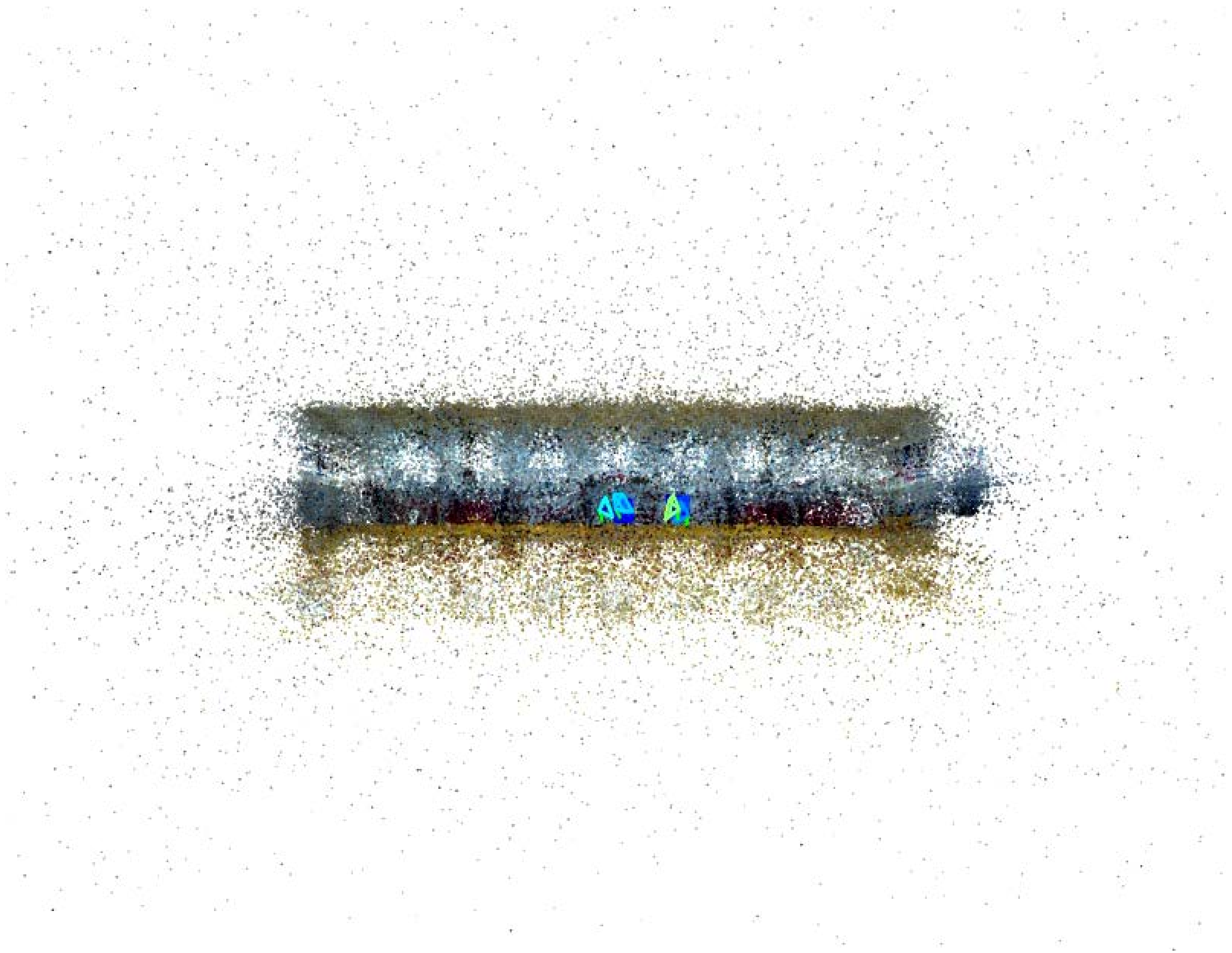
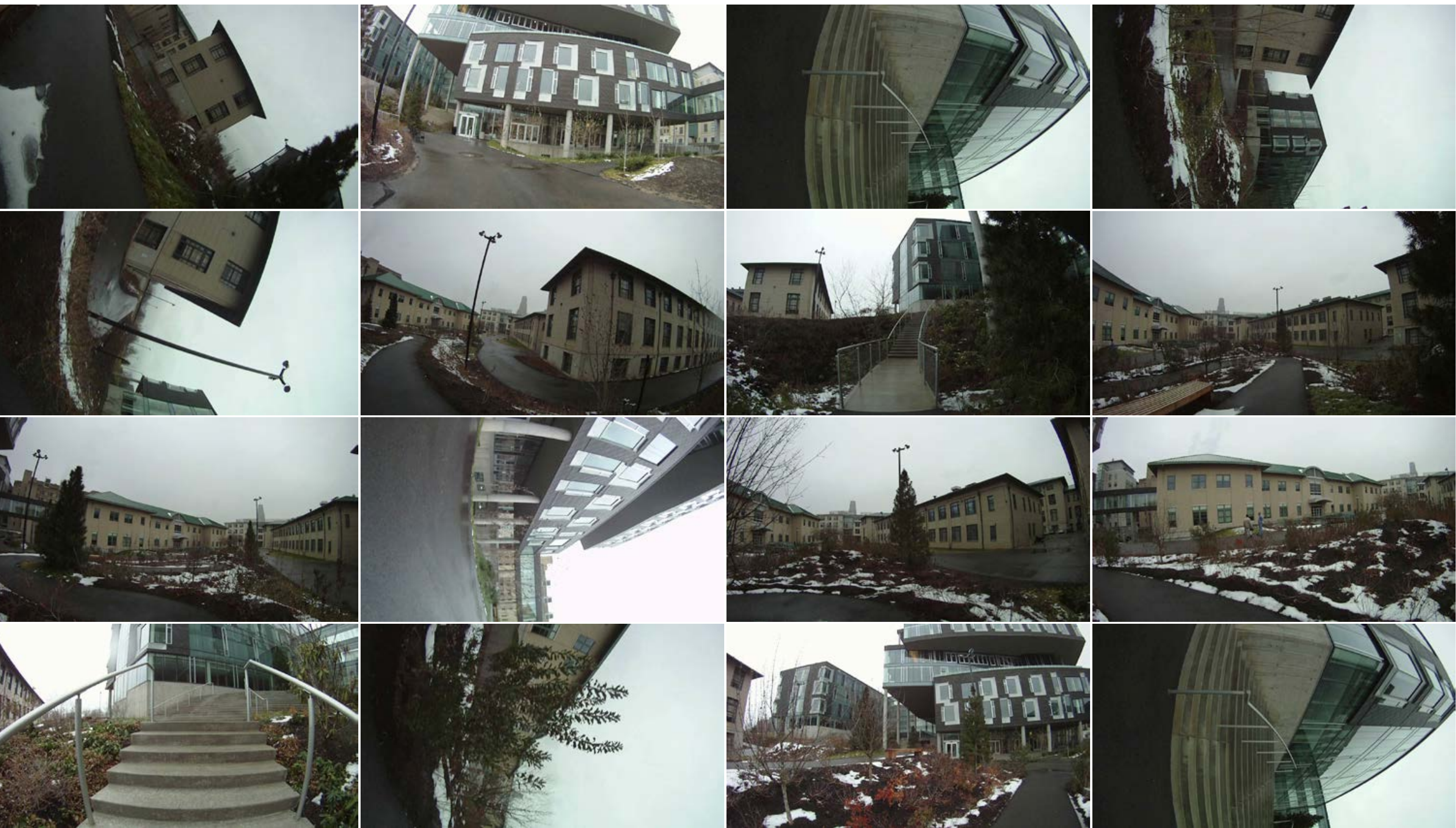
Structure from Motion Pipeline

Where am I (camera)?
Where are they (points)?

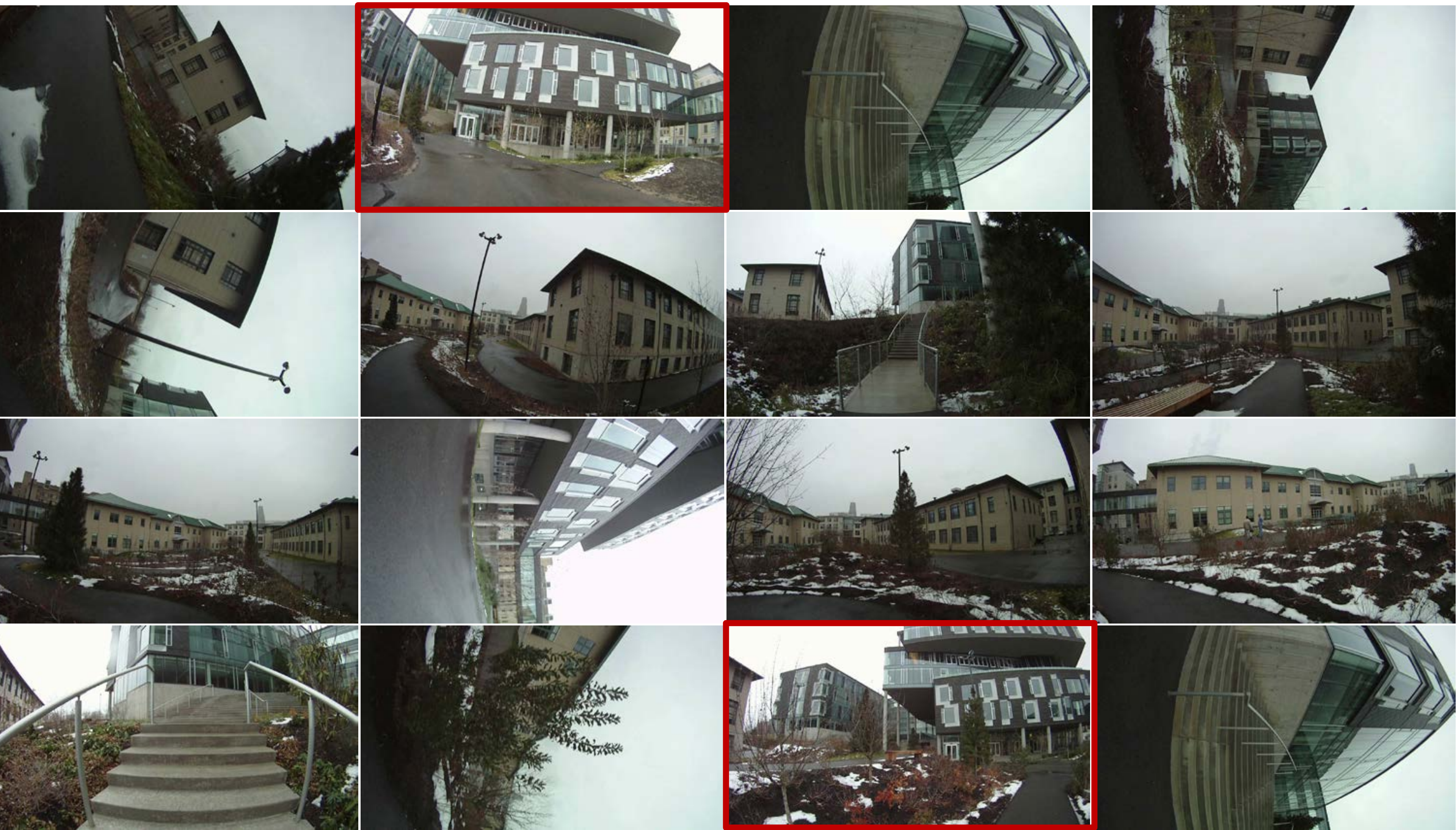

GO TARTANS



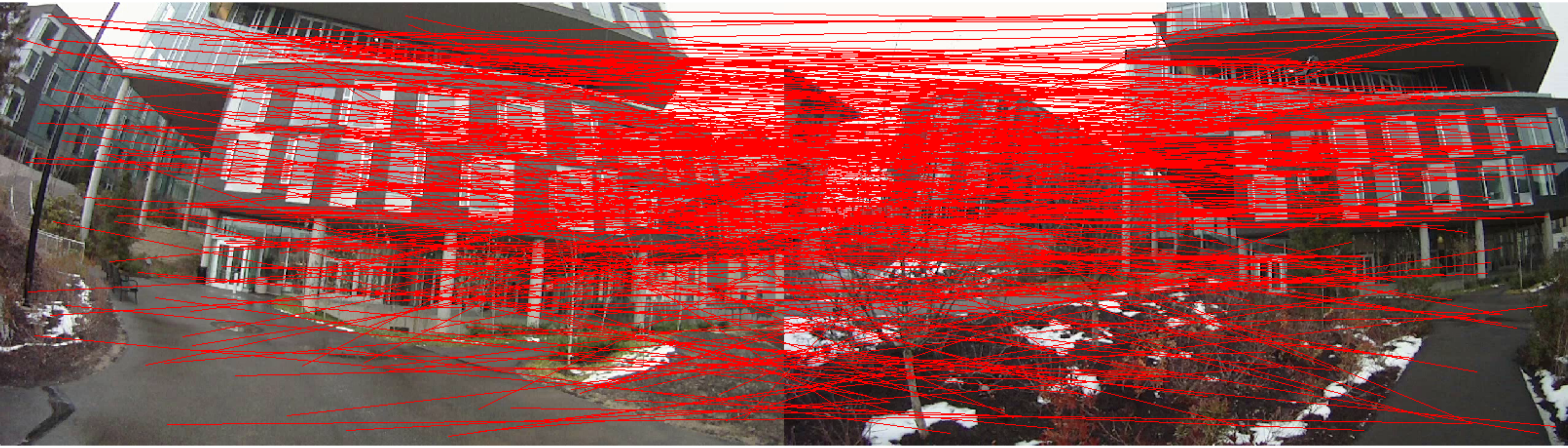
Input Images

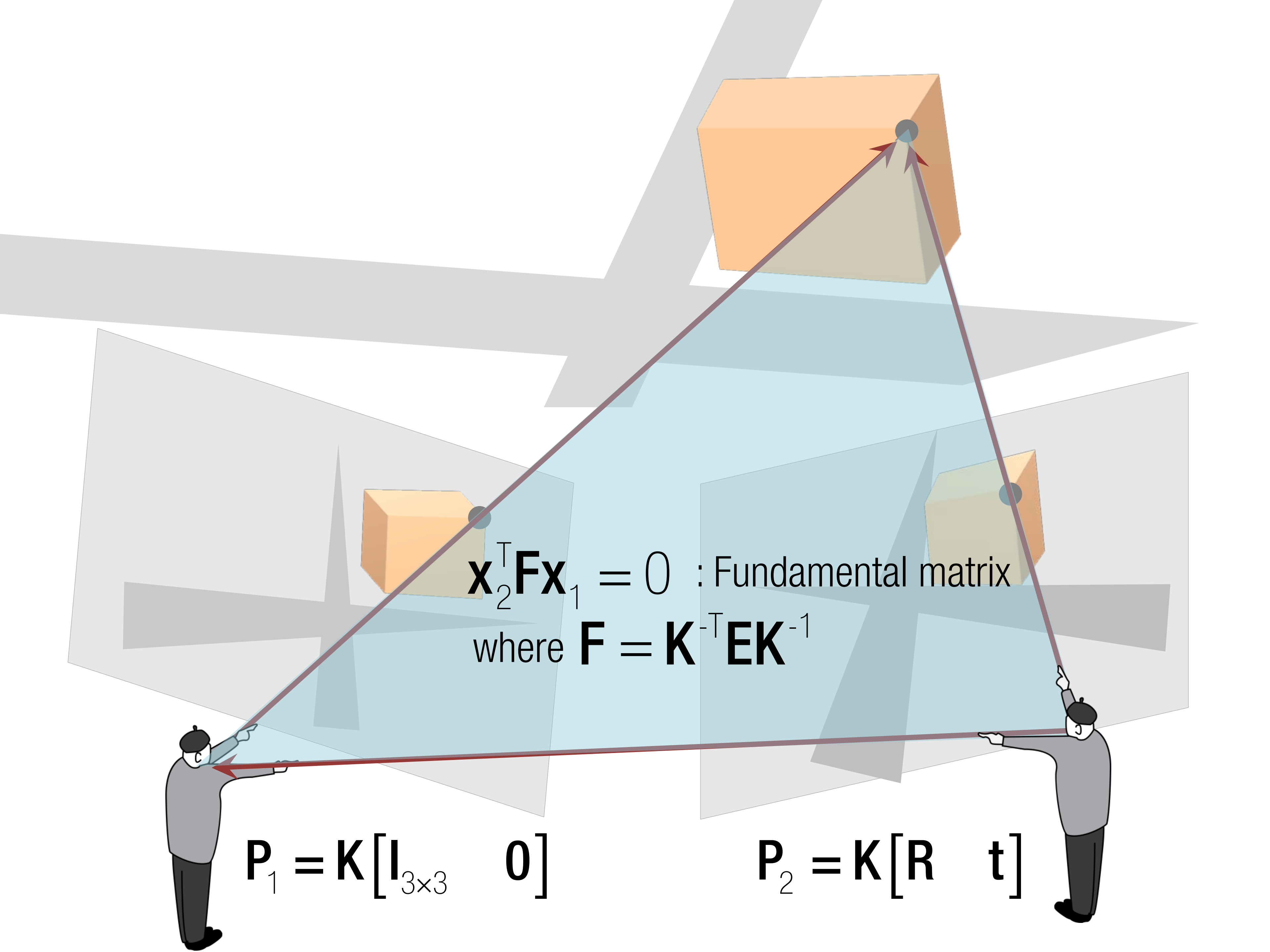


Initial Pair Images



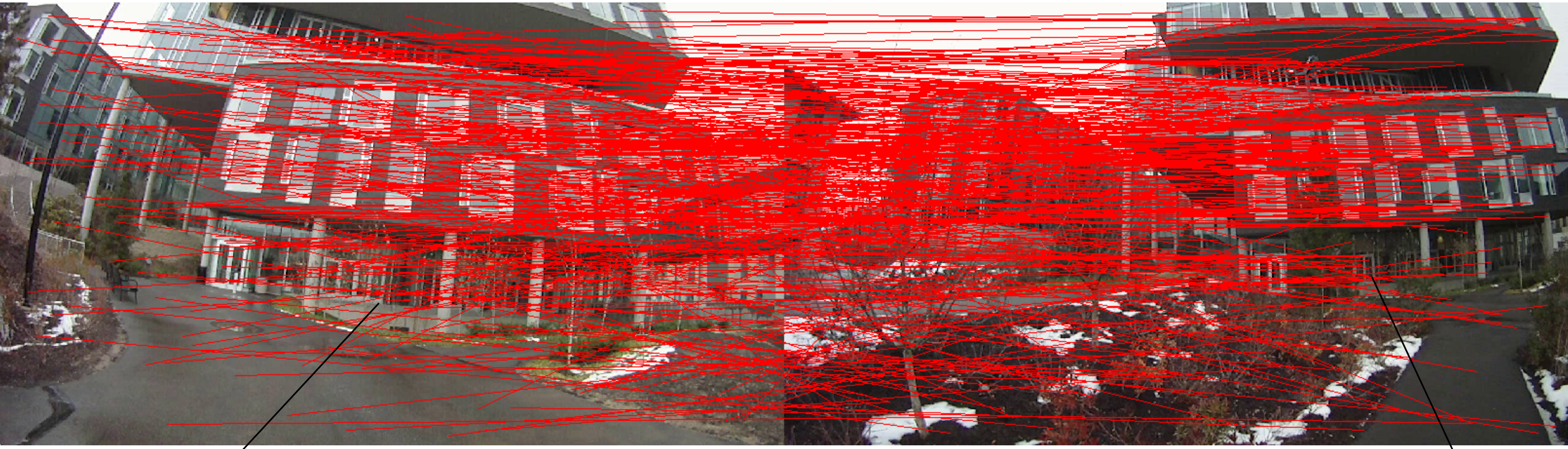
1. Pairwise Image Feature Matching





$\mathbf{x}_2^T \mathbf{F} \mathbf{x}_1 = 0$: Fundamental matrix
where $\mathbf{F} = \mathbf{K}^{-T} \mathbf{E} \mathbf{K}^{-1}$
$\mathbf{P}_1 = \mathbf{K}\begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{0} \end{bmatrix}$
$\mathbf{P}_2 = \mathbf{K}\begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix}$

1. Pairwise Image Feature Matching



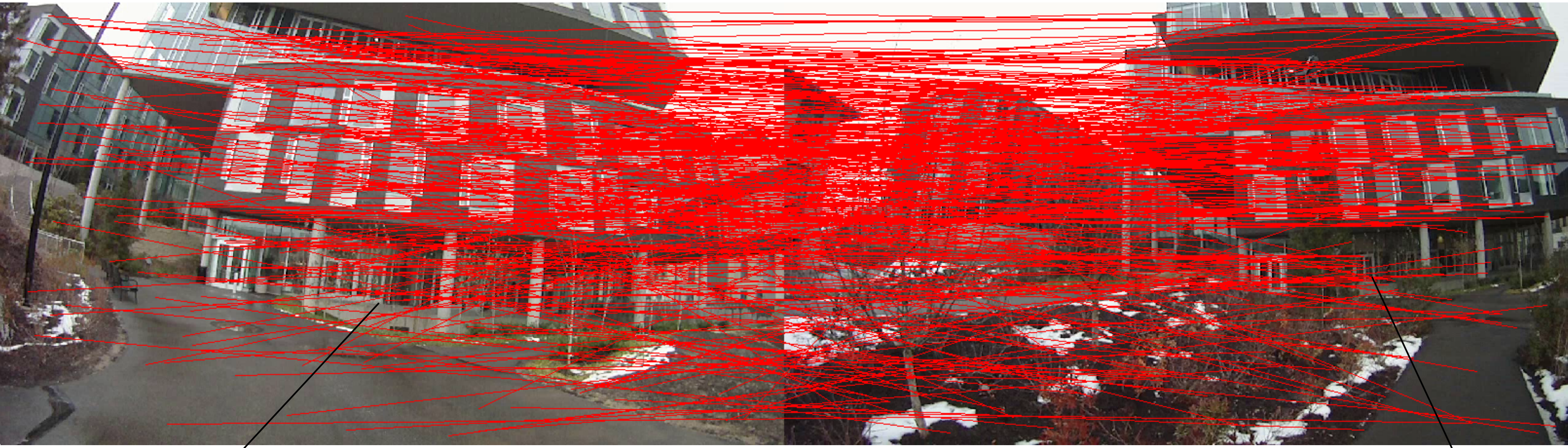
$\mathbf{x}_1$

$\mathbf{x}_2$

$$\mathbf{x}_2^{\mathsf{T}}\mathbf{F}\mathbf{x}_1 = 0$$

Epipolar constraint

1. Pairwise Image Feature Matching



$\mathbf{x}_1$

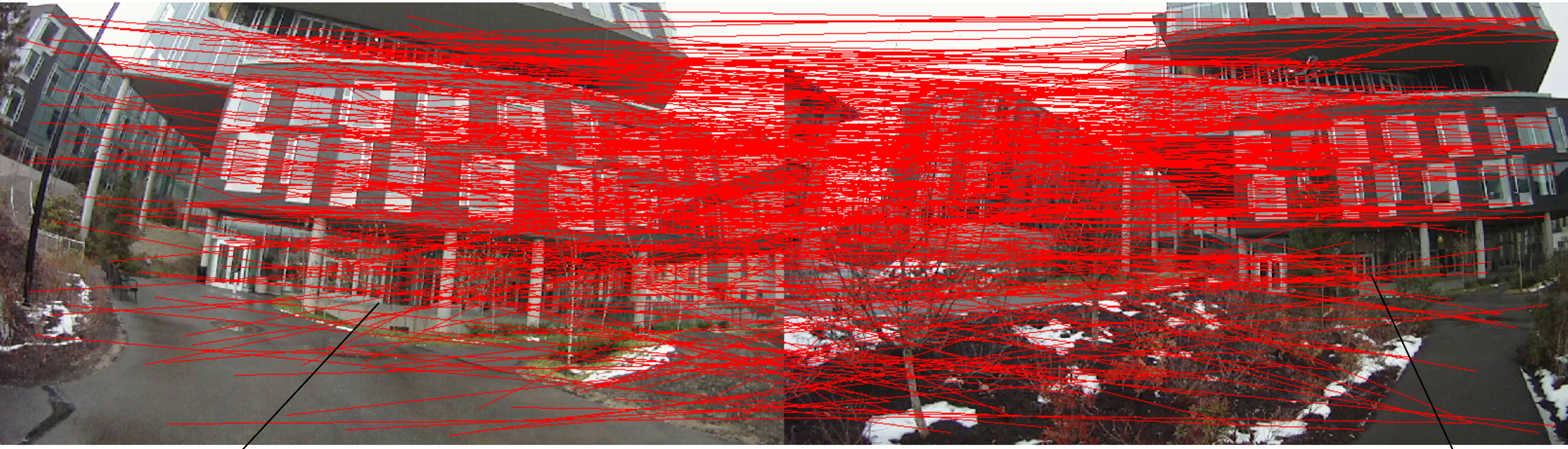
$\mathbf{x}_2$

$$\mathbf{x}_2^{\mathsf{T}}\mathbf{F}\mathbf{x}_1 = 0$$

Epipolar constraint

# of unknowns: 8
# of required equations: 8

1. Pairwise Image Feature Matching



$\mathbf{x}_1$

$\mathbf{x}_2$

$$\mathbf{x}_2^{\mathsf{T}} \mathbf{F} \mathbf{x}_1 = 0$$

Epipolar constraint

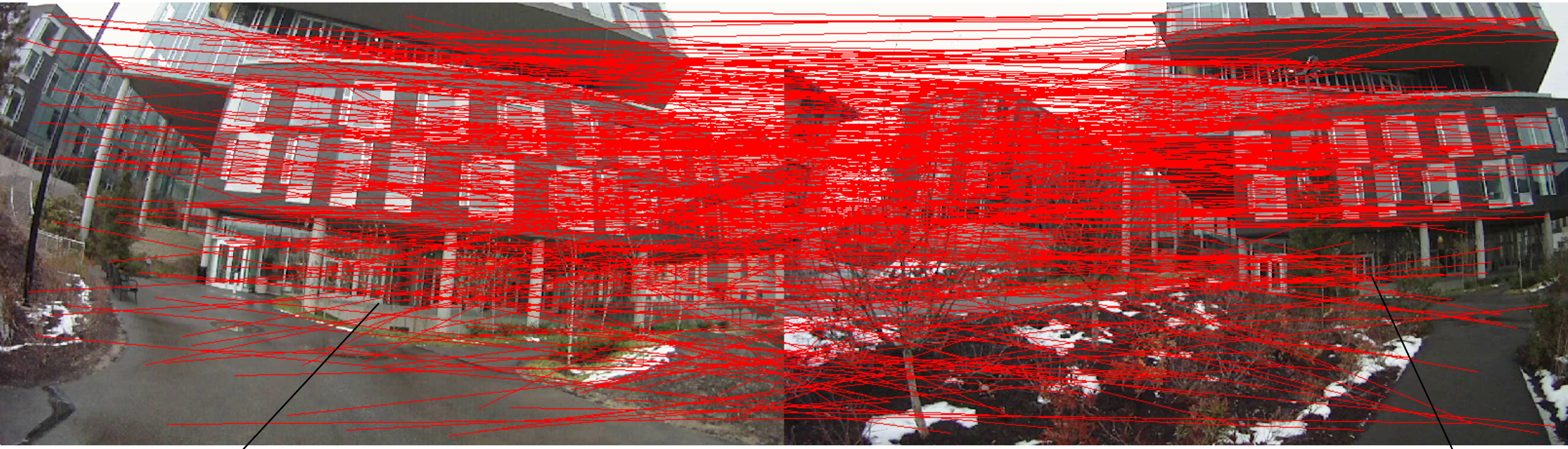
# of unknowns: 8
# of required equations: 8

$$\mathbf{x}_{2,1}^{\mathsf{T}} \mathbf{F} \mathbf{x}_{1,1} = 0$$

$$\vdots$$

$$\mathbf{x}_{2,8}^{\mathsf{T}} \mathbf{F} \mathbf{x}_{1,8} = 0$$

1. Pairwise Image Feature Matching



$\mathbf{x}_1$ $\mathbf{x}_2$

$$\mathbf{x}_2^{\mathsf{T}}\mathbf{F}\mathbf{x}_1 = 0$$

Epipolar constraint

$$\underbrace{\begin{bmatrix} u_1^1u_1^2 & u_1^1v_1^2 & u_1^1 & v_1^1u_1^2 & v_1^1v_1^2 & v_1^1 & u_1^2 & v_1^2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_8^1u_8^2 & u_8^1v_8^2 & u_8^1 & v_8^1u_8^2 & v_8^1v_8^2 & v_8^1 & u_8^2 & v_8^2 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ \vdots \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix}}_{\mathbf{x}} = \mathbf{0}$$

8 Point Algorithm

- Construct 8x9 matrix **A**.
- Solving linear homogeneous equations via SVD:

  $\mathbf{x} = \mathbf{V}_{:,8}$ where $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^{\mathrm{T}}$

  $\mathbf{F} = \mathrm{reshape}(\mathbf{x}, 3, 3)$: constructing matrix from vector.
- Applying rank constraint, i.e., $\mathrm{rank}(\mathbf{F}) = 2$.

$\mathbf{F}_{\mathrm{rank2}} = \mathbf{U}\tilde{\mathbf{D}}\mathbf{V}^{\mathrm{T}}$ where $\tilde{\mathbf{D}}$ $:\mathbf{D}$ with the last element zero.

SVD cleanup

Epipolar Geometry

$\mathbf{x}_1$

$\mathbf{x}_2$

$$\mathbf{x}_2^\top \mathbf{F} \mathbf{x}_1 = 0$$

2. Outlier Rejection via RANSAC

$\mathbf{x}_1$

$\mathbf{x}_2$

$$\mathbf{x}_2^{\top}\mathbf{F}\mathbf{x}_1 = 0$$

Random sampling

Model building

Thresholding

Inlier counting

2. Outlier Rejection via RANSAC

$\mathbf{x}_1$

$\mathbf{x}_2$

$$\mathbf{x}_2^{\mathsf{T}}\mathbf{F}\mathbf{x}_1 = 0$$

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2. Outlier Rejection via RANSAC

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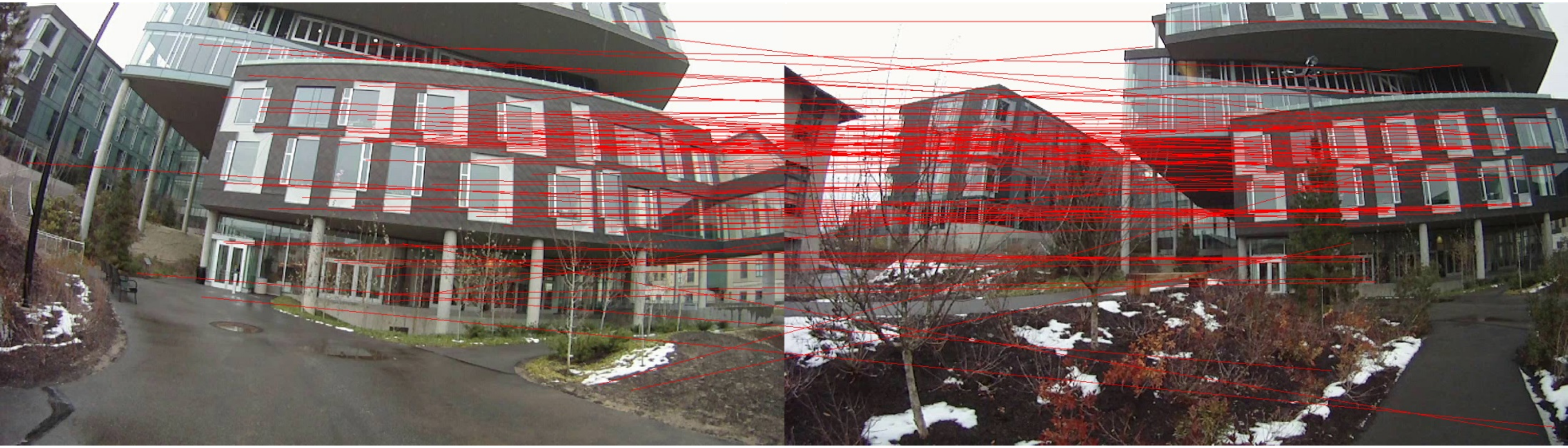


$$\mathbf{x}_2^{\mathsf{T}}\mathbf{F}\mathbf{x}_1 = 0$$

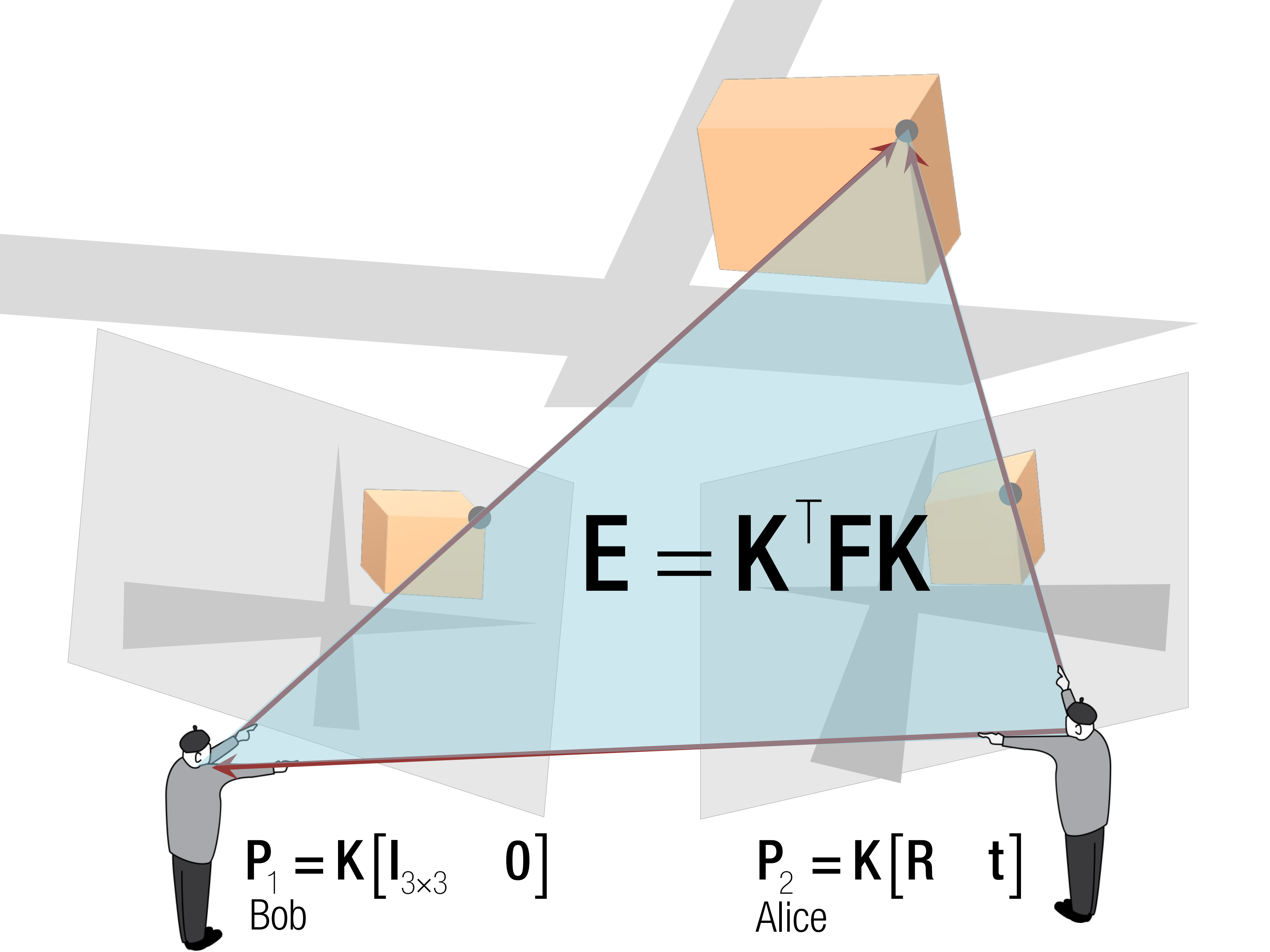
# of inliers: 253

Random sampling

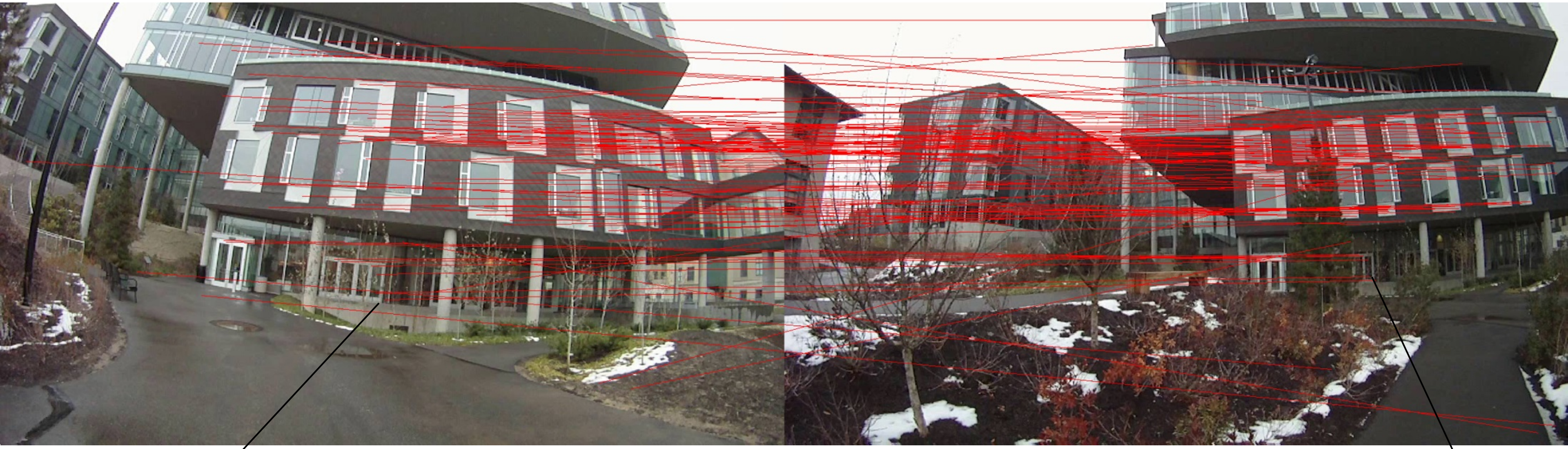
Model building

Thresholding

Inlier counting


$E = K^{\top}FK$
$P_1 = K[I_{3\times3} \quad 0]$
Bob
$P_2 = K[R \quad t]$
Alice

3. Essential Matrix Computation



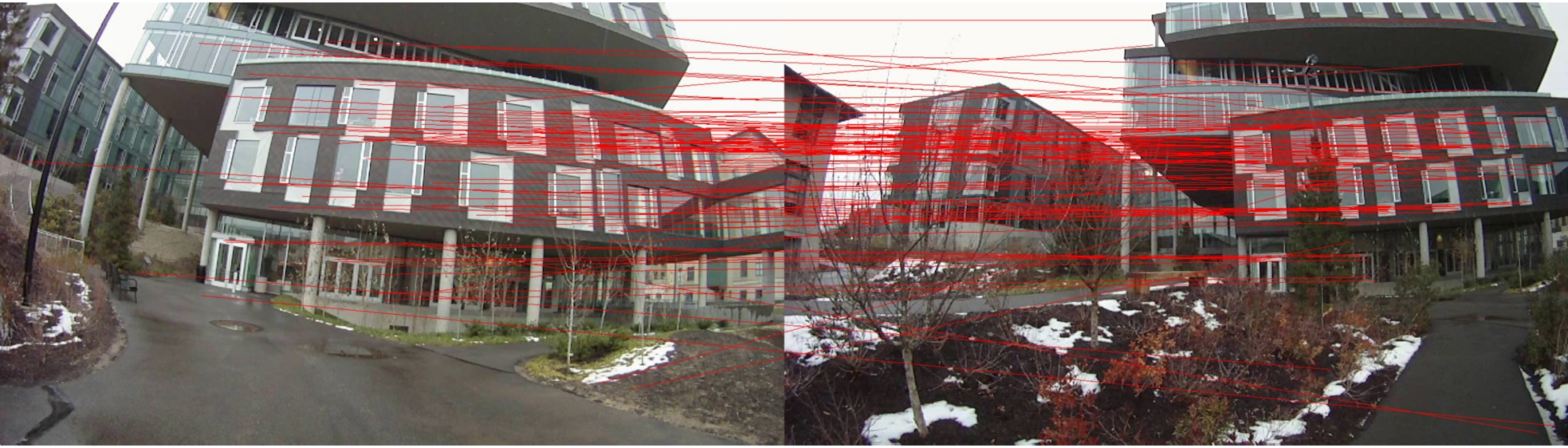
$\mathbf{x}_1$ $\mathbf{x}_2$

$$\mathbf{x}_2^T \mathbf{K}^{-T} \mathbf{E} \mathbf{K}^{-1} \mathbf{x}_1 = 0$$

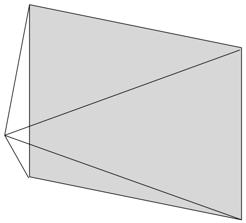
$$\mathbf{x}_2^T \mathbf{F} \mathbf{x}_1 = 0$$

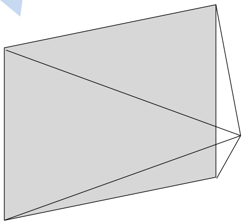
$$\longrightarrow \quad \mathbf{K}^T \mathbf{F} \mathbf{K} = \mathbf{E}$$

4. Relative Transform from Essential Matrix



$$E = [t]_\times R$$

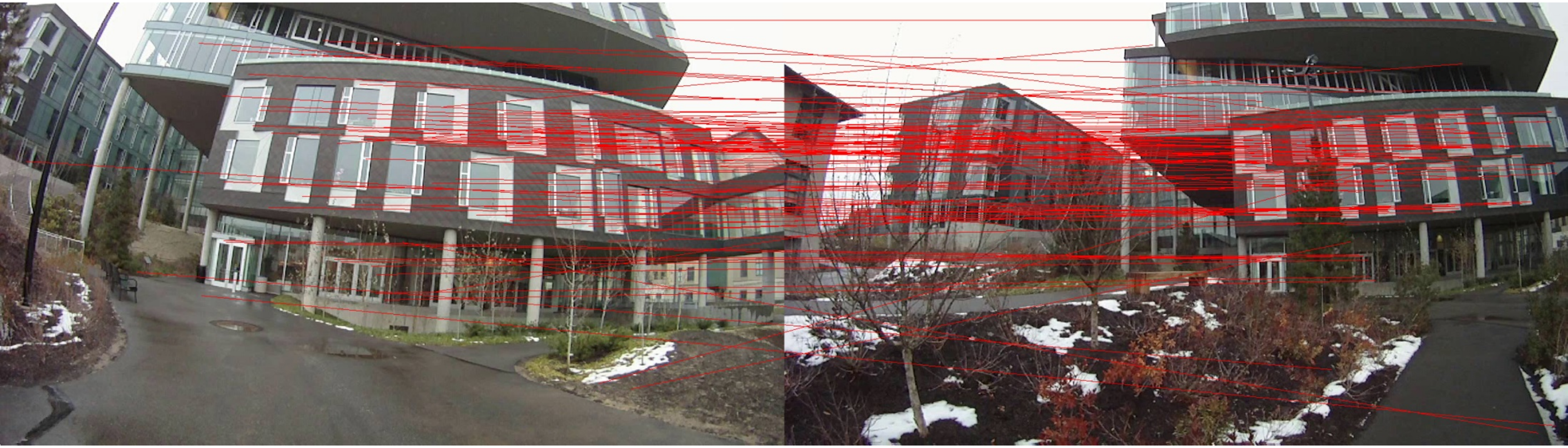


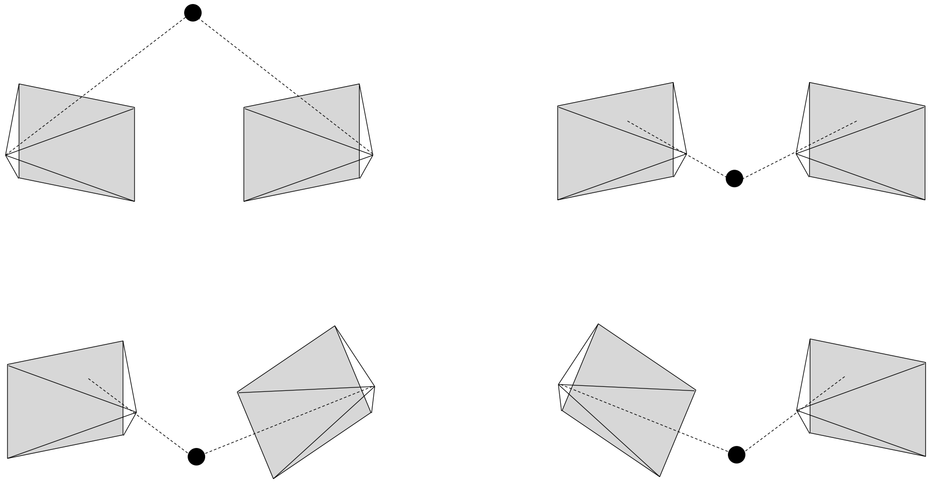


$$P_1 = [I_{3\times 3} \mid 0_3] \qquad P_2 = [R \mid t]$$

$$P_2 = [UYV^\top \mid u_3]$$
$$[UY^\top V^\top \mid u_3]$$
$$[UYV^\top \mid -u_3]$$
$$[UY^\top V^\top \mid -u_3]$$

4. Relative Transform from Essential Matrix





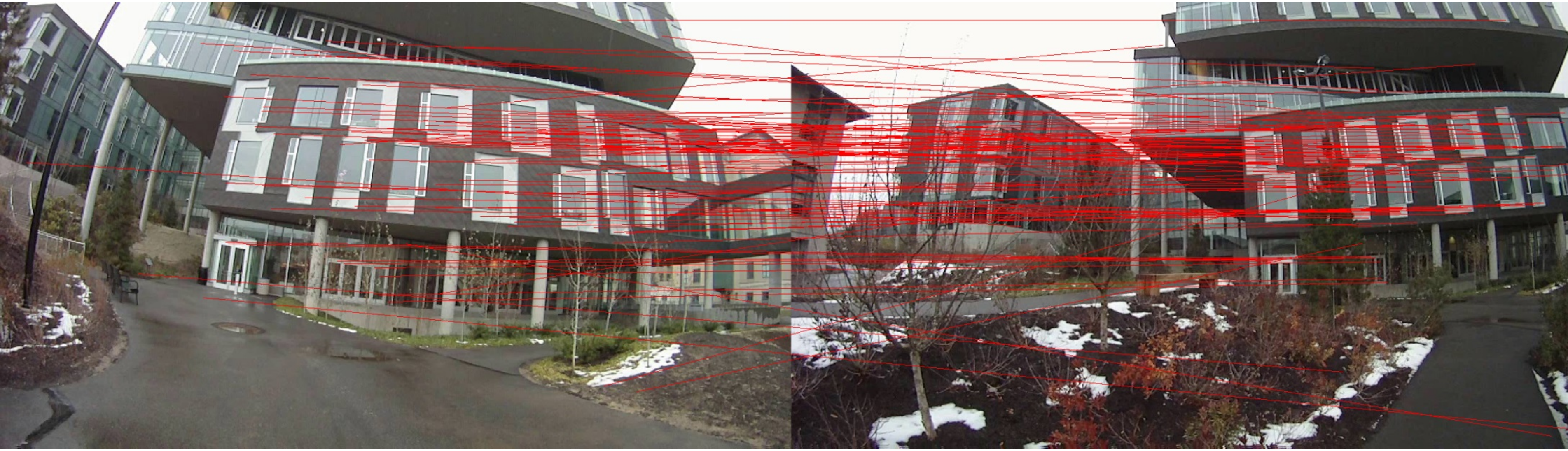
$$P_2 = \left[UYV^\top \mid u_3 \right]$$

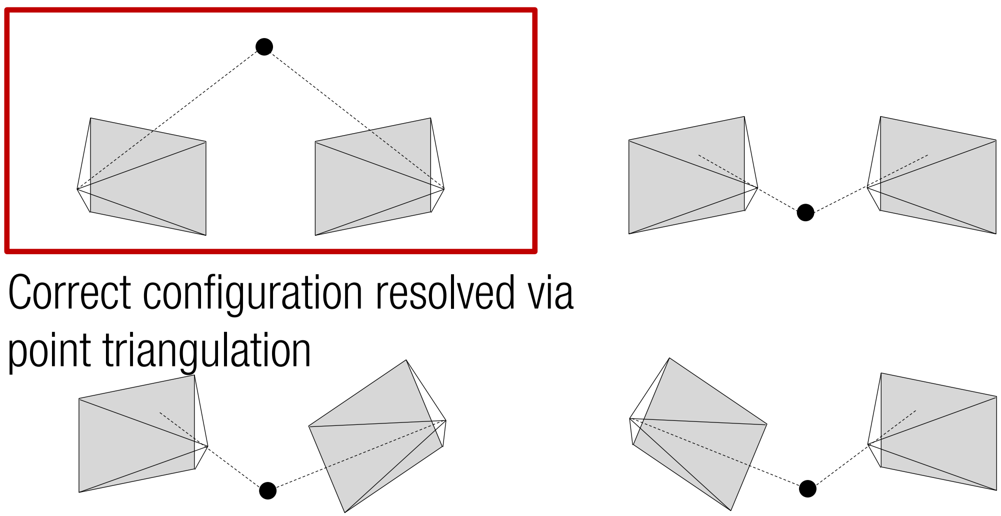
$$\left[UY^\top V^\top \mid u_3 \right]$$

$$\left[UYV^\top \mid -u_3 \right]$$

$$\left[UY^\top V^\top \mid -u_3 \right]$$

4. Relative Transform from Essential Matrix

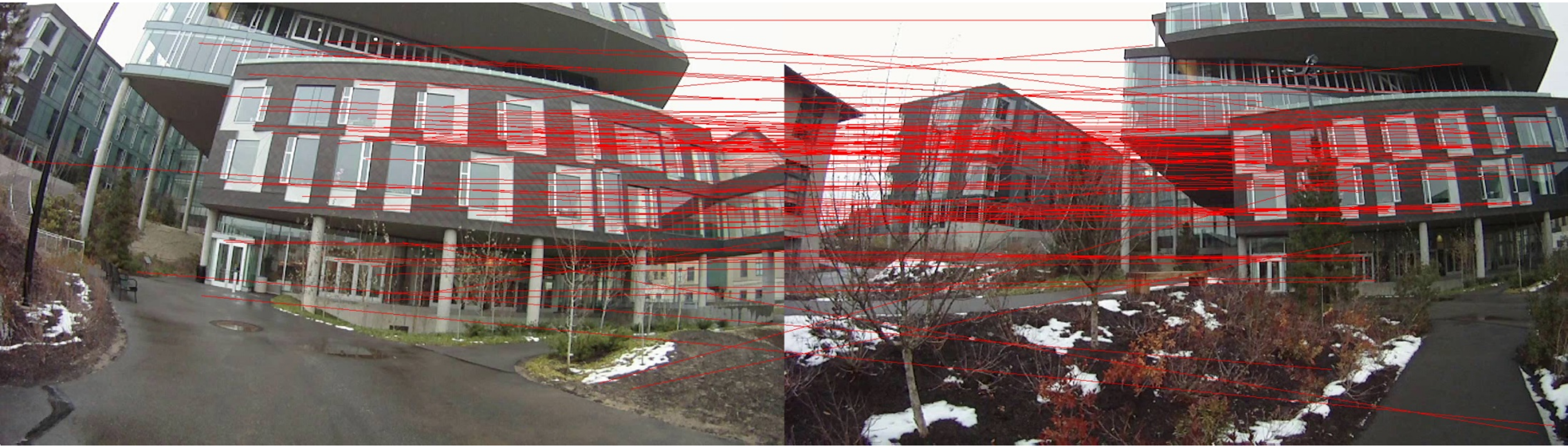


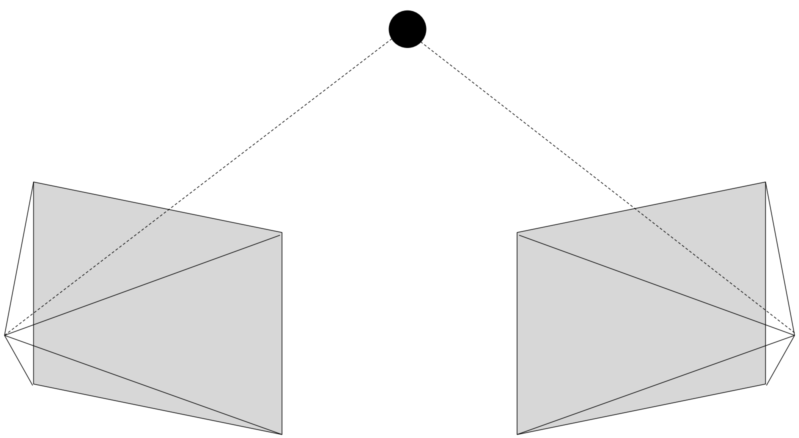


Correct configuration resolved via point triangulation

$$
\begin{aligned}
P_2 = &\left[\, UYV^{\top} \;\middle|\; u_3 \,\right] \\
&\left[\, UY^{\top}V^{\top} \;\middle|\; u_3 \,\right] \\
&\left[\, UYV^{\top} \;\middle|\; -u_3 \,\right] \\
&\left[\, UY^{\top}V^{\top} \;\middle|\; -u_3 \,\right]
\end{aligned}
$$

5. Point Triangulation

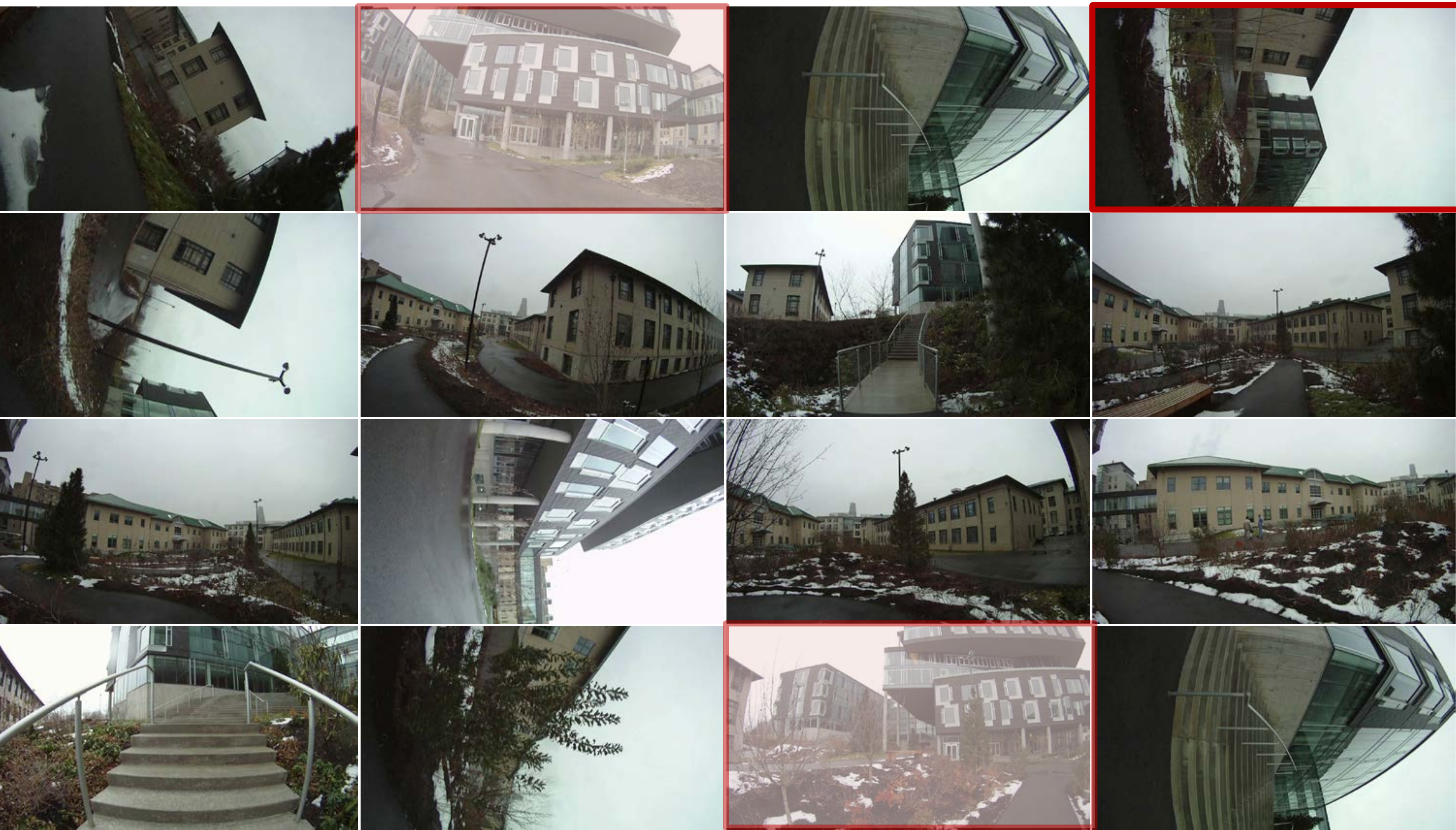




$$\begin{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ 1 \end{bmatrix}_\times \mathbf{P}_1 \\ \begin{bmatrix} \mathbf{x}_2 \\ 1 \end{bmatrix}_\times \mathbf{P}_2 \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{0}$$

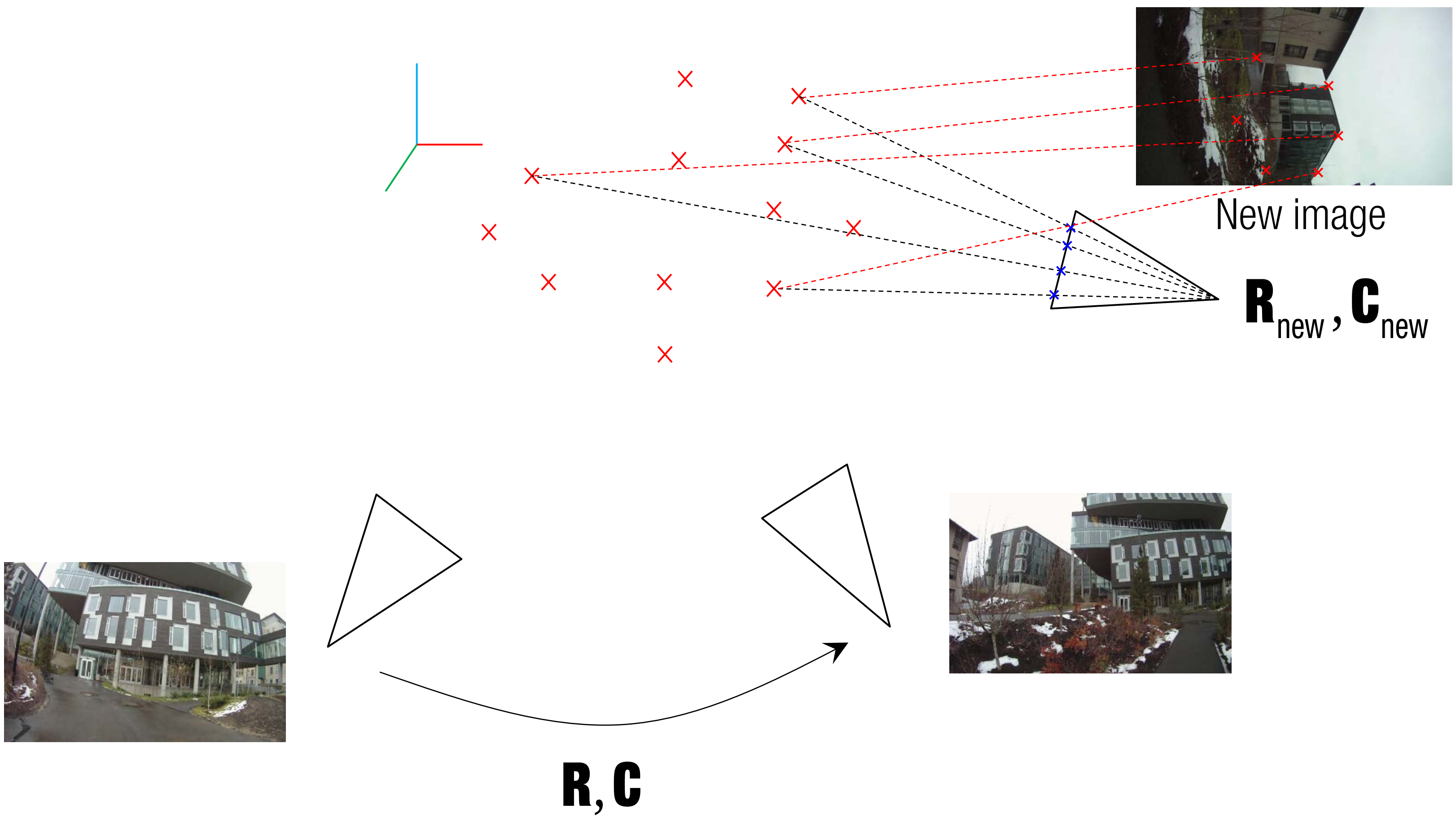
Initial Pair Images

Adding New Image



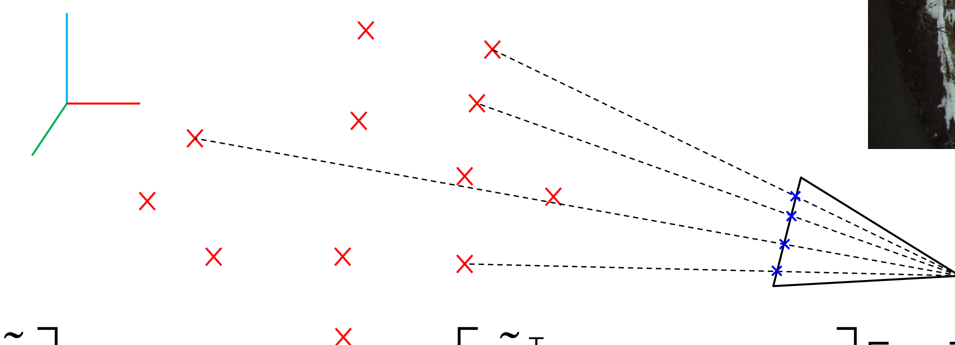
6. New Camera Registration

Perspective-n-point



6. New Camera Registration

Perspective-n-point



New image

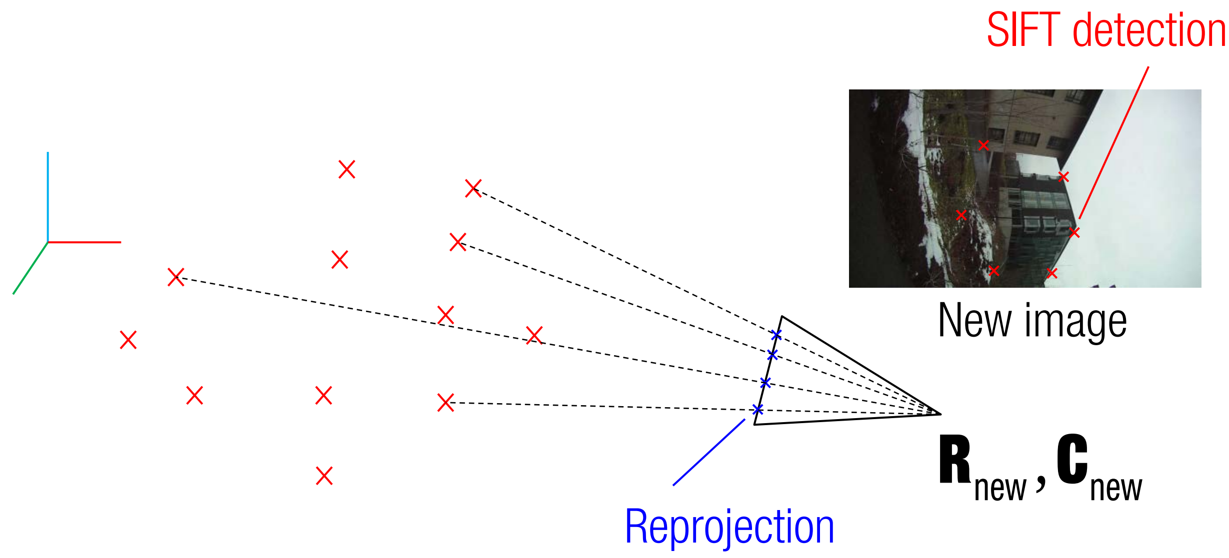
$\mathbf{R}_{\text{new}}, \mathbf{C}_{\text{new}}$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix}_\times \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{bmatrix} \tilde{\mathbf{X}} = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}_\times \begin{bmatrix} \mathbf{P}_1 \tilde{\mathbf{X}} \\ \mathbf{P}_2 \tilde{\mathbf{X}} \\ \mathbf{P}_3 \tilde{\mathbf{X}} \end{bmatrix} = \begin{bmatrix} 0 & -1 & v \\ 1 & 0 & -u \\ -v & u & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{X}}^{\mathsf{T}} & \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 4} \\ \mathbf{0}_{1\times 4} & \tilde{\mathbf{X}}^{\mathsf{T}} & \mathbf{0}_{1\times 4} \\ \mathbf{0}_{1\times 4} & \mathbf{0}_{1\times 4} & \tilde{\mathbf{X}}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \mathbf{P}_1^{\mathsf{T}} \\ \mathbf{P}_2^{\mathsf{T}} \\ \mathbf{P}_3^{\mathsf{T}} \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{0}_{1\times 4} & -\tilde{\mathbf{X}}^{\mathsf{T}} & v\tilde{\mathbf{X}}^{\mathsf{T}} \\ \tilde{\mathbf{X}}^{\mathsf{T}} & \mathbf{0}_{1\times 4} & -u\tilde{\mathbf{X}}^{\mathsf{T}} \\ -v\tilde{\mathbf{X}}^{\mathsf{T}} & u\tilde{\mathbf{X}}^{\mathsf{T}} & \mathbf{0}_{1\times 4} \end{bmatrix} \begin{bmatrix} \mathbf{P}_1^{\mathsf{T}} \\ \mathbf{P}_2^{\mathsf{T}} \\ \mathbf{P}_3^{\mathsf{T}} \end{bmatrix} = \mathbf{0}$$

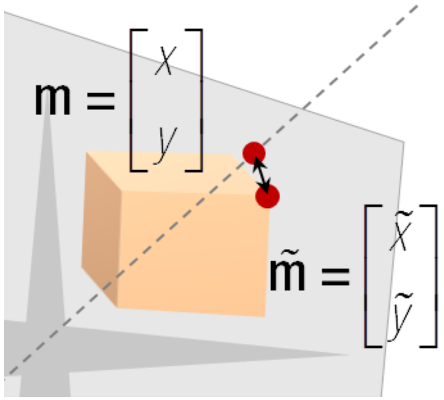
3x12 matrix

7. Bundle Adjustment

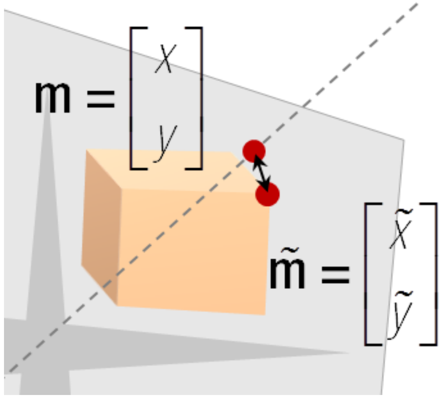


Reprojection error

$$e = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} - \begin{bmatrix} u/w \\ v/w \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{P}\begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I}_{3\times 3} & -\mathbf{C} \end{bmatrix}\begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$



Reprojection error



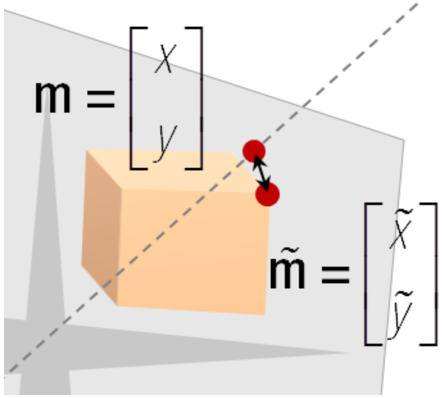
$$e = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} - \begin{bmatrix} u/w \\ v/w \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{P}\begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I}_{3\times3} & -\mathbf{C} \end{bmatrix}\begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

$$\underset{\mathbf{q},\mathbf{C},\mathbf{X}}{\text{minimize}} \left\| \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} - \begin{bmatrix} u(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) / w(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) \\ v(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) / w(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) \end{bmatrix} \right\|^2 = \underset{\mathbf{q},\mathbf{C},\mathbf{X}}{\text{minimize}} \left\| \mathbf{b} - \mathbf{f}(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) \right\|^2$$

$$\mathbf{f}(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) = \begin{bmatrix} u/w \\ u/w \end{bmatrix}$$

Reprojection error

$$e = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} - \begin{bmatrix} u/w \\ v/w \end{bmatrix} \quad \text{where} \quad \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \mathbf{P}\begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix} = \mathbf{K}\mathbf{R}\begin{bmatrix} \mathbf{I}_{3\times 3} & -\mathbf{C} \end{bmatrix}\begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

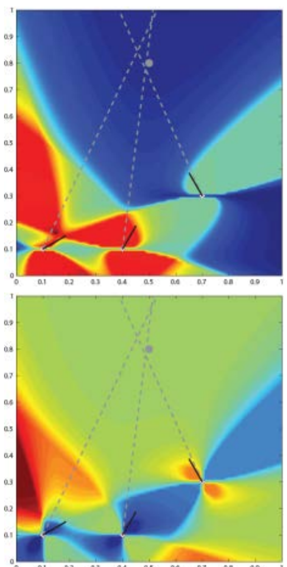


$$\underset{\mathbf{q},\mathbf{C},\mathbf{X}}{\text{minimize}} \left\| \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} - \begin{bmatrix} u(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) / w(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) \\ v(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) / w(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) \end{bmatrix} \right\|^2 = \underset{\mathbf{q},\mathbf{C},\mathbf{X}}{\text{minimize}} \left\| \mathbf{b} - \mathbf{f}(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) \right\|^2$$

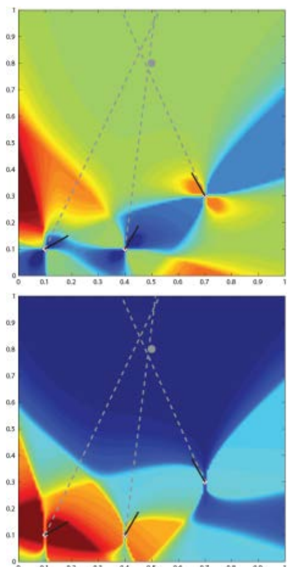
$$\mathbf{f}(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X}) = \begin{bmatrix} u/w \\ u/w \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \underbrace{\frac{\partial \mathbf{f}(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X})}{\partial \mathbf{R}}}_{2\text{x}9} \underbrace{\frac{\partial \mathbf{R}}{\partial \mathbf{q}}}_{9\text{x}4} & \underbrace{\frac{\partial \mathbf{f}(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X})}{\partial \mathbf{C}}}_{2\text{x}3} & \underbrace{\frac{\partial \mathbf{f}(\mathbf{R}(\mathbf{q}),\mathbf{C},\mathbf{X})}{\partial \mathbf{X}}}_{2\text{x}3} \end{bmatrix}$$

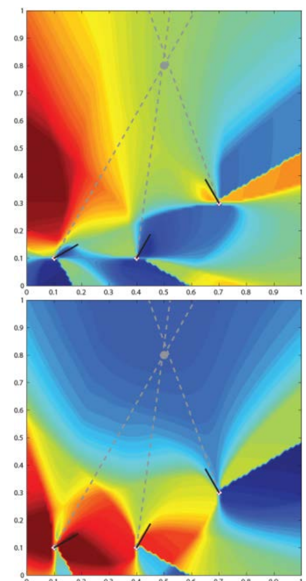
$$\mathbf{J}^{\top}\mathbf{J}\Delta\mathbf{x} = \mathbf{J}^{\top}\left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$$



$\Delta X +$

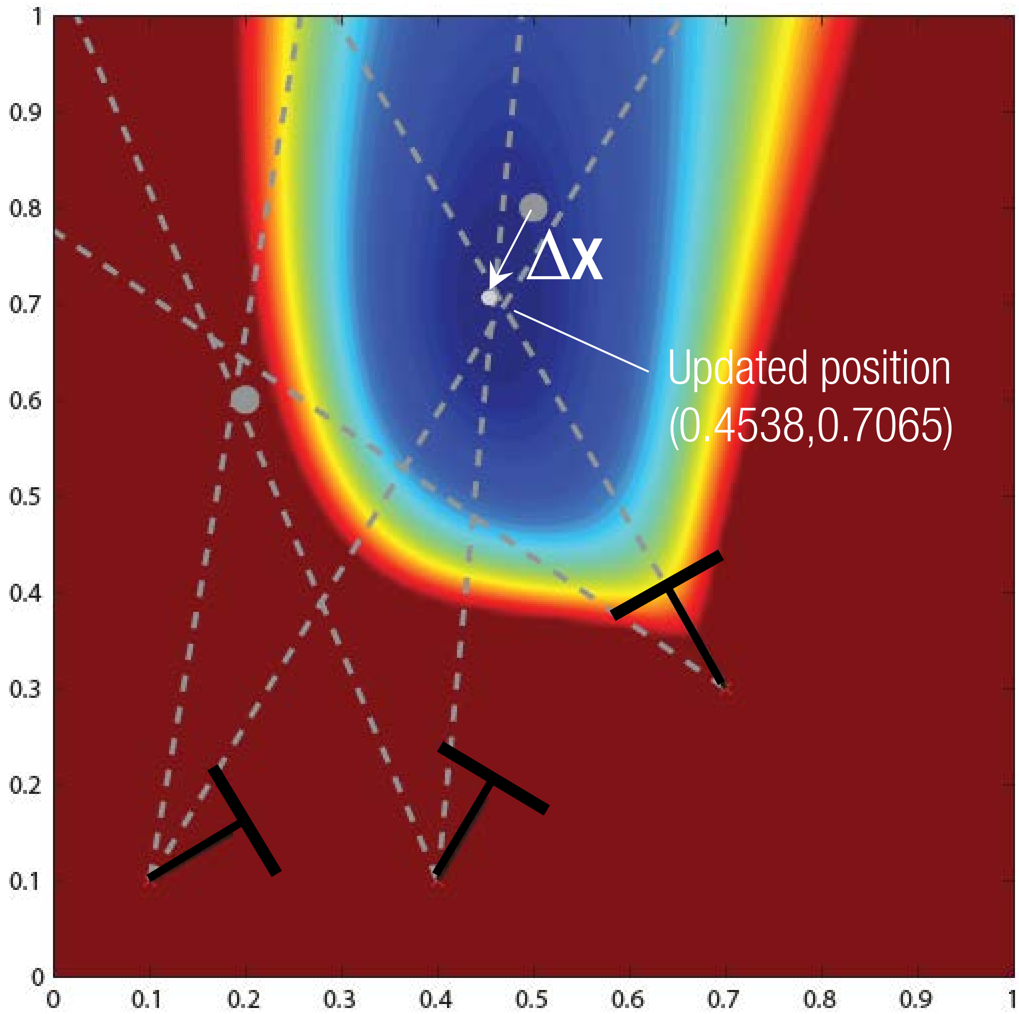


$\Delta Y =$



where $\Delta\mathbf{x} = \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix}$

$$\Delta \mathbf{x} = \left(\mathbf{J}^{\mathsf{T}}\mathbf{J}\right)^{-1}\mathbf{J}^{\mathsf{T}}\left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$$



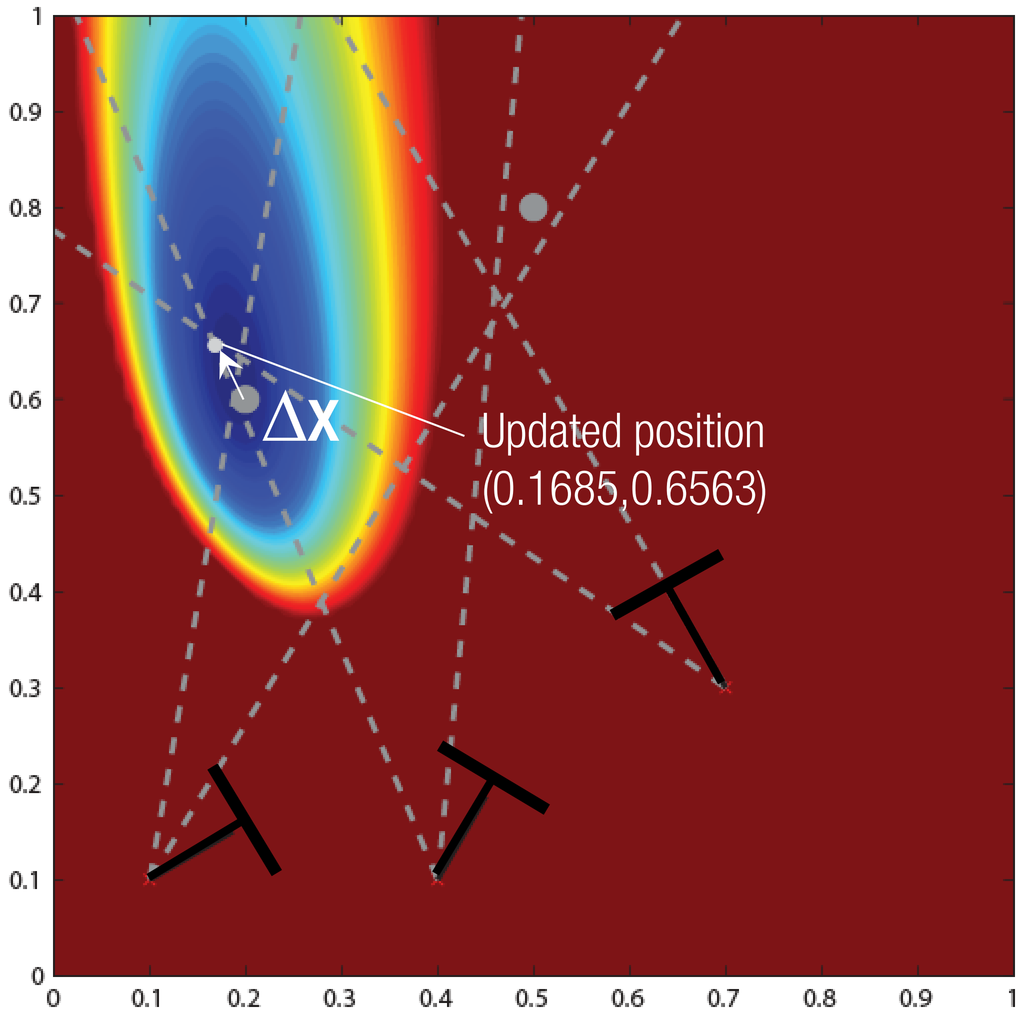
$$\sum_{i=1}^{3}\left\|\tilde{x}_{i1} - f_{i1}(\mathbf{X})\right\|^2$$

$$= \sum_{i=1}^{3}\left\|\tilde{x}_{i1} - u_{i1}/w_{i1}\right\|^2$$

$$\tilde{x}_{i1}$$

Camera index Point index

$$\Delta \mathbf{x} = \left(\mathbf{J}^{\mathrm{T}}\mathbf{J}\right)^{-1}\mathbf{J}^{\mathrm{T}}\left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$$

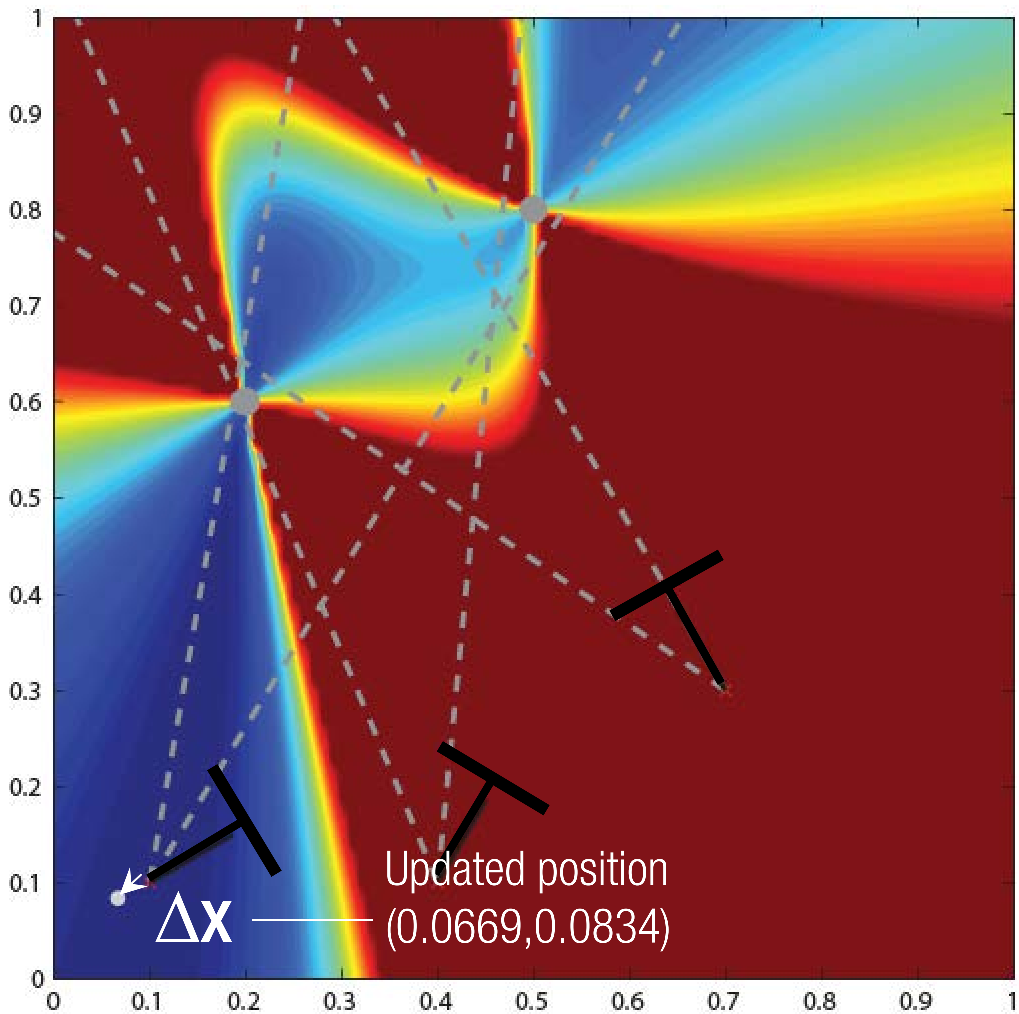


$$\sum_{i=1}^{3}\left\|\tilde{x}_{i2} - f_{i2}(\mathbf{X})\right\|^2$$

$$= \sum_{i=1}^{3}\left\|\tilde{x}_{i2} - u_{i2} / w_{i2}\right\|^2$$

$$\tilde{x}_{\underbrace{i}_{\text{Camera index}}\underbrace{2}_{\text{Point index}}}$$

$$\Delta \mathbf{X} = \left(\mathbf{J}^{T}\mathbf{J}\right)^{-1}\mathbf{J}^{T}\left(\mathbf{b}-\mathbf{f}(\mathbf{X})\right)$$



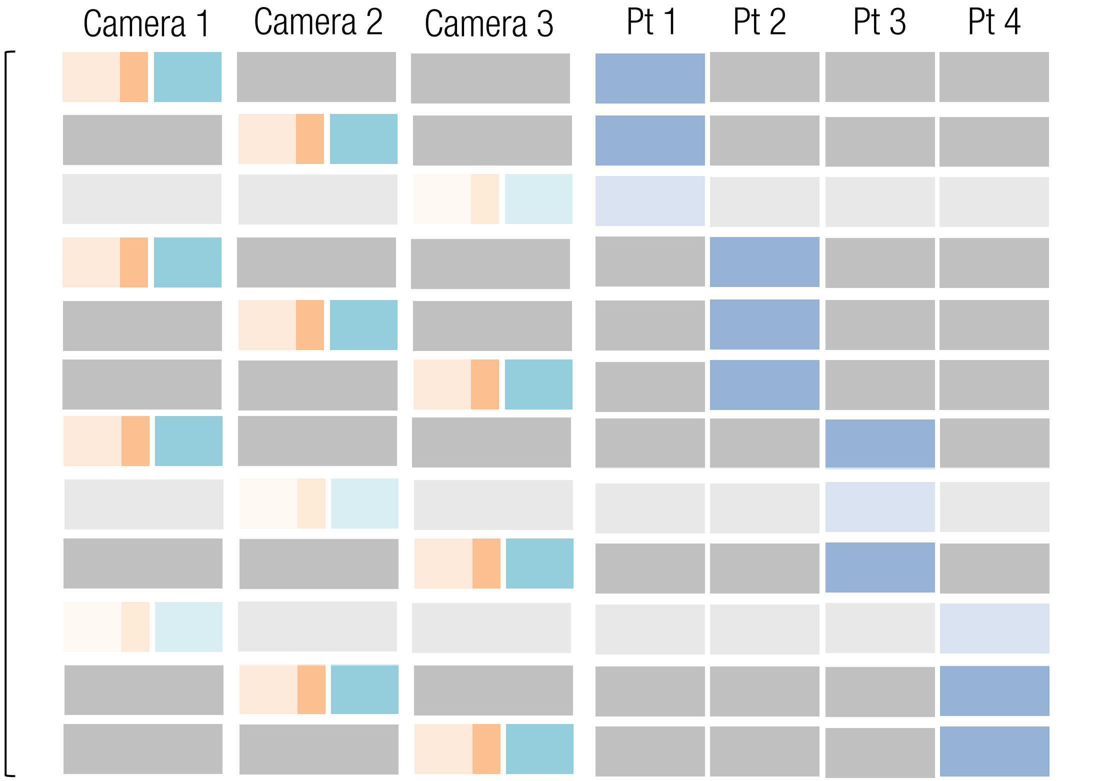
$$\sum_{j=1}^{2}\left\|\tilde{X}_{1j} - f_{1j}(\mathbf{X})\right\|^{2}$$

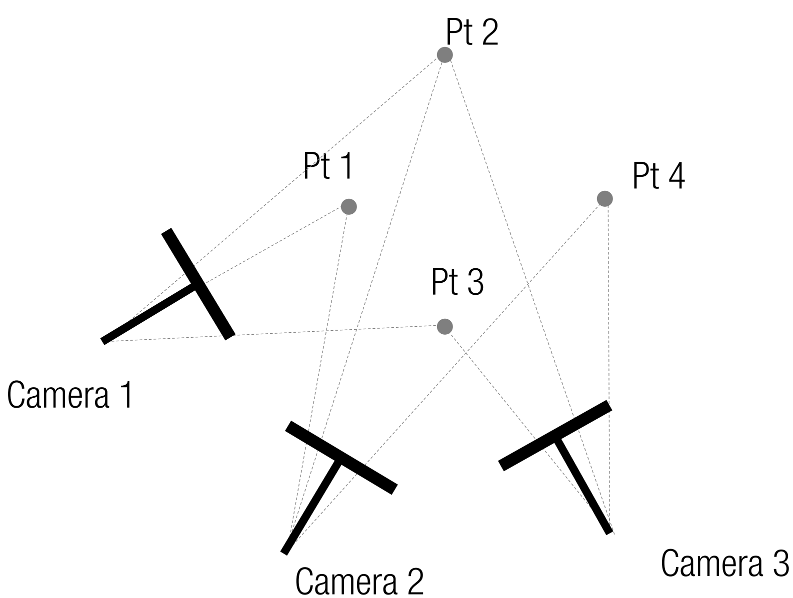
$$= \sum_{j=1}^{2}\left\|\tilde{X}_{1j} - u_{1j} / w_{1j}\right\|^{2}$$

$$\tilde{X}_{1j}$$

Camera index Point index

$\mathbf{J} =$

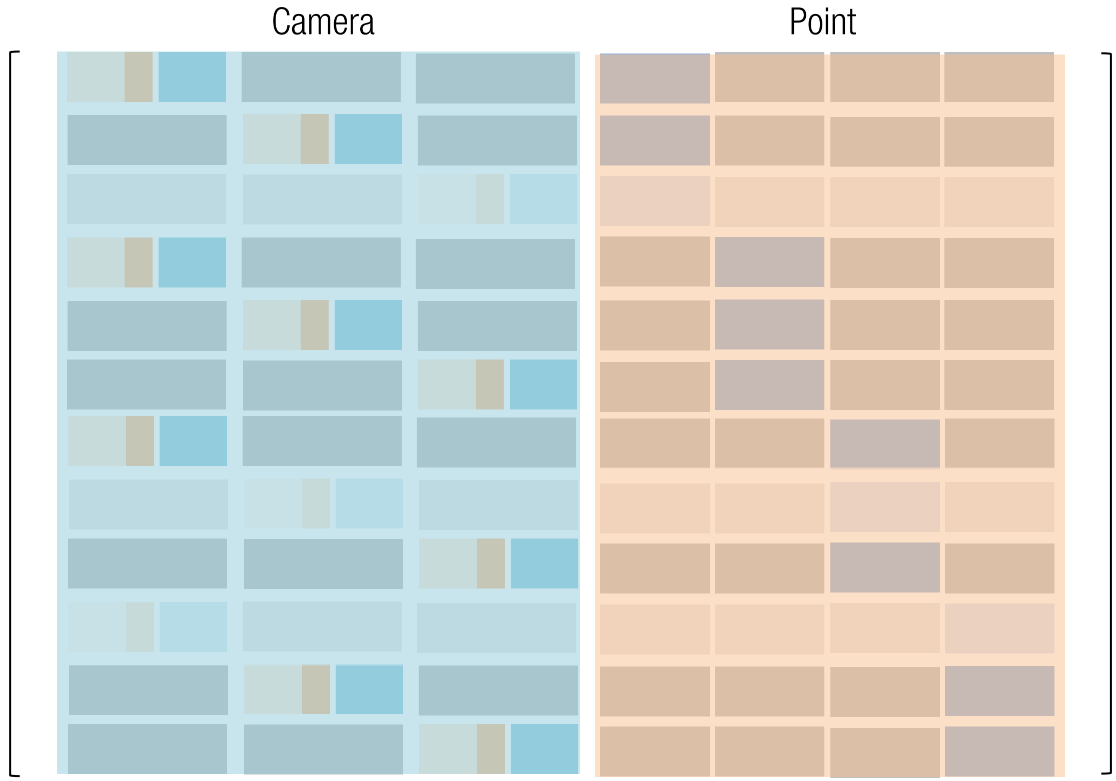


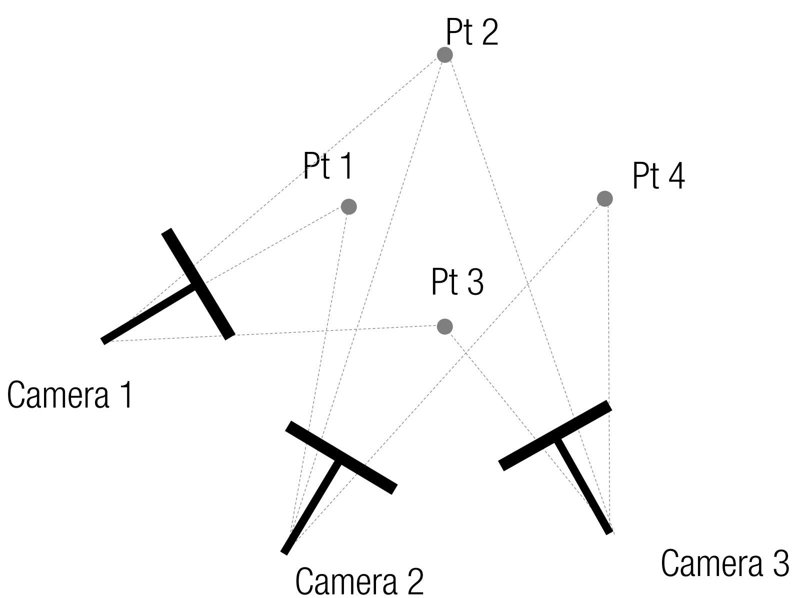


$$\Delta \mathbf{x} = \underline{\left(\mathbf{J}^{T}\mathbf{J}\right)^{-1}}\,\mathbf{J}^{T}\left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$$

Main computational bottle neck

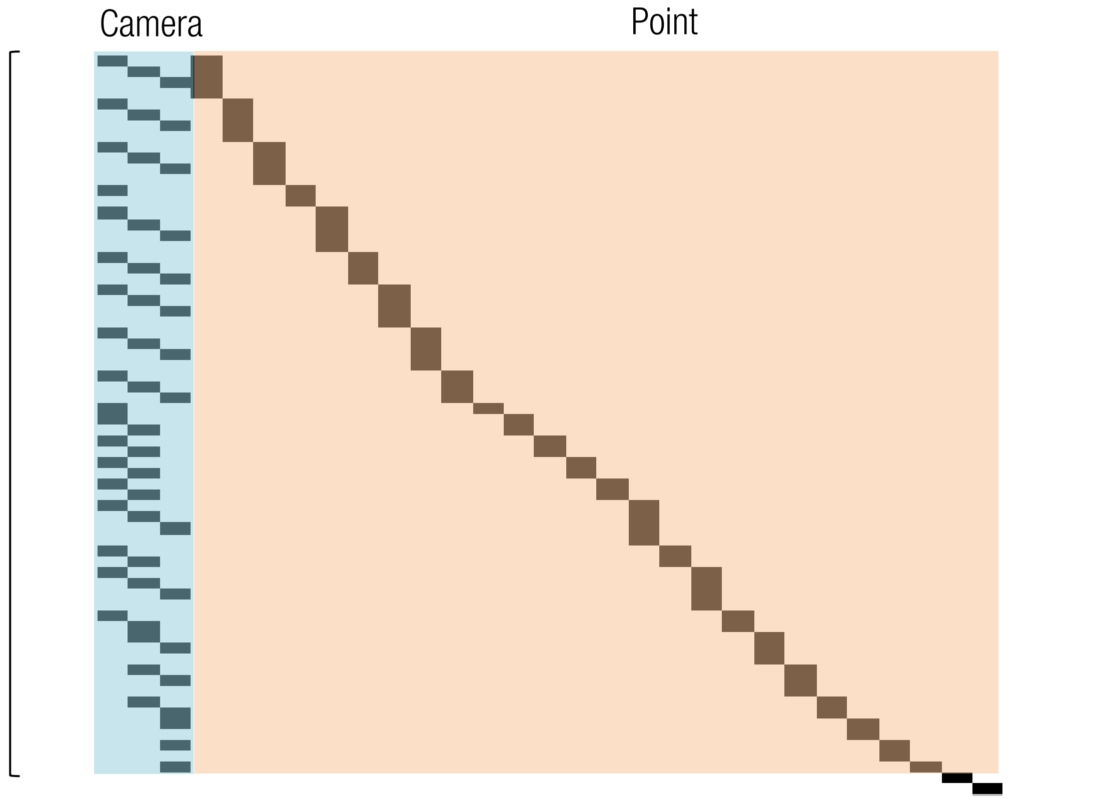
$\mathbf{J} =$



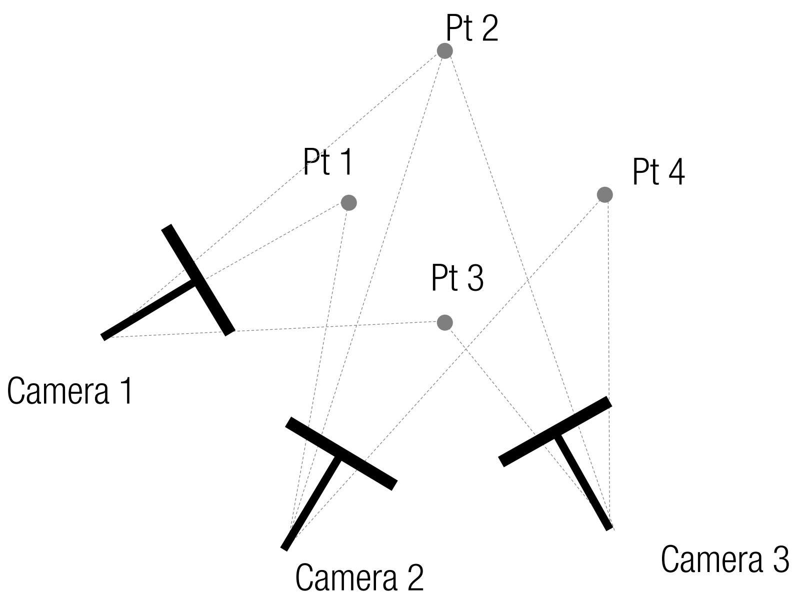


$$\Delta\mathbf{x} = \underbrace{\left(\mathbf{J}^{T}\mathbf{J}\right)^{-1}}_{\text{Main computational bottle neck}} \mathbf{J}^{T}\left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$$

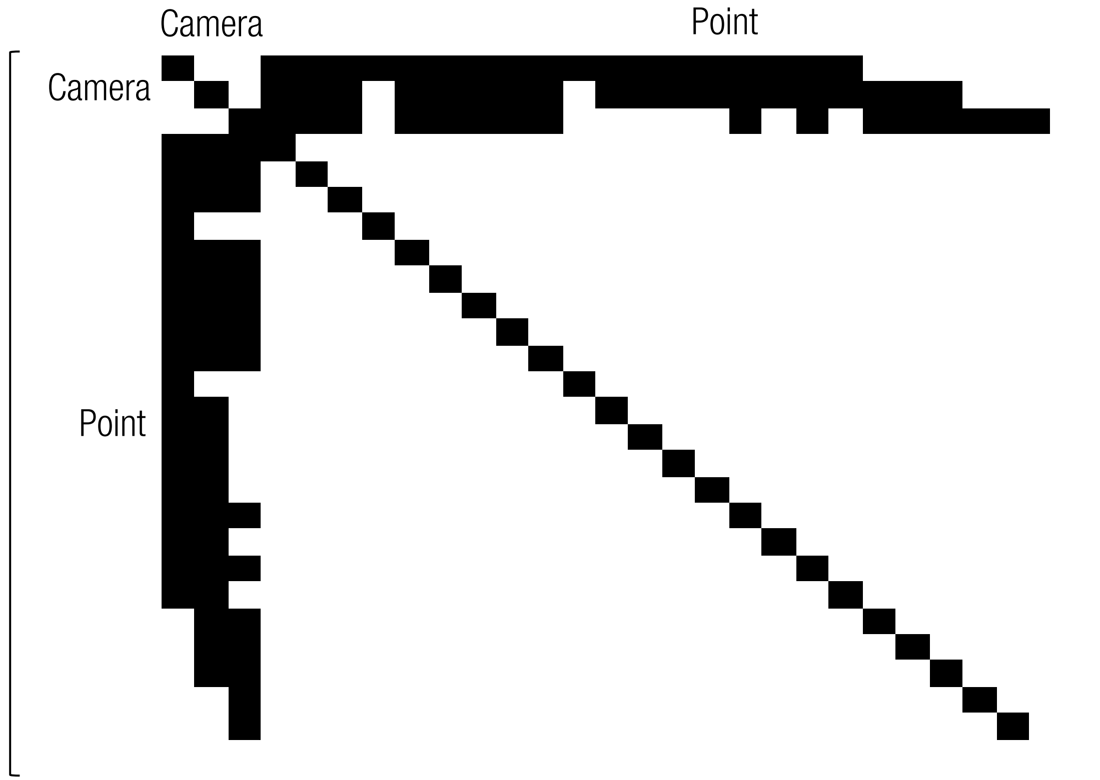
$\mathbf{J} =$

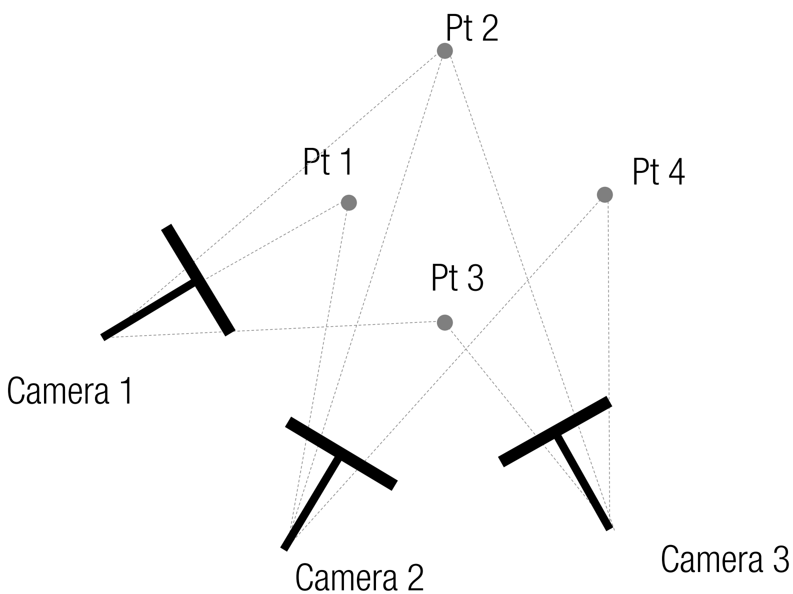


$$\Delta \mathbf{x} = \underbrace{\left(\mathbf{J}^{T}\mathbf{J}\right)^{-1}}_{\text{Main computational bottle neck}} \mathbf{J}^{T}\left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$$



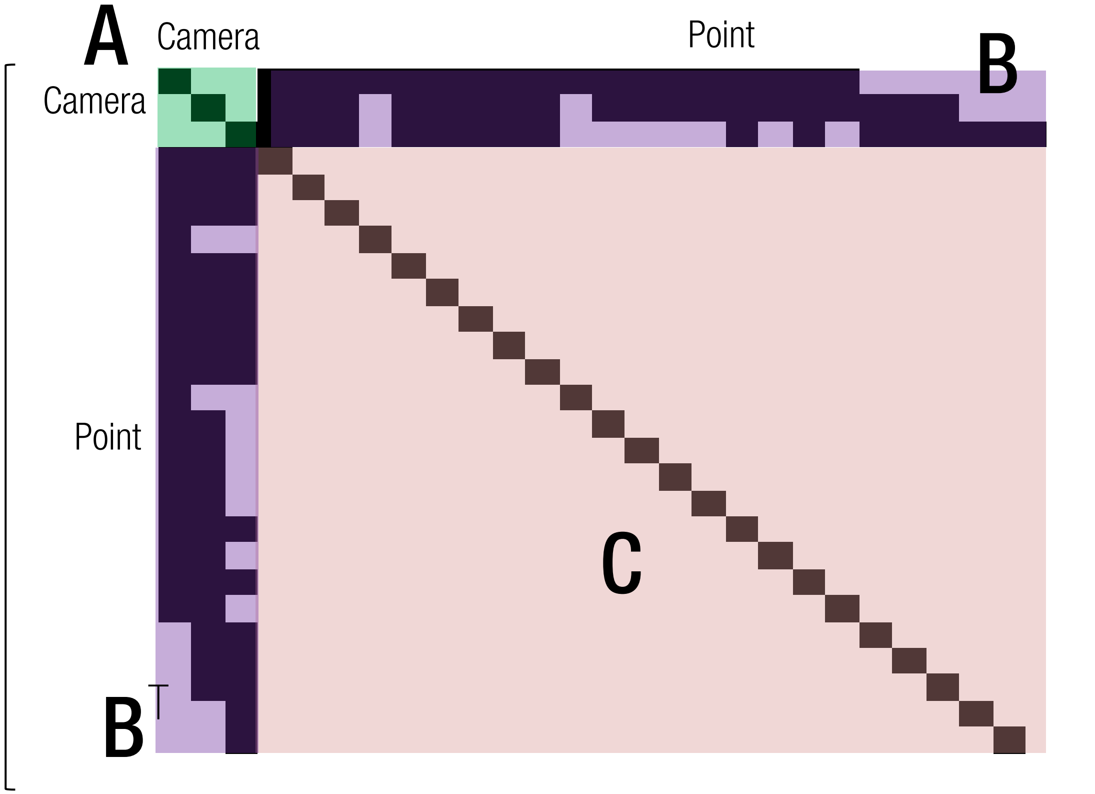
$$\mathbf{J}^{T}\mathbf{J} =$$





$$\Delta \mathbf{x} = \underbrace{\left(\mathbf{J}^{T}\mathbf{J}\right)^{-1}}_{\text{Main computational bottle neck}} \mathbf{J}^{T}\left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$$

$$\mathbf{J}^T\mathbf{J} =$$

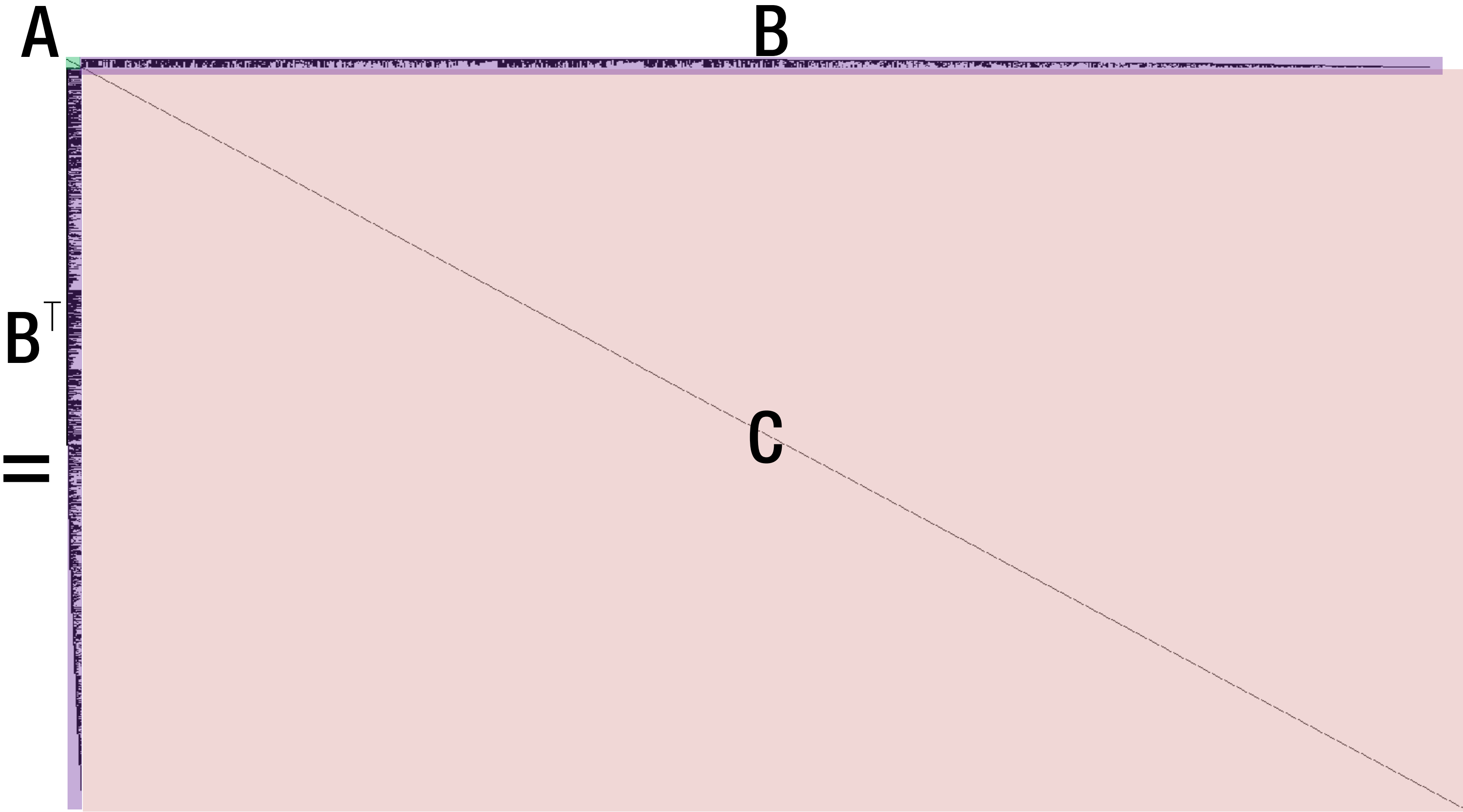


$$\Delta \mathbf{x} = \underbrace{\left(\mathbf{J}^T\mathbf{J}\right)}_{\text{Main computational bottle neck}}^{-1} \mathbf{J}^T\left(\mathbf{b} - \mathbf{f}(\mathbf{x})\right)$$

$$\left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T\right)\Delta \mathbf{x}_c = \mathbf{e}_c - \mathbf{B}\mathbf{C}^{-1}\mathbf{e}_p$$: Reduced system

$$\mathbf{C}\Delta \mathbf{x}_p = \mathbf{e}_p - \mathbf{B}^T\Delta \mathbf{x}_c$$: Back substitution

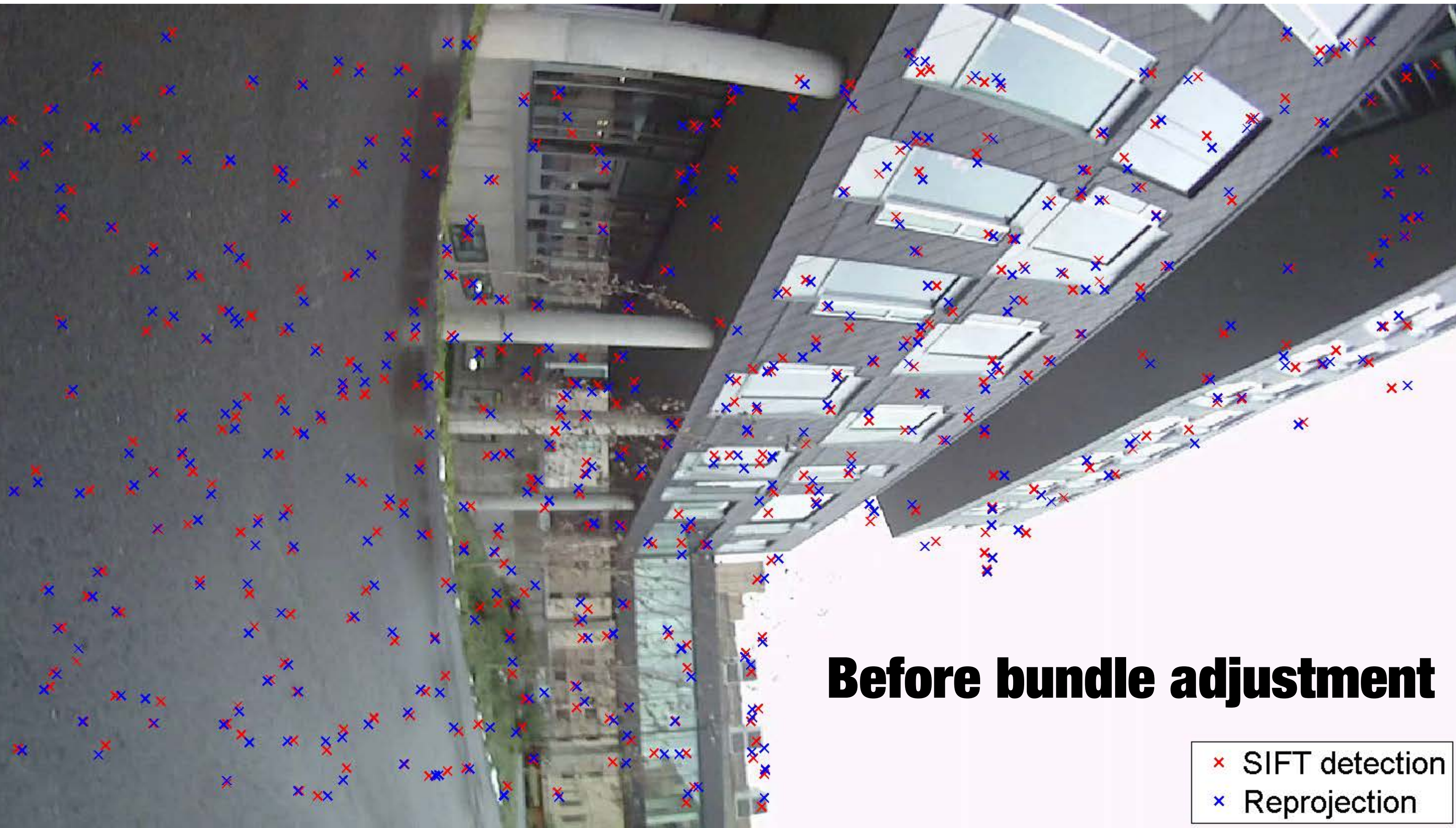
$$\mathbf{J}^\top\mathbf{J} =$$



$$\left(\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^\top\right)\Delta\mathbf{x}_c = \mathbf{e}_c - \mathbf{B}\mathbf{C}^{-1}\mathbf{e}_p$$: Reduced system

$$\mathbf{C}\Delta\mathbf{x}_p = \mathbf{e}_p - \mathbf{B}^\top\Delta\mathbf{x}_c$$: Back substitution

7. Bundle Adjustment



7. Bundle Adjustment