Symbolic dynamics

Xuanqi Wang

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This is my original proof of how homoclinic tangle leads to symbolic dynamics in non-wandering set. It has been a question for our group (since everyone knew this conclusion but cannot rigorously prove it) for more than three years, so I was so thrilled to write this up. This full version will not be included in our non-linear paper, but I still think it worth a separate document, as it might help more people in the future.

The main idea is what Prof. Cvitanovic described as "once a point leaves the optimal cover of non-wandering set, it will never come back again." Thus, all the points outside non-wandering set either came from infinity or will go to infinity. And once we prove this property of optimal cover (which is like the setting of $\bigcap_{s \in S} B_M(s)$ for all symbols s in Sterling98, but with different ideas), we proceed with Cantor's intersection theorem.

For hyperbolic dynamics, the locally maximal invariant set is constructed by

$$\Omega = \bigcap_{n=-\infty}^{\infty} f^n(U) \tag{1}$$

for some open set $U \subset \mathcal{M}$ and $f : \mathcal{M} \to \mathcal{M}$ invertible. We construct a sequence of nested compact (i.e. close and bounded in $\mathbb{R}^2$) sets $\{\Omega_k\}$ such that

$$\Omega_k = f^k(N) \cap N \cap f^{-k}(N) \tag{2}$$

where $N \subset \mathcal{M}$ is the compact optimal cover that we choose, whose boundary is given by invariant manifolds. We will prove that $\{\Omega_k\}$ is indeed a sequence of nested compact sets, and this sequence will be very useful in understanding Ω. The key of this proof lies in the fact that

$$\Omega_k = \bigcap_{n=-k}^{k} f^n(N) \tag{3}$$

We begin by proving the following lemma.

Lemma 1: $\Omega_m \subset \Omega_n$ for all $m > n$.

Proof: It suffices to prove that $\Omega_{n+1} \subset \Omega_n$. Assume there is $x \in \mathcal{M}$ such that $x \in \Omega_{n+1}$ and $x \notin \Omega_n$. Then, without loss of generality assume $x \in f^{n+1}(N) \backslash f^n(N)$. By invertibility of f, we can conclude that $f^{-(n+1)}(x) \in$

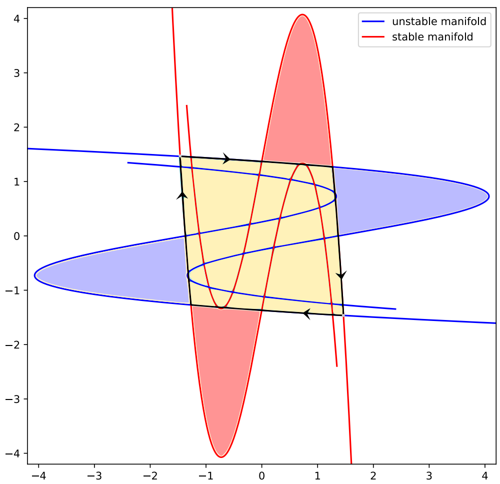

Figure 1: A visualized aid to understand $f(N)\backslash N$, with example comes from ϕ^4 theory $\mu^2 = 3.5$

$f(N)\backslash N$, which is the colored region shown in Fig.1 ($f(N)\backslash N$ is shaded in blue, while $f^{-1}(N)\backslash N$ is in red). However, $f(N)\backslash N$ lies on the "wrong" side of stable manifolds W_1^s and W_2^s, which means that $f(N)\backslash N$ never intersects N in any forward in time image of f. Therefore, $f^{n+1}(f^{-(n+1)}(x)) = x \notin N$, contradicts with the assumption $x \in \Omega_{n+1}$.

This argument is valid for any smooth f that can produce a Smale's horseshoe, as the boundary of optimal cover is given by sections of invariant manifold that are 'close' to the fixed points, which guarantees a simple geometry (straight without many twists and wiggling).

In fact, the reason that any forward in time image of $f(N)\backslash N$ would never intersect N (similarly any backward image of $f^{-1}(N)\backslash N$ would never intersect N) is clearer when looking at the direction of its boundary (which is piecewisely given by three invariant manifolds) of the "smallest" regions in $\Omega_0\backslash\Omega_1$ whose boundaries are given by $\partial f(N) \cup \partial f^{-1}(N)$.

Intuitively, this lemma suggests that if a point $x \in N$ runs out of N (i.e. $f^k(x) \notin N\ \exists k \in \mathbb{Z}$), then it can never return to N. The important condition here is that x has to start out in N, because points that are outside N can enter, but such entrance is allow for only once.

Corollary: $\Omega_k = \bigcap_{n=-k}^{k} f^n(N)$

Proof: It is the direct result by applying Lemma 1.

$$\bigcap_{n=-k}^{k} f^n(N) = \bigcap_{n=-k}^{k} f^n(N) \cap N = f^k(N) \cap N \cap f^{-k}(N) = \Omega_k \tag{4}$$

as $f^k(N) \cap N \cap f^{-k}(N) \subset f^m(N) \cap N \cap f^{-m}(N)$ by Lemma 1.

The structure of invariant set $\Omega = \lim_{n\to\infty} \bigcap \Omega_n$ is revealed by the sequence of $\{\Omega_n\}$. Write $\Omega_n = (f^n(N) \cap N) \cap (f^{-n}(N) \cap N)$, it is clear that $f^n(N) \cap N$ resembles the construction of Smale's horseshoe map, where the existence of a transversally homoclinic point guarantees that $f^n(N) \cap N$ is a disjoint union of m^n regions (where m is a positive integer), each being compact and connected. Therefore, we can label the $2m^n$ disjoint connected regions in Ω_n by a length 2n string with m symbols.

$$\Omega_n = \bigsqcup_{s\in S} \Omega_n^s\ ,\ S = \mathcal{A}^{2n}\ ,\ |\mathcal{A}| = m \tag{5}$$

We arrange the symbol strings in such a way that if $s = s_{-n+1}s_{-n+2}...s_0s_1...s_n$ and $s' = s'_{-m+1}s'_{-m+2}...s'_0s'_1...s'_m$ $(n > m)$ and $s_k = s'_k$ $(\forall |k| \le m)$, then $\Omega_n^s \subset \Omega_m^{s'}$. With this arrangement, for each bi-infinite string $s = ...s_{-1}s_0s_1s_2... \in \mathcal{A}^{\mathbb{Z}}$, we can construct a sequence of nested compact sets $\{\Omega_n^{s^n}\}$ where $s^n = s_{-n+1}...s_0s_1...s_n$ denote the finitely truncated substring of s of length 2n. Let $\Omega^s = \lim_{n\to\infty} \bigcap \Omega_n^{s^n}$, by Cantor's intersection theorem Ω^s is non-empty. Since Ω is the disjoint union of Ω^s

$$\Omega = \bigsqcup_{s\in\mathcal{A}^{\mathbb{Z}}} \Omega^s \tag{6}$$

and Ω is, by its nature, a totally disconnected, whose connected components are only singletons, we just need to prove that Ω^s is connected to establish the one-to-one correspondence between Ω and $\mathcal{A}^{\mathbb{Z}}$ and prove that the dynamics is conjugated to a m-symbol full shift.

Lemma 2: Ω^s is connect

Proof: We prove by contradiction. Assume that Ω^s is disconnected, covered by $U, V \subset \mathcal{M}$ open, then we construct a new sequence of nested sets $\{\Omega_n^{s^s}\backslash(U \sqcup V)\}$. Since U, V are open, all the sets in this sequence are still compact. And as $\Omega_n^{s^n}$ are connected, $\Omega_n^{s^s}\backslash(U \sqcup V)$ much be non-empty for each n. Then, there exists $x \in \bigcap \Omega_n^{s^n}\backslash(U \sqcup V) = \Omega^s\backslash(U \sqcup V)$, contradicts with the assumption that $U \sqcup V$ covers Ω^s.

Therefore, we conclude that the locally maximal invariant set Ω has its points in one-to-one correspondence with $\mathcal{A}^{\mathbb{Z}}$. This construction natrually manifests the fact that dynamics in Ω is conjugated to a m-symbol full shift, which is the important conclusion we will use in shadow state method.