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1 Linear-inequality feasibility

1.1 Solution Description

Suppose we attempt to find an feasible solution to the following linear program L.

$$\begin{array}{lcccccccccc} s.t. & a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n & \leq & b_1 \\ & a_{21}x_1 & + & a_{22}x_2 & + & ... & + & a_{2n}x_n & \leq & b_2 \\ & & & & & ... & & & & \\ & a_{m1}x_1 & + & a_{m2}x_2 & + & ... & + & a_{mn}x_n & \leq & b_m \\ & x_1 & , & x_2 & , & ... & , & x_n & \geq & 0 \end{array}$$

The linear-inequality feasibility problem can be solved by constructing a auxiliary program L_{aux} as follows.

$$\begin{array}{lcccccccccccc} \min & & & & & & & & & x_0 & & \\ s.t. & a_{11}x_1 & + & a_{12}x_2 & + & ... & + & a_{1n}x_n & - & x_0 & \leq & b_1 \\ & a_{21}x_1 & + & a_{22}x_2 & + & ... & + & a_{2n}x_n & - & x_0 & \leq & b_2 \\ & & & & & ... & & & & & & \\ & a_{m1}x_1 & + & a_{m2}x_2 & + & ... & + & a_{mn}x_n & - & x_0 & \leq & b_m \\ & x_1 & , & x_2 & , & ... & , & x_n & , & x_0 & \geq & 0 \end{array}$$

We say the original problem L is feasible if the optimal solution to the above auxiliary problem satisfies $OBJ(L_{aux}) = 0$.

1.2 Proof

1. Suppose that L has a feasible solution: $(x_1, x_2, ..., x_n)$
2. Expand this solution by combining with $x_0 = 0$: we get $(0, x_1, x_2, ..., x_n)$
3. The expanded solution is a feasible solution to L_{aux}
4. As $x_0 \leq 0$, $x_0 = 0$ is the optimal objective value to L_{aux}

Thus, if L has a feasible solution, then it can be converted to the optimal solution to L_{aux} with the minimalist objective value 0.

Conversely, if the auxiliary problem achieves the minimalist value, then by substituting the LP equation with its optimal solution $(0, x_1, x_2, ..., x_n)$, we have $a_{i1}x_1 + a_{i2}x_2 + ... + a_{ij}x_j + ... + a_{in}x_n \leq b_i$, which means that $(x_1, x_2, ..., x_n)$ is a feasible solution to the given **Linear-nequality feasibility** problem.

2 Interval Scheduling Problem

Let x_i indicate whether class i is selected in the schedule. It's easy to figure out that the objective function of this problem is $\max \sum_{i=1}^{n} x_i$. The constraints on x_i can be described as: at any time, the amount of class scheduled is not larger than the number of classrooms m.

To mathematically describe this problem, we first sort $S_1, F_1, ..., S_n, F_n$ in ascending order. The problem matrix A is constructed by the following rule: for class i, suppose S_i, F_i is at position j, k in the sorted array, then we mark elements $a_{i,j}, a_{i,j+1}, ..., a_{i,k}$ as 1. The constraints on x_i is $a_{i,1}x_1 + a_{i,2}x_2 + ... + a_{i,n}x_n \leq m$.

2.1 ILP Equation

$$
\begin{array}{rrcrcccrcl}
\max & x_1 & + & x_2 & + & ... & + & x_n & & \\
s.t. & a_{1,1}x_1 & + & a_{1,2}x_2 & + & ... & + & a_{1,n}x_n & \leq & m \\
 & a_{2,1}x_1 & + & a_{2,2}x_2 & + & ... & + & a_{2,n}x_n & \leq & m \\
 & & & & & ... & & & & \\
 & a_{2n,1}x_1 & + & a_{2n,2}x_2 & + & ... & + & a_{2n,n}x_n & \leq & m \\
 & x_1 & , & x_2 & , & ... & , & x_n & \geq & 0
\end{array}
$$

2.2 Instance and Solve

We use glpk to solve the following instance.

$$m = 2, n = 4 \tag{1}$$
$$S_1 = 1, F_1 = 4 \tag{2}$$
$$S_2 = 2, F_2 = 7 \tag{3}$$
$$S_3 = 3, F_3 = 5 \tag{4}$$
$$S_4 = 6, F_4 = 8 \tag{5}$$

Convert it to LP matrix, we have:

$$
A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad b = [2, 2, 2, 2]^T
$$

The problem is modeled as the following:

```
// GLPK MathProg Model
var x1, binary;
var x2, binary;
var x3, binary;
var x4, binary;

maximize z: x1 + x2 + x3 + x4;

s.t. a0: 1 <= 2;
s.t. a1: x1 + x2 <= 2;
s.t. a2: x1 + x2 + x3 <= 2;
s.t. a3: x1 + x2 + x3 <= 2;
s.t. a4: x2 + x3 <= 2;
s.t. a5: x2 + x4 <= 2;
s.t. a6: x2 + x4 <= 2;
```

```
s.t. a7: x4 <= 2;

end;
```

The output of glpsol suggests that the optimal solution is $(1, 1, 0, 1)$ with objective value $z = 3$, which is easy to verify to be true.

3 Gas Station Placement

According to the problem description, we can model it in the following way.

$$Target : \min \max_{1 \le i \le n-1} \{x_{i+1} - x_i\}$$
$$Constraints : 0 \le xi \le d_n,\ |x_i - d_i| \le r$$

To convert it to the standard form, we introduce a assistant variable t such that $t \ge x_{i+1} - x_i$. Now the problem can be described as a LP problem.

$$Target : \min t$$
$$Constraints : 0 \le xi \le d_n,\ |x_i - d_i| \le r,\ t \ge x_{i+1} - x_i$$