Final report

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1 Introduction

We frequently encounter the concept of wave in almost every area of physics. Mechanic waves, electromagnetic waves, sound waves, and even sometimes we deal with gravitational waves. But look back to the start of physics, what is the first example of wave? It's water wave, isn't it? We are all so familiar to it that no one would doubt that water wave is the prototype of wave. However, do we really understand water wave now, after we have learnt so much about waves? Is water wave traverse wave of longitudinal wave?

From our research, we found that water wave is defined as surface wave, or Rayleigh wave. Surface wave is a mechanical wave that propagates along the interface between differing media (From Wikipedia). In 1885, Rayleigh in his article prove that such wave has both longitudinal and traversal component, and gave a beautiful solution. After that, most of the analysis about surface wave focused on the application. Most of the articles were about how to use Rayleigh wave to understand seismic wave, and how to utilize it in non destructive evaluation. Therefore, except for Rayleigh's article, most works focused on the application of surface wave, instead of it's nature. And Rayleigh's article didn't focus on the physics of such wave neither. Its emphasis was the solution of wave equation. But surface wave is very different from other waves we have seen, and also very interesting. So our group decide to choose surface wave as our topic for this final report, with the focus on the physics nature of such phenomena.

In this report, we will try to explore the propagation of a pulse on the surface of some elastic material. Such pulse is a very good example of surface wave, and we will mainly study in three different ways: theory, numerical simulation, and experiment. To give more physics insight, we start with only Newtonian in the theory part, instead of using any exist formula about Rayleigh wave or surface wave. With some basic assumptions we try to establish a system differential equations and find its analytic solution. To make the result more plausible, we compare the solution obtained with result from numerical analysis and with Rayleigh's description in his article "On Waves Propagated along the Plane Surface of an Elastic Solid". Finally we will apply such result to real experiment and see how well it works. Limited by the online collaboration, our experiment part can only presented in a video form, conducted by one student, and data is sent to Deeplabcut for analysis. Finally the experimental result is compared with the solution from theoretical part, and some conclusion is reached.

2 Theories about surface wave

So we will use Newtonian to analyze the motion of some elastic solid. We will compress the z-direction, which is not of our interest, using vertical and horizontal dimension. The pulse is given vertically, and we look at the horizontal surface in which the surface wave propagates. Here we use $\chi_x(x,t)$ to denote the horizontal displacement of some mass at time t, whose undisturbed position is x. Similarly, we use $\chi_y(x,t)$ to denote vertical displacement.

Now, let's analyze $\chi_y(x,t)$ first. Assume that there is some tension in the solid, and consider some small mass of length dx at initial position x_0. It has mass $dm = \lambda dx$, where λ is the 'equivalent linear density', or mass per unit length near the surface, for this elastic solid. Its motion strictly follows Newtonian equation $\mathbf{F} = m\mathbf{a}$. Let's break this into horizontal and vertical direction. Denote by θ_1 the angle made by this small mass and its previous one with horizontal axis, and θ_2 the angle made by this small mass and the next one with horizontal axis. If $d\chi_y$ is small compare to dx, which is usually true in elastic solid, we can easily find that $\theta_1 = \frac{\partial\chi_y}{\partial x}|_{x_0,t}$, and $\theta_2 = \frac{\partial\chi_y}{\partial x}|_{x_0+dx,t}$. With simple geometry, we find that the equation for $\chi_y(x,t)$ will be :

$$\mathrm{T}\cdot\sin\theta_2 - \mathrm{T}\cdot\sin\theta_1 = dm\cdot a_y = \lambda dx\frac{\partial^2\chi_y}{\partial t^2}$$

Use small angle approximation, we have $\sin\theta_1 \sim \theta_1$, $\sin\theta_2 \sim \theta_2$, thus the equation is reduced to

$$\mathrm{T}\cdot\frac{\partial\chi_y}{\partial x}|_{x_0+dx,t} - \mathrm{T}\cdot\frac{\partial\chi_y}{\partial x}|_{x_0,t} = \lambda dx\frac{\partial^2\chi_y}{\partial t^2}$$

Divide both sides by dx, and we have

$$\frac{\partial^2\chi_y}{\partial x^2} = \frac{\lambda}{T}\frac{\partial^2\chi_y}{\partial t^2}$$

and this is the wave equation we are familiar with, whose solution is $\chi_y(x,t) = A\cos(\omega t - kx + \phi)$

Now we turn to the horizontal displacement $\chi_x(x,t)$. The force that drives (small)horizontal displacement is pressure results from compression. The compression increases local density, and thus induce pressure. There is a nice linear relation between induced pressure and change in density, express in $P' = \kappa\rho'$ when such change is small. Consider some mass of length dx at position x_0 and has density ρ. In the motion, its position and length might change, but its mass conserves. So if at some time t, when it has certain displacement, its length is

$$dx' = [x_0 + dx + \chi_x(x_0+dx,t)] - [x_0 + \chi_x(x_0,t)] = (1 + \frac{\partial\chi_x}{\partial x}|_{x_0})dx$$

For mass conserves, we have $dx'\cdot(\rho+\rho') = dx\cdot\rho$

Combine these two equations and we get $(1+\frac{\partial\chi_x}{\partial x}|_{x_0})(\rho+\rho')=\rho$. If we neglect $\frac{\partial\chi_x}{\partial x}|_{x_0}\rho'$, because we have assumed that ρ' is small, and so is displacement. Reduce the equation and we have

$$\rho' = -\rho\frac{\partial\chi_x}{\partial x}|_{x_0}$$

With this relation in mind, we try to establish our Newtonian equation. As this time we consider horizontal displacement, the force will be the difference in pressure. Thus our equation will be

$$P(x_0,t)-P(x_0+dx,t)=\lambda dx\frac{\partial^2\chi_x}{\partial x^2}$$

where λ is still the 'equivalent linear density'. Use the same trick by dividing both sides by dx, we get

$$\frac{\partial P}{\partial x}|_{x_0,t}=\lambda\frac{\partial^2\chi_x}{\partial x^2}$$

Notice that we can express pressure as $P=P_0+P'$, and P_0 should be constant in the solid, therefore $\frac{\partial P}{\partial x}=\frac{\partial P'}{\partial x}$. Take the linear relation between induced pressure and change in density, and the relation we deduced just now, we have our Newtonian equation in horizontal direction:

$$\frac{\partial^2\chi_x}{\partial x^2}=\frac{\lambda}{\kappa}\frac{\partial^2\chi_x}{\partial x^2}$$

Notice that it is, again, a standard wave equation, and its solution is

$$\chi_x(x,t)=A'\cos(\omega t-k'x+\phi')$$

Now, we have solved by vertical and horizontal displacement, and they are both sinusoidal functions. From our knowledge about polarized light, we know that the superposition of two sinusoidal functions in perpendicular directions can result in an ellipse.

To summarize, we found two partial differential equations to account for the displacement in vertical and horizontal directions.

$$\frac{\partial^2\chi_x}{\partial x^2}=\frac{\lambda}{\kappa}\frac{\partial^2\chi_x}{\partial x^2}$$
$$\frac{\partial^2\chi_x}{\partial x^2}=\frac{\lambda}{\kappa}\frac{\partial^2\chi_x}{\partial x^2}$$

So we predict that the trajectory of each particle should be an ellipse ideally. Here we compare with the current explanation of surface wave and Rayleigh's article, and found that the result generally agrees. But we also notice that there are some problems with the reasoning. First is that we totally neglected the z direction, by assuming that particles will always stay in x-y plane. There might be complex interactions in z direction that would influence the result. Also, to account for the pressure in elastic solid, tensor operation should be introduced. But now such simple differential equations should be enough. We will verify the two solutions with our result from numerical simulation, and finally we will compare with experiment.

3 Numerical simulation

In this section we will introduce our result from numerical simulation. The software we use is Comsol multiphysics. We first made a solid by stretching a rectangle in z direction, and then upload it to Comsol. The Young's modules was set at 10^6, and Poisson ratio 0.3. We give our elastic solid a pulse and collect data in a time interval that is sufficient for the propagation. Finally use

Excel to analyze the data we got. The result figure is shown below:

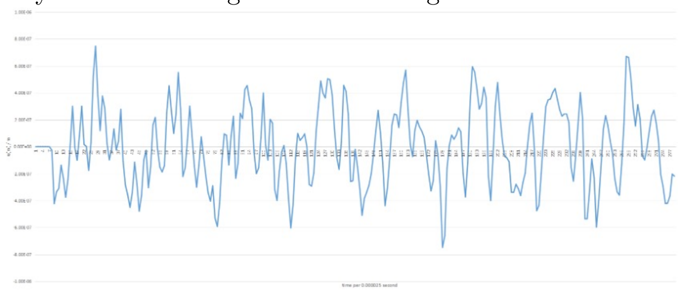


Figure 1 vertical displacement vs time

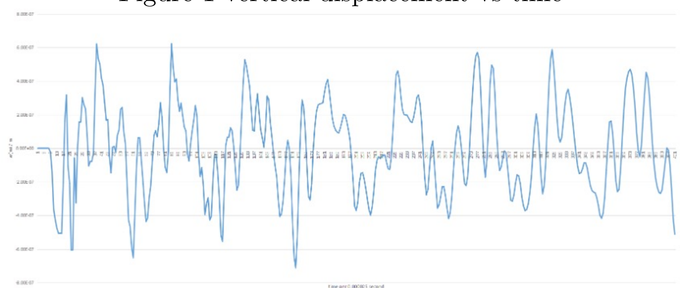


Figure 2 horizontal displacement vs time

Figure 1 and 2 both reveal the movement of a specific point during the transmission of surface wave. Particularly, figure 1 represents horizontal movement and figure 2 represents vertical movement. As shown above, after a period of zero value readings, both figure 1 and 2 begin to fluctuate between positive and negative values. Although the is significant fluctuation, we can still identify some typical sinusoidal pattern of the wave. This agrees with our solution of the partial differential equations from last section. The significant fluctuation might come from the random motion in z direction that we ignored in our theory. As the particles might not move in a perfect plane, and the complex interaction of particles in different planes can result in such fluctuation.

Such motion demonstrates that the point is undergoing an elliptical motion. Before deep study of the wave transfer in water, we hardly believe that water is not only traversal wave but also longitudinal wave. When the water surface is disturbed, there are two restoring forces that make the water surface returns to balanced position, one is gravity and the other is surface tension. In water waves, the restoring force exerted on the particles near the surface of the water is dominantly surface tension.

Experiment

1. Purpose

Elastic surface waves can travel along solid surfaces. In the previous part, we used COMSOL to simulate the propagation of surface waves on elastic objects, and used the simulation data to draw the trajectory of a specific point. But reality is different from simulation software. In reality, there are many factors that will affect the results of our experiments, such as the shaking of the table, the airflow generated by arm waving, and even the dust on the object. We hope that our model can be used in reality, not just in simulation software.

2. Procedure

We chose a elastic mass and marked four black dots on it to observe their movement trajectory. Next, we applied a pulse to this object and used camera to recorded the movement of the block. Then, we used Tracker to analyze the records we got and digitize them. Finally, we import the data into Excel, build tables and draw them into images.

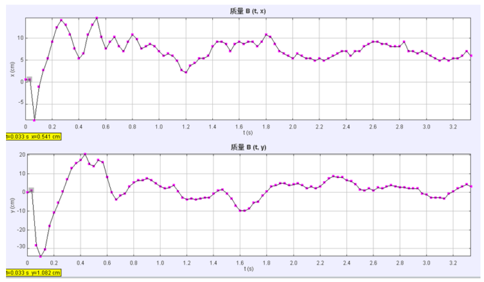
3. Data Collection

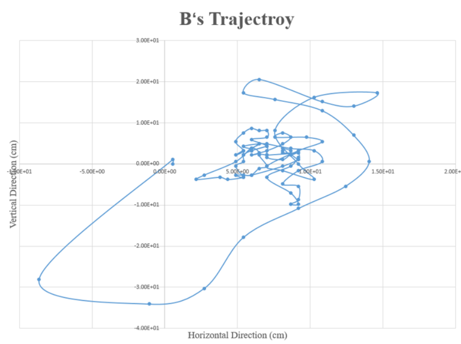
We import the video into Tracker. According to the positions of different points, take their initial position as the origin to establish a coordinate system. Tracker helped us get the position of the point in each frame, and drew the image of time vs. distance. We denote the four points on the block, from left to right, as A, B, C, D, and we only shows the result of B's according to the limit on the number of images.

Experiment image

4. Data Analysis

From the image below, we can see that, similar to our simulation, the two images represent the movement in the horizontal direction and the movement in the vertical direction, respectively. And, they also fluctuate between positive and negative values, which positively indicates that the point is moving in an ellipse. We imported the data from Tracker into Excel, and based on these data, we got these images, which represent the trajectory of these points. The picture below shows the trajectory of B. We can see that B has a trend of elliptical motion, but it does not match our simulated data as we assumed.





Experiment data of point C

5. Error and Uncertainty

During the experiment, we made some mistakes. Because of the occlusion of obstacles, we cannot accurately observe A's trajectory, which also means that we cannot obtain A's data. In addition, during the experiment, the point C was detached from the elastic block due to vibration. Fortunately, we can still see the traces left by C, and use this to track the trajectory of C. At the same time, due to the objective reasons of the experimental apparatus, our experiment has certain errors. As we slowed down the video, there was a certain degree of frame drop. When we track the movement of the points, the dropped frames make the continuity of our observations decrease. In addition, in using Tracker, because we have to click the position of the point with the mouse, in each frame, this inevitably produces uncertainty.

6. The Connection between Simulation and Experiment.

In our simulation, we tracked the movement of the point during the entire time period to see whether its motion agrees with our theory. And in the experiment we mainly focused on the trajectory. The logic is that in reality, the pulse cannot be too violent, or it might break our material. So the impact of the pulse disappear very fast (in only about 1 to 2 periods). Thus it would be unwise to use such data to test whether the motion is sinusoidal, as the main characteristic of sine function is its period. So we mainly focused our attend on the trajectory, which is also easier to track, and used a numerical simulation to amend such shortcoming.

4 Conclusion

Now we have seen that both simulation and experiment agrees with our theory. So even though we neglected some aspects, our solution for surface wave is generally successful. However, as mentioned in each chapter , there are much more to refine. We will introduce more realistic differential equations, which will be more complex but can explain better what's really happened inside an elastic solid. Furthermore, we will look for more suitable material and try to apply more advanced techniques in data analysis.