\ \frac{1}{2} & \frac{1}{2} & & \\ & & \frac{1}{2} & \frac{1}{2} \\ 0 & & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

After measurement, correlation matrix is
$$\begin{pmatrix} \frac{1}{2} & 0 & & 0 \\ 0 & \frac{1}{2} & & \\ & & \frac{1}{2} & 0 \\ 0 & & 0 & \frac{1}{2} \end{pmatrix}$$

The problem is that if we take submatrices C_{13} or C_{24} to be $C_{13} = \begin{pmatrix} C_{11} & C_{13} \\ C_{31} & C_{33} \end{pmatrix}$ $C_{24} = \begin{pmatrix} C_{22} & C_{24} \\ C_{42} & C_{44} \end{pmatrix}$, it doesn't change at all.