CentraleSupélec

Projet S8 - Rapport

Parametric optimization of an acoustic liner

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2 The Model

2.1 Linearized Euler equations and notations

This part is mainly taken from [3], we introduce the equations, hypothesis and notations of [3].

As in [3], the hypothesis taken is the one of an ideal fluid :

- no friction (no viscous forces)
- no heat conduction
- no heat production

This implies that the flow is isentropic. The equations for the fluid are then the Euler Equations :

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{v} + m \qquad \text{Mass conservation} \tag{1}$$

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \boldsymbol{f} \qquad \text{Momentum conservation} \tag{2}$$

Where we denote

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$$

With

- $\boldsymbol{v}(\boldsymbol{r},t)$ the fluid velocity
- $\rho(\boldsymbol{r},t)$ the density
- $p(\boldsymbol{r},t)$ the (static) fluid pressure
- $s(\boldsymbol{r},t)$ the specific entropy
- $m(\boldsymbol{r},t)$ the specific mass source per unit time
- $\boldsymbol{f}(\boldsymbol{r},t)$ the force per unit mass exerting on the fluid (neglecting gravity)

$\boldsymbol{r} = (x, y, z)$ being the position and t the time

We consider, as in [3] the actual flow is the sum of a stationnary flow (denoted X_0) and a perturbation of negligible contribution (denoted X', and $X' \ll X^0$) generated by the mass m and force $\boldsymbol{f}$ injected by the fluid:

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + \rho'(\mathbf{r}, t) \quad \text{fluid density} \tag{3}$$
$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v_0}(\mathbf{r}) + \mathbf{v}'(\mathbf{r}, t) \quad \text{velocity} \tag{4}$$
$$s(\mathbf{r}, t) = s_0(r) + s'(\mathbf{r}, t) \quad \text{specific entropy} \tag{5}$$
$$p(\mathbf{r}, t) = p_0(\mathbf{r}) + p'(\mathbf{r}, t) \quad \text{(static) fluid pressure} \tag{6}$$
$$c^2(\mathbf{r}, t) = c_0^2(\mathbf{r}) + (c^2)'(\mathbf{r}, t) \quad \text{squared speed of sound} \tag{7}$$

The speed of sound is defined thanks to the equation of state $p = p^{\#}(\rho, s)$ in the adiabatic case :

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \underbrace{\left.\frac{\partial p^{\#}}{\partial \rho}\right|_s}_{c^2(\rho,s)} \frac{\mathrm{D}\rho}{\mathrm{D}t} \tag{8}$$

The stationnary part of the flow verifies the following equalities : (resolve equations (1), (2) and (8) with $\frac{\partial .}{\partial t} = 0$, $m = 0$, $\boldsymbol{f} = 0$ -as $\boldsymbol{f}, m$ generates the perturbation-, for $(p_0, \boldsymbol{v_0}, \rho_0)$ the unperturbed flow)

$$\begin{aligned} (\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\rho_0 &= -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v_0} \\ (\rho_0 \boldsymbol{v_0} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} &= -\boldsymbol{\nabla} p_0 \\ (\boldsymbol{v_0} \cdot \boldsymbol{\nabla})p_0 &= c_0^2(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\rho_0 \end{aligned} \tag{9}$$

The perturbation verifies the Linearized Euler equations (mass conservation, momentum conservation and state equation) :

$$\begin{aligned} \frac{\mathrm{D}_c \rho'}{\mathrm{D}t} + (\boldsymbol{v'} \cdot \boldsymbol{\nabla})\rho_0 &= -\rho' \boldsymbol{\nabla} \cdot \boldsymbol{v_0} - \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v'} + m \\ \frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} + (\boldsymbol{v'} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} &= \frac{\rho'}{\rho_0^2} \boldsymbol{\nabla} p_0 - \frac{1}{\rho_0} \nabla p' + \boldsymbol{f} \\ \frac{\mathrm{D}_c p'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} p_0 &= c_0^2 \left(\frac{\mathrm{D}_c \rho'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \nabla \rho_0 \right) + (c^2)' \boldsymbol{v_0} \cdot \boldsymbol{\nabla} \rho_0 \end{aligned} \tag{10}$$

With

$$\frac{\mathrm{D}_c}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v_0} \cdot \nabla \tag{11}$$

Proof. (mass conversation : resolving for the actual flow)

$$\frac{\partial \rho'}{\partial t} + (\boldsymbol{v_0} + \boldsymbol{v'}) \cdot \boldsymbol{\nabla}(\rho_0 + \rho') = -(\rho_0 + \rho')\boldsymbol{\nabla} \cdot (\boldsymbol{v_0} + \boldsymbol{v'}) + m$$

Only keeping first order terms, and account for $\frac{\partial \rho_0}{\partial t} = 0$, we end up with

$$\frac{\partial \rho'}{\partial t} + \boldsymbol{v_0} \cdot \boldsymbol{\nabla}\rho_0 + \boldsymbol{v_0} \cdot \boldsymbol{\nabla}\rho' + \boldsymbol{v'} \cdot \boldsymbol{\nabla}\rho_0 = -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v_0} - \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v'} - \rho' \boldsymbol{\nabla} \cdot \boldsymbol{v_0} + m$$

Regrouping the terms

$$\frac{\mathrm{D}_c\rho'}{\mathrm{D}t} + \boldsymbol{v'}\cdot\boldsymbol{\nabla}\rho_0 = -\underbrace{(\boldsymbol{v_0}\cdot\boldsymbol{\nabla}\rho_0 + \rho_0\boldsymbol{\nabla}\cdot\boldsymbol{v_0})}_{=0\ (9)} - \rho_0\boldsymbol{\nabla}\cdot\boldsymbol{v'} - \rho'\boldsymbol{\nabla}\cdot\boldsymbol{v_0} + m$$

$$\boxed{\frac{\mathrm{D}_c\rho'}{\mathrm{D}t} + \boldsymbol{v'}\cdot\boldsymbol{\nabla}\rho_0 = -\rho_0\boldsymbol{\nabla}\cdot\boldsymbol{v'} - \rho'\boldsymbol{\nabla}\cdot\boldsymbol{v_0} + m}$$

(momentum conservation)

$$(\rho_0+\rho')\left(\underbrace{\frac{\partial \boldsymbol{v_0}}{\partial t}}_{=0} + \frac{\partial \boldsymbol{v'}}{\partial t} + ((\boldsymbol{v_0}+\boldsymbol{v'})\cdot\boldsymbol{\nabla})(\boldsymbol{v_0}+\boldsymbol{v'})\right) = -\boldsymbol{\nabla}p_0 - \boldsymbol{\nabla}p' + \rho_0\boldsymbol{f} + \underbrace{\rho'\boldsymbol{f}}_{\substack{\boldsymbol{f}\text{ induces a}\\ \text{small perturbation}}}$$

As $\boldsymbol{f}$ only induces a small perturbation, it is also a first order term so $\rho'\boldsymbol{f}$ is a second order term that is negligible. Developping the left hand side and keeping only terms up to first order :

$$\rho_0\frac{\partial \boldsymbol{v'}}{\partial t} + \rho_0(\boldsymbol{v_0}\cdot\boldsymbol{\nabla})\boldsymbol{v_0} + \rho_0(\boldsymbol{v_0}\cdot\boldsymbol{\nabla})\boldsymbol{v'} + \rho_0(\boldsymbol{v'}\cdot\boldsymbol{\nabla})\boldsymbol{v_0} + \rho'(\boldsymbol{v_0}\cdot\boldsymbol{\nabla})\boldsymbol{v_0} = -\boldsymbol{\nabla}p_0 - \boldsymbol{\nabla}p' + \rho_0\boldsymbol{f}$$

Regrouping the terms, and using (9) to replace $(\boldsymbol{v_0}\cdot\boldsymbol{\nabla})\boldsymbol{v_0} = -\frac{1}{\rho_0}\boldsymbol{\nabla}p_0$,

$$\rho_0\frac{\mathrm{D}_c\boldsymbol{v'}}{\mathrm{D}t} - \boldsymbol{\nabla}p_0 + \rho_0(\boldsymbol{v'}\cdot\boldsymbol{\nabla})\boldsymbol{v_0} - \frac{\rho'}{\rho_0}\boldsymbol{\nabla}p_0 = -\boldsymbol{\nabla}p_0 - \boldsymbol{\nabla}p' - \rho_0\boldsymbol{f}$$

Thus, dividing by ρ_0 yields :

$$\boxed{\frac{\mathrm{D}_c\boldsymbol{v'}}{\mathrm{D}t} + (\boldsymbol{v'}\cdot\boldsymbol{\nabla})\boldsymbol{v_0} = \frac{\rho'}{\rho_0^2}\boldsymbol{\nabla}p_0 - \frac{1}{\rho_0}\boldsymbol{\nabla}p' + \boldsymbol{f}}$$

For the state equation, we first simplify each of the terms

$$\begin{aligned}\frac{\mathrm{D}_c p}{\mathrm{D}t} &= \frac{\partial p'}{\partial t} + (\boldsymbol{v_0}+\boldsymbol{v'})\cdot\boldsymbol{\nabla}(p_0+p')\\ &= \frac{\partial p'}{\partial t} + \boldsymbol{v_0}\cdot\boldsymbol{\nabla}p_0 + \boldsymbol{v_0}\cdot\boldsymbol{\nabla}p' + \boldsymbol{v'}\cdot\boldsymbol{\nabla}p_0\\ &= \frac{\mathrm{D}_c p'}{\mathrm{D}t} + \underbrace{\boldsymbol{v_0}\cdot\boldsymbol{\nabla}p_0}_{=c_0^2(\boldsymbol{v_0}\cdot\boldsymbol{\nabla})\rho_0\quad(9)} + \boldsymbol{v'}\cdot\boldsymbol{\nabla}p_0\end{aligned}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = c^2 = c_0^2 + (c^2)'$$

$$\begin{aligned}\frac{\mathrm{D}_c \rho}{\mathrm{D}t} &= \frac{\partial \rho'}{\partial t} + (\boldsymbol{v_0}+\boldsymbol{v'})\cdot\boldsymbol{\nabla}(\rho_0+\rho')\\ &= \frac{\partial \rho'}{\partial t} + \boldsymbol{v_0}\cdot\boldsymbol{\nabla}\rho_0 + \boldsymbol{v_0}\cdot\boldsymbol{\nabla}\rho' + \boldsymbol{v'}\cdot\boldsymbol{\nabla}\rho_0\\ &= \frac{\mathrm{D}_c \rho'}{\mathrm{D}t} + \underbrace{\boldsymbol{v_0}\cdot\boldsymbol{\nabla}\rho_0}_{=-\rho_0\boldsymbol{\nabla}\cdot\boldsymbol{v_0}\quad(9)} + \boldsymbol{v'}\cdot\boldsymbol{\nabla}\rho_0\end{aligned}$$

Thus

$$\left(\frac{\mathrm{D}_c p'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} p_0\right) + c_0^2 (\boldsymbol{v_0} \cdot \boldsymbol{\nabla}) \rho_0 = (c_0^2 + \underbrace{(c^2)'}_{*}) \left(\frac{\mathrm{D}_c \rho'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} \rho_0\right) + (c_0^2 + (c^2)')(\boldsymbol{v_0} \cdot \boldsymbol{\nabla}) \rho_0$$

The (*) term is a first order term multiplied by another first order term, we can thus neglect it. Removing it and simplyfing the $c_0^2(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\rho_0$ term gives the result.

□

2.2 The Convected Helmholtz equation

In the case of a background flow velocity $\boldsymbol{v_0}$ is a non vanishing constant, then $\boldsymbol{\nabla} p_0 = 0$ and $\boldsymbol{v_0} \cdot \boldsymbol{\nabla} \rho_0 = 0$ Then taking $\frac{d}{dt}$(mass conservation) $- \boldsymbol{\nabla} \cdot$ (momentum conservation) :

$$\frac{\mathrm{D}_c}{\mathrm{D}t}\left(\frac{1}{c_0^2}\frac{\mathrm{D}_c p'}{\mathrm{D}t}\right) - \rho_0 \boldsymbol{\nabla} \cdot \left(\frac{1}{\rho_0} \boldsymbol{\nabla} p'\right) = \frac{\mathrm{D}_c m}{\mathrm{D}t} - \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{f} \tag{12}$$

Proof. As $\boldsymbol{v_0}$ is a non vanishing constant, the equations (9) give us

$$0 = \rho_0 (\boldsymbol{v_0} \cdot \boldsymbol{\nabla}) \boldsymbol{v_0} = \boldsymbol{\nabla p_0} \Rightarrow \boldsymbol{\nabla} p_0 = 0$$

$$(\boldsymbol{v_0} \cdot \boldsymbol{\nabla}) \rho_0 = -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v_0} = 0 \rightarrow (\boldsymbol{v_0} \cdot \boldsymbol{\nabla}) \rho_0 = 0$$

Then, we write the mass conversation, momentum conversation and state equation (10) and simplify them.

$$\frac{\mathrm{D}_c \rho'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} \rho_0 = -\underbrace{\rho' \boldsymbol{\nabla} \cdot \boldsymbol{v_0}}_{=0} - \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v'} + m$$

$$\frac{\mathrm{D}_c \rho'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} \rho_0 = -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v'} + m$$

For momentum conservation :

$$\frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} + \underbrace{(\boldsymbol{v'} \cdot \boldsymbol{\nabla}) \boldsymbol{v_0}}_{=0} = \underbrace{-\frac{\rho'}{\rho_0^2} \boldsymbol{\nabla} p_0}_{=0} - \frac{1}{\rho_0} \boldsymbol{\nabla} p' + \boldsymbol{f}$$

$$\frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} = -\frac{1}{\rho_0} \boldsymbol{\nabla} p' + \boldsymbol{f}$$

For the state equation :

$$\frac{\mathrm{D}_c p'}{\mathrm{D}t} + \underbrace{\boldsymbol{v'} \cdot \boldsymbol{\nabla} p_0}_{=0} = c_0^2 \left(\frac{\mathrm{D}_c p'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} \rho_0\right) + (c^2)' \underbrace{\boldsymbol{v_0} \cdot \boldsymbol{\nabla} \rho_0}_{=0}$$

$$\frac{\mathrm{D}_c p'}{\mathrm{D}t} = c_0^2 \left(\frac{\mathrm{D}_c p'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} \rho_0\right)$$

Then, with these three new equations, we have

$$\begin{aligned}
\frac{\mathrm{D}_c}{\mathrm{D}t}\left(\frac{1}{c_0^2}\frac{\mathrm{D}_c p'}{\mathrm{D}t}\right) &= \frac{\mathrm{D}_c}{\mathrm{D}t}\left(\frac{\mathrm{D}_c p'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} \rho_0\right) \\
&= \frac{\mathrm{D}_c}{\mathrm{D}t}(m - \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v'}) \\
&= \frac{\mathrm{D}_c m}{\mathrm{D}t} - \rho_0 \frac{\mathrm{D}_c}{\mathrm{D}t}(\boldsymbol{\nabla} \cdot \boldsymbol{v'})
\end{aligned}$$

And

$$-\rho_0 \boldsymbol{\nabla} \cdot \left(\frac{1}{\rho_0} \boldsymbol{\nabla} p' \right) = -\rho_0 \boldsymbol{\nabla} \cdot \left(\boldsymbol{f} - \frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} \right)$$
$$= -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{f} + \rho_0 \boldsymbol{\nabla} \cdot \left(\frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} \right)$$

Summing these two expression to get the left side of the convected Helmotz equation :

$$\frac{\mathrm{D}_c}{\mathrm{D}t} \left(\frac{1}{c_0^2} \frac{\mathrm{D}_c p'}{\mathrm{D}t} \right) - \rho_0 \boldsymbol{\nabla} \cdot \left(\frac{1}{\rho_0} \boldsymbol{\nabla} p' \right) = \frac{\mathrm{D}_c m}{\mathrm{D}t} - \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{f} + \rho_0 \underbrace{\left(\boldsymbol{\nabla} \cdot \frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} - \frac{\mathrm{D}_c}{\mathrm{D}t} (\boldsymbol{\nabla} \cdot \boldsymbol{v'}) \right)}_{=A}$$

What is left to do is to prove that $A = 0$

$$A = \underbrace{\left(\boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{v'}}{\partial t} - \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{v'}}{\partial t} \right)}_{=0} + \underbrace{(\boldsymbol{\nabla} \cdot ((\boldsymbol{v_0} \cdot \boldsymbol{\nabla}) \boldsymbol{v'}) - \boldsymbol{v_0} \cdot \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{v'}))}_{=B}$$

$B = (\boldsymbol{\nabla} \cdot ((\boldsymbol{v_0} \cdot \boldsymbol{\nabla}) \boldsymbol{v'}) - \boldsymbol{v_0} \cdot \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{v'}))$

Denoting $\frac{\partial}{\partial x_i}$the derivation along the i-th coordinate and x_i the i-th component of$x$

$$B = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left[\sum_{j=1}^{3} v_{0j} \frac{\partial v_i'}{\partial x_j} \right] - \sum_{i=1}^{3} v_{0i} \frac{\partial}{\partial x_i} \left[\sum_{j=1}^{3} \frac{\partial v_j'}{\partial x_j} \right]$$
$$= \sum_{1 \leq i,j \leq 3} \underbrace{\frac{\partial}{\partial x_i} \left(v_{0,j} \frac{\partial v_i'}{\partial x_j} \right)} - v_{0i} \frac{\partial^2 v_j'}{\partial x_i \partial y_j}$$
$$= \frac{\partial v_{0,j}}{\partial x_i} \frac{\partial v_i'}{\partial x_j} + v_{0j} \frac{\partial^2 v_i'}{\partial x_i \partial y_i}$$
$$= v_{0j} \frac{\partial^2 v_i'}{\partial x_i \partial y_i}$$
$$= \sum_{1 \leq i,j \leq 3} v_{0j} \frac{\partial^2 v_i'}{\partial x_i \partial x_j} - v_{0i} \frac{\partial^2 v_j'}{\partial x_i \partial x_j}$$

with a change of indices $i \leftrightarrow j$ and using $\frac{\partial^2 v_i'}{\partial x_j \partial x_i} = \frac{\partial^2 v_i'}{\partial x_i \partial x_j}$

$$B = \sum_{1 \leq i,j \leq 3} v_{0j} \frac{\partial^2 v_i'}{\partial x_i \partial x_j} - v_{0j} \frac{\partial^2 v_i'}{\partial x_i \partial x_j}$$

$B = 0$

Then $A = 0$ □

As seen in [3], applying this convected wave equation to our problem, with the conditions $\mathbf{v_0} = (u_0, 0, 0), \mathbf{f} = 0, m = 0$ (it thus means that the source of the perturbation, m and $\boldsymbol{f}$ are outside of the volume we consider, but still create the perturbation), and u_0 constant, we have

$$\frac{\partial^2 p'}{\partial t^2} + 2u_0 \frac{\partial^2 p'}{\partial x \partial t} + u_0^2 \frac{\partial^2 p'}{\partial x^2} - c_0^2 \Delta p' = 0$$

We seek harmonic solutions, so we have the equation (9) from the ref [3].

$$\Delta_{\perp}\hat{p}' + k_0^2\hat{p}' + (1 - M_0^2)\frac{\partial^2\hat{p}'}{\partial x^2} - 2ik_0M_0\frac{\partial\hat{p}'}{\partial x} = 0 \tag{13}$$

With the following conventions :

- dependance in $e^{+i\omega t}$
- $M_0 = \frac{u_0}{c_0}$ is the Mach number
- $k_0 = \frac{\omega}{c_0}$ is the wave number (without flow)
- $\Delta_{\perp} = \Delta - \partial_x^2$

We can write this equation in a simpler form, as follow :

$$\Delta\hat{p}' + k_0^2\left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)^2\hat{p}' = 0 \tag{14}$$

2.3 Geometry of the studied problem

We model this problem in a rectangular 2D domain Ω with a border $\partial\Omega$.

$\Omega = \{(x, y), 0 < x < L, 0 < y < l\}$; $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_D \cup \Gamma_N$.
With

- Γ_1 and Γ_2 the border with the acoustic liner
- Γ_D the left part (inflow condition)
- Γ_N the right part (outgoing flow)

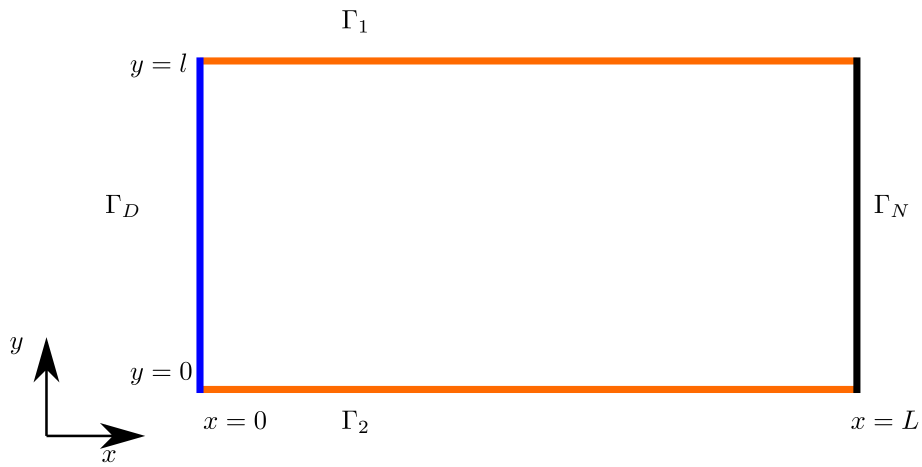

Figure 1: The model problem for acoustic liners

2.4 The Boundary condition on the hard walls

We denote Γ_1 and Γ_2 the two surfaces (at rest) of the liners.

2.4.1 On the liner

To obtain the boundary condition, we use the continuity of the normal displacement at the surface of the liner. We denote by ξ_1 the normal displacement at the surface of the wall. Using the impedance of the liner $Z = \frac{p'}{\mathbf{v'}\cdot\mathbf{n}}$ we have the following condition for ξ_1:

$$\frac{\partial \xi_1}{\partial t} = \mathbf{v}' \cdot \mathbf{n} = \frac{p'}{Z}$$

Where $Z = Z(\omega)$ is the complex normal impedance of the liner (and, as seen in [1], the effect of fluid motion is that $Z' = Z(1 + M_0 \sin(\phi))$ with ϕ the angle of incidence, $\phi = 0$ in normal incidence so the impedance Z we have to consider is the complex normal impedance of the liner at rest with no fluid flow).

We denote by ξ_2 the normal displacement of a fluid particle on the boundary of the domain. This particle's motion satisfies the Navier-Stokes equation using the assumptions of the model, it yields :

$$\rho_0(\partial_t + u_0\partial_x)^2\xi_2 = -\frac{\partial p'}{\partial \mathbf{n}}$$

Finally writing $\xi_1 = \xi_2$ yields the temporal boundary condition. As we are seeking the harmonic solutions of the system with frequency f $= 2\pi\omega$, we can rewrite the equations for $\hat{\xi}_1$, $\hat{\xi}_2$ and $\hat{p}'$ (the harmonic form of ξ_1, ξ_2 and p').

$$\begin{cases} i\omega\hat{\xi}_1 &= \frac{\hat{p}'}{Z} \\ \rho_0(i\omega + u_0\partial_x)^2\hat{\xi}_2 &= -\frac{\partial \hat{p}'}{\partial \mathbf{n}} \end{cases}$$

Setting ξ_1 equal to ξ_2 yields :

$$Z\frac{\partial \hat{p}'}{\partial \mathbf{n}} + i\rho\omega(1 - i\frac{u_0}{\omega}\partial_x)^2\hat{p}' = 0$$

Using the physical quantities $Z_0 = \rho_0 c_0$ (fluid impedance), $M_0 = \frac{u_0}{c_0}$ (Mach number of the flow) and $k_0 = \frac{\omega}{c_0}$ (wave vector of the harmonic acoustic wave) we get our final boundary condition :

$$Z\frac{\partial \hat{p}'}{\partial \mathbf{n}} + iZ_0k_0(1 - i\frac{M_0}{k_0}\partial_x)^2\hat{p}' = 0 \tag{15}$$

2.4.2 On the hard wall

In the limit case of a hard wall (Z goes to infinity), we can use the same method but this time ξ_1 the normal displacement on the wall is zero. It yields

$$\frac{\partial \hat{p}'}{\partial \mathbf{n}} = 0 \tag{16}$$

2.4.3 Final form of the boundary condition on $\Gamma_1 \cup \Gamma_2$

We add a term $\chi \in L^2(\Gamma_1 \cup \Gamma_2)$ such that

$$\forall z \in \Gamma_1 \cup \Gamma_2, \chi(z) = \begin{cases} 1 & \text{if we placed a liner placed at this emplacement} \\ 0 & \text{otherwise} \end{cases}$$

This function χ is the liner distribution on the surfaces $\Gamma_1 \cup \Gamma_2$, the boundary condition on the hard wall and on the liner can be given by :

$$Z\frac{\partial \hat{p}'}{\partial \mathbf{n}} + iZ_0\chi k_0(1 - i\frac{M_0}{k_0}\partial_x)^2\hat{p}' = 0 \quad \text{on}\, \Gamma_1 \cup \Gamma_2 \tag{17}$$

2.5 More general boundary condition

We can extend this boundary condition to non-rectangular geometries, as is done below :

We start from the following equation.

$$\rho_0(i\omega + u_0\partial_x)\hat{\mathbf{v}}' \cdot \mathbf{n} = -\frac{\partial \hat{p}'}{\partial \mathbf{n}} \text{ on } \Gamma_1 \cup \Gamma_2 \tag{18}$$

The second equation we need comes from [2], the hypothesis taken is that the surface of the liner is deformed by the incident sound field from the fluid (or the inverse situation). The continuity of the normal speed is expressed on the moving surface of the liner $S(t)$. After some calculations, the starting boundary condition can be expressed on the mean surface of the liner S_0, (equation (11) from [2]). As we are seeking harmonic solutions (the deformation of the liner is also supposed to be harmonic), we have the equation (14) from [2].

Rewriting this equation with our notations ($\mathbf{n}$ as the outward pointing normal to our domain), we have

$$\hat{\mathbf{v}}' \cdot \mathbf{n} = \frac{\hat{p}'}{Z} + \frac{1}{i\omega}\left(\mathbf{v}_0 \cdot \nabla\left(\frac{\hat{p}}{Z}\right)\right) + \frac{\hat{p}'}{i\omega Z}\left(\mathbf{n} \cdot (\mathbf{n} \cdot \nabla \mathbf{v_0})\right) \text{ on } \Gamma_1 \cup \Gamma_2$$

Where $Z = Z(\omega)$ is the complex normal impedance of the liner (and, as seen in [1], the effect of fluid motion is that $Z' = Z(1 + M_0 \sin(\phi))$ with ϕ the angle of incidence, $\phi = 0$ in normal incidence so the impedance Z we have to consider is the complex normal impedance of the liner at rest with no fluid flow).

This equation covers a larger range of problems, particularly with non rectangular geometries and could be used to extend this work to a more realistic geometry.

In our case, we can verify we obtain the same result by simplifying this equation, and we obtain (the second term cancels out) :

$$i\omega Z\hat{\mathbf{v}}' \cdot \mathbf{n} = \left(i\omega + u_0\frac{\partial}{\partial x}\right)\hat{p} \text{ on } \Gamma_1 \cup \Gamma_2 \tag{19}$$

This condition can also be written in the temporal domain as follow :

$$\frac{\partial}{\partial t}(\mathbf{v}' \cdot \mathbf{n}) = \frac{\mathrm{D}_c}{\mathrm{D}t}(Z^{-1} *_t p')$$

with $*_t$ the temporal convolution

Combining equations (18) and (19) (eliminating $\hat{\mathbf{v}}' \cdot \mathbf{n}$) and using the definition of $Z_0 = \rho_0 c_0$ the fluid impedance, we have

$$Z\frac{\partial \hat{p}'}{\partial \mathbf{n}} + ik_0 Z_0\left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)^2 \hat{p}' = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \tag{20}$$

2.6 Inflow and outflow boundary condition

We impose a non homogeneous Dirichlet on one side Γ_D and a corresponding condition on Γ_R to eliminate the reflection.

The acoustic pressure on Γ_D will have a dependency in

$$\hat{p}' = s(y)e^{-ik'x}, \mathrm{Re}(k') > 0$$

On Γ_R, we set a boundary condition so as to only allow as a dependency in x the one of the imposed mode on the other part of the duct, that is to say

$$\frac{\partial \hat{p}'}{\partial n} = \frac{\partial \hat{p}'}{\partial x} = -ik'\hat{p}' \text{ on } \Gamma_R$$

2.7 Final Model problem

We point out now that in this section we did not take into account χ, so in what follows this is what we would obtain with $\chi = 1$ on $\Gamma_1 \cup \Gamma_2$ Regrouping all the conditions we set : We model this problem in a 2D domain Ω with a border $\partial\Omega$.

$\Omega = \{(x, y), 0 < x < L, 0 < y < l\}$; $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_D \cup \Gamma_N$ where

- $\Delta \hat{p}' + k_0^2 \left(1 - i\dfrac{M_0}{k_0}\dfrac{\partial}{\partial x}\right)^2 \hat{p}' = 0$ in Ω
- $\hat{p}'|_{\Gamma_D} = s(y)$ on Γ_D (inflow condition)
- $Z\dfrac{\partial \hat{p}'}{\partial \mathbf{n}} + ik_0 Z_0 \left(1 - i\dfrac{M_0}{k_0}\dfrac{\partial}{\partial x}\right)^2 \hat{p}'|_{\Gamma_1 \cup \Gamma_2} = 0$ on $\Gamma_1 \cup \Gamma_2$ (acoustic liner boundary condition)
- $\dfrac{\partial \hat{p}'}{\partial \mathbf{n}} + ik'\ \hat{p}'|_{\Gamma_N} = 0$ on Γ_N (radiation condition,outgoing flow)

To simplify the notations, we introduce the operator A, defined as follow :

$$A = k_0 \left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right) \tag{21}$$

Then the equation in Ω rewrites to

$$\Delta \hat{p}' + A^2 \hat{p}' = 0 \text{ in } \Omega \tag{22}$$

And the boundary condition on the liner rewrites to :

$$Z\frac{\partial \hat{p}'}{\partial \boldsymbol{n}} + i\frac{Z_0}{k_0} A^2 \hat{p}' = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \tag{23}$$

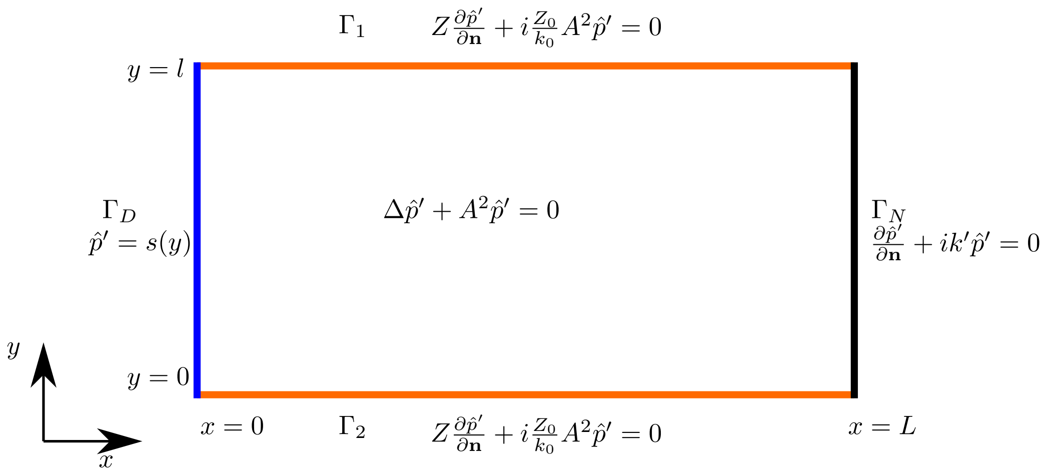

Figure 2: The model problem for acoustic liners

3 The Variational formula

3.1 The weak formulation

The condition on Γ_D being non homogeneous isn't practical, we thus rewrite our problem with the form

$$\hat{p}' = u + g$$

(u is a new variable, and not anymore the pression) With g veryfing

$$\begin{cases} \Delta g & = 0 \text{ in } \Omega \\ g|_{\Gamma_D} & = s(y) \\ \dfrac{\partial g}{\partial n} & = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_N \end{cases}$$

Thus we have

$$\begin{cases} \Delta u + A^2 u = \underbrace{-A^2 g}_{f} \\ u|_{\Gamma_D} = 0 \\ \dfrac{\partial u}{\partial n} + \dfrac{iZ_0}{Zk_0} A^2 u = \underbrace{-\dfrac{iZ_0}{Zk_0} A^2 g}_{\eta} \text{ on } \Gamma_1 \cup \Gamma_2 \\ \dfrac{\partial u}{\partial n} + ik' u = \underbrace{-ik' g}_{\gamma} \end{cases}$$

Let's then introduce a few notations and lemmas that will be used throughout this paper.

$$T(\Omega) = H^1(\Omega) \cap \{u \in L^2(\Omega), u(x) = 0, \forall x \in \Gamma_D\}$$
$$\forall u \in T(\Omega), \|u\|_{T(\Omega)} = \|\nabla u\|_{L^2}$$

Recall that

$$\begin{cases} A : T(\Omega) \to L^2(\Omega) \\ A = k_0 \left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right) \end{cases}$$

Let's see one useful lemma that can be proven with Green's formula for integration by part on A.

Lemma 3.1. *Let $u \in T(\Omega)$ and v be two functions on which the green formulas can be applied, it holds:*

$$\int_\Omega A(u)\bar{v} = \int_\Omega u A^* \bar{v} - iM_0 \int_{\Gamma_N} u\bar{v}$$

with

$$A^* = k_0 \left(1 + \frac{iM_0}{k_0}\frac{\partial}{\partial x}\right)$$

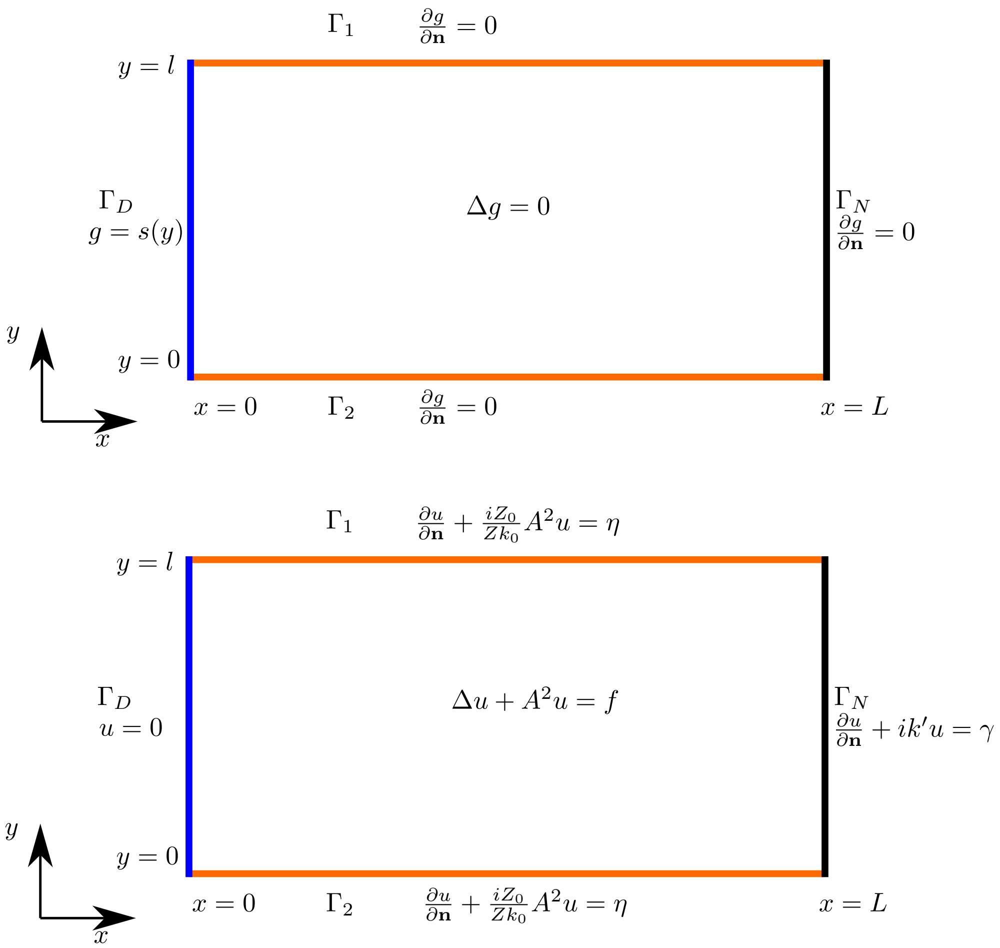

Figure 3: The new model problem for g and p

Proof.

$$\begin{aligned}\int_\Omega A(u)\bar{v}d\Omega &= \int_\Omega k_0 u\bar{v}d\Omega - iM_0\int_\Omega \frac{\partial u}{\partial x}\bar{v}d\Gamma \\ &= \int_\Omega k_0 u\bar{v}d\Omega - iM_0\left(\int_{\partial\Omega} u\bar{v}\boldsymbol{e_x}\cdot\boldsymbol{n}d\Gamma - \int_\Omega u\frac{\partial \bar{v}}{\partial x}d\Omega\right) \\ &= \int_\Omega u\left(k_0\bar{v} + iM_0\frac{\partial \bar{v}}{\partial x}\right)\Gamma - iM_0\int_{\Gamma_N} u\bar{v}d\Gamma \\ &= \int_\Omega uA^*(\bar{v})d\Omega - iM_0\int_{\Gamma_N} u\bar{v}d\Gamma\end{aligned}$$

□

We are now ready to write the variational formula associated to the convected Helmoltz equation. Given $\varphi \in T(\Omega)$, we have the following equation:

Lemma 3.2. *We have, from the convected Helmoltz equation*

$$\forall\varphi \in T(\Omega), a(u,\varphi) = l(\varphi)$$

with:

$$\begin{cases} a(u,\varphi) &= \int_\Omega \nabla u\nabla\bar{\varphi} - Au\overline{A\varphi}d\Omega + \int_{\Gamma_1\cup\Gamma_2}\frac{iZ_0}{Zk_0}A^2(u)\bar{\varphi}d\Gamma + \int_{\Gamma_N}\left[iM_0A(u)\bar{\varphi} + ik'u\bar{\varphi}\right]d\Gamma \\ l(\varphi) &= -\int_\Omega f\bar{\varphi}d\Omega + \int_{\Gamma_1\cup\Gamma_2}\eta\bar{\varphi}d\Gamma + \int_{\Gamma_N}\gamma\bar{\varphi}d\Gamma \end{cases}$$

3.2 Properties

We will try to apply the Lax-Milgram theorem (as a first approach at resolution), we thus gather some results.

Theorem 3.1. *a is a bilinear form, well defined on $T(\Omega)$ and l is a linear continuous form on $T(\Omega)$.*

With a homegeneous boundary dirichlet boundary condition, we have the Poincarré inequality, which will be useful to prove the continuity of a.

Lemma 3.3 (Poincarré Inequality)**.**

$$\forall u \in T(\Omega), \|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{T(\Omega)}$$

A simple majoration yelds, $\forall v \in H^1(\Omega), \|\nabla v\|_{L^2(\Omega)} \geq \left\|\frac{\partial v}{\partial x}\right\|_{L^2(\Omega)}$. This result allows us to prove the continuity of A.

Lemma 3.4. *The operator A is continuous from $T(\Omega)$ to L^2.*

Proof. $\forall u \in T(\Omega), \|Au\|_{L^2} \leq k_0\|u\|_{L^2} + M_0\|\nabla u\|_{L^2} \leq (k_0C + M_0)\|u\|_{T(\Omega)}$ □

3.3 Existence and unicity of the solution for u

The Lax-Milgram theorem does not allow us to conclude, we will use the first Fredholm theorem instead.

Theorem 3.2. *Let A be a linear compact operator in a Banach space X. The following assertions are equivalent*

1. *$\forall f \in X, x = Ax + f$ has a solution in X*

2. $z - Az = 0$ (homogeneous equation) has only the trivial solution $z = 0$

3. $\forall g \in X^*, y - A^*y = 0$ has a solution in X^*

4. homogeneous equation $\phi - A^*\phi = 0$ has only the trivial solution $\phi = 0$

Furthermore, if one of the assertion 1 to 4 holds, then there exists the inverse operators $(I - A)^{-1}$ and $(I - A^*)^{-1}$ which are continuous

First, let us remember the lemma 3.2 on u :

$$a(u,\varphi) = \int_\Omega \underbrace{\nabla u \nabla \bar{\varphi}}_{(1)} + \underbrace{(-Au\overline{A\varphi})}_{(2)}\, d\Omega + \int_{\Gamma_1\cup\Gamma_2} \underbrace{\frac{iZ_0}{Zk_0} A^2(u)\bar{\varphi}\, d\Gamma}_{(3)} + \int_{\Gamma_N} [\underbrace{iM_0A(u)\bar{\varphi}}_{(4)} + \underbrace{ik'u\bar{\varphi}}_{(5)}] d\Gamma$$

We will have to redevelop the terms in A, A^*, A^2. As a reminder, we have

$$A = k_0\left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)$$

$$A^* = k_0\left(1 + i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)$$

$$\begin{aligned}(2) &= -k_0^2\left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)(u)\left(1 + i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)(\overline{\varphi}) \\ &= -k_0^2\left(u\overline{\varphi} + i\frac{M_0}{k_0}\left(u\frac{\partial\overline{\varphi}}{\partial x} - \frac{\partial u}{\partial x}\overline{\varphi}\right) + \frac{M_0^2}{k_0^2}\frac{\partial u}{\partial x}\frac{\partial\overline{\varphi}}{\partial x}\right)\end{aligned}$$

What we want to change is the term in $u\dfrac{\partial\overline{\varphi}}{\partial x}$:

$$\int_\Omega u\frac{\partial\overline{\varphi}}{\partial x}d\Omega = -\int_\Omega \frac{\partial u}{\partial x}\overline{\varphi}d\Omega + \int_{\Gamma_N} u\overline{\varphi}d\Gamma$$

Thus

$$\int_\Omega (2)d\Omega = -k_0^2\langle u \mid \varphi\rangle_{L^2(\Omega)} - M_0^2\left\langle \frac{\partial u}{\partial x}\middle| \frac{\partial\varphi}{\partial x}\right\rangle_{L^2(\Omega)} + 2iM_0k_0\left\langle\frac{\partial u}{\partial x}\middle|\varphi\right\rangle_{L^2(\Omega)} - iM_0k_0\langle u \mid \varphi\rangle_{L^2(\Gamma_N)}$$

For (3):

$$\begin{aligned}\int_{\Gamma_1\cup\Gamma_2}(3)d\Omega &= \frac{iZ_0\overline{Z(\omega)}}{|Z(\omega)|^2k_0}\int_\Omega k_0^2\left(1 - 2i\frac{M_0}{k_0}\frac{\partial}{\partial x} - \frac{M_0^2}{k_0^2}\frac{\partial^2}{\partial x^2}\right)(u)\overline{\varphi}d\Omega \\ &= \frac{iZ_0\overline{Z(\omega)}k_0}{|Z(\omega)|^2}\left(\langle u \mid \varphi\rangle_{L^2(\Gamma_1\cup\Gamma_2)} - \frac{2iM_0}{k_0}\left\langle\frac{\partial u}{\partial x}\middle|\varphi\right\rangle_{L^2(\Gamma_1\cup\Gamma_2)} - \frac{M_0^2}{k_0^2}\left\langle\frac{\partial^2 u}{\partial x^2}\middle|\varphi\right\rangle_{L^2(\Gamma_1\cup\Gamma_2)}\right)\end{aligned}$$

For (4):

$$\begin{aligned}\int_{\Gamma_N}(4)d\Omega &= \int_{\Gamma_N} iM_0k_0\left(u\overline{\varphi} - i\frac{M_0}{k_0}\frac{\partial u}{\partial x}\overline{\varphi}\right)d\Omega \\ &= iM_0k_0\langle u \mid \varphi\rangle_{L^2(\Gamma_N)} + M_0^2\left\langle\frac{\partial u}{\partial x}\middle|\varphi\right\rangle_{L^2(\Gamma_N)}\end{aligned}$$

If we rewrite our variational formulation with scalar product, we thus end up with

$$
\begin{aligned}
a(u,\varphi) = {} & \langle \nabla u | \nabla \varphi \rangle_{L^2(\Omega)} && (1)\\
& - M_0^2 \left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Omega)} - k_0^2 \langle u \mid \varphi \rangle_{L^2(\Omega)} + 2iM_0k_0 \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Omega)} \boldsymbol{- i M_0 k_0 \langle u \mid \varphi \rangle_{L^2(\Gamma_N)}} && (2)\\
& + \frac{iZ_0\overline{Z(\omega)}k_0}{|Z(\omega)|^2} \left(\langle u \mid \varphi \rangle_{L^2(\Gamma_1 \cup \Gamma_2)} - \frac{2iM_0}{k_0} \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} - \frac{M_0^2}{k_0^2} \left\langle \frac{\partial^2 u}{\partial x^2} \middle| \varphi \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} \right) && (3)\\
& \boldsymbol{+ i M_0 k_0 \langle u \mid \varphi \rangle_{L^2(\Gamma_N)}} + M_0^2 \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Gamma_N)} && (4)\\
& + ik' \langle u | \varphi \rangle_{L^2(\Gamma_N)} && (5)
\end{aligned}
$$

The two terms in (2) and (4) in bold cancel out, so we end up this final fom :

$$
\begin{aligned}
a(u,\varphi) = {} & \langle \nabla u | \nabla \varphi \rangle_{L^2(\Omega)} && (1)\\
& - M_0^2 \left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Omega)} - k_0^2 \langle u \mid \varphi \rangle_{L^2(\Omega)} + 2iM_0k_0 \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Omega)} && (2)\\
& + \frac{iZ_0\overline{Z(\omega)}k_0}{|Z(\omega)|^2} \left(\langle u \mid \varphi \rangle_{L^2(\Gamma_1 \cup \Gamma_2)} - \frac{2iM_0}{k_0} \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} - \frac{M_0^2}{k_0^2} \left\langle \frac{\partial^2 u}{\partial x^2} \middle| \varphi \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} \right) && (3)\\
& + M_0^2 \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Gamma_N)} && (4)\\
& + ik' \langle u | \varphi \rangle_{L^2(\Gamma_N)} && (5)
\end{aligned}
$$

As we are searching the weak formulation, by derivating in the distribution sense, we have

Lemma 3.5.

$$
- \left\langle \frac{\partial^2 u}{\partial x^2} \middle| \varphi \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} = + \left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)}
$$

and we can rewrite the term (3) and (4). Note that we change $l(\varphi)$ by adding a term

$$
-M_0^2 \int_{\Gamma_N} \gamma \bar{\varphi} d\Gamma
$$

$$
\begin{aligned}
a(u,\varphi) = {} & \langle \nabla u | \nabla \varphi \rangle_{L^2(\Omega)} && (1)\\
& - M_0^2 \left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Omega)} - k_0^2 \langle u \mid \varphi \rangle_{L^2(\Omega)} + 2iM_0k_0 \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Omega)} && (2)\\
& + \frac{iZ_0\overline{Z(\omega)}k_0}{|Z(\omega)|^2} \left(\langle u \mid \varphi \rangle_{L^2(\Gamma_1 \cup \Gamma_2)} - \frac{2iM_0}{k_0} \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} + \frac{M_0^2}{k_0^2} \left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} \right) && (3)\\
& + (1 - M_0^2) ik' \langle u | \varphi \rangle_{L^2(\Gamma_N)} && (4)+(5)
\end{aligned}
$$

We also have $l(\varphi)$

$$
l(\varphi) = - \int_{\Omega} f \bar{\varphi} d\Omega + \int_{\Gamma_1 \cup \Gamma_2} \eta \bar{\varphi} d\Gamma + (1 - M_0^2) \int_{\Gamma_N} \gamma \bar{\varphi} d\Gamma
$$

$$\forall \varphi \in V(\Omega), a(u,\varphi) = l(\varphi)$$

defines our variational formula.

So as to get some results on the compacity of the operator, we define $V(\Omega)$ a subset of $T(\Omega)$:

$$V(\Omega) = \{H^1(\Omega) | \Delta u \in L^2(\Omega), \text{ + weak boundary conditions}\}$$

We need to define a new scalar product in $V(\Omega)$:

$$\langle u \mid v \rangle_{V(\Omega)} = \langle \nabla u \mid \nabla v \rangle_{L^2(\Omega)} - M_0^2 \left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial v}{\partial x} \right\rangle$$

(ok under the hypothesis of $(1 - M_0^2) > 0$)

3.3.1 Rewriting the scalar products

We have to remember in what follows that we ommited to put the trace operator on the scalar products on the boundary Γ_N and $\Gamma_1 \cup \Gamma_2$ in the variational formulation.

We write $i_{L^2(\Omega)} : V(\Omega) \to L^2(\Omega)$ the linear compact injection from $V(\Omega)$ to $L^2(\Omega)$

$$\exists S : L^2(\Omega) \to V(\Omega), \forall \psi \in L^2(\Omega), \forall \varphi \in V(\Omega), \left\langle \psi \middle| \varphi \right\rangle_{L^2(\Omega)} = \left\langle S\psi \middle| \varphi \right\rangle_{V(\Omega)}$$

$$\exists T : L^2(\Gamma_1 \cup \Gamma_2) \to V(\Omega), \forall \psi \in L^2(\Gamma_1 \cup \Gamma_2), \forall \varphi \in V(\Omega), \left\langle \psi \middle| \operatorname{Tr}(\varphi) \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} = \left\langle T\psi \middle| \varphi \right\rangle_{V(\Omega)}$$

$$\exists U : L^2(\Gamma_N) \to V(\Omega), \forall \psi \in L^2(\Gamma_N), \forall \varphi \in V(\Omega), \left\langle \psi \middle| \operatorname{Tr}(\varphi) \right\rangle_{L^2(\Gamma_N)} = \left\langle U\psi \middle| \varphi \right\rangle_{V(\Omega)}$$

We also have to take into account the term in $\left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)}$ so we define a new operator :

$$\exists R : \underbrace{H_0^1(\Gamma_1 \cup \Gamma_2)}_{\operatorname{Tr}(f)|_{x=0}=0} \to V(\Omega)$$

$$\forall \psi \in H_0^1(\Omega), \forall \varphi \in V(\Omega), \left\langle \frac{\partial \psi}{\partial x} \middle| \operatorname{Tr} \circ \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} = \left\langle R(\psi) \middle| \varphi \right\rangle_{V(\Omega)}$$

$$\begin{aligned}
\forall u, \varphi \in V(\Omega), \quad a(u,\varphi) = {} & \left\langle u \middle| \varphi \right\rangle_{V(\Omega)} && \text{(1) and first term of (2)} \\
& - k_0^2 \left\langle S \circ i_{L^2(\Omega)}(u) \middle| \varphi \right\rangle_{V(\Omega)} && \text{second term of (2)} \\
& + 2iM_0 k_0 \left\langle S \circ i_{L^2(\Omega)} \circ \frac{\partial}{\partial x}(u) \middle| \varphi \right\rangle_{V(\Omega)} && \text{last term of (2)} \\
& + \frac{iZ_0 \overline{Z(\omega)} k_0}{|Z(\omega)|^2} \left\langle T \circ \operatorname{Tr}_{L^2(\Gamma_1 \cup \Gamma_2)}(u) \middle| \varphi \right\rangle_{V(\Omega)} && \text{first term of (3)} \\
& + \frac{2M_0 Z_0 \overline{Z(\omega)}}{|Z(\omega)|^2} \left\langle T \circ \operatorname{Tr}_{L^2(\Gamma_1 \cup \Gamma_2)} \circ \frac{\partial}{\partial x}(u) \middle| \varphi \right\rangle_{V(\Omega)} && \text{second term of (3)} \\
& + \frac{iZ_0 \overline{Z(\omega)} M_0^2}{|Z(\omega)|^2 k_0} \left\langle R \circ \operatorname{Tr}_{L^2(\Gamma_1 \cup \Gamma_2)}(u) \middle| \varphi \right\rangle_{V(\Omega)} && \text{third term of (3)} \\
& + ik'(1 - M_0^2) \left\langle U \circ \operatorname{Tr}_{L^2(\Gamma_N)}(u) \middle| \varphi \right\rangle_{V(\Omega)} && (4) + (5)
\end{aligned}$$

By denoting $R' = R \circ \mathrm{Tr}_{L^2(\Gamma_1 \cup \Gamma_2)}$,$S' = S \circ i_{L^2(\Omega)}$, $T' = T \circ \mathrm{Tr}_{L^2(\Gamma_1 \cup \Gamma_2)}$ and $U' = U \circ \mathrm{Tr}_{L^2(\Gamma_N)}$ we can write in a compact form

$$\begin{aligned}
&\forall u, \varphi \in V(\Omega) \\
&\quad a(u,\varphi) = \left\langle \left(I - k_0^2 S' + \frac{2M_0 Z_0 \overline{Z(\omega)}}{|Z(\omega)|^2} T' \circ \partial_x + 2iM_0 k_0 S' \circ \partial_x + \frac{iZ_0 \overline{Z(\omega)} k_0}{|Z(\omega)|^2} T' \right.\right. \\
&\qquad\qquad \left.\left. + \frac{iZ_0 \overline{Z(\omega)} M_0^2}{|Z(\omega)|^2 k_0} R' + ik'(1-M_0^2) U' \right)(u) \middle| \varphi \right\rangle_{V(\Omega)}
\end{aligned}$$

We end up, denoting

$$P = k_0^2 S' - \frac{2M_0 Z_0 \overline{Z(\omega)}}{|Z(\omega)|^2} T' \circ \partial_x - 2iM_0 k_0 S' \circ \partial_x - \frac{iZ_0 \overline{Z(\omega) k_0}}{|Z(\omega)|^2} T' - \frac{iZ_0 \overline{Z(\omega)} M_0^2}{|Z(\omega)|^2 k_0} R' - ik'(1-M_0^2) U'$$

with

$$\forall u, \varphi \in V(\Omega), a(u,\varphi) = \langle (I-P)(u) | \, \varphi \rangle_{V(\Omega)}$$

4 Compacity study

4.1 R', S', T', U'

Let us prove that S' is compact. As $i_{L^2(\Omega)} : V(\Omega) \to L^2(\Omega)$ is compact, we only have to prove that $S : L^2(\Omega) \to V(\Omega)$ is continuous.

We have

$$\exists S : L^2(\Omega) \to V(\Omega), \forall \psi \in L^2(\Omega), \forall \varphi \in V(\Omega), \langle \psi | \, \varphi \rangle_{L^2(\Omega)} = \langle S\psi | \, \varphi \rangle_{V(\Omega)}$$

so

$$\begin{aligned}
\forall \varphi \in L^2(\Omega), \|S(\varphi)\|_{V(\Omega)}^2 &= | \langle S(\varphi) | \, S(\varphi) \rangle_{V(\Omega)} | \\
&= | \langle i_{L^2(\Omega)} \circ S(\varphi) | \, \varphi \rangle_{L^2(\Omega)} | \\
&\leq \|S(\varphi)\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \\
&\leq C(\Omega) \|\nabla S(\varphi)\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \\
&\leq \frac{C(\Omega)}{1-M_0^2} \|S(\varphi)\|_{V(\Omega)} \|\varphi\|_{L^2(\Omega)}
\end{aligned}$$

Thus S is continuous, and S' compact.
In the last lign, we used that

$$\begin{aligned}
\|u\|_{V(\Omega)}^2 &= (1-M_0^2) \left\| \frac{\partial u}{\partial x} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial y} \right\|_{L^2(\Omega)}^2 \\
&\geq (1-M_0^2) \left(\left\| \frac{\partial u}{\partial x} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial u}{\partial y} \right\|_{L^2(\Omega)}^2 \right) \\
&\geq (1-M_0^2) \|\nabla u\|_{L^2(\Omega)}^2
\end{aligned}$$

We prove in the same way that R', T' and U' are compact as well, using the continuity of the trace operator. For T

$$\begin{aligned}\forall u \in L^2(\Gamma_1 \cup \Gamma_2), \|T(u)\|^2_{V(\Omega)} &= \langle u | \operatorname{Tr} \circ T(u) \rangle_{L^2(\Omega)} \\ &\leq \|u\|_{L^2(\Gamma_1 \cup \Gamma_2)} \| \operatorname{Tr} \circ T(u)\|_{L^2(\Gamma_1 \cup \Gamma_2)} \\ &\leq \|u\|_{L^2(\Gamma_1 \cup \Gamma_2)} K \|T(u)\|_{V(\Omega)}\end{aligned}$$

4.2 partial derivative

The operator $\partial_x : V(\Omega) \to L^2(\Omega)$ is continuous so every term of the form

$$A' \circ \partial_x, \quad A' \in \{S', T', U'\}$$

is compact as a composition of a continuous function with a compact function.

We have prooved the compacity of our operators, but did not manage to proove the existence and unicity of the solution using the Fredholm theorem.

5 Optimisation problem

In what follows, we take into account the term χ, defined at section 2.4.3, that we recall here : We add a term $\chi \in L^2(\Gamma_1 \cup \Gamma_2)$ such that

$$\forall z \in \Gamma_1 \cup \Gamma_2, \chi(z) = \begin{cases} 1 & \text{if we placed a liner placed at this emplacement} \\ 0 & \text{otherwise} \end{cases}$$

In our variational formulation, it can be simply integrated by replacing

$$Z_0 \to Z_0 \chi(z)$$

What we will do is try to solve the following optimisation problem :
Find $\chi \in L^2(\Gamma_1 \cup \Gamma_2)$, and

$$\int_{\Gamma_1 \cup \Gamma_2} \chi(z) dz = \beta \lambda(\Gamma_1 \cup \Gamma_2)$$

$\beta \in]0, 1[$, the proportion of acoustic liner we can put on the wall such that the acoustic energy in $L^2(\Omega)$ is minimised fir a given set of conditions $(\omega, s(y), M_0, Z_0, Z(\omega), c_0, \beta)$

6 Numerical implementation

6.1 Finite differences scheme

We rewrite the problem in its differential form :

- $\dfrac{\partial^2 \hat{p}'}{\partial y^2} + k_0^2 \hat{p}' + (1 - M_0^2) \dfrac{\partial^2 \hat{p}'}{\partial x^2} - 2ik_0 M_0 \dfrac{\partial \hat{p}'}{\partial x} = 0$ in Ω
- $\hat{p}' = s(y)$ on $\Gamma_D (x = 0)$ (inflow condition)
- $\dfrac{\partial \hat{p}'}{\partial y} + \dfrac{\chi_1(x) Z_0 \overline{Z}}{|Z|^2} \left(ik_0 \hat{p}' + 2M_0 \dfrac{\partial \hat{p}'}{\partial x} - i \dfrac{M_0^2}{k_0} \dfrac{\partial^2 \hat{p}'}{\partial x^2} \right) = 0$ on Γ_1 (y=L)
- $-\dfrac{\partial \hat{p}'}{\partial y} + \dfrac{\chi_2(x) Z_0 \overline{Z}}{|Z|^2} \left(ik_0 \hat{p}' + 2M_0 \dfrac{\partial \hat{p}'}{\partial x} - i \dfrac{M_0^2}{k_0} \dfrac{\partial^2 \hat{p}'}{\partial x^2} \right) = 0$ on Γ_2 (y=0)

- $\dfrac{\partial \hat{p}'}{\partial x} + ik'\hat{p}' = 0$ on $\Gamma_N(x = L)$ (radiation condition,outgoing flow)

We discretize the domain in $(i, j) \in \{0, N_y\}\times\{0, N_x\}$, note $p_{i,j} = p(jh_x, ih_y)$ the solution and $h_x = \frac{l}{N_x}, h_y = \frac{L}{N_y}$

- $\forall(i, j) \in \{1, N_y - 1\} \times \{1, N_x - 1\}$

$$\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{h_y^2} + k_0^2 p_{i,j} + (1-M_0^2)\frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{h_x^2} - 2iM_0k_0\frac{p_{i,j+1} - p_{i,j-1}}{2h_x} = 0$$

- $\forall i \in \{0, N_y\}, p_{i,0} = s(h_y i)$
- $\forall j \in \{1, N_x - 1\}$,

$$\frac{p_{N_y,j} - p_{N_y-1,j}}{h_y} + \frac{\chi_1(j)Z_0\overline{Z}}{|Z|^2}\left(ik_0 p_{N_y,j} + 2M_0\frac{p_{N_y,j+1} - p_{N_y,j-1}}{2h_x} - i\frac{M_0^2}{k_0}\frac{p_{N_y,j+1} - 2p_{N_y,j} + p_{N_y,j-1}}{h_x^2}\right) = 0$$

- $\forall j \in \{1, N_x - 1\}$,

$$\frac{p_{0,j} - p_{1,j}}{h_y} + \frac{\chi_2(j)Z_0\overline{Z}}{|Z|^2}\left(ik_0 p_{0,j} + 2M_0\frac{p_{0,j+1} - p_{0,j-1}}{2h_x} - i\frac{M_0^2}{k_0}\frac{p_{0,j+1} - 2p_{0,j} + p_{0,j-1}}{h_x^2}\right) = 0$$

- $\forall i \in \{0, N_y\}$

$$\frac{p_{i,N_x} - p_{i,N_x-1}}{h_x} + ik'p_{i,N_x} = 0$$

6.2 Genetic algorithm

The nature of this problem makes it perfectly fit for a genetic algorithm. The liner can be represented by a sequence of bits. 1 means that there is actually some liner, and 0 means that we didn't place any liner at this spot. We can run an optimisation loop applying the main steps of a genetic algorithm after making an initial population.

1. Evaluate all the liner presences (sequence of bits) using a certain cost function.
2. Select the best individuals according to the cost function we have chosen.
3. Make a crossover on them in order to create a new individual, hopefully better.
4. With a small probability, mutate the individual by swapping a few of its bits.
5. Project the individuals to obtain a density of β.

In theory, using as our cost function the energy inside the domain is alright, but it is practically impossible. The reason is that the algorithm could return a list that would look for instance like [0,1,0,1,0,1,0,1]. This means that we have to cut our liner is several small pieces, which are arbitrarily small. To prevent this issue, we can add a second term to our cost function. If We denote the liner as $[b_1, b_2, ..., b_N]$, with b_i being either 1 or 0, we can count the number of consecutive 1 between each 0 (We will note them $I_1, .., I_k$). Our additional term can be expressed as $\displaystyle\sum_{i=1}^{k} \frac{\sigma}{I_i^\alpha}$, σ represents the importance of having only a few cuts. If $f(\chi) = \sigma$ is 0, it doesn't matter. α represents how much we want to penalize small groups of 1 compared to large groups. If α is 0, we penalize all the groups with the same coefficient. We can then use this cost function $C(\chi)$

$$C(\chi) = \int_\Omega u^2(\chi)d\Omega + f(\chi)$$

With α set to 1.5 and σ set to 1 we get the result below (black means there is a liner, white means there is no liner).

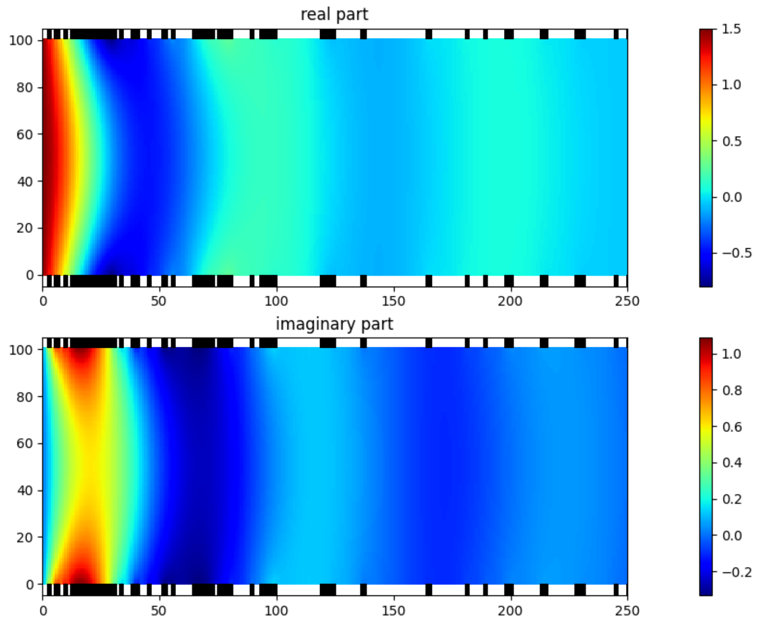

Figure 4: Optimized chi with $\alpha = 1.5$ and $\sigma = 1$

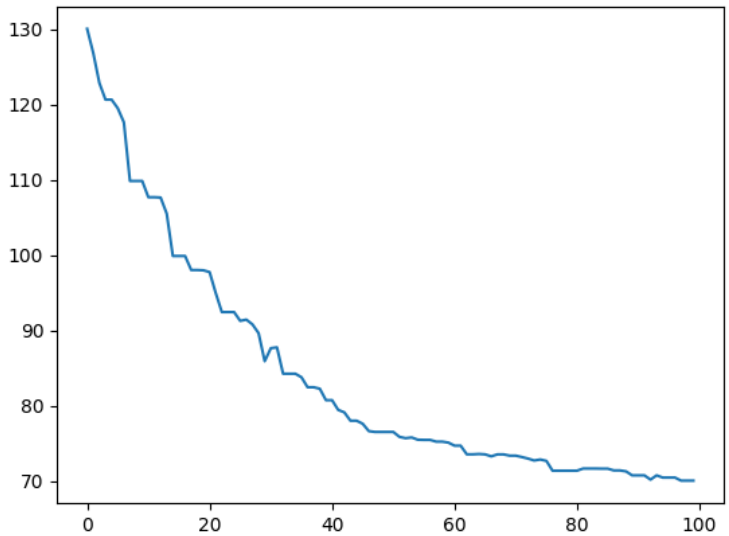

Figure 5: Cost after each generation

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