# Projet S8 - Rapport

*Parametric optimization of an acoustic liner*

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# 1 Introduction

Airplanes are well-known for producing bothersome noise with their engines, the most popular one being the turbofan engine. This type of propeller is nosier than its older counterpart, the turbo-reactor engine, mainly because of its fan. Cities near airports are filing more and more frequently for noise disturbance (see Toulouse in France with 10,000 complaints in February 2020). This issue has then become a subject for research in aeronautics in the recent years. Currently, the widespread method is to place liner on the hard walls inside the engine to absorb noise energy. Because of the harsh conditions in the combustion jet (pressure and temperature) liners cannot be dispatched there, we are left with the red areas of the cold jet part in the following figure.

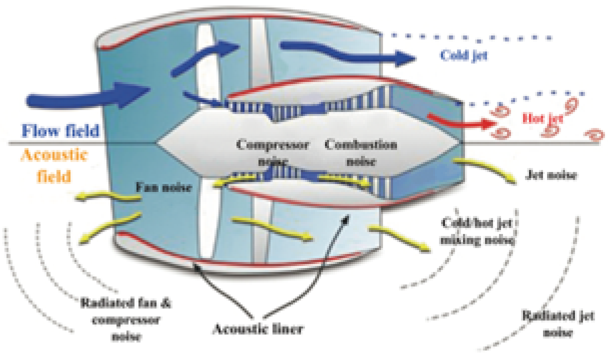



Figure 1: Turbofan engine

Liners are pierced into the engine walls, therefore we cannot place liner all over the available areas (it would jeopardize the engine structure). We will give ourselves a budget for liner as a fraction of the available areas to place liner onto. Our work will consist in developing a numerical tool to solve for the partition of liner on the walls so as to minimize the acoustic energy inside the domain.

# 2 The Model

## 2.1 Linearized Euler equations and notations

This part is mainly taken from [8], we introduce the equations, hypothesis and notations of [8].

As in [8], the hypothesis taken is the one of an ideal fluid :

- no friction (no viscous forces)
- no heat conduction
- no heat production

This implies that the flow is isentropic. The equations for the fluid are then the Euler Equations :

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{v} + m \qquad \text{Mass conservation} \qquad (1)$$

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho} \boldsymbol{\nabla} p + \boldsymbol{f} \qquad \text{Momentum conservation} \qquad (2)$$

Where we denote

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$$

With

- $\boldsymbol{v}(\boldsymbol{r}, t)$ the fluid velocity
- $\rho(\boldsymbol{r}, t)$ the density
- $p(\boldsymbol{r}, t)$ the (static) fluid pressure
- $s(\boldsymbol{r}, t)$ the specific entropy
- $m(\boldsymbol{r}, t)$ the specific mass source per unit time
- $\boldsymbol{f}(\boldsymbol{r}, t)$ the force per unit mass exerting on the fluid (neglecting gravity)

$\boldsymbol{r} = (x, y, z)$ being the position and $t$ the time

We consider, as in [8] the actual flow is the sum of a stationnary flow (denoted $X_0$) and a perturbation of negligible contribution (denoted $X'$, and $X' \ll X^0$) generated by the mass $m$ and force $\boldsymbol{f}$ injected by the fluid:

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + \rho'(\mathbf{r}, t) \qquad \text{fluid density} \tag{3}$$
$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v_0}(\mathbf{r}) + \mathbf{v}'(\mathbf{r}, t) \qquad \text{velocity} \tag{4}$$
$$s(\mathbf{r}, t) = s_0(r) + s'(\mathbf{r}, t) \qquad \text{specific entropy} \tag{5}$$
$$p(\mathbf{r}, t) = p_0(\mathbf{r}) + p'(\mathbf{r}, t) \qquad \text{(static) fluid pressure} \tag{6}$$
$$c^2(\mathbf{r}, t) = c_0^2(\mathbf{r}) + (c^2)'(\mathbf{r}, t) \qquad \text{squared speed of sound} \tag{7}$$

The speed of sound is defined thanks to the equation of state $p = p^{\#}(\rho, s)$ in the adiabatic case :

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \underbrace{\left.\frac{\partial p^{\#}}{\partial \rho}\right|_s}_{c^2(\rho,s)} \frac{\mathrm{D}\rho}{\mathrm{D}t} \tag{8}$$

The stationnary part of the flow verifies the following equalities : (resolve equations (1), (2) and (8) with $\frac{\partial .}{\partial t} = 0$, $m = 0$, $\boldsymbol{f} = 0$ -as $\boldsymbol{f}, m$ generates the perturbation-, for $(p_0, \boldsymbol{v_0}, \rho_0)$ the unperturbed flow)

$$\begin{aligned}
(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\rho_0 &= -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v_0} \\
(\rho_0 \boldsymbol{v_0} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} &= -\boldsymbol{\nabla} p_0 \\
(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})p_0 &= c_0^2 (\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\rho_0
\end{aligned} \tag{9}$$

The perturbation verifies the Linearized Euler equations (mass conservation, momentum conservation and state equation) :

$$\begin{aligned}
\frac{\mathrm{D}_c \rho'}{\mathrm{D}t} + (\boldsymbol{v'} \cdot \boldsymbol{\nabla})\rho_0 &= -\rho' \boldsymbol{\nabla} \cdot \boldsymbol{v_0} - \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v'} + m \\
\frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} + (\boldsymbol{v'} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} &= \frac{\rho'}{\rho_0^2} \boldsymbol{\nabla} p_0 - \frac{1}{\rho_0} \nabla p' + \boldsymbol{f} \\
\frac{\mathrm{D}_c p'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} p_0 &= c_0^2 \left( \frac{\mathrm{D}_c \rho'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \nabla \rho_0 \right) + (c^2)' \boldsymbol{v_0} \cdot \boldsymbol{\nabla} \rho_0
\end{aligned} \tag{10}$$

With

$$\frac{\mathrm{D}_c}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v_0} \cdot \nabla \tag{11}$$

*Proof.* (mass conversation : resolving for the actual flow)

$$\frac{\partial \rho'}{\partial t} + (\boldsymbol{v_0} + \boldsymbol{v'}) \cdot \boldsymbol{\nabla}(\rho_0 + \rho') = -(\rho_0 + \rho')\boldsymbol{\nabla} \cdot (\boldsymbol{v_0} + \boldsymbol{v'}) + m$$

Only keeping first order terms, and account for $\frac{\partial \rho_0}{\partial t} = 0$, we end up with

$$\frac{\partial \rho'}{\partial t} + \boldsymbol{v_0} \cdot \boldsymbol{\nabla}\rho_0 + \boldsymbol{v_0} \cdot \boldsymbol{\nabla}\rho' + \boldsymbol{v'} \cdot \boldsymbol{\nabla}\rho_0 = -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v_0} - \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v'} - \rho' \boldsymbol{\nabla} \cdot \boldsymbol{v_0} + m$$

Regrouping the terms

$$\frac{\mathrm{D}_c \rho'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla}\rho_0 = -\underbrace{(\boldsymbol{v_0} \cdot \boldsymbol{\nabla}\rho_0 + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v_0})}_{=0 \ (9)} - \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v'} - \rho' \boldsymbol{\nabla} \cdot \boldsymbol{v_0} + m$$

$$\boxed{\frac{\mathrm{D}_c \rho'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla}\rho_0 = -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v'} - \rho' \boldsymbol{\nabla} \cdot \boldsymbol{v_0} + m}$$

(momentum conservation)

$$(\rho_0 + \rho')\left(\underbrace{\frac{\partial \boldsymbol{v_0}}{\partial t}}_{=0} + \frac{\partial \boldsymbol{v'}}{\partial t} + ((\boldsymbol{v_0} + \boldsymbol{v'}) \cdot \boldsymbol{\nabla})(\boldsymbol{v_0} + \boldsymbol{v'})\right) = -\boldsymbol{\nabla} p_0 - \boldsymbol{\nabla} p' + \rho_0 \boldsymbol{f} + \underbrace{\rho' \boldsymbol{f}}_{\substack{\boldsymbol{f} \text{ induces a} \\ \text{small perturbation}}}$$

As $\boldsymbol{f}$ only induces a small perturbation, it is also a first order term so $\rho' \boldsymbol{f}$ is a second order term that is negligible. Developping the left hand side and keeping only terms up to first order :

$$\rho_0 \frac{\partial \boldsymbol{v'}}{\partial t} + \rho_0(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} + \rho_0(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\boldsymbol{v'} + \rho_0(\boldsymbol{v'} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} + \rho'(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} = -\boldsymbol{\nabla} p_0 - \boldsymbol{\nabla} p' + \rho_0 \boldsymbol{f}$$

Regrouping the terms, and using (9) to replace $(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} = -\frac{1}{\rho_0}\boldsymbol{\nabla} p_0$,

$$\rho_0 \frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} - \boldsymbol{\nabla} p_0 + \rho_0(\boldsymbol{v'} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} - \frac{\rho'}{\rho_0}\boldsymbol{\nabla} p_0 = -\boldsymbol{\nabla} p_0 - \boldsymbol{\nabla} p' - \rho_0 \boldsymbol{f}$$

Thus, dividing by $\rho_0$ yields :

$$\boxed{\frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} + (\boldsymbol{v'} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} = \frac{\rho'}{\rho_0^2}\boldsymbol{\nabla} p_0 - \frac{1}{\rho_0}\boldsymbol{\nabla} p' + \boldsymbol{f}}$$

For the state equation, we first simplify each of the terms

$$\begin{aligned}
\frac{\mathrm{D}_c p}{\mathrm{D}t} &= \frac{\partial p'}{\partial t} + (\boldsymbol{v_0} + \boldsymbol{v'}) \cdot \boldsymbol{\nabla}(p_0 + p') \\
&= \frac{\partial p'}{\partial t} + \boldsymbol{v_0} \cdot \boldsymbol{\nabla} p_0 + \boldsymbol{v_0} \cdot \boldsymbol{\nabla} p' + \boldsymbol{v'} \cdot \boldsymbol{\nabla} p_0 \\
&= \frac{\mathrm{D}_c p'}{\mathrm{D}t} + \underbrace{\boldsymbol{v_0} \cdot \boldsymbol{\nabla} p_0}_{=c_0^2(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\rho_0 \quad (9)} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} p_0
\end{aligned}$$

$$\left(\frac{\partial p}{\partial \rho}\right)_s = c^2 = c_0^2 + (c^2)'$$

$$\begin{aligned}\frac{\mathrm{D}_c\rho}{\mathrm{D}t} &= \frac{\partial \rho'}{\partial t} + (\boldsymbol{v_0} + \boldsymbol{v'}) \cdot \boldsymbol{\nabla}(\rho_0 + \rho') \\ &= \frac{\partial \rho'}{\partial t} + \boldsymbol{v_0} \cdot \boldsymbol{\nabla}\rho_0 + \boldsymbol{v_0} \cdot \boldsymbol{\nabla}\rho' + \boldsymbol{v'} \cdot \boldsymbol{\nabla}\rho_0 \\ &= \frac{\mathrm{D}_c\rho'}{\mathrm{D}t} + \underbrace{\boldsymbol{v_0} \cdot \boldsymbol{\nabla}\rho_0}_{=-\rho_0\boldsymbol{\nabla}\cdot\boldsymbol{v_0} \quad (9)} + \boldsymbol{v'} \cdot \boldsymbol{\nabla}\rho_0\end{aligned}$$

Thus

$$\left(\frac{\mathrm{D}_c p'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla}p_0\right) + c_0^2(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\rho_0 = (c_0^2 + \underbrace{(c^2)'}_{*})\left(\frac{\mathrm{D}_c\rho'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla}\rho_0\right) + (c_0^2 + (c^2)')(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\rho_0$$

The (*) term is a first order term multiplied by another first order term, we can thus neglect it. Removing it and simplyfing the $c_0^2(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\rho_0$ term gives the result.

□

## 2.2 The Convected Helmholtz equation

In the case of a background flow velocity $\boldsymbol{v_0}$ is a non vanishing constant, then $\boldsymbol{\nabla}p_0 = 0$ and $\boldsymbol{v_0} \cdot \boldsymbol{\nabla}\rho_0 = 0$ Then taking $\frac{d}{dt}$(mass conservation) $- \boldsymbol{\nabla} \cdot$ (momentum conservation) :

$$\frac{\mathrm{D}_c}{\mathrm{D}t}\left(\frac{1}{c_0^2}\frac{\mathrm{D}_c p'}{\mathrm{D}t}\right) - \rho_0\boldsymbol{\nabla} \cdot \left(\frac{1}{\rho_0}\boldsymbol{\nabla}p'\right) = \frac{\mathrm{D}_c m}{\mathrm{D}t} - \rho_0\boldsymbol{\nabla} \cdot \boldsymbol{f} \tag{12}$$

*Proof.* As $\boldsymbol{v_0}$ is a non vanishing constant, the equations (9) give us

$$0 = \rho_0(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\boldsymbol{v_0} = \boldsymbol{\nabla p_0} \Rightarrow \boldsymbol{\nabla}p_0 = 0$$

$$(\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\rho_0 = -\rho_0\boldsymbol{\nabla} \cdot \boldsymbol{v_0} = 0 \rightarrow (\boldsymbol{v_0} \cdot \boldsymbol{\nabla})\rho_0 = 0$$

Then, we write the mass conversation, momentum conversation and state equation (10) and simplify them.

$$\frac{\mathrm{D}_c\rho'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla}\rho_0 = -\underbrace{\rho'\boldsymbol{\nabla} \cdot \boldsymbol{v_0}}_{=0} - \rho_0\boldsymbol{\nabla} \cdot \boldsymbol{v'} + m$$

$$\frac{\mathrm{D}_c\rho'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla}\rho_0 = -\rho_0\boldsymbol{\nabla} \cdot \boldsymbol{v'} + m$$

For momentum conservation :

$$\frac{\mathrm{D}_c\boldsymbol{v'}}{\mathrm{D}t} + \underbrace{(\boldsymbol{v'} \cdot \boldsymbol{\nabla})\boldsymbol{v_0}}_{=0} = \underbrace{-\frac{\rho'}{\rho_0^2}\boldsymbol{\nabla}p_0}_{=0} - \frac{1}{\rho_0}\boldsymbol{\nabla}p' + \boldsymbol{f}$$

$$\frac{\mathrm{D}_c\boldsymbol{v'}}{\mathrm{D}t} = -\frac{1}{\rho_0}\boldsymbol{\nabla}p' + \boldsymbol{f}$$

For the state equation :

$$\frac{\mathrm{D}_c p'}{\mathrm{D}t} + \underbrace{\boldsymbol{v'} \cdot \boldsymbol{\nabla}p_0}_{=0} = c_0^2\left(\frac{\mathrm{D}_c p'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla}\rho_0\right) + (c^2)'\underbrace{\boldsymbol{v_0} \cdot \boldsymbol{\nabla}\rho_0}_{=0}$$

$$\frac{\mathrm{D}_c p'}{\mathrm{D}t} = c_0^2 \left( \frac{\mathrm{D}_c p'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} \rho_0 \right)$$

Then, with these three new equations, we have

$$\begin{aligned} \frac{\mathrm{D}_c}{\mathrm{D}t} \left( \frac{1}{c_0^2} \frac{\mathrm{D}_c p'}{\mathrm{D}t} \right) &= \frac{\mathrm{D}_c}{\mathrm{D}t} \left( \frac{\mathrm{D}_c p'}{\mathrm{D}t} + \boldsymbol{v'} \cdot \boldsymbol{\nabla} \rho_0 \right) \\ &= \frac{\mathrm{D}_c}{\mathrm{D}t} \left( m - \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v'} \right) \\ &= \frac{\mathrm{D}_c m}{\mathrm{D}t} - \rho_0 \frac{\mathrm{D}_c}{\mathrm{D}t} (\boldsymbol{\nabla} \cdot \boldsymbol{v'}) \end{aligned}$$

And

$$\begin{aligned} -\rho_0 \boldsymbol{\nabla} \cdot \left( \frac{1}{\rho_0} \boldsymbol{\nabla} p' \right) &= -\rho_0 \boldsymbol{\nabla} \cdot \left( \boldsymbol{f} - \frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} \right) \\ &= -\rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{f} + \rho_0 \boldsymbol{\nabla} \cdot \left( \frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} \right) \end{aligned}$$

Summing these two expression to get the left side of the convected Helmotz equation :

$$\frac{\mathrm{D}_c}{\mathrm{D}t} \left( \frac{1}{c_0^2} \frac{\mathrm{D}_c p'}{\mathrm{D}t} \right) - \rho_0 \boldsymbol{\nabla} \cdot \left( \frac{1}{\rho_0} \boldsymbol{\nabla} p' \right) = \frac{\mathrm{D}_c m}{\mathrm{D}t} - \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{f} + \rho_0 \underbrace{\left( \boldsymbol{\nabla} \cdot \frac{\mathrm{D}_c \boldsymbol{v'}}{\mathrm{D}t} - \frac{\mathrm{D}_c}{\mathrm{D}t} (\boldsymbol{\nabla} \cdot \boldsymbol{v'}) \right)}_{=A}$$

What is left to do is to prove that $A = 0$

$$A = \underbrace{\left( \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{v'}}{\partial t} - \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{v'}}{\partial t} \right)}_{=0} + \underbrace{(\boldsymbol{\nabla} \cdot ((\boldsymbol{v_0} \cdot \boldsymbol{\nabla}) \boldsymbol{v'}) - \boldsymbol{v_0} \cdot \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{v'}))}_{=B}$$

$B = (\boldsymbol{\nabla} \cdot ((\boldsymbol{v_0} \cdot \boldsymbol{\nabla}) \boldsymbol{v'}) - \boldsymbol{v_0} \cdot \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{v'}))$

Denoting $\dfrac{\partial}{\partial x_i}$the derivation along the i-th coordinate and $x_i$ the i-th component of$x$

$$\begin{aligned} B &= \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left[ \sum_{j=1}^{3} v_{0j} \frac{\partial v'_i}{\partial x_j} \right] - \sum_{i=1}^{3} v_{0i} \frac{\partial}{\partial x_i} \left[ \sum_{j=1}^{3} \frac{\partial v'_j}{\partial x_j} \right] \\ &= \sum_{1 \le i,j \le 3} \underbrace{\frac{\partial}{\partial x_i} \left( v_{0,j} \frac{\partial v'_i}{\partial x_j} \right)} - v_{0i} \frac{\partial^2 v'_j}{\partial x_i \partial y_j} \\ &\qquad = \frac{\partial v_{0,j}}{\partial x_i} \frac{\partial v'_i}{\partial x_j} + v_{0j} \frac{\partial^2 v'_i}{\partial x_i \partial y_i} \\ &\qquad = v_{0j} \frac{\partial^2 v'_i}{\partial x_i \partial y_i} \\ &= \sum_{1 \le i,j \le 3} v_{0j} \frac{\partial^2 v'_i}{\partial x_i \partial x_j} - v_{0i} \frac{\partial^2 v'_j}{\partial x_i \partial x_j} \end{aligned}$$

with a change of indices $i \leftrightarrow j$ and using $\dfrac{\partial^2 v'_i}{\partial x_j \partial x_i} = \dfrac{\partial^2 v'_i}{\partial x_i \partial x_j}$

$$B = \sum_{1 \le i,j \le 3} v_{0j} \frac{\partial^2 v'_i}{\partial x_i \partial x_j} - v_{0j} \frac{\partial^2 v'_i}{\partial x_i \partial x_j}$$

$B = 0$

Then $A = 0$ $\square$

As seen in [8], applying this convected wave equation to our problem, with the conditions $\mathbf{v_0} = (u_0, 0, 0), \mathbf{f} = 0, m = 0$ (it thus means that the source of the perturbation, $m$ and $\boldsymbol{f}$ are outside of the volume we consider, but still create the perturbation), and $u_0$ constant, we have

$$\frac{\partial^2 p'}{\partial t^2} + 2u_0 \frac{\partial^2 p'}{\partial x \partial t} + u_0^2 \frac{\partial^2 p'}{\partial x^2} - c_0^2 \Delta p' = 0$$

We seek harmonic solutions, so we have the equation (9) from the ref [8].

$$\Delta_\perp \hat{p}' + k_0^2 \hat{p}' + (1 - M_0^2) \frac{\partial^2 \hat{p}'}{\partial x^2} - 2ik_0 M_0 \frac{\partial \hat{p}'}{\partial x} = 0 \tag{13}$$

With the following conventions :

- dependance in $e^{+i\omega t}$
- $M_0 = \frac{u_0}{c_0}$ is the Mach number
- $k_0 = \frac{\omega}{c_0}$ is the wave number (without flow)
- $\Delta_\perp = \Delta - \partial_x^2$

We can write this equation in a simpler form, as follow :

$$\Delta \hat{p}' + k_0^2 \left(1 - i \frac{M_0}{k_0} \frac{\partial}{\partial x}\right)^2 \hat{p}' = 0 \tag{14}$$

## 2.3 Geometry of the studied problem

We model this problem in a rectangular 2D domain $\Omega$ with a border $\partial\Omega$.

$\Omega = \{(x, y), 0 < x < L, 0 < y < l\}$; $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_D \cup \Gamma_N$.
With

- $\Gamma_1$ and $\Gamma_2$ the border with the acoustic liner
- $\Gamma_D$ the left part (inflow condition)
- $\Gamma_N$ the right part (outgoing flow)

## 2.4 The Boundary condition on the hard walls

We denote $\Gamma_1$ and $\Gamma_2$ the two surfaces (at rest) of the liners.

### 2.4.1 On the liner

To obtain the boundary condition, we use the continuity of the normal displacement at the surface of the liner. We denote by $\xi_1$ the normal displacement at the surface of the wall. Using the impedance of the liner $Z = \frac{p'}{\mathbf{v}' \cdot \mathbf{n}}$ we have the following condition for $\xi_1$:

$$\frac{\partial \xi_1}{\partial t} = \mathbf{v}' \cdot \mathbf{n} = \frac{p'}{Z}$$

Where $Z = Z(\omega)$ is the complex normal impedance of the liner (and, as seen in [1], the effect of fluid motion is that $Z' = Z(1 + M_0 \sin(\phi))$ with $\phi$ the angle of incidence, $\phi = 0$ in

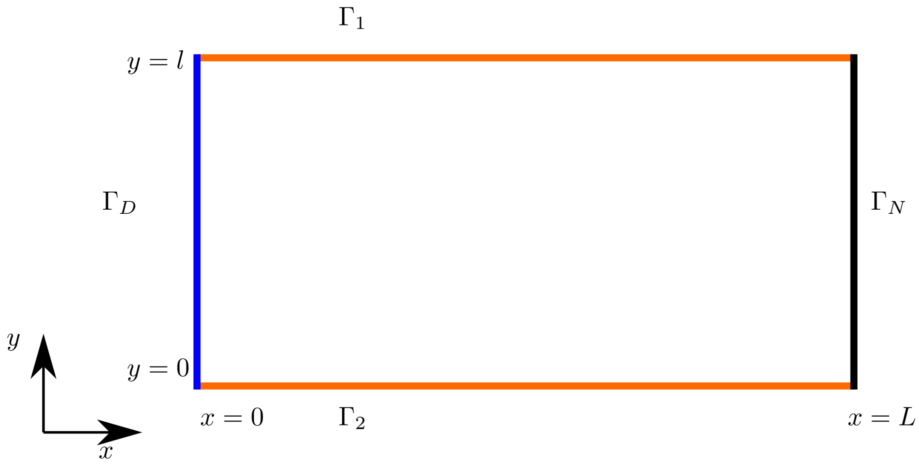



Figure 2: The model problem for acoustic liners

normal incidence so the impedance $Z$ we have to consider is the complex normal impedance of the liner at rest with no fluid flow).

We denote by $\xi_2$ the normal displacement of a fluid particle on the boundary of the domain. This particle's motion satisfies the Navier-Stokes equation using the assumptions of the model, it yields :

$$\rho_0(\partial_t + u_0\partial_x)^2\xi_2 = -\frac{\partial p'}{\partial \mathbf{n}}$$

Finally writing $\xi_1 = \xi_2$ yields the temporal boundary condition. As we are seeking the harmonic solutions of the system with frequency f $= 2\pi\omega$, we can rewrite the equations for $\hat{\xi}_1$, $\hat{\xi}_2$ and $\hat{p}'$ (the harmonic form of $\xi_1$, $\xi_2$ and $p$').

$$\begin{cases} i\omega\hat{\xi}_1 &= \frac{\hat{p}'}{Z} \\ \rho_0(i\omega + u_0\partial_x)^2\hat{\xi}_2 &= -\frac{\partial \hat{p}'}{\partial \mathbf{n}} \end{cases}$$

Setting $\xi_1$ equal to $\xi_2$ yields :

$$Z\frac{\partial \hat{p}'}{\partial \mathbf{n}} + i\rho\omega(1 - i\frac{u_0}{\omega}\partial_x)^2\hat{p}' = 0$$

Using the physical quantities $Z_0 = \rho_0 c_0$ (fluid impedance), $M_0 = \frac{u_0}{c_0}$ (Mach number of the flow) and $k_0 = \frac{\omega}{c_0}$ (wave vector of the harmonic acoustic wave) we get our final boundary condition :

$$Z\frac{\partial \hat{p}'}{\partial \mathbf{n}} + iZ_0k_0(1 - i\frac{M_0}{k_0}\partial_x)^2\hat{p}' = 0 \tag{15}$$

#### 2.4.2 On the hard wall

In the limit case of a hard wall (Z goes to infinity), we can use the same method but this time $\xi_1$ the normal displacement on the wall is zero. It yields

$$\frac{\partial \hat{p}'}{\partial \mathbf{n}} = 0 \tag{16}$$

### 2.4.3 Final form of the boundary condition on $\Gamma_1 \cup \Gamma_2$

We add a term $\chi \in L^2(\Gamma_1 \cup \Gamma_2)$ such that

$$\forall z \in \Gamma_1 \cup \Gamma_2, \chi(z) = \begin{cases} 1 & \text{if we placed a liner placed at this emplacement} \\ 0 & \text{otherwise} \end{cases}$$

This function $\chi$ is the liner distribution on the surfaces $\Gamma_1 \cup \Gamma_2$, the boundary condition on the hard wall and on the liner can be given by :

$$Z\frac{\partial \hat{p}'}{\partial \mathbf{n}} + iZ_0\chi k_0(1 - i\frac{M_0}{k_0}\partial_x)^2\hat{p}' = 0 \quad \text{on}\, \Gamma_1 \cup \Gamma_2 \tag{17}$$

## 2.5 More general boundary condition

We can extend this boundary condition to non-rectangular geometries, as is done below :

We start from the following equation.

$$\rho_0(i\omega + u_0\partial_x)\hat{\mathbf{v}}' \cdot \mathbf{n} = -\frac{\partial \hat{p}'}{\partial \mathbf{n}} \text{ on } \Gamma_1 \cup \Gamma_2 \tag{18}$$

The second equation we need comes from [4], the hypothesis taken is that the surface of the liner is deformed by the incident sound field from the fluid (or the inverse situation). The continuity of the normal speed is expressed on the moving surface of the liner $S(t)$. After some calculations, the starting boundary condition can be expressed on the mean surface of the liner $S_0$, (equation (11) from [4]). As we are seeking harmonic solutions (the deformation of the liner is also supposed to be harmonic), we have the equation (14) from [4].

Rewriting this equation with our notations ($\mathbf{n}$ as the outward pointing normal to our domain), we have

$$\hat{\mathbf{v}}' \cdot \mathbf{n} = \frac{\hat{p}'}{Z} + \frac{1}{i\omega}\left(\mathbf{v}_0 \cdot \nabla\left(\frac{\hat{p}}{Z}\right)\right) + \frac{\hat{p}'}{i\omega Z}\left(\mathbf{n} \cdot (\mathbf{n} \cdot \nabla \mathbf{v_0})\right) \text{ on } \Gamma_1 \cup \Gamma_2$$

Where $Z = Z(\omega)$ is the complex normal impedance of the liner (and, as seen in [1], the effect of fluid motion is that $Z' = Z(1 + M_0 \sin(\phi))$ with $\phi$ the angle of incidence, $\phi = 0$ in normal incidence so the impedance $Z$ we have to consider is the complex normal impedance of the liner at rest with no fluid flow).

This equation covers a larger range of problems, particularly with non rectangular geometries and could be used to extend this work to a more realistic geometry.

In our case, we can verify we obtain the same result by simplifying this equation, and we obtain (the second term cancels out as $\nabla \boldsymbol{v_0} = \mathbf{0}$) :

$$\begin{aligned}
\hat{\mathbf{v}}' \cdot \mathbf{n} &= \frac{\hat{p}'}{Z} + \frac{1}{i\omega}\left(\mathbf{v}_0 \cdot \nabla\left(\frac{\hat{p}}{Z}\right)\right) \\
&= \frac{\hat{p}'}{Z} + \frac{1}{i\omega Z}\left(u_0 \boldsymbol{u_x} \cdot \nabla \hat{p}'\right) \\
&= \frac{\hat{p}'}{Z} + \frac{u_0}{i\omega Z}\frac{\partial \hat{p}'}{\partial x}
\end{aligned}$$

$$i\omega Z\hat{\mathbf{v}}' \cdot \mathbf{n} = \left(i\omega + u_0\frac{\partial}{\partial x}\right)\hat{p} \text{ on } \Gamma_1 \cup \Gamma_2 \tag{19}$$

This condition can also be written in the temporal domain as follow :

$$\frac{\partial}{\partial t}(\mathbf{v}' \cdot \mathbf{n}) = \frac{\mathrm{D}_c}{\mathrm{D}t}(Z^{-1} *_t p')$$

with $*_t$ the temporal convolution

Combining equations (18) and (19) (eliminating $\hat{\mathbf{v}}' \cdot \mathbf{n}$) and using the definition of $Z_0 = \rho_0 c_0$ the fluid impedance, we have

$$Z\frac{\partial \hat{p}'}{\partial \mathbf{n}} + ik_0 Z_0 \left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)^2 \hat{p}' = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \tag{20}$$

## 2.6 Inflow and outflow boundary condition

We impose a non homogeneous Dirichlet on one side $\Gamma_D$ and a corresponding condition on $\Gamma_R$ to eliminate the reflection.

The acoustic pressure on $\Gamma_D$ will have a dependency in

$$\hat{p}' = s(y)e^{-ik'x}, \mathrm{Re}(k') > 0$$

On $\Gamma_R$, we set a boundary condition so as to only allow as a dependency in $x$ the one of the imposed mode on the other part of the duct, that is to say

$$\frac{\partial \hat{p}'}{\partial n} = \frac{\partial \hat{p}'}{\partial x} = -ik'\hat{p}' \text{ on } \Gamma_R$$

## 2.7 Final Model problem

We point out now that in this section we did not take into account $\chi$, so in what follows this is what we would obtain with $\chi = 1$ on $\Gamma_1 \cup \Gamma_2$ Regrouping all the conditions we set : We model this problem in a 2D domain $\Omega$ with a border $\partial\Omega$.

$\Omega = \{(x, y), 0 < x < L, 0 < y < l\}$; $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_D \cup \Gamma_N$ where

- $\Delta \hat{p}' + k_0^2 \left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)^2 \hat{p}' = 0$ in $\Omega$
- $\hat{p}'|_{\Gamma_D} = s(y)$ on $\Gamma_D$ (inflow condition)
- $Z\frac{\partial \hat{p}'}{\partial \mathbf{n}} + ik_0 Z_0 \left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)^2 \hat{p}'|_{\Gamma_1 \cup \Gamma_2} = 0$ on $\Gamma_1 \cup \Gamma_2$ (acoustic liner boundary condition)
- $\frac{\partial \hat{p}'}{\partial \mathbf{n}} + ik'\, \hat{p}'|_{\Gamma_N} = 0$ on $\Gamma_N$ (radiation condition,outgoing flow)

To simplify the notations, we introduce the operator $A$, defined as follow :

$$A = k_0 \left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right) \tag{21}$$

Then the equation in $\Omega$ rewrites to

$$\Delta \hat{p}' + A^2 \hat{p}' = 0 \text{ in } \Omega \tag{22}$$

And the boundary condition on the liner rewrites to :

$$Z\frac{\partial \hat{p}'}{\partial \boldsymbol{n}} + i\frac{Z_0}{k_0}A^2 \hat{p}' = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \tag{23}$$

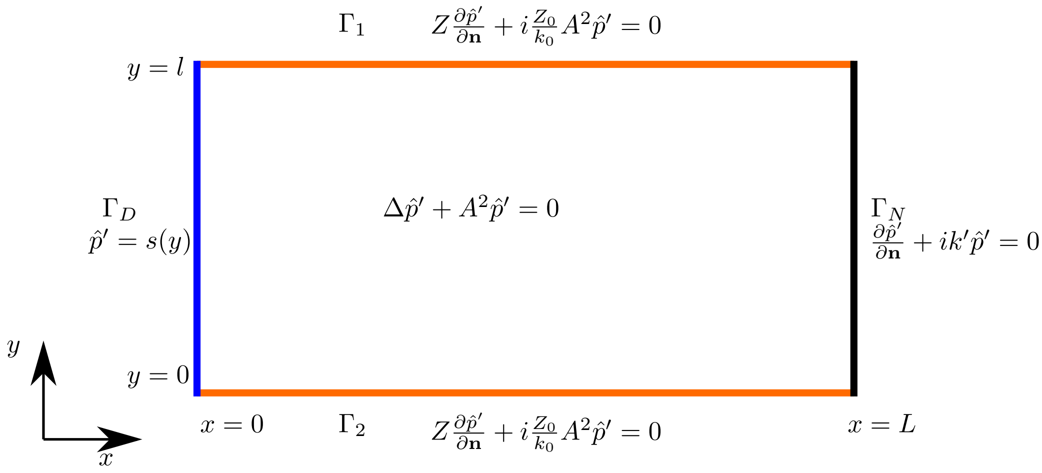



Figure 3: The model problem for acoustic liners

# 3 The Variational formulation

## 3.1 Rewriting the problem

The condition on $\Gamma_D$ being non homogeneous isn't practical, we thus rewrite our problem with the form

$$\hat{p}' = u + g$$

($u$ is a new variable, and not anymore the pression) With $g$ veryfing

$$\begin{cases} \Delta g &= 0 \text{ in } \Omega \\ g|_{\Gamma_D} &= s(y) \\ \dfrac{\partial g}{\partial n} &= 0 \text{ on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_N \end{cases}$$

Thus we have

$$\begin{cases} \Delta u + A^2 u = \underbrace{-A^2 g}_{f} \\ u|_{\Gamma_D} = 0 \\ \dfrac{\partial u}{\partial n} + \dfrac{iZ_0}{Zk_0}A^2 u = \underbrace{-\dfrac{iZ_0}{Zk_0}A^2 g}_{\eta} \text{ on } \Gamma_1 \cup \Gamma_2 \\ \dfrac{\partial u}{\partial n} + ik'u = \underbrace{-ik'g}_{\gamma} \end{cases}$$

## 3.2 Notations and variational formula

Let's then introduce a few notations and lemmas that will be used throughout this paper.

$$T(\Omega) = H^1(\Omega) \cap \{u \in L^2(\Omega), u(x) = 0, \forall x \in \Gamma_D\}$$
$$\forall u \in T(\Omega), \|u\|_{T(\Omega)} = \|\nabla u\|_{L^2}$$

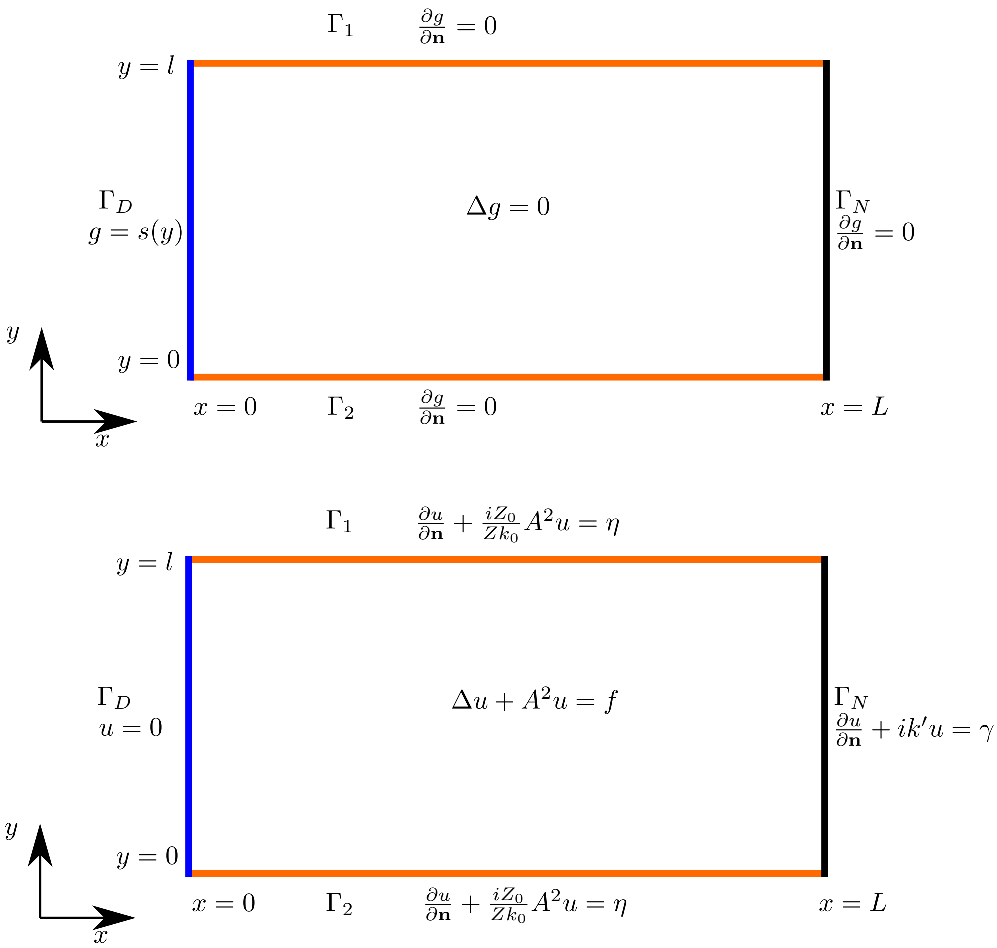



Figure 4: The new model problem for $g$ and $u$

Recall that

$$\begin{cases} A : T(\Omega) \to L^2(\Omega) \\ A = k_0 \left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right) \end{cases}$$

Let's see one useful lemma that can be proven with Green's formula for integration by part on $A$.

**Lemma 3.1.** *Let $u \in T(\Omega)$ and $v$ be two functions on which the green formulas can be applied, it holds:*

$$\int_\Omega A(u)\bar{v} = \int_\Omega uA^*\bar{v} - iM_0 \int_{\Gamma_N} u\bar{v}$$

*with*

$$A^* = k_0 \left(1 + \frac{iM_0}{k_0}\frac{\partial}{\partial x}\right)$$

*Proof.*

$$\begin{aligned} \int_\Omega A(u)\bar{v}d\Omega &= \int_\Omega k_0 u\bar{v}d\Omega - iM_0 \int_\Omega \frac{\partial u}{\partial x}\bar{v}d\Gamma \\ &= \int_\Omega k_0 u\bar{v}d\Omega - iM_0 \left(\int_{\partial\Omega} u\bar{v}\boldsymbol{e_x}\cdot\boldsymbol{n}d\Gamma - \int_\Omega u\frac{\partial\bar{v}}{\partial x}d\Omega\right) \\ &= \int_\Omega u\left(k_0\bar{v} + iM_0\frac{\partial\bar{v}}{\partial x}\right)\Gamma - iM_0\int_{\Gamma_N} u\bar{v}d\Gamma \\ &= \int_\Omega uA^*(\bar{v})d\Omega - iM_0\int_{\Gamma_N} u\bar{v}d\Gamma \end{aligned}$$

□

By integrating and using the preceding lemma, we obtain the following result

**Lemma 3.2.** *We have, from the convected Helmoltz equation the following equality*

$$\forall\varphi \in T(\Omega), a(u,\varphi) = l(\varphi)$$

*with:*

$$\begin{cases} a(u,\varphi) &= \int_\Omega \nabla u\nabla\bar{\varphi} - Au\overline{A\varphi}d\Omega + \int_{\Gamma_1\cup\Gamma_2}\frac{iZ_0}{Zk_0}A^2(u)\bar{\varphi}d\Gamma + \int_{\Gamma_N}\left[iM_0A(u)\bar{\varphi} + ik'u\bar{\varphi}\right]d\Gamma \\ l(\varphi) &= -\int_\Omega f\bar{\varphi}d\Omega + \int_{\Gamma_1\cup\Gamma_2}\eta\bar{\varphi}d\Gamma + \int_{\Gamma_N}\gamma\bar{\varphi}d\Gamma \end{cases}$$

This is not yet a variational formulation (term in $A^2(u)$ on $\Gamma_1 \cup \Gamma_2$). We will thus develop this term (and also see how we can arrange the rest of the terms).

$$a(u,\varphi) = \int_\Omega \underbrace{\nabla u\nabla\bar{\varphi}}_{(1)} + \underbrace{(-Au\overline{A\varphi})}_{(2)}d\Omega + \int_{\Gamma_1\cup\Gamma_2}\underbrace{\frac{iZ_0}{Zk_0}A^2(u)\bar{\varphi}}_{(3)}d\Gamma + \int_{\Gamma_N}[\underbrace{iM_0A(u)\bar{\varphi}}_{(4)} + \underbrace{ik'u\bar{\varphi}}_{(5)}]d\Gamma$$

We will have to redevelop the terms in $A, A^*, A^2$. As a reminder, we have

$$A = k_0\left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)$$

$$A^* = k_0\left(1 + i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)$$

$$
\begin{aligned}
(2) &= -k_0^2 \left(1 - i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)(u)\left(1 + i\frac{M_0}{k_0}\frac{\partial}{\partial x}\right)(\overline{\varphi}) \\
&= -k_0^2 \left(u\overline{\varphi} + i\frac{M_0}{k_0}\left(u\frac{\partial\overline{\varphi}}{\partial x} - \frac{\partial u}{\partial x}\overline{\varphi}\right) + \frac{M_0^2}{k_0^2}\frac{\partial u}{\partial x}\frac{\partial \overline{\varphi}}{\partial x}\right)
\end{aligned}
$$

What we want to change is the term in $u\dfrac{\partial\overline{\varphi}}{\partial x}$:

$$
\int_\Omega u\frac{\partial\overline{\varphi}}{\partial x}d\Omega = -\int_\Omega \frac{\partial u}{\partial x}\overline{\varphi}d\Omega + \int_{\Gamma_N} u\overline{\varphi}d\Gamma
$$

Thus

$$
\int_\Omega (2)d\Omega = -k_0^2\langle u \mid \varphi\rangle_{L^2(\Omega)} - M_0^2\left\langle \frac{\partial u}{\partial x}\middle|\frac{\partial \varphi}{\partial x}\right\rangle_{L^2(\Omega)} + 2iM_0k_0\left\langle\frac{\partial u}{\partial x}\middle|\varphi\right\rangle_{L^2(\Omega)} - iM_0k_0\langle u \mid \varphi\rangle_{L^2(\Gamma_N)}
$$

For (3):

$$
\begin{aligned}
\int_{\Gamma_1\cup\Gamma_2}(3)d\Omega &= \frac{iZ_0\overline{Z(\omega)}}{|Z(\omega)|^2k_0}\int_\Omega k_0^2\left(1 - 2i\frac{M_0}{k_0}\frac{\partial}{\partial x} - \frac{M_0^2}{k_0^2}\frac{\partial^2}{\partial x^2}\right)(u)\overline{\varphi}d\Omega \\
&= \frac{iZ_0\overline{Z(\omega)}k_0}{|Z(\omega)|^2}\left(\langle u \mid \varphi\rangle_{L^2(\Gamma_1\cup\Gamma_2)} - \frac{2iM_0}{k_0}\left\langle\frac{\partial u}{\partial x}\middle|\varphi\right\rangle_{L^2(\Gamma_1\cup\Gamma_2)} - \frac{M_0^2}{k_0^2}\left\langle\frac{\partial^2 u}{\partial x^2}\middle|\varphi\right\rangle_{L^2(\Gamma_1\cup\Gamma_2)}\right)
\end{aligned}
$$

For (4):

$$
\begin{aligned}
\int_{\Gamma_N}(4)d\Omega &= \int_{\Gamma_N} iM_0k_0\left(u\overline{\varphi} - i\frac{M_0}{k_0}\frac{\partial u}{\partial x}\overline{\varphi}\right)d\Omega \\
&= iM_0k_0\langle u \mid \varphi\rangle_{L^2(\Gamma_N)} + M_0^2\left\langle\frac{\partial u}{\partial x}\middle|\varphi\right\rangle_{L^2(\Gamma_N)}
\end{aligned}
$$

If we rewrite $a(u,\varphi)$ with scalar products, we thus end up with

$$
\begin{aligned}
a(u,\varphi) = \ &\langle\nabla u|\,\nabla\varphi\rangle_{L^2(\Omega)} &(1)\\
&- M_0^2\left\langle \frac{\partial u}{\partial x}\middle|\frac{\partial \varphi}{\partial x}\right\rangle_{L^2(\Omega)} - k_0^2\langle u \mid \varphi\rangle_{L^2(\Omega)} + 2iM_0k_0\left\langle\frac{\partial u}{\partial x}\middle|\varphi\right\rangle_{L^2(\Omega)} \boldsymbol{-iM_0k_0\langle u \mid \varphi\rangle_{L^2(\Gamma_N)}} &(2)\\
&+ \frac{iZ_0\overline{Z(\omega)}k_0}{|Z(\omega)|^2}\left(\langle u \mid \varphi\rangle_{L^2(\Gamma_1\cup\Gamma_2)} - \frac{2iM_0}{k_0}\left\langle\frac{\partial u}{\partial x}\middle|\varphi\right\rangle_{L^2(\Gamma_1\cup\Gamma_2)} - \frac{M_0^2}{k_0^2}\left\langle\frac{\partial^2 u}{\partial x^2}\middle|\varphi\right\rangle_{L^2(\Gamma_1\cup\Gamma_2)}\right) &(3)\\
&\boldsymbol{+iM_0k_0\langle u \mid \varphi\rangle_{L^2(\Gamma_N)}} + M_0^2\left\langle\frac{\partial u}{\partial x}\middle|\varphi\right\rangle_{L^2(\Gamma_N)} &(4)\\
&+ ik'\,\langle u|\,\varphi\rangle_{L^2(\Gamma_N)} &(5)
\end{aligned}
$$

The two terms in (2) and (4) in bold cancel out, so we end up this final fom :

$$
\begin{aligned}
a(u,\varphi) =\ & \langle \nabla u | \nabla \varphi \rangle_{L^2(\Omega)} && (1)\\
& - M_0^2 \left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Omega)} - k_0^2 \langle u \mid \varphi \rangle_{L^2(\Omega)} + 2iM_0k_0 \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Omega)} && (2)\\
& + \frac{iZ_0\overline{Z(\omega)}k_0}{|Z(\omega)|^2} \left( \langle u \mid \varphi \rangle_{L^2(\Gamma_1 \cup \Gamma_2)} - \frac{2iM_0}{k_0} \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} - \frac{M_0^2}{k_0^2} \left\langle \frac{\partial^2 u}{\partial x^2} \middle| \varphi \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} \right) && (3)\\
& + M_0^2 \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Gamma_N)} && (4)\\
& + ik' \langle u | \varphi \rangle_{L^2(\Gamma_N)} && (5)
\end{aligned}
$$

We still have a term in (3) in $\partial_{x,x} u$, as we are searching for a weak formulation (derivation in the distribution sense), we have

**Lemma 3.3.**

$$
- \left\langle \frac{\partial^2 u}{\partial x^2} \middle| \varphi \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} = + \left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)}
$$

and we can rewrite the term (3) and (4). Note that we change $l(\varphi)$ by adding a term

$$
-M_0^2 \int_{\Gamma_N} \gamma \bar{\varphi} d\Gamma
$$

Now, we have our variational formulation :

**Theorem 3.1.** *Variational formulation*

$$
\forall \varphi \in T(\Omega), a(u,\varphi) = l(\varphi)
$$

*With*

$$
\begin{aligned}
a(u,\varphi) =\ & \langle \nabla u | \nabla \varphi \rangle_{L^2(\Omega)} && (1)\\
& - M_0^2 \left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Omega)} - k_0^2 \langle u \mid \varphi \rangle_{L^2(\Omega)} + 2iM_0k_0 \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Omega)} && (2)\\
& + \frac{iZ_0\overline{Z(\omega)}k_0}{|Z(\omega)|^2} \left( \langle u \mid \varphi \rangle_{L^2(\Gamma_1 \cup \Gamma_2)} - \frac{2iM_0}{k_0} \left\langle \frac{\partial u}{\partial x} \middle| \varphi \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} + \frac{M_0^2}{k_0^2} \left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} \right) && (3)\\
& + (1 - M_0^2) ik' \langle u | \varphi \rangle_{L^2(\Gamma_N)} && (4) + (5)
\end{aligned}
$$

*and*

$$
l(\varphi) = - \int_\Omega f \bar{\varphi} d\Omega + \int_{\Gamma_1 \cup \Gamma_2} \eta \bar{\varphi} d\Gamma + (1 - M_0^2) \int_{\Gamma_N} \gamma \bar{\varphi} d\Gamma
$$

## 3.3 Some properties

**Theorem 3.2.** *a is a sesquilinear form, well defined on $T(\Omega)$ and $l$ is a linear continuous form on $T(\Omega)$.*

With a homegeneous boundary dirichlet boundary condition, we have the Poincarré inequality, which will be useful to prove the continuity of $a$.

**Lemma 3.4** (Poincarré Inequality)**.**

$$
\forall u \in T(\Omega), \|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{T(\Omega)}
$$

A simple majoration yields, $\forall v \in H^1(\Omega), \|\nabla v\|_{L^2(\Omega)} \geq \left\|\frac{\partial v}{\partial x}\right\|_{L^2(\Omega)}$. This result allows us to prove the continuity of A.

**Lemma 3.5.** *The operator $A$ is continuous from $T(\Omega)$ to $L^2$.*

*Proof.* $\forall u \in T(\Omega), \|Au\|_{L^2} \leq k_0 \|u\|_{L^2} + M_0 \|\nabla u\|_{L^2} \leq (k_0 C + M_0) \|u\|_{T(\Omega)}$ □

### 3.4 Existence and unicity of the solution for $u$

The Lax-Milgram theorem does not allow us to conclude on the existence and unicity of the solution (coercivity hypothesis), we will use the first Fredholm theorem instead.

**Theorem 3.3.** *Let $A$ be a linear compact operator in a Banach space $X$. The following assertions are equivalent*

1. *$\forall f \in X, x = Ax + f$ has a solution in $X$*
2. *$z - Az = 0$ (homogeneous equation) has only the trivial solution $z = 0$*
3. *$\forall g \in X^*, y - A^*y = 0$ has a solution in $X^*$*
4. *homogeneous equation $\phi - A^*\phi = 0$ has only the trivial solution $\phi = 0$*

*Furthermore, if one of the assertion 1 to 4 holds, then there exists the inverse operators $(I - A)^{-1}$ and $(I - A^*)^{-1}$ which are continuous*

So as to get some results on the compacity of the operator, we define $V(\Omega)$ a subset of $T(\Omega)$:

$$V(\Omega) = \{H^1(\Omega) | \Delta u \in L^2(\Omega), \text{ + weak boundary conditions}\}$$

We need to define a new scalar product in $V(\Omega)$ :

$$\langle u \mid v \rangle_{V(\Omega)} = \langle \nabla u \mid \nabla v \rangle_{L^2(\Omega)} - M_0^2 \left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial v}{\partial x} \right\rangle$$

This is a scalar product only under the hypothesis of $(1 - M_0^2) > 0$, so we will take $0 < M_0 < 1$ in what follows.

#### 3.4.1 Rewriting the scalar products

We have to remember in what follows that we ommited to put the trace operator on the scalar products on the boundary $\Gamma_N$ and $\Gamma_1 \cup \Gamma_2$ in the variational formulation.

We write $i_{L^2(\Omega)} : V(\Omega) \to L^2(\Omega)$ the linear compact injection from $V(\Omega)$ to $L^2(\Omega)$

$$\exists S : L^2(\Omega) \to V(\Omega), \forall \psi \in L^2(\Omega), \forall \varphi \in V(\Omega), \langle \psi | \varphi \rangle_{L^2(\Omega)} = \langle S\psi | \varphi \rangle_{V(\Omega)}$$

$$\exists T : L^2(\Gamma_1 \cup \Gamma_2) \to V(\Omega), \forall \psi \in L^2(\Gamma_1 \cup \Gamma_2), \forall \varphi \in V(\Omega), \langle \psi | \operatorname{Tr}(\varphi) \rangle_{L^2(\Gamma_1 \cup \Gamma_2)} = \langle T\psi | \varphi \rangle_{V(\Omega)}$$

$$\exists U : L^2(\Gamma_N) \to V(\Omega), \forall \psi \in L^2(\Gamma_N), \forall \varphi \in V(\Omega), \langle \psi | \operatorname{Tr}(\varphi) \rangle_{L^2(\Gamma_N)} = \langle U\psi | \varphi \rangle_{V(\Omega)}$$

We also have to take into account the term in $\left\langle \frac{\partial u}{\partial x} \middle| \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)}$ so we define a new operator :

$$\exists R : \underbrace{H_0^1(\Gamma_1 \cup \Gamma_2)}_{\mathrm{Tr}(f)|_{x=0}=0} \to V(\Omega)$$

$$\forall \psi \in H_0^1(\Omega), \forall \varphi \in V(\Omega),\ \left\langle \frac{\partial \psi}{\partial x} \middle| \mathrm{Tr} \circ \frac{\partial \varphi}{\partial x} \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} = \langle R(\psi) | \, \varphi \rangle_{V(\Omega)}$$

$$\begin{aligned}
\forall u, \varphi \in V(\Omega), \quad a(u,\varphi) = {} & \langle u | \, \varphi \rangle_{V(\Omega)} && \text{(1) and first term of (2)} \\
& - k_0^2 \left\langle S \circ i_{L^2(\Omega)}(u) \middle| \, \varphi \right\rangle_{V(\Omega)} && \text{second term of (2)} \\
& + 2iM_0k_0 \left\langle S \circ i_{L^2(\Omega)} \circ \frac{\partial}{\partial x}(u) \middle| \varphi \right\rangle_{V(\Omega)} && \text{last term of (2)} \\
& + \frac{iZ_0\overline{Z(\omega)}k_0}{|Z(\omega)|^2} \left\langle T \circ \mathrm{Tr}_{L^2(\Gamma_1 \cup \Gamma_2)}(u) \middle| \, \varphi \right\rangle_{V(\Omega)} && \text{first term of (3)} \\
& + \frac{2M_0Z_0\overline{Z(\omega)}}{|Z(\omega)|^2} \left\langle T \circ \mathrm{Tr}_{L^2(\Gamma_1 \cup \Gamma_2)} \circ \frac{\partial}{\partial x}(u) \middle| \varphi \right\rangle_{V(\Omega)} && \text{second term of (3)} \\
& + \frac{iZ_0\overline{Z(\omega)}M_0^2}{|Z(\omega)|^2k_0} \left\langle R \circ \mathrm{Tr}_{L^2(\Gamma_1 \cup \Gamma_2)}(u) \middle| \, \varphi \right\rangle_{V(\Omega)} && \text{third term of (3)} \\
& + ik'(1-M_0^2) \left\langle U \circ \mathrm{Tr}_{L^2(\Gamma_N)}(u) \middle| \, \varphi \right\rangle_{V(\Omega)} && (4) + (5)
\end{aligned}$$

By denoting $R' = R \circ \mathrm{Tr}_{L^2(\Gamma_1 \cup \Gamma_2)}$ ,$S' = S \circ i_{L^2(\Omega)}$, $T' = T \circ \mathrm{Tr}_{L^2(\Gamma_1 \cup \Gamma_2)}$ and $U' = U \circ \mathrm{Tr}_{L^2(\Gamma_N)}$ we can write in a compact form

$$\forall u, \varphi \in V(\Omega)$$

$$\begin{aligned}
a(u,\varphi) = \Bigg\langle \Bigg( & I - k_0^2 S' + \frac{2M_0Z_0\overline{Z(\omega)}}{|Z(\omega)|^2} T' \circ \partial_x + 2iM_0k_0 S' \circ \partial_x + \frac{iZ_0\overline{Z(\omega)}k_0}{|Z(\omega)|^2} T' \\
& + \frac{iZ_0\overline{Z(\omega)}M_0^2}{|Z(\omega)|^2k_0} R' + ik'(1-M_0^2) U' \Bigg) (u) \Bigg| \varphi \Bigg\rangle_{V(\Omega)}
\end{aligned}$$

We end up, denoting

$$P = k_0^2 S' - \frac{2M_0Z_0\overline{Z(\omega)}}{|Z(\omega)|^2} T' \circ \partial_x - 2iM_0k_0 S' \circ \partial_x - \frac{iZ_0\overline{Z(\omega)k_0}}{|Z(\omega)|^2} T' - \frac{iZ_0\overline{Z(\omega)}M_0^2}{|Z(\omega)|^2k_0} R' - ik'(1-M_0^2) U'$$

with

$$\forall u, \varphi \in V(\Omega), a(u,\varphi) = \langle (I-P)(u) | \, \varphi \rangle_{V(\Omega)}$$

# 4 Compacity study

## 4.1 $R', S', T', U'$

Let us prove that $S'$ is compact. As $i_{L^2(\Omega)} : V(\Omega) \to L^2(\Omega)$ is compact, we only have to prove that $S : L^2(\Omega) \to V(\Omega)$ is continuous.

We have

$$\exists S : L^2(\Omega) \to V(\Omega), \forall \psi \in L^2(\Omega), \forall \varphi \in V(\Omega),\ \langle \psi | \, \varphi \rangle_{L^2(\Omega)} = \langle S\psi | \, \varphi \rangle_{V(\Omega)}$$

so

$$
\begin{aligned}
\forall \varphi \in L^2(\Omega), \|S(\varphi)\|^2_{V(\Omega)} &= |\, \langle S(\varphi)|\, S(\varphi)\rangle_{V(\Omega)} \,| \\
&= |\, \left\langle i_{L^2(\Omega)} \circ S(\varphi) \middle|\, \varphi \right\rangle_{L^2(\Omega)} \,| \\
&\leq \|S(\varphi)\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \\
&\leq C(\Omega) \|\nabla S(\varphi)\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \\
&\leq \frac{C(\Omega)}{1 - M_0^2} \|S(\varphi)\|_{V(\Omega)} \|\varphi\|_{L^2(\Omega)}
\end{aligned}
$$

Thus $S$ is continuous, and $S'$ compact.
In the last line, we used that

$$
\begin{aligned}
\|u\|^2_{V(\Omega)} &= (1 - M_0^2) \left\| \frac{\partial u}{\partial x} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial u}{\partial y} \right\|^2_{L^2(\Omega)} \\
&\geq (1 - M_0^2) \left( \left\| \frac{\partial u}{\partial x} \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial u}{\partial y} \right\|^2_{L^2(\Omega)} \right) \\
&\geq (1 - M_0^2) \|\nabla u\|^2_{L^2(\Omega)}
\end{aligned}
$$

We prove in the same way that $R'$, $T'$ and $U'$ are compact as well, using the continuity of the trace operator. For $T$

$$
\begin{aligned}
\forall u \in L^2(\Gamma_1 \cup \Gamma_2), \|T(u)\|^2_{V(\Omega)} &= \langle u|\, \mathrm{Tr} \circ T(u)\rangle_{L^2(\Omega)} \\
&\leq \|u\|_{L^2(\Gamma_1 \cup \Gamma_2)} \|\, \mathrm{Tr} \circ T(u)\|_{L^2(\Gamma_1 \cup \Gamma_2)} \\
&\leq \|u\|_{L^2(\Gamma_1 \cup \Gamma_2)} K \|T(u)\|_{V(\Omega)}
\end{aligned}
$$

### 4.2 partial derivative

The operator $\partial_x : V(\Omega) \to L^2(\Omega)$ is continuous so every term of the form

$$
A' \circ \partial_x, \quad A' \in \{S', T', U'\}
$$

is compact as a composition of a continuous function with a compact function.

We have proved the compacity of our operators, but did not manage to prove the existence and unicity of the solution using the Fredholm theorem. One potential lead would be to find a

## 5 Numerical implementation

### 5.1 Finite difference scheme

We rewrite the problem in its differential form :

- $\dfrac{\partial^2 \hat{p}'}{\partial y^2} + k_0^2 \hat{p}' + (1 - M_0^2) \dfrac{\partial^2 \hat{p}'}{\partial x^2} - 2ik_0 M_0 \dfrac{\partial \hat{p}'}{\partial x} = 0$ in $\Omega$
- $\hat{p}' = s(y)$ on $\Gamma_D (x = 0)$ (inflow condition)
- $\dfrac{\partial \hat{p}'}{\partial y} + \dfrac{\chi_1(x) Z_0 \overline{Z}}{|Z|^2} \left( ik_0 \hat{p}' + 2M_0 \dfrac{\partial \hat{p}'}{\partial x} - i \dfrac{M_0^2}{k_0} \dfrac{\partial^2 \hat{p}'}{\partial x^2} \right) = 0$ on $\Gamma_1$ (y=L)

- $-\dfrac{\partial \hat{p}'}{\partial y} + \dfrac{\chi_2(x) Z_0 \overline{Z}}{|Z|^2}\left(ik_0\hat{p}' + 2M_0\dfrac{\partial \hat{p}'}{\partial x} - i\dfrac{M_0^2}{k_0}\dfrac{\partial^2 \hat{p}'}{\partial x^2}\right) = 0$ on $\Gamma_2$ (y=0)
- $\dfrac{\partial \hat{p}'}{\partial x} + ik'\hat{p}' = 0$ on $\Gamma_N(x = L)$ (radiation condition,outgoing flow)

We discretize the domain in $(i, j) \in \{0, N_y\}\times\{0, N_x\}$, note $p_{i,j} = p(jh_x, ih_y)$ the solution and $h_x = \frac{l}{N_x}, h_y = \frac{L}{N_y}$

- $\forall (i, j) \in \{1, N_y - 1\} \times \{1, N_x - 1\}$

$$\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{h_y^2} + k_0^2 p_{i,j} + (1-M_0^2)\frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{h_x^2} - 2iM_0k_0\frac{p_{i,j+1} - p_{i,j-1}}{2h_x} = 0$$

- $\forall i \in \{0, N_y\}, p_{i,0} = s(h_y i)$
- $\forall j \in \{1, N_x - 1\}$,

$$\frac{p_{N_y,j} - p_{N_y-1,j}}{h_y} + \frac{\chi_1(j)Z_0\overline{Z}}{|Z|^2}\left(ik_0 p_{N_y,j} + 2M_0\frac{p_{N_y,j+1} - p_{N_y,j-1}}{2h_x} - i\frac{M_0^2}{k_0}\frac{p_{N_y,j+1} - 2p_{N_y,j} + p_{N_y,j-1}}{h_x^2}\right) = 0$$

- $\forall j \in \{1, N_x - 1\}$,

$$\frac{p_{0,j} - p_{1,j}}{h_y} + \frac{\chi_2(j)Z_0\overline{Z}}{|Z|^2}\left(ik_0 p_{0,j} + 2M_0\frac{p_{0,j+1} - p_{0,j-1}}{2h_x} - i\frac{M_0^2}{k_0}\frac{p_{0,j+1} - 2p_{0,j} + p_{0,j-1}}{h_x^2}\right) = 0$$

- $\forall i \in \{0, N_y\}$

$$\frac{p_{i,N_x} - p_{i,N_x-1}}{h_x} + ik'p_{i,N_x} = 0$$

We implement this finite differences scheme with sparse matrix to speed up the process and optimize the memory consumption of our program.

# 6 Optimisation problem

In what follows, we take into account the term $\chi$, defined at section 2.4.3, that we recall here : We add a term $\chi \in L^2(\Gamma_1 \cup \Gamma_2)$ such that

$$\forall z \in \Gamma_1 \cup \Gamma_2, \chi(z) = \begin{cases} 1 & \text{if we placed a liner placed at this emplacement} \\ 0 & \text{otherwise} \end{cases}$$

In our boundary condition, it can be simply integrated by replacing

$$Z_0 \to Z_0\chi(z)$$

Beware that integrating this term in the variational formulation we also need to replace

$$\eta \to \chi\eta$$

This comes from the rewriting of the problem we did at section 2.1

$$\frac{\partial u}{\partial n} + \frac{iZ_0\boldsymbol{\chi}}{Zk_0}A^2u = \boldsymbol{\chi}\underbrace{\left(-\frac{iZ_0}{Zk_0}A^2g\right)}_{\eta} \text{ on } \Gamma_1 \cup \Gamma_2$$

We note that we will restraint ourselves to symmetrical solutions (same $\chi$ on $\Gamma_1$ than on $\Gamma_2$) as our problem is a cut of a cylindrical duct.

For $\beta \in ]0,1[$, we define

$$B_\beta = \left\{ \chi \in L^2(\Gamma_1 \cup \Gamma_2), \chi(\Gamma_1 \cup \Gamma_2) = \{0,1\}, \int_{\Gamma_1 \cup \Gamma_2} \chi(z)dz = \beta\lambda(\Gamma_1 \cup \Gamma_2) \right\}$$

This $\beta$ term signify we only place the liner on some part of the boundary (budget constraint).

We also denote $p(\chi, \omega)$ the numerical solution for a given $\chi$ and $\omega$ (and a certain set of conditions $(Z(\omega), s(y), M_0, Z_0, c_0)$)

## 6.1 Optimisation problem $(OP_1)$

Given :

- $\omega$ the frequency we want to optimize the liner for.
- $s(y)$ the form of the incident wave for that frequency (we will take $s(y) = 1$ for most if not all our calculation, it corresponds to one of the modes of propagation in a cylindrical duct with hard walls)
- $Z(\omega)$ the normal impedance of the liner (generally already optimized for one frequency)
- $\beta \in ]0,1[$ (budget constraint)
- $(M_0, Z_0, c_0)$ the physical characteristics of the incoming flow

$$\text{Find } \chi \in B_\beta \text{ that } \underset{\chi \in B_\beta}{\text{minimizes}} \|p(\chi, \omega)\|_{L^2(\Omega)}$$

We can improve the following approach (centered on optimizing for one frequency and then verifying the performances on other frequencies -thus the need for the data of $s(y)$ for different frequencies, we will take $s(y) = 1$ as a first approach-)

This constant acoustic pressure correspond to one of the modes of propagative wave in a cylindrical duct with hard walls (no liner), as can be seen in [6]. In this situation, (for a cylindrical duct with no liner), we also have

$$k' = \frac{k_0}{1 + M_0}$$

as this mode correspond to the solution :

$$p = P_0 e^{i(k'x - \omega t)}$$

and we easily verify that such a solution is valid for $\chi = 0$. In what follows, we will impose such a value of $k'$.

In practice we will verify the performance on a frequency band after the optimization step $(OP_1)$ on one frequency.

# 7 Algorithms for $(OP_1)$

## 7.1 Genetic algorithm for $(OP_1)$

The nature of this problem makes it perfectly fit for a genetic algorithm. The liner can be represented by a sequence of bits. 1 means that there is actually some liner, and 0 means that we didn't place any liner at this spot. We can run an optimisation loop applying the main steps of a genetic algorithm after making an initial population.

1. Evaluate all the liner presences (sequence of bits) using a certain cost function.
2. Select the best individuals according to the cost function we have chosen.
3. Make a crossover on them in order to create a new individual, hopefully better.
4. With a small probability, mutate the individual by swapping a few of its bits.
5. Project the individuals to obtain a density of $\beta$.

In theory, using as our cost function the energy inside the domain is alright, but it is practically impossible. The reason is that the algorithm could return a list that would look for instance like [0,1,0,1,0,1,0,1]. This means that we have to cut our liner is several small pieces, which are arbitrarily small. To prevent this issue, we can add a second term to our cost function. If We denote the liner as $[b_1, b_2, ..., b_N]$, with $b_i$ being either 1 or 0, we can count the number of consecutive 1 between each 0 (We will note them $I_1, .., I_k$). Our additional term can be expressed as $\sum_{i=1}^{k} \frac{\sigma}{I_i^\alpha}$, $\sigma$ represents the importance of having only a few cuts. If $f(\chi) = \sigma$ is 0, it doesn't matter. $\alpha$ represents how much we want to penalize small groups of 1 compared to large groups. If $\alpha$ is 0, we penalize all the groups with the same coefficient. We can then use this cost function $C(\chi)$

$$C(\chi) = \int_\Omega u^2(\chi) d\Omega + f(\chi)$$

With $\alpha$ set to 1.5 and $\sigma$ set to 1 we get the result below (black means there is a liner, white means there is no liner).

The complete set of parameters used in the following illustration are as follow

$$\begin{cases} f = 2\,kHz \\ L = 0.5\,m \\ l = 0.2\,m \\ N_x = 250, N_y = 100 \\ M_0 = 0.2 \\ c = 340\,m.s^{-1} \\ Z_0 = 428.3\,Pa.s/m \text{(acoustic impedance of air at } 0°C) \\ k' = \frac{k_0}{1+M_0} \\ p|_{\Gamma_D} = p_0 = 1.5\,Pa \\ Z(\omega) = Z_0(1-i) \\ \beta = 0.4 \end{cases}$$

We compare the results with the one obtained for a liner placed at the start of the duct, over a band of frequencies. Please note that the genetic algorithm picks random placements for $\chi$ at the initialization, so this 'chi at the start' is not used in the initialization phase of the algorithm.

We also note that we do **not** observe in the final distribution of the liner a characteristic length $\frac{\lambda}{2}$. We can try to explain this phenomenon :

- There is no reflection in the duct, the acoustic perturbation goes only in the forward direction
- The choice of impedance is the most important factor in the reduction of a given frequency (see the last part on the optimal impedance).

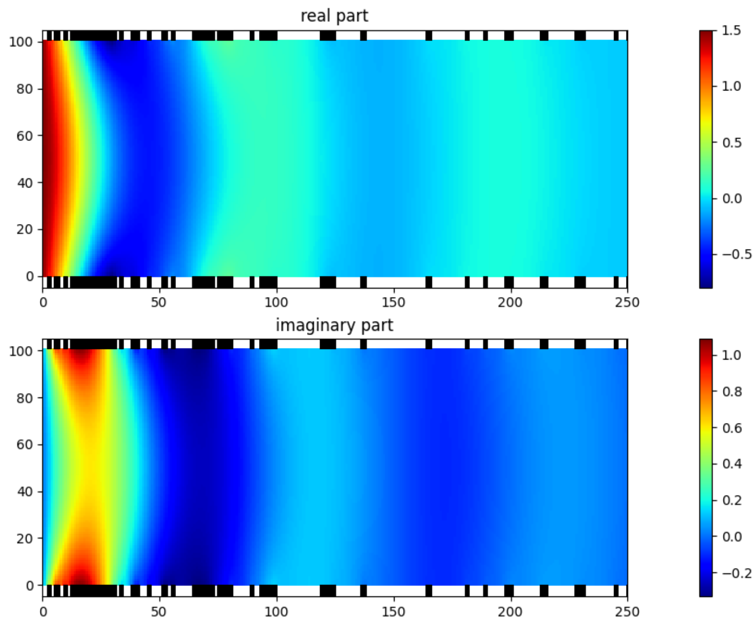



Figure 5: Optimized chi with $\alpha = 1.5$ and $\sigma = 1$

## 7.2 Energy derivative for $(OP_1)$

### 7.2.1 Real variational formula

We reuse the developed and rewritten equations on $u$ (with a homogeneous boundary condition on $\Gamma_D$). We have to note that, as said before, we have a $\chi\eta$ term on $\Gamma_1 \cup \Gamma_2$

$$\begin{cases} \Delta u + k_0^2 u - 2iM_0 k_0 \dfrac{\partial u}{\partial x} - M_0^2 \dfrac{\partial^2 u}{\partial x^2} = f & \Omega \\ u = 0 & \Gamma_D \\ \dfrac{\partial u}{\partial n} + \dfrac{\chi Z_0 \bar{Z}}{|Z|^2} \left( ik_0 u + 2M_0 \dfrac{\partial u}{\partial x} - i \dfrac{M_0^2}{k_0} \dfrac{\partial^2 u}{\partial x^2} \right) = \chi\eta & \Gamma_1 \cup \Gamma_2 \\ \dfrac{\partial u}{\partial n} + ik'u = \gamma & \Gamma_N \end{cases}$$

We use Mathematica to separate the real parts and imaginary part easily (with symbolic calculations)

Listing 1: Separation with Mathematica of the equation in $\Omega$ ($Du* = \Delta u_*, dxu* = \partial_x u_*, I = i$)

```
LP = Dur + I*Dui + k0^2 *(ur + I*Ui) − 2*I*M0*k0*(dxur + I*dxui) − (M0^2)*(dxxur + I*dxxui)
ComplexExpand[LP]
Dur + 2 dxui k0 M0 − dxxur M0^2 + + k0^2 ur +
I (Dui − 2 dxur k0 M0 − dxxui M0^2 + k0^2 Ui)
```

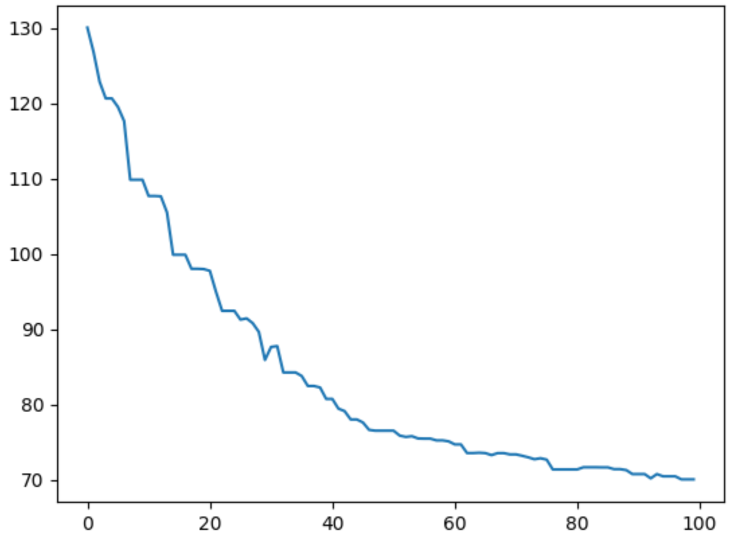



Figure 6: Cost after each generation

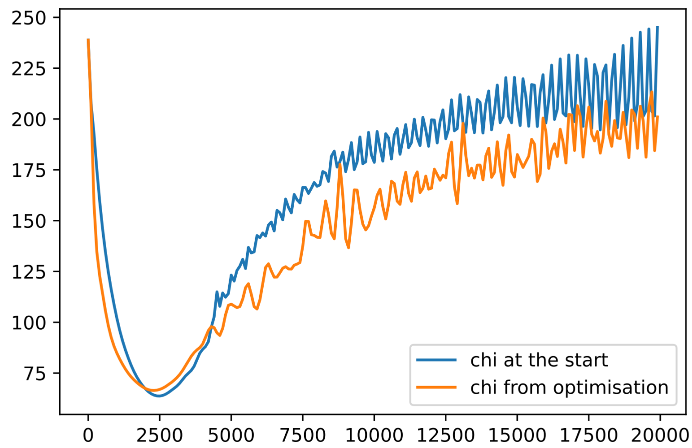



Figure 7: Frequency response for the optimised $\chi$ and a placement of liner at the start of the duct

Listing 2: Separation with Mathematica of the equation on $\Gamma_1 \cup \Gamma_2$ $(dxu* = \partial_x u_*, I = i; dnu* = \partial_n u_*, K = \frac{\chi Z_0 \overline{Z}}{|Z|^2})$

```
G1LP = (dnur + I*dnui) + (K)*(Zr - I*Zi)*(I*ko*(ur + I*ui) + 2*M0*(dxur + I*dxui) - I* (M0^2/k0
      )*(dxxur + I*dxxui))

ComplexExpand[G1LP]
dnur + 2 dxui K M0 Zi - (dxxur K M0^2 Zi)/k0 + K ko ur Zi + 2 dxur K M0 Zr + (dxxui K M0^2 Zr)
      /k0 - K ko ui Zr +
I (dnui - 2 dxur K M0 Zi - (dxxui K M0^2 Zi)/k0 + K ko ui Zi + 2 dxui K M0 Zr - (dxxur K M0^2 Zr)/
   k0 + K ko ur Zr)
```

Real part :

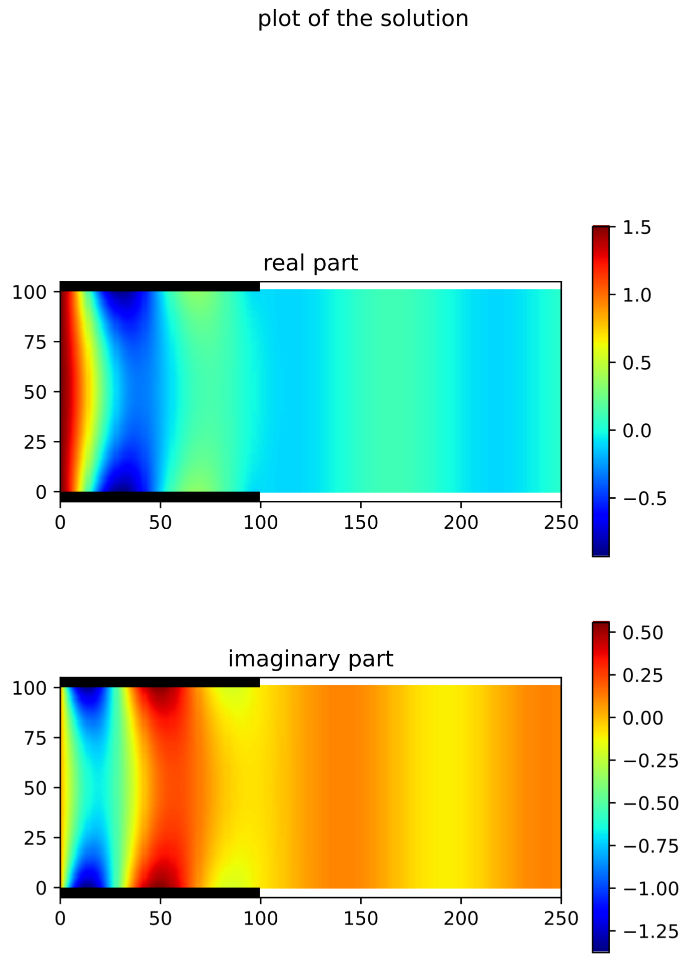



Figure 8: placement of the chi at the start and solution for $f = 2kHz$

$$
\begin{cases}
\Delta u_r + k_0^2 u_r + 2M_0 k_0 \dfrac{\partial u_i}{\partial x} - M_0^2 \dfrac{\partial^2 u_r}{\partial x^2} = f_r & \Omega \\
u_r = 0 & \Gamma_D \\
\dfrac{\partial u_r}{\partial n} + \dfrac{\chi Z_0}{|Z|^2}\Bigg( Z_i k_0 u_r - Z_r k_0 u_i + 2Z_i M_0 \dfrac{\partial u_i}{\partial x} + 2Z_r M_0 \dfrac{\partial u_r}{\partial x} & \\
+ \dfrac{Z_r M_0^2}{k_0}\dfrac{\partial^2 u_i}{\partial x^2} - \dfrac{Z_i M_0^2}{k_0}\dfrac{\partial^2 u_r}{\partial x^2}\Bigg) = \chi \eta_r & \Gamma_1 \cup \Gamma_2 \\
\dfrac{\partial u_r}{\partial n} - k' u_i = \gamma_r & \Gamma_N
\end{cases}
$$

We take test functions that are equal to 0 on $\Gamma_D$ :

$$
A_r(\chi, u_r, u_i, \varphi_r, \varphi_i) = \int_\Omega \left[ (1 - M_0^2) \underbrace{\frac{\partial^2 u_r}{\partial x^2}}_{(1)} + \underbrace{\frac{\partial^2 u_r}{\partial y^2}}_{(2)} + k_0^2 u_r + 2M_0 k_0 \frac{\partial u_i}{\partial x} - f_r \right] \varphi_r d\Omega
$$

We have to integrate by parts to verify the boundary conditions, we recall that

$$\int_\Omega \frac{\partial^2 u}{\partial x_i^2} v = \int_{\partial\Omega} \frac{\partial u}{\partial x_i} v n_i - \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$$

For (1):

$$\begin{aligned}\int_\Omega \frac{\partial^2 u_r}{\partial x^2}\varphi_r &= \int_{\Gamma_N} \frac{\partial u_r}{\partial x}\varphi_r d\Gamma - \int_\Omega \frac{\partial u_r}{\partial x}\frac{\partial \varphi_r}{\partial x} \\ &= \int_{\Gamma_N} (\gamma_r + k' u_i)\varphi_r d\Gamma - \int_\Omega \frac{\partial u_r}{\partial x}\frac{\partial \varphi_r}{\partial x}\end{aligned}$$

And for (2)

$$\begin{aligned}\int_\Omega \frac{\partial^2 u_r}{\partial y^2}\varphi_r &= \int_{\Gamma_1} +\frac{\partial u_r}{\partial y}\varphi_r + \int_{\Gamma_2} -\frac{\partial u_r}{\partial y}\varphi_r - \int_\Omega \frac{\partial u_r}{\partial y}\frac{\partial \varphi_r}{\partial y} \\ &= \int_{\Gamma_1\cup\Gamma_2} \varphi_r \frac{\partial u_r}{\partial n} - \int_\Omega \frac{\partial u_r}{\partial y}\frac{\partial \varphi_r}{\partial y} \\ &= \int_{\Gamma_1\cup\Gamma_2} \varphi_r \Bigg[ -\frac{\chi Z_0}{|Z|^2}\Bigg( Z_i k_0 u_r - Z_r k_0 u_i + 2 Z_i M_0 \frac{\partial u_i}{\partial x} \\ &+ 2 Z_r M_0 \frac{\partial u_r}{\partial x} + \frac{Z_r M_0^2}{k_0}\frac{\partial^2 u_i}{\partial x^2} - \frac{Z_i M_0^2}{k_0}\frac{\partial^2 u_r}{\partial x^2}\Bigg) + \chi \eta_r \Bigg] \\ &- \int_\Omega \frac{\partial u_r}{\partial y}\frac{\partial \varphi_r}{\partial y}\end{aligned}$$

We end up with

$$\begin{aligned}A_r(\chi, u_r, u_i, \varphi_r, \varphi_i) = &\int_\Omega \left( -(1-M_0^2)\frac{\partial u_r}{\partial x}\frac{\partial \varphi_r}{\partial x} - \frac{\partial u_r}{\partial y}\frac{\partial \varphi_r}{\partial y} + k_0^2 u_r \varphi_r + 2 M_0 k_0 \frac{\partial u_i}{\partial x}\varphi_r - f_r \varphi_r \right) \\ &+ \int_{\Gamma_N} (1-M_0^2)(\gamma_r + k' u_i)\varphi_r \\ &+ \int_{\Gamma_1\cup\Gamma_2} \varphi_r \Bigg[ -\frac{\chi Z_0}{|Z|^2}\Bigg( Z_i k_0 u_r - Z_r k_0 u_i + 2 Z_i M_0 \frac{\partial u_i}{\partial x} \\ &+ 2 Z_r M_0 \frac{\partial u_r}{\partial x} + \frac{Z_r M_0^2}{k_0}\frac{\partial^2 u_i}{\partial x^2} - \frac{Z_i M_0^2}{k_0}\frac{\partial^2 u_r}{\partial x^2}\Bigg) + \chi \eta_r \Bigg] d\Gamma\end{aligned}$$

Let us note that we do not have yet a variational formulation, due to the second derivatives on $\Gamma_1 \cup \Gamma_2$. As before, we place ourselves in $V(\Omega)$, with weak boundary conditions, and we have :

$$\forall (i,j) \in \{r,i\}^2 \quad \left\langle \frac{\partial^2 u_i}{\partial x^2} \middle| \varphi_j \right\rangle_{L^2(\Gamma_1\cup\Gamma_2)} = -\left\langle \frac{\partial u_i}{\partial x} \middle| \frac{\partial \varphi_j}{\partial x} \right\rangle_{L^2(\Gamma_1\cup\Gamma_2)}$$

We obtain the variational formulation $FV_r$ for the real part :

$$\begin{aligned}FV_r(\chi, u_r, u_i, \varphi_r, \varphi_i) = &\int_\Omega \left( -(1-M_0^2)\frac{\partial u_r}{\partial x}\frac{\partial \varphi_r}{\partial x} - \frac{\partial u_r}{\partial y}\frac{\partial \varphi_r}{\partial y} + k_0^2 u_r \varphi_r + 2 M_0 k_0 \frac{\partial u_i}{\partial x}\varphi_r - f_r \varphi_r \right) \\ &+ \int_{\Gamma_N} (1-M_0^2)(\gamma_r + k' u_i)\varphi_r \\ &+ \int_{\Gamma_1\cup\Gamma_2} \varphi_r \left[ -\frac{\chi Z_0}{|Z|^2}\left( Z_i k_0 u_r - Z_r k_0 u_i + 2 Z_i M_0 \frac{\partial u_i}{\partial x} + 2 Z_r M_0 \frac{\partial u_r}{\partial x}\right) + \chi \eta_r \right] \\ &+ \frac{\chi Z_0}{|Z|^2}\frac{\partial \varphi_r}{\partial x}\left( + \frac{Z_r M_0^2}{k_0}\frac{\partial u_i}{\partial x} - \frac{Z_i M_0^2}{k_0}\frac{\partial u_r}{\partial x}\right) d\Gamma\end{aligned}$$

Imaginary part :

$$\begin{cases} \Delta u_i + k_0^2 u_i - 2k_0 M_0 \dfrac{\partial u_r}{\partial x} - M_0^2 \dfrac{\partial^2 u_i}{\partial x^2} = f_i & \Omega \\ u_i = 0 & \Gamma_D \\ \dfrac{\partial u_i}{\partial n} + \dfrac{\chi Z_0}{|Z|^2}\left(+k_0 Z_i u_i + k_0 u_r Z_r + 2Z_r M_0 \dfrac{\partial u_i}{\partial x} - 2Z_i M_0 \dfrac{\partial u_r}{\partial x} - \dfrac{Z_r M_0^2}{k_0}\dfrac{\partial^2 u_r}{\partial x^2} - \dfrac{Z_i M_0^2}{k_0}\dfrac{\partial^2 u_i}{\partial x^2}\right) = \chi \eta_i & \Gamma_1 \cup \Gamma_2 \\ \dfrac{\partial u_i}{\partial n} + k' u_r = \gamma_i & \Gamma_N \end{cases}$$

In the same way :

$$A_i(\chi, u_r, u_i, \varphi_r, \varphi_i) = \int_\Omega \left[ (1 - M_0^2) \underbrace{\frac{\partial^2 u_i}{\partial x^2}}_{(1)} + \underbrace{\frac{\partial^2 u_i}{\partial y}}_{(2)} + k_0^2 u_i - 2k_0 M_0 \frac{\partial u_r}{\partial x} - f_i \right] \varphi_i d\Omega$$

For (1) :

$$\begin{aligned} \int_\Omega \frac{\partial^2 u_i}{\partial x^2}\varphi_i &= \int_{\Gamma_N} \frac{\partial u_i}{\partial x}\varphi_i d\Gamma - \int_\Omega \frac{\partial u_i}{\partial x}\frac{\partial \varphi_i}{\partial x} \\ &= \int_{\Gamma_N} (\gamma_i - k' u_r)\varphi_i d\Gamma - \int_\Omega \frac{\partial u_i}{\partial x}\frac{\partial \varphi_i}{\partial x} \end{aligned}$$

For (2) :

$$\begin{aligned} \int_\Omega \frac{\partial^2 u_i}{\partial y^2}\varphi_i &= \int_{\Gamma_1} +\frac{\partial u_i}{\partial y}\varphi_i + \int_{\Gamma_2} -\frac{\partial u_i}{\partial y}\varphi_i - \int_\Omega \frac{\partial u_i}{\partial y}\frac{\partial \varphi_i}{\partial y} \\ &= \int_{\Gamma_1 \cup \Gamma_2} \varphi_i \frac{\partial u_i}{\partial n} - \int_\Omega \frac{\partial u_i}{\partial y}\frac{\partial \varphi_i}{\partial y} \\ &= \int_{\Gamma_1 \cup \Gamma_2} \varphi_i \Bigg[ -\frac{\chi Z_0}{|Z|^2}\Bigg( +k_0 Z_i u_i + k_0 u_r Z_r + 2Z_r M_0 \frac{\partial u_i}{\partial x} - 2Z_i M_0 \frac{\partial u_r}{\partial x} \\ &\quad - \frac{Z_r M_0^2}{k_0}\frac{\partial^2 u_r}{\partial x^2} - \frac{Z_i M_0^2}{k_0}\frac{\partial^2 u_i}{\partial x^2}\Bigg) + \chi \eta_i \Bigg] \\ &\quad - \int_\Omega \frac{\partial u_i}{\partial y}\frac{\partial \varphi_i}{\partial y} \end{aligned}$$

Hence

$$\begin{aligned} A_i(\chi, u_r, u_i, \varphi_r, \varphi_i) &= \int_\Omega \left( -(1 - M_0^2)\frac{\partial u_i}{\partial x}\frac{\partial \varphi_i}{\partial x} - \frac{\partial u_i}{\partial y}\frac{\partial \varphi_i}{\partial y} + k_0^2 u_i \varphi_i - 2k_0 M_0 \frac{\partial u_r}{\partial x}\varphi_i - f_i \varphi_i \right) \\ &+ \int_{\Gamma_N} (1 - M_0^2)(\gamma_i - k' u_r)\varphi_i \\ &+ \int_{\Gamma_1 \cup \Gamma_2} \varphi_i \Bigg[ -\frac{\chi Z_0}{|Z|^2}\Bigg( +k_0 Z_i u_i + k_0 u_r Z_r + 2Z_r M_0 \frac{\partial u_i}{\partial x} - 2Z_i M_0 \frac{\partial u_r}{\partial x} \\ &- \frac{Z_r M_0^2}{k_0}\frac{\partial^2 u_r}{\partial x^2} - \frac{Z_i M_0^2}{k_0}\frac{\partial^2 u_i}{\partial x^2}\Bigg) + \chi \eta_i \Bigg] d\Gamma \end{aligned}$$

We rewrite the second derivatives and we obtain the variational formulation $FV_i$ for the imaginary part :

$$\begin{aligned}
FV_i(\chi, u_r, u_i, \varphi_r, \varphi_i) = &\int_\Omega \left( -(1-M_0^2)\frac{\partial u_i}{\partial x}\frac{\partial \varphi_i}{\partial x} - \frac{\partial u_i}{\partial y}\frac{\partial \varphi_i}{\partial y} + k_0^2 u_i \varphi_i - 2k_0 M_0 \frac{\partial u_r}{\partial x}\varphi_i - f_i \varphi_i \right) \\
&+ \int_{\Gamma_N} (1-M_0^2)(\gamma_i - k' u_r)\varphi_i \\
&+ \int_{\Gamma_1 \cup \Gamma_2} \varphi_i \left[ -\frac{\chi Z_0}{|Z|^2} \left( +k_0 Z_i u_i + k_0 u_r Z_r + 2 Z_r M_0 \frac{\partial u_i}{\partial x} - 2 Z_i M_0 \frac{\partial u_r}{\partial x} \right) + \chi \eta_i \right] \\
&+ \frac{\chi Z_0}{|Z|^2}\frac{\partial \varphi_i}{\partial x} \left( -\frac{Z_r M_0^2}{k_0}\frac{\partial u_r}{\partial x} - \frac{Z_i M_0^2}{k_0}\frac{\partial u_i}{\partial x} \right) d\Gamma
\end{aligned}$$

Finally, we define

$$FV(\chi, u_r, u_i, \varphi_r, \varphi_i) = FV_r(\chi, u_r, u_i, \varphi_r, \varphi_i) - FV_i(\chi, u_r, u_i, \varphi_r, \varphi_i)$$

We recall the expression for $FV_r$ :

$$\begin{aligned}
FV_r(\chi, u_r, u_i, \varphi_r, \varphi_i) = &\int_\Omega \left( -(1-M_0^2)\frac{\partial u_r}{\partial x}\frac{\partial \varphi_r}{\partial x} - \frac{\partial u_r}{\partial y}\frac{\partial \varphi_r}{\partial y} + k_0^2 u_r \varphi_r + 2 M_0 k_0 \frac{\partial u_i}{\partial x}\varphi_r - f_r \varphi_r \right) \\
&+ \int_{\Gamma_N} (1-M_0^2)(\gamma_r + k' u_i)\varphi_r \\
&+ \int_{\Gamma_1 \cup \Gamma_2} \varphi_r \left[ -\frac{\chi Z_0}{|Z|^2} \left( Z_i k_0 u_r - Z_r k_0 u_i + 2 Z_i M_0 \frac{\partial u_i}{\partial x} + 2 Z_r M_0 \frac{\partial u_r}{\partial x} \right) + \chi \eta_r \right] \\
&+ \frac{\chi Z_0}{|Z|^2}\frac{\partial \varphi_r}{\partial x} \left( +\frac{Z_r M_0^2}{k_0}\frac{\partial u_i}{\partial x} - \frac{Z_i M_0^2}{k_0}\frac{\partial u_r}{\partial x} \right) d\Gamma
\end{aligned}$$

We end up with

$$\begin{aligned}
FV(\chi, u_r, u_i, \varphi_r, \varphi_i) = &\int_\Omega \Bigg[ (1-M_0^2)\left( \frac{\partial u_i}{\partial x}\frac{\partial \varphi_i}{\partial x} - \frac{\partial u_r}{\partial x}\frac{\partial \varphi_r}{\partial x} \right) + \left( \frac{\partial u_i}{\partial y}\frac{\partial \varphi_i}{\partial y} - \frac{\partial u_r}{\partial y}\frac{\partial \varphi_r}{\partial y} \right) \\
&+ k_0^2 (u_r \varphi_r - u_i \varphi_i) + 2 M_0 k_0 \left( \frac{\partial u_i}{\partial x}\varphi_r + \frac{\partial u_r}{\partial x}\varphi_i \right) - f_r \varphi_r - f_i \varphi_i \Bigg] d\Omega \\
&+ \int_{\Gamma_N} (1-M_0^2) \left[ (\gamma_r \varphi_r - \gamma_i \varphi_i) + k'(u_i \varphi_r + u_r \varphi_i) \right] d\Gamma \\
&+ \int_{\Gamma_1 \cup \Gamma_2} \Bigg[ \chi \left( \varphi_r \eta_r - \varphi_i \eta_i \right) \\
&+ \frac{\chi Z_0}{|Z|^2} \Bigg( -\varphi_r \left( Z_i k_0 u_r - Z_r k_0 u_i + 2 Z_i M_0 \frac{\partial u_i}{\partial x} + 2 Z_r M_0 \frac{\partial u_r}{\partial x} \right) \\
&+ \frac{\partial \varphi_r}{\partial x} \left( \frac{Z_r M_0^2}{k_0}\frac{\partial u_i}{\partial x} - \frac{Z_i M_0^2}{k_0}\frac{\partial u_r}{\partial x} \right) \\
&+ \varphi_i \left( +k_0 Z_i u_i + k_0 u_r Z_r + 2 Z_r M_0 \frac{\partial u_i}{\partial x} - 2 Z_i M_0 \frac{\partial u_r}{\partial x} \right) \\
&+ \frac{\partial \varphi_i}{\partial x} \left( \frac{Z_r M_0^2}{k_0}\frac{\partial u_r}{\partial x} + \frac{Z_i M_0^2}{k_0}\frac{\partial u_i}{\partial x} \right) \Bigg) \Bigg]
\end{aligned}$$

### 7.2.2 Application of the Lagrangian method

We define the Lagrangian :

$$\mathcal{L}(\chi, w_r, w_i, q_r, q_i) = \int_\Omega (w_r^2 + w_i^2) d\Omega + FV(\chi, w_r, w_i, q_r, q_i)$$

The adjoint problem is Find $p_r, p_i \in V(\Omega)$ such that

$$\begin{cases} \forall \Phi_r, \left\langle \dfrac{\partial \mathcal{L}}{\partial w_r}(\chi, w_r, w_i, p_r, p_i) \middle| \Phi_r \right\rangle = 0 \\ \forall \Phi_i, \left\langle \dfrac{\partial \mathcal{L}}{\partial w_i}(\chi, w_r, w_i, p_r, p_i) \middle| \Phi_i \right\rangle = 0 \end{cases}$$

For the first equation, we have

$$\begin{aligned} \forall \Phi_r, \left\langle \frac{\partial \mathcal{L}}{\partial w_r}(\chi, w_r, w_i, p_r, p_i) \middle| \Phi_r \right\rangle = & \int_\Omega 2w_r\Phi_r + \left[(1-M_0^2)\left(-\frac{\partial \Phi_r}{\partial x}\frac{\partial p_r}{\partial x}\right) - \frac{\partial \Phi_r}{\partial y}\frac{\partial p_r}{\partial y} \right. \\ & \left. + k_0^2\Phi_r p_r + 2M_0k_0\frac{\partial \Phi_r}{\partial x}p_i\right] d\Omega \\ & + \int_{\Gamma_N} (1-M_0^2)k'\Phi_r p_i d\Gamma \\ & + \int_{\Gamma_1\cup\Gamma_2} \frac{\chi Z_0}{|Z|^2}\left[p_r\left(-Z_ik_0\Phi_r - 2Z_rM_0\frac{\partial \Phi_r}{\partial x}\right) - \frac{Z_iM_0^2}{k_0}\frac{\partial p_r}{\partial x}\frac{\partial \Phi_r}{\partial x}\right. \\ & \left. + p_i\left(k_0Z_r\Phi_r - 2Z_iM_0\frac{\partial \Phi_r}{\partial x}\right) + \frac{Z_rM_0^2}{k_0}\frac{\partial p_i}{\partial x}\frac{\partial \Phi_r}{\partial x}\right] d\Gamma \end{aligned}$$

We integrate by parts :

$$\begin{aligned} \forall \Phi_r, \left\langle \frac{\partial \mathcal{L}}{\partial w_r}(\chi, w_r, w_i, p_r, p_i) \middle| \Phi_r \right\rangle = & \int_\Omega 2w_r\Phi_r + \left[(1-M_0^2)\left(\Phi_r\frac{\partial^2 p_r}{\partial x^2}\right) + \Phi_r\frac{\partial^2 p_r}{\partial y^2} + k_0^2\Phi_r p_r - 2M_0k_0\Phi_r\frac{\partial p_i}{\partial x}\right] d\Omega \\ & - \int_{\partial\Omega} (1-M_0^2)\frac{\partial p_r}{\partial x}\Phi_r n_x + \frac{\partial p_r}{\partial y}\Phi_r n_y - 2M_0k_0\Phi_r p_i n_x d\Gamma \\ & + \int_{\Gamma_N} (1-M_0^2)k'\Phi_r p_i d\Gamma \\ & + \int_{\Gamma_1\cup\Gamma_2} \frac{\chi Z_0}{|Z|^2}\left[p_r\left(-Z_ik_0\Phi_r - 2Z_rM_0\frac{\partial \Phi_r}{\partial x}\right) - \frac{Z_iM_0^2}{k_0}\frac{\partial p_r}{\partial x}\frac{\partial \Phi_r}{\partial x}\right. \\ & \left. + p_i\left(k_0Z_r\Phi_r - 2Z_iM_0\frac{\partial \Phi_r}{\partial x}\right) + \frac{Z_rM_0^2}{k_0}\frac{\partial p_i}{\partial x}\frac{\partial \Phi_r}{\partial x}\right] d\Gamma \end{aligned}$$

$$\begin{aligned} \forall \Phi_r, \left\langle \frac{\partial \mathcal{L}}{\partial w_r}(\chi, w_r, w_i, p_r, p_i) \middle| \Phi_r \right\rangle = & \int_\Omega \Phi_r\left[2w_r + (1-M_0^2)\frac{\partial^2 p_r}{\partial x^2} + \frac{\partial^2 p_r}{\partial y^2} + k_0^2 p_r - 2M_0k_0\frac{\partial p_i}{\partial x}\right] d\Omega \\ & - \int_{\partial\Omega} \Phi_r\left((1-M_0^2)\frac{\partial p_r}{\partial x}n_x + \frac{\partial p_r}{\partial y}n_y - 2M_0k_0p_i n_x\right) d\Gamma \\ & + \int_{\Gamma_N} (1-M_0^2)k'\Phi_r p_i d\Gamma \\ & + \int_{\Gamma_1\cup\Gamma_2} \frac{\chi Z_0}{|Z|^2}\left[p_r\left(-Z_ik_0\Phi_r - 2Z_rM_0\frac{\partial \Phi_r}{\partial x}\right) - \frac{Z_iM_0^2}{k_0}\frac{\partial p_r}{\partial x}\frac{\partial \Phi_r}{\partial x}\right. \\ & \left. + p_i\left(k_0Z_r\Phi_r - 2Z_iM_0\frac{\partial \Phi_r}{\partial x}\right) + \frac{Z_rM_0^2}{k_0}\frac{\partial p_i}{\partial x}\frac{\partial \Phi_r}{\partial x}\right] d\Gamma \end{aligned}$$

By setting to zero the preceding expression $\forall \Phi_r \in H_0^1(\Omega)$, by density of $H_0^1(\Omega)$ in $L^2(\Omega)$ (in practice we can also concluse by taking, $\Phi_r \in C_c^\infty(\Omega)$ ),we obtain that

$$(1-M_0^2)\frac{\partial^2 p_r}{\partial x^2} + \frac{\partial^2 p_r}{\partial y^2} + k_0^2 p_r - 2M_0k_0\frac{\partial p_i}{\partial x} = -2w_r$$

What is left is thus

$$0 = \int_{\Gamma_1 \cup \Gamma_2} -\frac{\partial p_r}{\partial n}\Phi_r + \frac{\chi Z_0}{|Z|^2}\Bigg[p_r\left(-Z_i k_0 \Phi_r - 2Z_r M_0 \frac{\partial \Phi_r}{\partial x}\right) - \frac{Z_i M_0^2}{k_0}\frac{\partial p_r}{\partial x}\frac{\partial \Phi_r}{\partial x} + p_i\left(k_0 Z_r \Phi_r - 2Z_i M_0 \frac{\partial \Phi_r}{\partial x}\right) + \frac{Z_r M_0^2}{k_0}\frac{\partial p_i}{\partial x}\frac{\partial \Phi_r}{\partial x}\Bigg] d\Gamma + \int_{\Gamma_N}\left[(1-M_0^2)\left(k' p_i - \frac{\partial p_r}{\partial x}\right) + 2M_0 k_0 p_i\right]\Phi_r d\Gamma$$

But there is an issue with the terms in $\partial_x \Phi_r$. We will try to put the derivatives on $p$ rather than on $\Phi_r$. To do so, we do to the first derivatives what we did to the second derivatives :

$$\forall (i,j) \in \{r,i\}^2 \quad \left\langle \frac{\partial \Phi_i}{\partial x} \middle| p_j \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} = -\left\langle \Phi_i \middle| \frac{\partial p_j}{\partial x} \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)}$$

$$\forall (i,j) \in \{r,i\}^2 \quad \left\langle \frac{\partial \Phi_i}{\partial x} \middle| \frac{\partial p_j}{\partial x} \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)} = -\left\langle \Phi_i \middle| \frac{\partial^2 p_j}{\partial x^2} \right\rangle_{L^2(\Gamma_1 \cup \Gamma_2)}$$

Hence

$$\int_{\Gamma_1 \cup \Gamma_2} -\frac{\partial p_r}{\partial n}\Phi_r + \frac{\chi Z_0}{|Z|^2}\Bigg[p_r\left(-Z_i k_0 \Phi_r - 2Z_r M_0 \frac{\partial \Phi_r}{\partial x}\right) - \frac{Z_i M_0^2}{k_0}\frac{\partial p_r}{\partial x}\frac{\partial \Phi_r}{\partial x} + p_i\left(k_0 Z_r \Phi_r - 2Z_i M_0 \frac{\partial \Phi_r}{\partial x}\right) + \frac{Z_r M_0^2}{k_0}\frac{\partial p_i}{\partial x}\frac{\partial \Phi_r}{\partial x}\Bigg] d\Gamma$$
$$= \int_{\Gamma_1 \cup \Gamma_2} \Phi_r \left[-\frac{\partial p_r}{\partial n} + \frac{\chi Z_0}{|Z|^2}\left(-Z_i k_0 p_r + 2Z_r M_0 \frac{\partial p_r}{\partial x} + \frac{Z_i M_0^2}{k_0}\frac{\partial^2 p_r}{\partial x^2} + k_0 Z_r p_i + 2Z_i M_0 \frac{\partial p_i}{\partial x} - \frac{Z_r M_0^2}{k_0}\frac{\partial^2 p_i}{\partial x^2}\right)\right]$$

We can now separate, by density of $B(\partial\Omega)$ in $L^2(\partial\Omega)$, separate $\Gamma_D$, $\Gamma_N$, and $\Gamma_1 \cup \Gamma_2$
We extract the boundary condition on $\Gamma_N$:

$$(1-M_0^2)\left(k' p_i - \frac{\partial p_r}{\partial x}\right) + 2M_0 k_0 p_i = 0$$

For $\Gamma_D$ :

$$p_r = 0$$

Finally, for $\Gamma_1 \cup \Gamma_2$ :

$$\frac{\partial p_r}{\partial n} - \frac{\chi Z_0}{|Z|^2}\left(-Z_i k_0 p_r + 2Z_r M_0 \frac{\partial p_r}{\partial x} + \frac{Z_i M_0^2}{k_0}\frac{\partial^2 p_r}{\partial x^2} + k_0 Z_r p_i + 2Z_i M_0 \frac{\partial p_i}{\partial x} - \frac{Z_r M_0^2}{k_0}\frac{\partial^2 p_i}{\partial x^2}\right) = 0$$

In practice we take $u_r$ and $u_i$ the real and imaginary part of the solution of the original variational formulation of the physical problem (in place of $w_r$ and $w_i$). This gives us :

$$\mathcal{L}(\chi, u_r, u_i, p_r, p_i) = \int_\Omega |u|^2$$

The adjoint problem for $p_r$ is

$$\begin{cases} (1-M_0^2)\dfrac{\partial^2 p_r}{\partial x^2} + \dfrac{\partial^2 p_r}{\partial y^2} + k_0^2 p_r - 2M_0 k_0 \dfrac{\partial p_i}{\partial x} = -2u_r & \Omega \\ p_r = 0 & \Gamma_D \\ (1-M_0^2)\left(k' p_i - \dfrac{\partial p_r}{\partial n}\right) + 2M_0 k_0 p_i = 0 & \Gamma_N \\ \dfrac{\partial p_r}{\partial n} - \dfrac{\chi Z_0}{|Z|^2}\left(-Z_i k_0 p_r + 2Z_r M_0 \dfrac{\partial p_r}{\partial x} + \dfrac{Z_i M_0^2}{k_0}\dfrac{\partial^2 p_r}{\partial x^2} + k_0 Z_r p_i + 2Z_i M_0 \dfrac{\partial p_i}{\partial x} - \dfrac{Z_r M_0^2}{k_0}\dfrac{\partial^2 p_i}{\partial x^2}\right) = 0 & \Gamma_1 \cup \Gamma_2 \end{cases}$$

For the second equation :

$$\forall\Phi_i,\ \left\langle \frac{\partial\mathcal{L}}{\partial w_i}(\chi,w_r,w_i,p_r,p_i)\middle|\ \Phi_i\right\rangle = \int_\Omega 2w_i\Phi_i + \left[(1-M_0^2)\left(\frac{\partial\Phi_i}{\partial x}\frac{\partial p_i}{\partial x}\right)+\frac{\partial\Phi_i}{\partial y}\frac{\partial p_i}{\partial y} - k_0^2\Phi_i p_i + 2M_0k_0\frac{\partial\Phi_i}{\partial x}p_r\right]d\Omega$$
$$+\int_{\Gamma_N}(1-M_0^2)k'\Phi_i p_r d\Gamma$$
$$+\int_{\Gamma_1\cup\Gamma_2}\frac{\chi Z_0}{|Z|^2}\left[-p_r\left(-Z_rk_0\Phi_i+2Z_iM_0\frac{\partial\Phi_i}{\partial x}\right)+\frac{Z_rM_0^2}{k_0}\frac{\partial\Phi_i}{\partial x}\frac{\partial p_r}{\partial x}\right.$$
$$\left.+p_i\left(k_0Z_i\Phi_i+2Z_rM_0\frac{\partial\Phi_i}{\partial x}\right)+\frac{Z_iM_0^2}{k_0}\frac{\partial p_i}{\partial x}\frac{\partial\Phi_i}{\partial x}\right]$$

We integrate by parts :

$$\forall\Phi_i,\ \left\langle \frac{\partial\mathcal{L}}{\partial w_i}(\chi,w_r,w_i,p_r,p_i)\middle|\ \Phi_i\right\rangle = \int_\Omega 2w_i\Phi_i + \left[-(1-M_0^2)\Phi_i\frac{\partial^2 p_i}{\partial x^2}-\Phi_i\frac{\partial^2 p_i}{\partial y^2} - k_0^2\Phi_i p_i - 2M_0k_0\frac{\partial p_r}{\partial x}\Phi_i\right]d\Omega$$
$$+\int_{\partial\Omega}(1-M_0^2)\frac{\partial p_i}{\partial x}n_x\Phi_i+\frac{\partial p_i}{\partial y}\Phi_i n_y + 2M_0k_0p_r\Phi_i n_x d\Gamma$$
$$+\int_{\Gamma_N}(1-M_0^2)k'\Phi_i p_r d\Gamma$$
$$+\int_{\Gamma_1\cup\Gamma_2}\frac{\chi Z_0}{|Z|^2}\left[-p_r\left(-Z_rk_0\Phi_i+2Z_iM_0\frac{\partial\Phi_i}{\partial x}\right)+\frac{Z_rM_0^2}{k_0}\frac{\partial\Phi_i}{\partial x}\frac{\partial p_r}{\partial x}\right.$$
$$\left.+p_i\left(k_0Z_i\Phi_i+2Z_rM_0\frac{\partial\Phi_i}{\partial x}\right)+\frac{Z_iM_0^2}{k_0}\frac{\partial p_i}{\partial x}\frac{\partial\Phi_i}{\partial x}\right]$$

Similarly, in the domain $\Omega$

$$(1-M_0)^2\frac{\partial^2 p_i}{\partial x^2}+\frac{\partial^2 p_i}{\partial y^2}+k_0^2p_i+2M_0k_0\frac{\partial p_r}{\partial x}=2w_i$$

We still have on $\Gamma_D$ : $p_i=0$
On the other boundaries, we have

$$0=\int_{\Gamma_N}\left[(1-M_0^2)\left(k'p_r+\frac{\partial p_i}{\partial x}\right)+2M_0k_0p_r\right]\Phi_i d\Gamma$$
$$+\int_{\Gamma_1\cup\Gamma_2}\Phi_i\left(\frac{\partial p_i}{\partial n}+\frac{\chi Z_0}{|Z|^2}\left[+Z_rk_0p_r+2Z_iM_0\frac{\partial p_r}{\partial x}-\frac{Z_rM_0^2}{k_0}\frac{\partial^2 p_r}{\partial x^2}+k_0Z_ip_i-2Z_rM_0\frac{\partial p_i}{\partial x}-\frac{Z_iM_0^2}{k_0}\frac{\partial^2 p_i}{\partial x^2}\right]\right)d\Gamma$$

Hence the two corresponding boundary conditions on $\Gamma_1\cup\Gamma_2$ and $\Gamma_N$. The imaginary part of the adjoint problem $p_i$ verifies

$$\begin{cases}
(1-M_0)^2\dfrac{\partial^2 p_i}{\partial x^2}+\dfrac{\partial^2 p_i}{\partial y^2}+k_0^2p_i+2M_0k_0\dfrac{\partial p_r}{\partial x}=2u_i & \Omega\\
p_i=0 & \Gamma_D\\
(1-M_0^2)\left(k'p_r+\dfrac{\partial p_i}{\partial n}\right)+2M_0k_0p_r=0 & \Gamma_N\\
\dfrac{\partial p_i}{\partial n}+\dfrac{\chi Z_0}{|Z|^2}\left[+Z_rk_0p_r+2Z_iM_0\dfrac{\partial p_r}{\partial x}-\dfrac{Z_rM_0^2}{k_0}\dfrac{\partial^2 p_r}{\partial x^2}+k_0Z_ip_i-2Z_rM_0\dfrac{\partial p_i}{\partial x}-\dfrac{Z_iM_0^2}{k_0}\dfrac{\partial^2 p_i}{\partial x^2}\right]=0 & \Gamma_1\cup\Gamma_2
\end{cases}$$

We resume our equations : For the real part

$$\begin{cases}
(1-M_0^2)\dfrac{\partial^2 p_r}{\partial x^2} + \dfrac{\partial^2 p_r}{\partial y^2} + k_0^2 p_r - 2M_0 k_0 \dfrac{\partial p_i}{\partial x} = -2u_r & \Omega \\
p_r = 0 & \Gamma_D \\
(1-M_0^2)\left(k' p_i - \dfrac{\partial p_r}{\partial n}\right) + 2M_0 k_0 p_i = 0 & \Gamma_N \\
\dfrac{\partial p_r}{\partial n} - \dfrac{\chi Z_0}{|Z|^2}\left(-Z_i k_0 p_r + 2Z_r M_0 \dfrac{\partial p_r}{\partial x} + \dfrac{Z_i M_0^2}{k_0}\dfrac{\partial^2 p_r}{\partial x^2} + k_0 Z_r p_i + 2 Z_i M_0 \dfrac{\partial p_i}{\partial x} - \dfrac{Z_r M_0^2}{k_0}\dfrac{\partial^2 p_i}{\partial x^2}\right) = 0 & \Gamma_1 \cup \Gamma_2
\end{cases}$$

For the imaginary part :

$$\begin{cases}
(1-M_0)^2\dfrac{\partial^2 p_i}{\partial x^2} + \dfrac{\partial^2 p_i}{\partial y^2} + k_0^2 p_i + 2M_0 k_0 \dfrac{\partial p_r}{\partial x} = 2u_i & \Omega \\
p_i = 0 & \Gamma_D \\
(1-M_0^2)\left(k' p_r + \dfrac{\partial p_i}{\partial n}\right) + 2M_0 k_0 p_r = 0 & \Gamma_N \\
\dfrac{\partial p_i}{\partial n} + \dfrac{\chi Z_0}{|Z|^2}\left[+Z_r k_0 p_r + 2Z_i M_0 \dfrac{\partial p_r}{\partial x} - \dfrac{Z_r M_0^2}{k_0}\dfrac{\partial^2 p_r}{\partial x^2} + k_0 Z_i p_i - 2 Z_r M_0 \dfrac{\partial p_i}{\partial x} - \dfrac{Z_i M_0^2}{k_0}\dfrac{\partial^2 p_i}{\partial x^2}\right] = 0 & \Gamma_1 \cup \Gamma_2
\end{cases}$$

We deduce the complex form of the adjoint problem :

$$\begin{cases}
(1-M_0^2)\dfrac{\partial^2 p}{\partial x^2} + \dfrac{\partial^2 p}{\partial y^2} + k_0^2 p + 2M_0 k_0 i \dfrac{\partial p}{\partial x} = -2\bar{u} & \Omega \\
p = 0 & \Gamma_D \\
(1-M_0^2)\left(\dfrac{\partial p}{\partial n} + ik' p\right) + 2M_0 k_0 i p = 0 & \Gamma_N \\
\dfrac{\partial p}{\partial n} + \dfrac{\chi Z_0 \overline{Z}}{|Z|^2}\left(ik_0 p - M_0 \dfrac{\partial p}{\partial x} - i\dfrac{M_0^2}{k_0}\dfrac{\partial^2 p}{\partial x^2}\right) = 0 & \Gamma_1 \cup \Gamma_2
\end{cases}$$

The boundary condition on $\Gamma_1 \cup \Gamma_2$ was verified with Mathematica (we searched for a boundary condition similar to what we had for $u$ to have the correct real part and verified the imaginary part with Mathematica)

Listing 3: Verification with Mathematica of the equation on $\Gamma_1 \cup \Gamma_2$ ($dxu* = \partial_x u_*, I = i; dnu* = \partial_n u_*, K = \frac{\chi Z_0 \overline{Z}}{|Z|^2}$)

```
G2LP = (dnpr + I*dnpi) + (K)*(Zr - I*Zi)*(I*ko*(pr + I*pi) - 2*M0*(dxpr + I*dxpi) - I* (M0^2/k0)*(
    dxxpr + I*dxxpi))
ComplexExpand[G2LP]
    dnpr - 2 dxpi K M0 Zi - (dxxpr K M0^2 Zi)/k0 + K ko pr Zi - 2 dxpr K M0 Zr + (dxxpi K M0^2 Zr)
        /k0 - K ko pi Zr
+ I (dnpi + 2 dxpr K M0 Zi - (dxxpi K M0^2 Zi)/k0 + K ko pi Zi - 2 dxpi K M0 Zr - (dxxpr K M0^2 Zr)
    /k0 + K ko pr Zr)
```

### 7.2.3 Energy derivative

We found the adjoint problem, so we can calculate the energy derivative. We have :

$$\langle J'(\chi) | \, \chi_0 \rangle = \left\langle \frac{\partial \mathcal{L}}{\partial \chi}(\chi, u_r, u_i, p_r, p_i) \middle| \, \chi_0 \right\rangle$$

We recall the expression of the variational formulation :

$$
\begin{aligned}
FV(\chi, u_r, u_i, \varphi_r, \varphi_i) = \int_\Omega \Bigg[ &(1 - M_0^2) \left( \frac{\partial u_i}{\partial x} \frac{\partial \varphi_i}{\partial x} - \frac{\partial u_r}{\partial x} \frac{\partial \varphi_r}{\partial x} \right) + \left( \frac{\partial u_i}{\partial y} \frac{\partial \varphi_i}{\partial y} - \frac{\partial u_r}{\partial y} \frac{\partial \varphi_r}{\partial y} \right) \\
&+ k_0^2 (u_r \varphi_r - u_i \varphi_i) + 2 M_0 k_0 \left( \frac{\partial u_i}{\partial x} \varphi_r + \frac{\partial u_r}{\partial x} \varphi_i \right) - f_r \varphi_r - f_i \varphi_i \Bigg] d\Omega \\
&+ \int_{\Gamma_N} (1 - M_0^2) \left[ (\gamma_r \varphi_r - \gamma_i \varphi_i) + k' (u_i \varphi_r + u_r \varphi_i) \right] d\Gamma \\
&+ \int_{\Gamma_1 \cup \Gamma_2} \Bigg[ \chi \left( \varphi_r \eta_r - \varphi_i \eta_i \right) \\
&+ \frac{\chi Z_0}{|Z|^2} \Bigg( -\varphi_r \left( Z_i k_0 u_r - Z_r k_0 u_i + 2 Z_i M_0 \frac{\partial u_i}{\partial x} + 2 Z_r M_0 \frac{\partial u_r}{\partial x} \right) \\
&+ \frac{\partial \varphi_r}{\partial x} \left( \frac{Z_r M_0^2}{k_0} \frac{\partial u_i}{\partial x} - \frac{Z_i M_0^2}{k_0} \frac{\partial u_r}{\partial x} \right) \\
&+ \varphi_i \left( + k_0 Z_i u_i + k_0 u_r Z_r + 2 Z_r M_0 \frac{\partial u_i}{\partial x} - 2 Z_i M_0 \frac{\partial u_r}{\partial x} \right) \\
&+ \frac{\partial \varphi_i}{\partial x} \left( \frac{Z_r M_0^2}{k_0} \frac{\partial u_r}{\partial x} + \frac{Z_i M_0^2}{k_0} \frac{\partial u_i}{\partial x} \right) \Bigg) \Bigg] d\Gamma
\end{aligned}
$$

We have

$$
\begin{aligned}
\langle J'(\chi) |\, \chi_0 \rangle = \int_{\Gamma_1 \cup \Gamma_2} \chi_0 \Bigg[ &\frac{Z_0}{|Z|^2} \Bigg( -p_r \Bigg( Z_i k_0 u_r - Z_r k_0 u_i + 2 Z_i M_0 \frac{\partial u_i}{\partial x} + 2 Z_r M_0 \frac{\partial u_r}{\partial x} \\
&+ \frac{Z_r M_0^2}{k_0} \frac{\partial^2 u_i}{\partial x^2} - \frac{Z_i M_0^2}{k_0} \frac{\partial^2 u_r}{\partial x^2} \Bigg) \\
&+ p_i \left( + k_0 Z_i u_i + k_0 u_r Z_r + 2 Z_r M_0 \frac{\partial u_i}{\partial x} - 2 Z_i M_0 \frac{\partial u_r}{\partial x} - \frac{Z_r M_0^2}{k_0} \frac{\partial^2 u_r}{\partial x^2} - \frac{Z_i M_0^2}{k_0} \frac{\partial^2 u_i}{\partial x^2} \right) \Bigg) \Bigg] \\
&+ \chi_0 (p_r \eta_r - p_i \eta_i) d\Gamma
\end{aligned}
$$

We have to calculate and use $\eta$ for the gradient descent algorithm. Fortunately, $\eta$ does not depend on $\chi$ so we only have to calculate $\eta$ once and keep it in memory. We only need to calculate $u$ and $p$ for each step.

Remark: there are certainly some simplifications to make to the expression of $\langle J'(\chi) |\, \chi_0 \rangle$, probably of the form Re(...) or similarly. We find

$$
\langle J'(\chi) |\, \chi_0 \rangle = \int_{\Gamma_1 \cup \Gamma_2} \chi_0 \left[ \mathrm{Re}(p\eta) + \frac{Z_0}{|Z|^2} \left( k_0 \, \mathrm{Im}(p \overline{Z} u) + 2 M_0 \, \mathrm{Re} \left( p \overline{Z} \frac{\partial u}{\partial x} \right) - \frac{M_0^2}{k_0} \, \mathrm{Im} \left( p \overline{Z} \frac{\partial^2 u}{\partial x^2} \right) \right) \right] d\Gamma
$$

This is a little more compact but not very simplified either.

# 8 Choice of the acoustic impedance for the liner

## 8.1 Models for liners

An acoustic liner is often represented as honeycombed structure with layers of of perforated sheets, this type of liner is passive (no circuit or human activation needed to absorb acoustical energy). On figure 9, (a) is a simple passive liner with honeycombed structure (SDOF : single degree of freedom) while (b) is more complex than (a) using a perforated sheet in the middle to enhance the absorption (DDOF : double degree of freedom). All passive liners use destructive interference of the sound wave in honeycombed cavities, therefore the size of the chambers is given for a specific frequency. It can be a problem to use this technology for high frequencies as they require a large depth of the chamber to be efficient. The resonant frequency of the chamber (the frequency when the absorption is maximized) is given by :

$$f = \frac{c}{2\pi}\sqrt{\frac{S}{V(l+\Delta l)}}$$

where $S$ is the orifice section, $l$ the orifice depth, $V$ is the cavity volume, $\Delta l$ the correction factor for compensating the vibrating mass of gases in the orifice and $c$ is the sound velocity (see [2]).

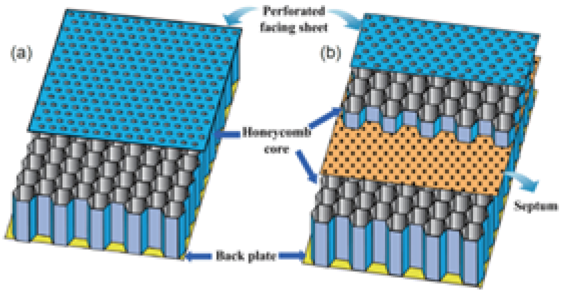


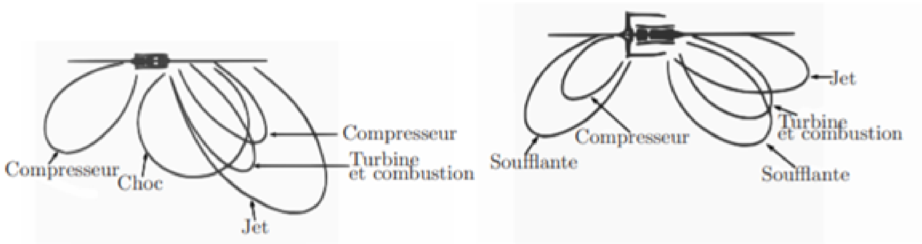

Figure 9: Two passive liners

Although conventional liners might be designed so as to target multiple tonal frequencies, their passive principle prevents the adaptation to varying engine speeds and therefore lowers their performance during flight, especially in the take-off and landing phases. Active acoustic liners on the other hand use adaptive technology to reduce noise on a broader spectrum. For example, bias acoustic liners consist of a skin made of loudspeakers and microphones that is used in [3]. The microphones detect the pressure variations while the loudspeakers generate sound waves specifically designed to interfere destructively with the incoming acoustic wave. Adaptive acoustic liners by tunable geometry dimension use piezoelectric composite diaphragm to change $S$ the orifice section on demand to match the cavity resonant frequency to the noise frequency (see the formula above).

The liners are defined mathematically by their acoustic impedance Z the ratio of the pressure variation on the liner and the liner normal velocity : $Z(\omega) = \frac{p'}{\mathbf{v}'\cdot\mathbf{n}}$. There exist several models for this physical quantity corresponding to the liner used. In [11] some impedance models are given for several liners both passive or active.

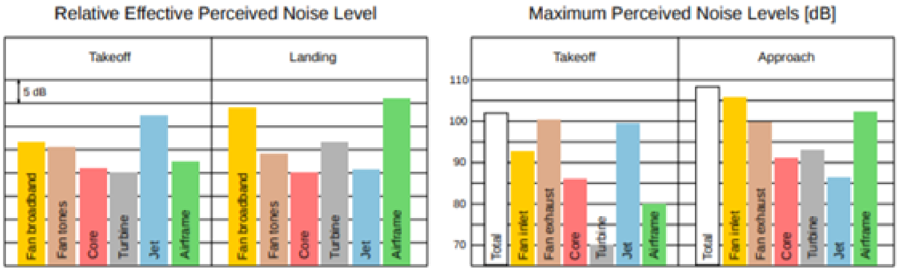
## 8.2 Noise in a turbofan

Many research papers were produced about the turbofan noise, from [7] 3 main sources of noise are identified : jet and combustion noise, rotating engine components noise and fan noise. The jet and combustion were the main sources of noise in turbo-reactors, however the turbofan uses cold jet on top of the combustion jet and with a dilution rate (ratio of cold over hot mass flow) around 10 to 15, dampening this noise. The rotating parts inside the engine make noise due to aerodynamic loads but those are also negligible if the speed is less than Mach 0.8. In the end, the fan noise is the main noise source for the aircraft at subsonic speeds. Their spatial distribution is given on figure 10. When we try to dampen acoustical waves we need to use specific liners that absorb the main frequencies. The noise spectrum is drawn in figure 12. For subsonic speeds, the main frequencies are the harmonics of the fan blades rotating frequency, as for supersonic speeds, the spectrum is much more complex (the noise from the rotating parts is no more negligible). These results are supported by another thesis [5].




(a) Moteur simple flux (b) Moteur double flux

Figure 10: Noise diagram (a) turbo-reactor engine (b) turbofan engine




(a) Data from German Aerospace Centre (DLR). (b) Data from NASA

Figure 11: Noise levels

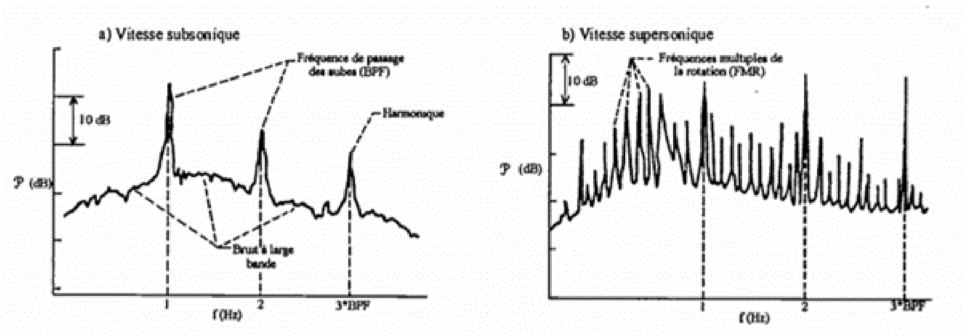



Figure 12: Noise spectrum (a) subsonic speed and (b) supersonic speed

## 8.3 Liner's optimal impedance

As shown beforehand, the choice of the liner, more precisely its acoustic impedance, is key to reduce noise, we need to target specific frequencies. Javier Rodriguez's thesis [9] is about optimizing the liner's impedance to reduce noise. The model is similar to ours : a 2 dimension rectangle, a uniform flow, two hard walls enclosing the domain ($\Gamma_1 \cup \Gamma 2$) ; the only difference is that the walls are completely covered by liner. After writing the equations, the paper considers the spatial mode of the acoustic wave to study the absorption : the harmonic (temporal) PDE is not directly solved but the temporal and spatial harmonic acoustic waves behavior are studied for a harmonic source on $\Gamma_D$.

After optimizing the impedance to reduce noise, it is found that for a reduced frequency $\omega = 2\pi f$ the optimum impedance is given by Tester's law (see Figure 11):

$$Z_T = (0.929 - 0.744i)\frac{\omega L_y \rho_0}{\pi}\frac{1}{(1+M)^2}$$

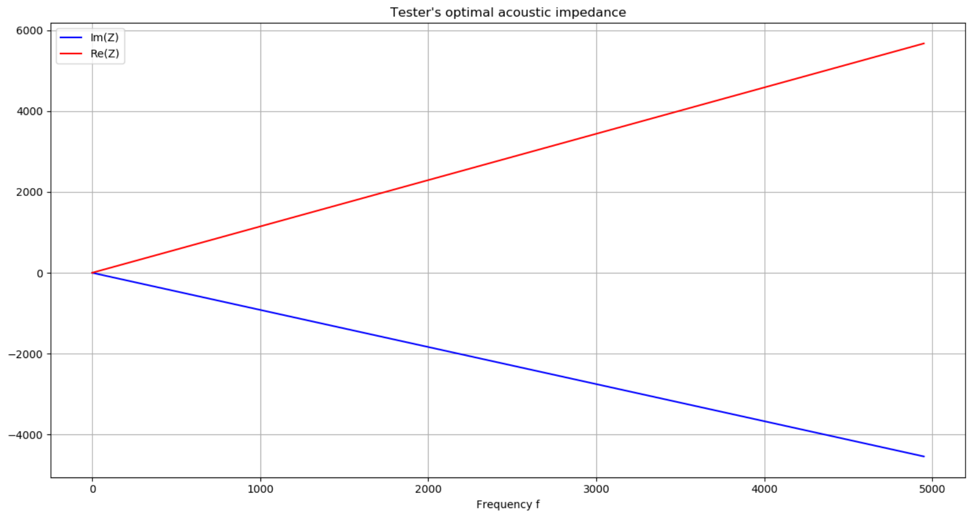



Figure 13: Tester's impedance as a function of the frequency f

As our model differs from this thesis, we will consider the result from Tester's original thesis [10], that also extends to circular duct (in which we will place the liner everywhere, at least in our testing).

$$Z_T = (0.88 - 0.38i)\frac{\omega L_y \rho_0}{\pi}\frac{1}{(1+M)^2}$$

## 8.4 Results

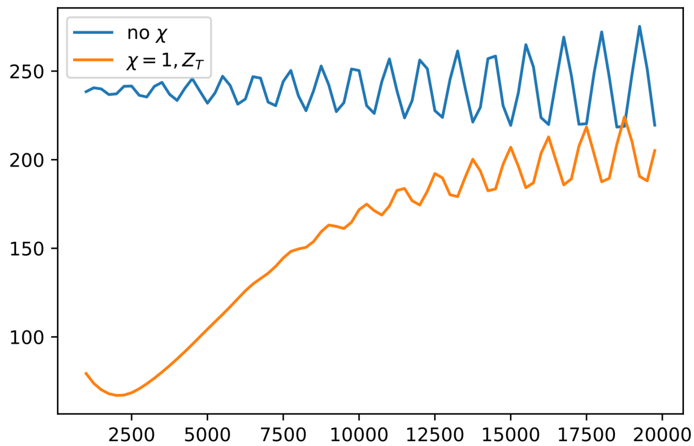



Figure 14: Comparison over a large band of frequency of the energy for a duct with no liner vs with $\chi = 1$ and $Z = Z_T$ for $f = 2kHz$

We see two things : with no acoustic liner, the response in frequency is somewhat flat, and the acoustic liner gives the system a huge reduction in noise, that is maximum for $f = 2kHz$ as expected. The reduction in noise decreases with larger frequencies, meaning

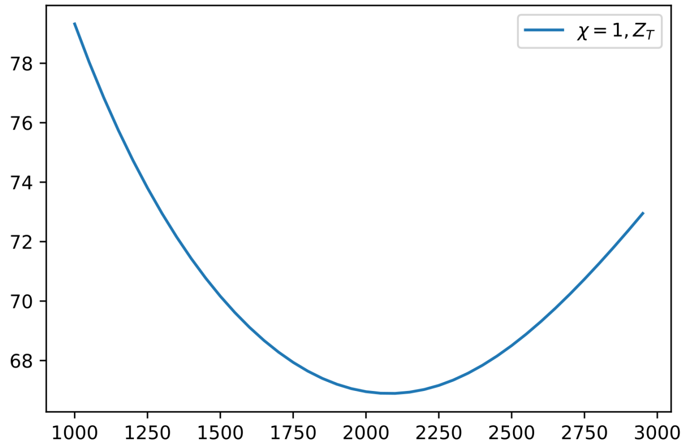



Figure 15: Energy for $\chi = 1$ and $Z = Z_T$ for $f = 2kHz$ around $2kHz$ to verify the minimum

we should target some frequencies. The absorption is efficient for a rather small range of frequencies: the noise is reduced by half for $f \in [1; 6]kHz$. It emphasizes the need of technical specifications for the noise frequencies to be reduced, as the noise frequencies depends on the aircraft speed, it is important to specify during which flight phase we want to reduce noise the most. Once the flight phases are identified (aircraft speeds), we can then choose the optimal impedance for the liner and finally use our numerical tool to conclude on its optimal placement.

Also, we experimentally find that the impedance we first used in the genetic algorithm in section 7.1 is close to the impedance $Z_T$ for $f = 2kHz$, and that justify why we obtained such a similar form of frequency response.

We also test the genetic algorithm for $(OP_1)$ with $Z_t$. We used some conservative parameters in this optimisation ($size = 16, selection\ size = 8, generations = 60$) to quickly get a result. This means this is what we obtain with a stricter time and computing constraint that what was obtained before.

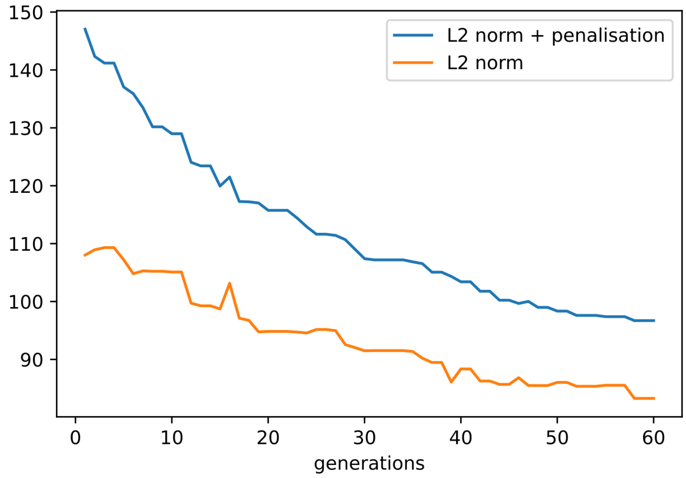



Figure 16: Energy over the generations with and without taking into account the penalization term

We see that the result is worse for $f = 2kHz$ (to be expected), that the overall frequency

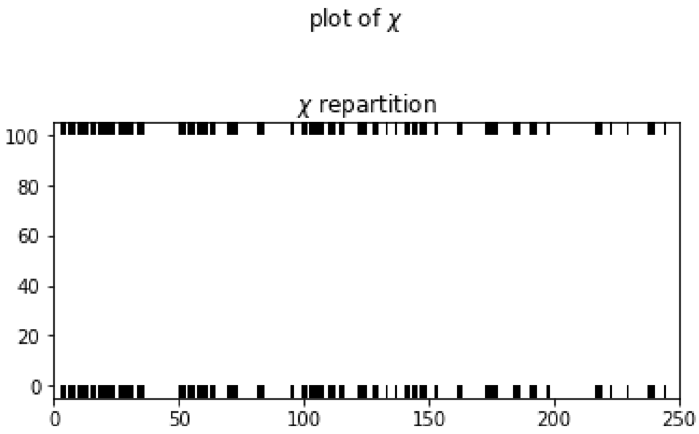



Figure 17: Optimised placement of $\chi$ for $Z_T$

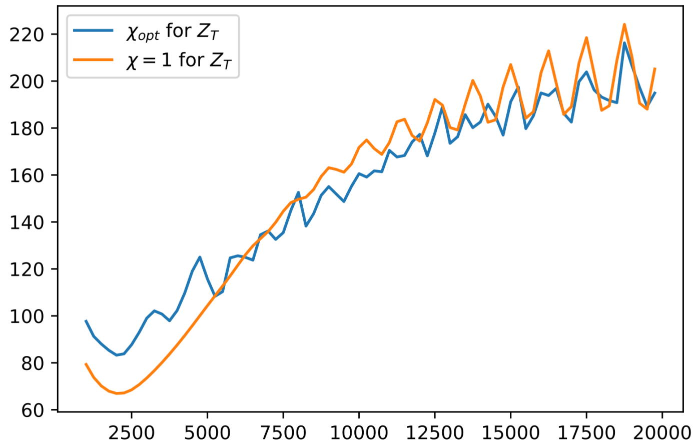



Figure 18: Frequency response of $\chi_{opt}$ compared to $\chi = 1$

response is for the most part similar, and worse for lower frequencies for $\chi_{opt}$

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