Local limits of random walks on a torus and random interlacements

Proof and Extension

Wang Xiaochen

Hong Kong University of Science and Technology

2023

Table of Contents

Table of Contents

Preliminaries

Consider the measurable space $(\overline{RW}, \overline{\mathscr{RW}})$. Here, $\overline{RW}$ is the set consisted of the sequence $w = (w_n)_{n>0} = (w_1, w_2, \ldots, w_n)$. And $\overline{\mathscr{RW}}$ is the σ-algebra generated by the coordinate map: $X_n : \overline{RW} \to \mathbb{Z}^d$, $X_n(w) = w_n$. The random sequence $(X_n)_{n\geq 0}$ is a Markov chain on $\mathbb{Z}^d$ under P_x, with initial position $x \in \mathbb{Z}^d$ and

Transition Probability

$$P_x\left[X_{n+1} = y | X_n = x\right] := \begin{cases} \frac{1}{2d}, \text{ if } |x-y| = 1, \\ 0, \text{ otherwise.} \end{cases}$$

The following random times arise naturally.

$$H_K(w) := \inf\{n \geq 0 : X_n(w) \in K\}, \text{"first entrance time"},$$
$$\tilde{H}_K(w) := \inf\{n \geq 1 : X_n(w) \in K\}, \text{ "first hitting time"},$$
$$T_K(w) := \inf\{n \geq 0 : X_n(K) \notin K\}, \text{ "first exit time"},$$
$$L_K(w) := \sup\{n \geq 0 : X_n(w) \in K\}, \text{ "time of last visit"}^*.$$

Preliminaries

Theorem

A simple random walk is recurrent for dimension $d \leq 2$ *and transient for dimension* $d \geq 3$ *[[2], Polya, 1921.]*

The topic of random interlacement is studied in a space with dimensions greater or equal to 3.

Preliminaries

The Green function $g(\cdot, \cdot)$ will be needed and it can be formulated as the following,

Green Function

$$g(x, y) = \sum_{n \geq 0} P_x[X_n = y] = E_x \left[\sum_{n \geq 0} \mathbb{1}_{\{X_n = y\}} \right], x, y \in \mathbb{Z}^d.$$

We also introduce the equilibrium measure e_K. For $K \subset\subset \mathbb{Z}^d$ and $x \in \mathbb{Z}$, define

$$e_K(x) := P_x[\tilde{H}_K = \infty] \cdot \mathbb{1}_{x \in K} = P_x[L_K = 0]\mathbb{1}_{x \in K},$$

its total mass,

$$\mathrm{cap}(K) := \sum_{x \in K} e_K(x).$$

is called the capacity of K. Note that e_K is supported on the interior boundary of K ($\partial_{int} K := \{x \in K : \exists y \notin K, |x - y| = 1\}$).

Preliminaries

Last exit decomposition:

Theorem

For $x \in \mathbb{Z}^d$ and $K \subset\subset \mathbb{Z}^d$

$$P_x[H_K < \infty] = \sum_{y \in K} g(x,y) e_K(y).$$

If $K = \{y\}(|K| = 1)$, the hitting probability of a single point y can be expressed as

$$P_x[H_y < \infty] = g(x,y) P_y[\tilde{H}_y = \infty]. \tag{1}$$

Preliminaries

.

For $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$, the canonical projection map is $\varphi : \mathbb{Z}^d \to \mathbb{T}_N^d$, and in the following part, we will use the bold letter to indicate quantities in the torus. For instance, $\varphi(K) = \mathbf{K} \subset \mathbb{T}_N^d$, $\mathbf{x} \in \mathbb{T}_N^d$. We use $\mathbf{P_x}$ for the law of simple random walk on $\mathbb{T}_N^d$ starting from $\mathbf{x}$, and

Probability Measure with Uniformly Chosen Starting Point

$$\mathbf{P} = \frac{1}{N^d} \sum_{\mathbf{x} \in \mathbb{T}_N^d} \mathbf{P_x},$$

for the law of simple random walk on $\mathbb{T}_N^d$ starting from a uniformly chosen point.

Table of Contents

Main Result

How can we interpret "Random Interlacements"?

How exactly does this φ work?

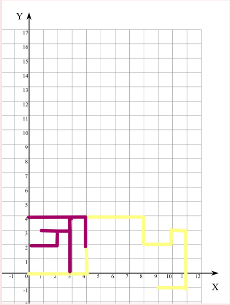


Figure: Yellow trace is the random walk in $\mathbb{Z}^2$, when projected to a torus $(\mathbb{Z}/4\mathbb{Z})^2$, it becomes the red trace.

Main Result

Theorem

For a given $K \subset\subset \mathbb{Z}^d$, $\varphi : \mathbb{Z}^d \to \mathbb{T}_N^d$,

$$\lim_{N\to\infty} \mathbf{P}[\{\mathbf{X}_0, \dots, \mathbf{X}_{\lfloor uN^d \rfloor}\} \cap \varphi(K) = \emptyset] = e^{-ucap(K)}.$$

Here u is a quantity that *governs the amount of trajectories which enter the picture*[[3], *Sznitman, 2010*]. (i.e. As u increases, more trajectories.) The right handside gives the precise probability that the random interlacements at level u do not intersect K.

Main Result

How are we going to prove this result? Subdivide the whole trajectory into small streches, and each strech is separated by a small proportion.

Methodology

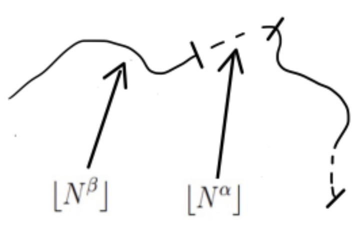


Figure: Parts of the whole trajectory.

For $2 < \alpha < \beta < d$, $L = 2\lfloor uN^d \rfloor$, We set

$$\ell^* = \lfloor N^{\beta} \rfloor + \lfloor N^{\alpha} \rfloor, \quad \ell = \lfloor N^{\beta} \rfloor, \quad \mathscr{K} = \lfloor L/\ell^* \rfloor - 1.$$

Main Result

Benefit:

- Small sub trajectory behaves well(ℓ),
- Assemble all the sub trajectories to retrieve the whole picture.

Problems:

- Can we really consider each part separately? Independence problem.

Main Result

Consider the sub-trajectory does hit $\varphi(K)$,

Lemma

Let $\beta \in (0, d)$, $N \geq 1$, and $n = \lfloor N^\beta \rfloor$. For any $K \subset\subset \mathbb{Z}^d$,

$$\lim_{N\to\infty} \frac{N^d}{n} \cdot \mathbf{P}[\{\mathbf{X}_0, \ldots, \mathbf{X}_n\} \cap \varphi(K) \neq \emptyset] = cap(K).$$

Interpretation: No hitting $\implies (1 - \frac{n}{N^d}\text{cap}(K))$

Proof: Simple random walks are reversible,

$$\begin{aligned}
\frac{N^d}{n}\mathbf{P}[\{\mathbf{X}_0, \ldots, \mathbf{X}_n\} \cap \mathbf{K} \neq \emptyset] &= \frac{N^d}{n}\sum_{t=0}^{n}\sum_{\mathbf{x}\in\mathbf{K}} \mathbf{P}[\mathbf{H}_\mathbf{K} = t, \mathbf{X}_{\mathbf{H}_\mathbf{K}} = x] \\
&= \frac{N^d}{n}\sum_{t=0}^{n}\sum_{\mathbf{x}\in\mathbf{K}} \mathbf{P}[\mathbf{X}_0 = \mathbf{x}, \tilde{\mathbf{H}}_\mathbf{K} > t] \\
&= \frac{1}{n}\sum_{t=0}^{n}\sum_{\mathbf{x}\in\mathbf{K}} \mathbf{P}_\mathbf{x}[\tilde{\mathbf{H}}_\mathbf{K} > t] = \frac{1}{n}\sum_{t=0}^{n}\sum_{\mathbf{x}\in\mathbf{K}} \mathbf{e}_\mathbf{K}(\mathbf{x}, t),
\end{aligned}$$

Main Result

we define

$$\mathbf{e_K}(\mathbf{x}, t) = \mathbf{P_x}[\mathbf{H_K} > t], \mathbf{x} \in \mathbb{T}_N^d$$

Consider the random walk on $\mathbb{Z}^d$, we already have that,

$$\lim_{N\to\infty} \frac{1}{n} \sum_{t=0}^{n} \sum_{x\in K} e_K(x,t) = cap(K), x \in \mathbb{Z}^d.$$

Therefore, we only need to prove

$$\lim_{N\to\infty} \max_{0\le t\le n} |e_K(x,t) - \mathbf{e_K}(\mathbf{x}, t)| = 0. \tag{3}$$

Main Result

Proving (3): $\varphi^{-1}(K)$

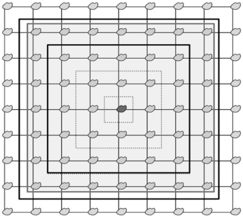
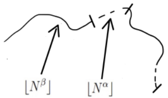


Figure: Adapted from [[1],*Drewitz,2014*]

- Prove that the random walk is contained in a box of radius $n^{1+\varepsilon}$(Grey Square)
- Decompose the box into union of annulus(Doted squares, outermost one is solid and bold).
- (1): $P_x[H_y < \infty] = g(x,y)P_y[\tilde{H}_y = \infty]$, Bound for Green function[1] Claim 1.9.

Main Result

Independence Problem:



About mixing time of lazy random walk [[4], *Welmer, 2009*]

$$\epsilon_n(N) = \sum_{y \in \mathbb{T}_N^d} \left|\mathbf{P_x}[\mathbf{Y_n} = \mathbf{y}] - \frac{1}{N^d}\right|, \text{ for any} x \in \mathbb{T}_N^d$$
$$\leq Ce^{-cN^{\delta-2}}.$$

Lemma

For $N \geq 1$, $1 \leq t_1 \leq t_2 \leq T$, $\mathscr{E} \in \sigma(\mathbf{Y}_0, \ldots, \mathbf{Y}_{t_1})$ and $\mathscr{E}_2 \in \sigma(\mathbf{Y}_{t_2}, \ldots, \mathbf{Y}_T)$, we have

$$|\mathbf{P}[\mathscr{E}_1 \cap \mathscr{E}_2] - \mathbf{P}[\mathscr{E}_1]\mathbf{P}[\mathscr{E}_2]| \leq \epsilon_{t_2 - t_1}(N).$$

Main Result

Assemble all the sub-trajectories: First,

$$0 \le \lim_{N\to\infty}\left(\mathbf{P}\left[\bigcap_{k=0}^{\mathscr{K}}\mathscr{E}_k\right]-\mathbf{P}\left[\{\mathbf{Y}_0,\ldots,\mathbf{Y}_{2u\lfloor N^d\rfloor}\}\cap\varphi(K)=\emptyset\right]\right)$$

$$\le \lim_{N\to\infty}\mathbf{P}\left[\bigcup_{k=0}^{\mathscr{K}}\bigcup_{t=k\ell^*+l}^{(k+1)\ell^*}\{\mathbf{Y}_t\in\mathbf{K}\}\right]\le\lim_{N\to\infty}\frac{|K|\cdot(\mathscr{K}+1)\cdot(\ell^*-\ell)}{N^d}=0.$$

Then,

$$\lim_{n\to\infty}\mathbf{P}\left[\bigcap_{k=0}^{\mathscr{K}}\mathscr{E}_k\right]=\lim_{N\to\infty}\prod_{k=0}^{\mathscr{K}}\mathbf{P}[\mathscr{E}_k]=\lim_{N\to\infty}(\mathbf{P}[\mathscr{E}_0])^{\mathscr{K}+1}$$

$$=\lim_{N\to\infty}\left(1-\frac{\lfloor N^\beta\rfloor}{2N^d}\mathrm{cap}(K)+\mathrm{o}(\frac{\lfloor N^\beta\rfloor}{2N^d})\right)^{\frac{2uN^d}{\lfloor N^\beta\rfloor}}$$

$$=e^{-u\mathrm{cap}(K)}$$

Table of Contents

Extension

Weight function:

$$\mu_{xy} = \begin{cases} > 0 & , |x-y|_1 = 1, |x|_1 = \sum_{i=1}^d |x_i| \\ 0 & , \text{otherwise,} \end{cases}$$

and bounded weight for all vertices x

$$\lambda < \mu_x = \sum_{y:|y-x|_1=1} \mu_{xy} < \Lambda$$

Transition Probability

$$\mathbf{P}[\mathbf{X}_{n+1} = \mathbf{y} | \mathbf{X}_n = \mathbf{x}] = \frac{\mu_{\mathbf{xy}}}{\mu_{\mathbf{x}}}.$$

Extension

Reversing the path?

$$\mathbf{P}[\mathbf{H_K} = t, \mathbf{X}_{\mathbf{H_K}} = x] \neq \mathbf{P}[\mathbf{X}_0 = \mathbf{x}, \tilde{\mathbf{H_K}} > t]$$

Adjusted result (Consider the bounded weight):

Theorem

For a given $K \subset\subset \mathbb{Z}^d$, any vertex $x \in \mathbb{Z}^d$ is assigned with weight μ_x, $\lambda \leq \mu_x \leq \Lambda$, and $c(\lambda, \Lambda) < C(\lambda, \Lambda)$

$$e^{-ucap(K)c(\lambda,\Lambda)} \leq \lim_{N\to\infty} \mathbf{P}[\{\mathbf{X}_0, \ldots, \mathbf{X}_{\lfloor uN^d \rfloor}\} \cap \varphi(K) = \emptyset] \leq e^{-ucap(K)C(\lambda,\Lambda)}.$$

Extention

Estimate error: Take $\mathbf{Z}_{\mathbf{x}_0\mathbf{x}_t} = \frac{\mu_{\mathbf{x}_0}}{\mu_{\mathbf{x}_t}}$, where $\mu_{\mathbf{x}_0} \in \mathbf{K}$, $\mu_{\mathbf{x}_t} \notin \mathbf{K}$.

$$\varepsilon_{n,t} := \frac{N^d}{n}\sum_{t=0}^{n}\sum_{\mathbf{x}\in\mathbf{K}}\mathbf{P}[\mathbf{H}_\mathbf{K} = t, \mathbf{X}_{\mathbf{H}_\mathbf{K}} = x] - \frac{N^d}{n}\sum_{t=0}^{n}\sum_{\mathbf{x}\in\mathbf{K}}\mathbf{P}[\mathbf{X}_0 = \mathbf{x}, \tilde{\mathbf{H}_\mathbf{K}} > t]$$

$$= \frac{1}{n}\sum_{t=0}^{n}\sum_{\mathbf{x}_0\in\mathbf{K}}\sum_{\mathbf{x}_t\notin\mathbf{K}}(\mathbf{Z}_{\mathbf{x}_0\mathbf{x}_t} - 1) \to \lim_{t\to\infty}\sum_{\mathbf{x}_0\in\mathbf{K}}\sum_{\mathbf{x}_t\notin\mathbf{K}}(\mathbf{Z}_{\mathbf{x}_0\mathbf{x}_t} - 1), \text{ as } n \to \infty$$

Where

$$\mathbf{E}(\mathbf{Z}_{\mathbf{x}_0\mathbf{x}_t} - 1) = \mathbf{E}(\frac{\mu_{\mathbf{x}_0}}{\mu_{\mathbf{x}_t}} - 1) = 0$$

Extension

Potential Extensions

- Use homogenized capacity instead of the capacity defined in the equilibrium measure?
- Consider if the subset K may expand with a controllable rate in other blocks?
- Random walk on the hexagonal $d - dimensional$ lattice? Will it have a similar property?
- Percolation theory, disconnection property, existence of infinite cluster...

Acknowledgment

I would like acknowledge the invaluable guidance offered by Prof. Maximilian NITZSCHNER throughout the project.

References I

- Alexander Drewitz, Balázs Ráth, and Artëm Sapozhnikov, *An introduction to random interlacements*, Springer, 2014.
- Georg Pólya, *Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz*, Mathematische Annalen **84** (1921), no. 1-2, 149–160.
- Alain-Sol Sznitman, *Vacant set of random interlacements and percolation*, Annals of Mathematics (2010).
- EL Wilmer, David A Levin, and Yuval Peres, *Markov chains and mixing times*, American Mathematical Soc., Providence (2009).