Čech Cohomology and its applications

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1 Introduction

Čech cohomology is a tool in algebraic topology that provides a way to study the properties of topological spaces using algebraic methods. It is particularly useful in the study of sheaf cohomology and complex manifolds.

2 Definition

Let X be a topological space and $\mathcal{U} = \{U_i\}$ be an open cover of X. For a sheaf $\mathcal{F}$ on X, we give the definition of Čech cohomology, starting with simplices and chain complex.

2.1 q-simplex

A q-simplex σ of $\mathcal{U}$ is an ordered collection of q+1 sets from $\mathcal{U}$ such that $\bigcap U_i \neq \varnothing$, and we denote by $|\sigma| = \bigcap U_i$ as the intersection (which is often called the support of σ).

Naturally follows from the definition of simplices, we define a sequence of free abelian groups C_q, which is spanned by q-simplices with formal sum as addition.

2.2 Boundary map

To establish a chain complex, we also need to define the boundary map $\partial : C_q \to C_{q-1}$. We shall start by defining the partial boundary map on a q-simplex

$$\partial_j \sigma = (U_i)_{i \in \{0,1,\ldots,q\} \setminus j} \tag{1}$$

which just excludes the jth element from the original ordered collection. With this notation, we can define the boundary map as

$$\partial \sigma = \sum_j (-1)^{j+1} \partial_j \sigma \tag{2}$$

Now, to check this is indeed a boundary map, we need $\partial^2 = 0$, and it is straight forward once we notice that $\partial_i\partial_j = \partial_j\partial_{i+1}$ for $j < i$ and $\partial_i\partial_{j+1} = \partial_j\partial_i$ for $j > i$.

$$\begin{aligned}\partial^2\sigma &= \sum_i (-1)^{i+1}\partial_i(\sum_{j\neq i}(-1)^{j+1}\partial_j\sigma) \\ &= \sum_i\sum_{j<i}(-1)^{i+j+1}(\partial_j\partial_i\sigma - \partial_i\partial_j\sigma) \\ &= 0\end{aligned} \tag{3}$$

In the above equation, sign changes because we have either $i < j$ or $i > j$, and without loss of generality, we can assume that $i < j$. In this case, after deleting U_j, the position of U_i is unchanged. However, when we do this reverse order, deleting U_i first, then the order of U_j will decrease by one, which results in the change of sign.

Indeed we have a chain complex (C_q, ∂) of the open cover $\mathcal{U}$, and next we construct the cochain (as its dual) from it.

2.3 q-cochain

A q-cochain of $\mathcal{U}$ with coefficients in $\mathcal{F}$ is a map which associates with each q-simplex σ an element of $\mathcal{F}(|\sigma|)$, and we denote the set of all q-cochains of $\mathcal{U}$ with coefficients in $\mathcal{F}$ by $C^q(\mathcal{U},\mathcal{F})$. $C^q(\mathcal{U},\mathcal{F})$ is an abelian group by pointwise addition.

2.4 Coboundary map

Now we define the coboundary map $\delta_q : C^q(\mathcal{U},\mathcal{F}) \to C^{q+1}(\mathcal{U},\mathcal{F})$ such that for each q-cochain $f \in C^q(\mathcal{U},\mathcal{F})$

$$\delta_q f(\sigma) = \sum_j (-1)^j res^{|\partial_j\sigma|}_{|\sigma|} f(\partial_j\sigma) \tag{4}$$

here we notice that $|\sigma| = |\partial_j\sigma| \bigcap U_j \subseteq |\partial_j\sigma|$ reversed the containment relation of $|\sigma|$ and $|\partial_j\sigma|$, and the restriction map is well defined. The proof that $\delta_q \circ \delta_{q-1} = 0$ is essentially the same as previous proof for boundary map, utilizing the fact that partial boundary commutes

$$\begin{aligned}\delta_q \circ \delta_{q-1}(\sigma) &= \sum_i\sum_j(-1)^{i+j} res^{|\partial_i\partial_j\sigma|}_{|\partial_j\sigma|} \circ res^{|\partial_j\sigma|}_{|\sigma|} f(\partial_i\partial_j\sigma) \\ &= \sum_i\sum_{j<i}(-1)^{i+j} res^{|\partial_i\partial_j\sigma|}_{|\sigma|}(f(\partial_i\partial_j\sigma) - f(\partial_j\partial_i\sigma)) \\ &= 0\end{aligned} \tag{5}$$

Here we used the property of restriction map of sheaf (actually presheaf suffices)

2.5 Cohomology

We define, just as in singular cohomology, the qth cohomology group of $\mathcal{U}$ with coefficient in $\mathcal{F}$ as

$$\check{C}^q(\mathcal{U}, \mathcal{F}) := ker(\delta_q)/Im(\delta_{q-1}) \tag{6}$$

From the above definition, we have

- The 0-th Čech cohomology group, denoted as $\check{C}^0(\mathcal{U}, \mathcal{F})$, is the set of global sections of $\mathcal{F}$ that are consistent on the intersections of the open sets in $\mathcal{U}$ (this is important and will be proved later).
- For $n > 0$, the n-th Čech cohomology group, denoted $\check{C}^n(\mathcal{U}, \mathcal{F})$, measures the obstruction to extending local sections of $\mathcal{F}$ over the n-fold intersections of the open sets in $\mathcal{U}$.

We should emphasize that this cohomology group $\check{C}^q$ depends on the open cover $\mathcal{U}$ of our topological space X, and all open covers are partially ordered by refinement. Therefore, we define Čech cohomology of X as the direct limit

$$\check{C}(X, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{C}(\mathcal{U}, \mathcal{F}) \tag{7}$$

We will use $\check{C}^j(X, \mathcal{F})$ to denote the jth Čech cohomology group.

2.6 With sheaf cohomology

Generally, there is a natural homomorphism between Čech cohomology groups and sheaf cohomology groups $\check{C}^j(X, E) \to H^j(X, E)$, which is isomorphism for $j \leq 1$. When the underlying topological space X is paracompact Hausdorff space, this homomorphism is an isomorphism.

More often, Čech cohomology is viewed as an approximation of sheaf cohomology, because Čech cohomology is computed only by considering the sections on finite intersection of open sets from $\mathcal{U}$. When every finite intersection of open sets in $\mathcal{U}$ has trivial Čech cohomology group for $j > 1$, then there exists a natural isomorphism between $\check{C}^j(\mathcal{U}, \mathcal{F})$ and the sheaf cohomology groups of $\mathcal{F}$ for every j.

2.7 With other cohomology theories

To connect Čech coohomology with singular cohomology and de Rham cohomology, we need to first extend the definition of Čech cohomology with coefficient in a set A. This can be done be define $\mathcal{F}_A$ to be the constant sheaf, whose stalk $\mathcal{F}_x = A$ for a $x \in X$. Then, for all CW-complex X, Čech cohomology group $\check{C}^j(X, \mathcal{F}_A)$ is naturally isomorphic to singular cohomology group $H^j(X, A)$ for all j.

Its connection with de Rham cohomology is more subtle. Firstly, if the underlying topological space X is a differentiable manifold, then $\check{C}^j(X, \mathbb{R})$ is isomorphic to de Rham cohomology group. More importantly, if $\mathcal{U}$ is a good cover of X (all $U_i \in \mathcal{U}$ are contractible, and so are all the finite intersections of U_i's), then $\check{C}^j(\mathcal{U}, \mathbb{R})$ is isomorphic to de Rham cohomology. This property make Čech cohomology very useful in the field of theoretical physics (especially in quantum field theory). We will expand this discussion in the application section.

3 Applications of Čech Cohomology

Čech cohomology has applications in various areas of mathematics and physics, including:

- The study of complex manifolds and vector bundles.
- Algebraic geometry, particularly in the study of line bundles and divisors.
- Topological aspects of quantum field theory.

Here we will focus on the application of Čech cohomology on quantum field theory. To be more specific, we will see how Čech cohomology can be used to explain the quantization of so-called coupling constant in quantum field theory.

3.1 Basic quantum field theory and its difficulty

We won't go into much details about the physics of quantum field theory. All the necessity to understand this part is to know that quantum field theory is an extension of classical physics, and the underlying spacetime is a differentiable manifold. One important quantity in this extension is Lagrangian, which can be imagined as modification of energy, which is given by

$$\mathcal{L} = \frac{1}{2}(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + A_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \tag{8}$$

The first two quantities are purely physical, which are generalizations of kinetic energy and potential energy in non-Euclidean space. The last term, which can be seen as a line integral along the particle trajectory, is in fact a topological term that concerns with the topology of the underlying space. A_μ is what called a vector potential in physics, and here we simply regard it as a non-singular vector field whose divergence has zero curl. Additionally, it has to be compatible with gauge transformation $\psi_{\alpha\beta}$ such that

$$d\psi_{\alpha\beta} = A_\alpha - A_\beta \tag{9}$$

on the intersection $U_\alpha \bigcap U_\beta$, where $\psi_{\alpha\beta}$ is a one-cocycle of Čech Cohomology with coefficients in one forms (denote by Ω^1 the constant sheaf of one forms), and d the exterior derivative in De Rahm Cohomology.

In classical mechanics, everything is defined in $\mathbb{R}^3$, but when we want to generalize physical theories to manifolds, the problem of vector potential arises.

$$
\begin{array}{cc|cccc}
\Omega^3 & 0 & 0 & & & \\
\Omega^2 & F & \{F_\alpha\} & 0 & & \\
\Omega^1 & & \{A_\alpha\} & \delta\{A\}=\{d\psi\} & 0 & \\
\Omega^0 & & & \{\psi_{\alpha\beta}\} & \{c_{\alpha\beta\gamma}\} & 0 \\
\hline
d & \uparrow & & & \{c_{\alpha\beta\gamma}\} & 0 \\
\delta & \rightarrow & C^0 & C^1 & C^2 & C^3
\end{array}
$$

Figure 1: Tic-tac-toe borad for Čech Cohomology and De Rahm Cohomology

For example, in S^2 we cannot define a non-singular vector field, thus it's impossible to define a vector potential on it. The solution is usually to find some alternative vector fields that is locally non-singular and piece them together, and this is where Čech cohomology is useful.

3.2 Čech Cohomology in Quantum Field Theory

Introduction of Čech Cohomology is to generalize this final term properly. The main result can be summarized very well by Fig.1. In this figure, the subscript is used to distinguish the level of Čech Cohomology group (which can be naively thought that $\check{C}^q(\mathcal{U}, \mathcal{F})$ is defined on the intersection of $q+1$ components of $\mathcal{U}$).

To explain in words, the coboundary operator δ of Čech cohomology takes cohomology group $\check{C}^p(\mathcal{U}, \Omega^q)$ horizontally to $\check{C}^{p+1}(\mathcal{U}, \Omega^q)$, while coboundary operator d of De Rahm Cohomology takes $\check{C}^p(\mathcal{U}, \Omega^q)$ vertically to $\check{C}^p(\mathcal{U}, \Omega^{q+1})$. Thus, to fully specify the generalized vector potential A_α, one just need to specify its exterior derivative $F_\alpha = dA_\alpha$ as a zero cocycle of Čech Cohomology with coefficient in Ω^2 and the gauge transformation $\psi_{\alpha\beta}$ that it satisfies. This indeed gives a way to genralize the notion of vector potential to abitrary differentiable manifold (here we require differentiability and $\mathcal{U}$ to be a good cover so that Č) ech cohomology groups are isomorphic to De Rahm cohomology groups for further convenience). But the magic of cohomology and quantum field theory doesn't end here. This generalization by Čech cohomology can easily adopt Dirac's condition of quantization.

3.3 Dirac's quantization condition

Quantization means discrete, and here quantization condition applies to the gauge transformation $\psi_{\alpha\beta}$. To see this, we have to consider another physical term called 'phase'. It is basically the argument ϕ of a complex number $e^{i\phi}$.

Let $p \in U_\alpha$, $q \in U_\beta$, $r \in U_\alpha \bigcap U_\beta$, then the line integral of A_μ $p \to q$ is defined as

$$I = \int_p^r A_\alpha dx + \psi_{\alpha\beta}(r) + \int_r^q A_\beta dx \tag{10}$$

For triple intersection $U_\alpha \bigcap U_\beta \bigcap U_\gamma$, it is replaced by the term $\psi_{\alpha\beta}(r) + \psi_{\beta\gamma}(r) + \psi_{\gamma\alpha}(r)$. Combining with definition for gauge transformation, we have

$$d(\psi_{\alpha\beta} + \psi_{\beta\gamma} + \psi_{\gamma\alpha}) = 0 \Rightarrow \psi_{\alpha\beta} + \psi_{\beta\gamma} + \psi_{\gamma\alpha} = c_{\alpha\beta\gamma} \tag{11}$$

where $c_{\alpha\beta\gamma} \in \check{C}^2(\mathcal{U}, \Omega^0)$ is a constant. And as we can loop back with $p = q$ for triple intersection, we must have $\phi = I = 2n\pi$ for this loop in order that the phase doesn't induced any problem in calculation. This is Dirac's quantization condition. Because of this, usually the gauge transformation is specified by $\delta(\psi_{\alpha\beta}) = c_{\alpha\beta\gamma}$ which can only take values in integer cohomology group.

There are other properties of Čech cohomology that is useful in quantum field theory (for example with finite dimensional manifold X only finitely many Čech Cohomology group is non-trivial), but they are with much details and I choose not to include in this paper (and I haven't finished reading it yet to be honest, but will finish it in winter).

4 Conclusion

In this paper we mainly introduced Čech Cohomology with a pedagogical approach and its application in physics. Very surprisingly, based on Čech Cohomology with some physical conditions, we can naturally deduce Dirac's quantization condition in quantum field theory. Together with De Rham Cohomology, Čech Cohomology gives a theoretic fundation that the notion of vector potential in classical physics can be generalized to quantum field theory with arbitrary underlying topological space, and I hope that its potential can be dig further by our generation, as the main physics paper I cited is written 40 years ago!

5 Reference

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