Tensors: O_3 and SO_3

$\vec{X} = \{x_1, x_2, x_3\} = x_i$ - vector

$\hat{A} = A_{ik}$ - matrix

$\hat{C} = \hat{A}\hat{B} \Leftrightarrow C_{ik} = A_{is} B_{sk}$

transponce: $(A^T)_{ik} = A_{ki}$; $(AB)^T = B^T A^T$

If $\det A \neq 0 \Rightarrow \exists A^{-1}: AA^{-1} = A^{-1}A = I$

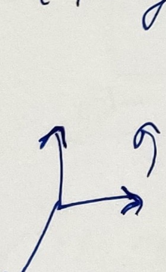
indexes: $I_{ik} = \delta_{ik} \Rightarrow BA = I \Leftrightarrow B_{is} A_{sk} = \delta_{ik}$

features: $(A^T)^{-1} = (A^{-1})^T \equiv A^{-1T}$; $(AB)^{-1} = B^{-1}A^{-1}$; $\det AB = \det A \cdot \det$

$\det A^T = \det A$

scalar product: $(Ax, y) = (x, A^T y)$

//

Rotation: 

point in new system

point in old system

$x' = \omega x \Leftrightarrow x'_i = \omega_{ik} x_k$

scalar product is invariant under rotations:

$$(x,y) = (x', y') = (\omega x, \omega y) = (x, \omega\omega^T y)$$

$$\Longrightarrow \omega^t \omega = 1 \Rightarrow \underbrace{\omega^T = \omega^{-1}}_{\text{definition of ortogonal matrixes.}}$$

1. 基础定义:

- $\vec{x} = \{x_1, x_2, x_3\} = x_i$ 表示向量
- $A = A_{ik}$ 表示矩阵
- $C = A\hat{B} \Leftrightarrow C_{ik} = A_{ij}B_{jk}$ 表示矩阵乘法

2. 重要性质:

- 转置: $(A^T)_{ik} = A_{ki}$ 且 $(AB)^T = B^T A^T$
- 逆矩阵: 当行列式不为零时，存在 $A^{-1} : AA^{-1} = A^{-1}A = I$
- 特征值: $I_{ik} = \delta_{ik} \Rightarrow BA = I \Leftrightarrow B_{ij}A_{jk} = \delta_{ik}$
- 其他特性: $(A^T)^{-1} = (A^{-1})^T = A^{-T}$ 和 $(AB)^{-1} = B^{-1}A^{-1}$

3. 标量积和旋转:

- 标量积: $(x, y) = (y, A^T y)$
- 旋转变换: $x' = \omega x \Leftrightarrow x'_i = \omega_{ik}x_k$
- 标量积在旋转下不变: $(x, y) = (x', y') = (\omega x, \omega y) = (x, \omega^T \omega y)$
- 正交矩阵性质: $\omega^T \omega = 1 \Rightarrow \omega^T = \omega^{-1}$

这些是张量分析中的基本概念和性质，对理解矩阵运算和坐标变换很重要。

$\det \underline{I} = 1 \cdot 1 \cdot 1 \cdot 1 \ldots 1 = 1$

$\omega^T \omega = 1 \mid \det \Rightarrow$

$(\det \omega)^2 = 1 \Rightarrow \det \omega = \pm 1.$

Pure rotation: any pure rotation can be continiously moved into $\omega = 1$, $\Rightarrow \det \omega = 1$ for pure rotation

ω orthogonal + $\det \omega = +1$ = SO_3 group.

Group:
1) multiplication, $a \cdot b \in G$ if $a \in G$ and $b \in G$
2) $\exists I: \; a \cdot I = I \cdot a = a$
3) $\forall a \; \exists a^{-1}: \; a^{-1} \cdot a = a \cdot a^{-1} = \underline{I}$, a^{-1} also $\in G$
4) $a(bc) = (ab)c$

$O(3)$ is group, multiplication is matrix multiplication.

~~Su~~ subgroup: part of elements, which is a group itself.

~~to~~ $\Rightarrow$ $\omega \in O(3)$, $\omega + \det \omega = +1 \in SO(3)$

$\omega + \det \omega = -1$ is not a subgroup: $a \cdot b = \overset{-1}{\overset{\shortparallel}{\det}} \cdot \overset{-1}{\overset{\shortparallel}{\det}} = +1$

//

& initial conditions.

$\det \omega = -1$: $\exists$ $P = -1$, reflection of space axis

$P x = -x$, i.e. $(Px)_i = -x_i$. $P \in O(3)$, $\det P = -1$.

Any reflection contains P, ω

$O_3 = SO_3 + P \cdot SO_3$

1. 对于正交矩阵 ω：

- $\omega^T\omega = 1$ 且 $detI = 1\cdot 1\cdot 1 ...\cdot 1 = 1$
- $(det\omega)^2 = 1 \Rightarrow det\omega = \pm 1$
- 纯旋转：可连续变换到单位矩阵，因此 $det\omega = 1$

2. 群的定义条件：

- 闭合性：$a, b \in G \Rightarrow ab \in G$
- 结合律：$a\cdot (b\cdot c) = (a\cdot b)\cdot c$
- 逆元：$\forall a \exists a^{-1} : a\cdot a^{-1} = a^{-1}\cdot a = I$
- a^{-1} 也属于群

3. $O(3)$ 群特性：

- 是矩阵乘法群
- ω 且 $det\omega = +1$ 构成 $SO(3)$ 子群
- $det\omega = -1$ 不是子群，例如 $a\cdot b = det\cdot det = +1$

4. 空间反射：

- $det\omega = -1$，$P = -1$ 表示空间轴反射
- $Px = -x$ 或写作 $(P)_{ij} = -r_i$
- $P \in O(3)$ 且 $detP = -1$
- $O_3 = SO_3 + P\cdot SO_3$（任何反射都包含 P）

// $\omega^{-1} = \omega^T \Leftrightarrow 1 = \omega^T \omega \Rightarrow \delta_{ik} = (\omega^T)_{is}\,\omega_{sk} =$

$$= \omega_{si}\,\omega_{sk}$$

$1 = \omega\omega^T \Rightarrow \delta_{ik} = \omega_{is}\,(\omega^T)_{sk} = \omega_{is}\,\omega_{ks}$ $\Bigg\} \Rightarrow$

$\omega_{\underline{s}i}\,\omega_{\underline{s}k} = \delta_{ik}$

$\omega_{i\underline{s}}\,\omega_{k\underline{s}} = \delta_{ik}$

Tensor.

tensor - some properties (rotation) under changing of frame of reference.

4 tensors under $SO(3)$

def: scalar: $\varphi' = \varphi$

vector: $A'_i = \omega_{ik} A_k$ (the same as rotation of the coordinats)

tensor: $T'_{i_1 \dots i_n} = \omega_{i_1 k_1} \dots \omega_{i_n k_n} T_{k_1 \dots k_n}$

NB: tensor is law of transformation (but no number of components):

if we consider 3 bodyes with mases m_1, m_2 and m_3

the set $\{m_1, m_2, m_3\}$ is not a tensor.

//

4 δ_{ij} and ε_{ijk}. Statement: both are tensors.

Definition $\delta_{ik} = \dots$, $\varepsilon_{ijk} = \dots$ in any frame of reference.

So, we should prove that $\delta_{ik} \to \delta_{ik}$ under the tensor changing law ($\omega \cdot \omega$):

1. 开头部分展示了一些数学关系：

```
ω=ω'(=) 1=ω'ω => δik=(ω')ij ωjk = ωij ωjk
1=ωωT => δik=ωij(ωT)jk=δij ωkj
```

   这导致了结论：`ωij ωjk=δik`

2. 图片主要讨论"Tensor"（张量）在参考系变换下的性质：

3. 张量在R(3)下的类型：

   - 标量(scalar): $\varphi'=\varphi$
   - 向量(vector): $A'_i=\omega_{ik} A_k$ (与坐标旋转相同)
   - 张量(tensor): $T'_{i_1\ldots i_n}=\omega_{i_1k_1}\ldots\omega_{i_nk_n} T_{k_1\ldots k_n}$

4. 重要说明(NB)：张量是变换定律（而不是分量的数目）

   - 举例说明：如果考虑3个具有质量m_1,m_2,m_3的物体
   - 集合$\{m_1,m_2,m_3\}$不是张量

5. 关于δ_{ij}和ε_{ijk}的说明：

   - 结论：两者都是张量
   - 定义δ_{ik}和ε_{ijk}在任何参考系中都成立
   - 需要证明δ_{ik}在张量变换定律(ω,ω')下转换为δ'_{ik}

这个图片主要在讨论张量的基本定义和性质，特别强调了张量不仅仅是一组数，而是要满足特定的变换规则。张量在物理和数学中是非常重要的概念，因为它们可以描述在坐标变换下保持不变的物理规律。

p. 18

$\# \delta_{ik}: \quad \delta'_{ik} = \omega_{is}\omega_{km}\delta_{sm} = \omega_{is}\omega_{ks} \overset{(*)}{=} \delta_{ik}$

$\# \varepsilon_{ijk}: \quad \varepsilon'_{\cancel{ijk}\ ikl} = \omega_{is}\omega_{kp}\omega_{lm}\varepsilon_{spm}$ def of ε: 1) antysymm 2) $\varepsilon_{123} = +1$

$\varepsilon'_{ikl} = \omega_{is}\omega_{kp}\omega_{lm}\varepsilon_{spm}$

$\varepsilon'_{kil} = \underbrace{\omega_{ks}\omega_{ip}}_{\shortparallel}\omega_{lm}\varepsilon_{spm}$

$\left.\omega_{ip}\omega_{ks}\cdot\omega_{lm}\underbrace{\varepsilon_{spm}}_{\overset{\shortparallel}{-\varepsilon_{psm}}} = -\varepsilon'_{ikl}\right\} \Rightarrow \varepsilon'_{kil} = -\varepsilon'_{ikl}$

$\varepsilon'_{ilk} = -\varepsilon'_{ikl}$ analogously.

2: $\varepsilon'_{123} = \omega_{1s}\omega_{2k}\omega_{3p}\varepsilon_{skp} \overset{\varepsilon_{112}=0}{=} \omega_{11}\cdot\omega_{22}\cdot\omega_{33} = \overset{\varepsilon}{\text{det}}\, \omega = +1$

$\Rightarrow \varepsilon'_{ikl}$ is ε-tensor. (and obtained from ε by $\omega\cdot\omega\cdot\omega$).

// if A = tensor^n, B = tensor^n $\Rightarrow$ $A+B$ = tensor of rank n

tensor product: $(\underset{\substack{\uparrow \\ \text{rank } n}}{A} \otimes \underset{\substack{\uparrow \\ \text{rank } m}}{B})_{i_1 \dots i_{n+m}} = A_{i_1 \dots i_n} B_{i_{n+1} \dots i_{n+m}}$

feature (follows from definition): $A \otimes B$ is also tensor.

// contraction: $\underbrace{T_{i_1 \dots i_k \dots i_k \dots i_n}}_{n \text{ indexes}} = \underbrace{T_{i_1 \dots i_{n-2}}}_{n-2 \text{ indexes}}$ ← this is also tensor.

e.g.: $T_{ikl} \to T_{ikk} = A_i$ $\Rightarrow A_s$ is <u>vector</u>.

$A'_i = T'_{ikk} = \omega_{is}\omega_{kp}\omega_{km}T_{spm} = \omega_{is}\delta_{pm}T_{spm} = \omega_{is}T_{spp} = \omega_{is}A_s$

1. δ_{ik}（克罗内克符号）的变换： $\delta'_{im} = \omega_{ij}\,\omega_{km}\,\delta_{km} = \omega_{ij}\,\omega_{kj} = \delta_{ik}$（标记为*）这证明了$\delta_{ik}$在坐标变换下保持不变，确认其张量性质。
2. ε_{ijk}（列维-奇维塔符号）的变换： $\varepsilon'_{ik\ell} = \omega_{ij}\,\omega_{kp}\,\omega_{\ell m}\,\varepsilon_{jpm}$

其中ε需满足两个条件：

1. antisymm（反对称性）
2. det ω = +1（变换矩阵行列式为1）

通过变换可以证明：
$\varepsilon'_{ki\ell} = -\varepsilon'_{ik\ell}$
$\varepsilon'_{i\ell k} = -\varepsilon'_{ik\ell}$ analogously（类似地）

3. 进一步的证明： $\varepsilon'_{123} = \omega_{1j}\,\omega_{2k}\,\omega_{3p}\,\varepsilon_{jkp}$
   其中 $\varepsilon_{412} = 0$ 结果显示：det ω = +1

结论：$\varepsilon'_{ik\ell}$ 是 E-tensor（通过ω.ω.ω获得）

4. 张量运算规则：

a) 加法规则：

- 如果 A是张量，B是n阶张量 => A+B 是n阶张量

b) 张量积（Tensor product）：
$(A\otimes B)_{i_1\ldots i_{n+m}} = A_{i_1\ldots i_n}\,B_{i_{n+1}\ldots i_{n+m}}$
其中rank n和rank m分别表示A和B的阶数
特征：A⊗B 也是张量

c) 缩并（Contraction）：
$T_{i_1\ldots i_k\ldots i_k\ldots i_n} = T_{i_1\ldots i_{n-2}}$
表示从n个指标缩减为n-2个指标，结果仍是张量

5. 具体例子： $T_{ik\ell} \to T_{inn} = A_i$ 验证A_i是向量： $A'_i = T'_{inn} = \omega_{ij}\,\omega_{kn}\,\omega_{\ell m}\,T_{p\ell m} = \omega_{ij}\,T_{pp} = \omega_{ij}\,A_j$

这些性质和运算规则展示了张量的几个重要特点：

1. 在坐标变换下的不变性
2. 运算封闭性（运算结果仍是张量）
3. 指标缩并后仍保持张量性质
4. 张量积的构造方法

这些性质使得张量成为描述物理规律的有力工具，特别是在需要保持物理定律在不同参考系下形式不变性的情况下。

Lemma: contraction of unknown with known:

if $X \cdot T_1 = T_2 \Rightarrow X$ is also tensor

$$X \cdot \underset{\uparrow\ \text{tensor}}{T_1} = \underset{\uparrow\ \text{tensor.}}{T_2}$$

e.g.: $\exists\, T_{is}$ is unknown quantity with 2 indexes (i and k).

$\exists\, A_i, B_i$ – two vectors

If $T_{ik} A_k = B_i \Rightarrow T_{ik}$ is tensor.

Let's ~~proof~~ prove: $A' = \omega_{ik} A_k$, $B' = \omega_{ik} B_k$ (A, B – vectors).

$$\left.\begin{array}{l} B_i' = T_{ik}' A_k' = T_{ik}' \omega_{ks} A_s \quad (\text{def } B:\ B_i = T_{ik} A_k) \\ B \text{ is vector}, \Rightarrow B_i' = \omega_{ik} B_k = \omega_{ik} T_{ks} A_s \end{array}\right\} \Longrightarrow$$

$$T_{ik}' \omega_{ks} = \omega_{ik} T_{ks} \quad | \, \omega_{ps}$$

$$\omega_{ik} \omega_{ps} T_{ks} = \underbrace{\omega_{ps} \omega_{ks}}_{\delta_{kp}} T_{ik}' \quad ; \quad \delta_{kp} T_{ik}' = T_{ip}'$$

$\Rightarrow T_{ip}' = \omega_{ik} \omega_{ps} T_{ks}$ – tensor rule.

q.e.d.

It's true for any tensors T.

Reflections:

true tensor: $T_{i_1 \dots i_n}' = (-1)^n T_{i_1 \dots i_n}$

pseudo tensor: $T_{i_1 \dots i_n}' = (-1)^{n+1} T_{i_1 \dots i_n}$

标题：张量缩并与已知量的引理

主要内容包括：

1. 基本引理：如果 $X \cdot T_1 = T_2$，其中 T_1、T_2 都是张量，则 X 也是张量。
2. 举例说明：

- J(T)ij 是一个带两个指标(i和k)的未知量
- A_i、B_i 是两个向量
- 如果 T_ik A_k = B_i，则 T_ik 是张量

3. 证明过程：

- 定义 A'=ω_ix A_i, B'=ω_ik B_ik (A,B为向量)
- B_i'=T_ik' A_k'=T_ik ω_kj A_j
- 利用张量变换规则：B_i'=ω_ip T_kj A_j
- 得到 T_ip'=ω_ix ω_pj T_kj

4. 结论：对于任意张量 T，该性质都成立。
5. 反射性质：

- 真张量：$T'\_i_1 \ldots i_n = (-1)^n T\_i_1 \ldots i_n$
- 赝张量：$T'\_i_1 \ldots i_n = (-1)^{n+1} T\_i_1 \ldots i_n$

δ_{ik} and ε_{ikl} has the same definitions in all frames,

$\Rightarrow$ δ_{ik} is pure tensor, ε_{ikl} - pseudo tensor.

tensor product: $(+1)\times(+1) = +1$

$(-1)\times(-1) = +1$

$(+1)\times(-1) = -1$

Tensor fields.

Tensor field is tensor defined at any space point x: $T_{i_1 \dots i_n}(x)$.

$\forall O_3$: $T'_{i_1 \dots i_n}(x') = \omega_{i_1 k_1} \dots \omega_{i_n k_n} T_{k_1 \dots k_n}(x)$

NB: geometrically x' is the same point as x
(but in another frame it has another coordinates).

scalar: $\varphi'(x') = \varphi(x)$ (*)
vector: $A'_i(x') = \omega_{ik} A_k(x)$

Example of scalar field: $\rho(x)$.

(*) is a statement that $\rho(x)$ do not depend of the frame of reference
(what is physical assumption: $\rho(x) \to Q$, which is measurable).

NB: $\not\Rightarrow$ ~~$\varphi(x)$~~ - $\varphi(x') = \varphi(x)$ — this is not invariance, thing!
this is spherical symmetry (complete another)

1. δ_{ij}和ε_{ijk}的定义：

- δ_{ij}是纯张量（也称为克罗内克函数）
- ε_{ijk}是偶张量（也称为列维-奇维塔张量）

2. 张量积规则：

- $(+1)\times(+1) = +1$
- $(-1)\times(-1) = +1$
- $(+1)\times(-1) = -1$

3. 张量场定义： 在任意空间点x定义的张量 $T\_i_1 \ldots i_n(x)$
4. 张量场变换定律： $T'\_i_1 \ldots i_n(x') = \omega\_i_1 k_1 \ldots \omega\_i_n k_n T\_k_1 \ldots k_n(x)$

重要注意事项：

- x'和x几何上是同一点(但在不同参考系有不同坐标)
- 标量场变换： $\varphi'(x') = \varphi(x)$
- 矢量场变换： $A'_i(x') = \omega\_ik\ A\_k(x)$

5. 标量场示例 $\rho(x)$：

- $\rho(x)$不依赖参考系的表述是物理假设
- 这种关系不仅仅是不变性，而是具有球对称性(这是更强的性质)

解释：

1. 这些定义和规则构成了相对论性电动力学中张量分析的基础。
2. δ_{ij} (克罗内克函数)：
   - 当i=j时值为1
   - 当i≠j时值为0 这是一个基本的数学工具，用于指标的升降和张量运算。
3. 张量场变换定律描述了在不同参考系之间物理量如何变换。这确保了物理规律在所有惯性参考系中都保持不变的形式。
4. 标量场(如电荷密度ρ)在不同参考系中的不变性反映了物理量的本质特性，这是相对论性要求的体现。
5. ω_ik代表坐标变换矩阵的元素，用于描述不同参考系之间的转换关系。

Differentiation

* $\partial/\partial x_i$. If $x \to x' = \omega x$, $\partial \to \partial' = \omega \partial$. i.e., $\partial \in O_3$

∄ $x' = Lx$, $\det L \neq 0$

∂: $\partial/\partial x'_i = \frac{\partial x_k}{\partial x'_i} \cdot \partial/\partial x_k$ $\qquad x' = Lx \Rightarrow x = L^{-1}x'$, i.e. $x_i = L^{-1}_{ik} x'_k$

$\Rightarrow \partial/\partial x'_i = L^{-1}_{ik} \cdot \partial/\partial x_k =$ $\qquad \Rightarrow \frac{\partial x_i}{\partial x'_k} = L^{-1}_{ik}$

$= L^{-1T}_{ki}\, \partial_k$ $\qquad (L^{-1})^T = (L^T)^{-1}$

$\Rightarrow x' = Lx \Rightarrow \underline{\partial' = L^{-1T}\partial}$

$\exists\, L \in O_3$ $(L = \omega)$: $\omega^T = \omega^{-1} \Rightarrow \partial' = \omega \partial$

$\Rightarrow \partial_i$ is vector ($\partial/\partial(-x) = -\partial/\partial x$, $\Rightarrow$ pure vector).

$\Rightarrow$ all things which are constructed with ∂ are also tensors

//

Maxwell: ρ = pure scalar (physical assumptions)

$\vec{x}, \vec{\partial}$ = pure vectors

$\operatorname{div} \vec{E} = \partial_i E_i = 4\pi\rho \Rightarrow \vec{E}$ is pure vector

$\operatorname{rot} E + 1/c\, \partial_t \vec{B} = 0 \Rightarrow \vec{B}$ is pseudo vector

$\operatorname{rot} \vec{B} - 1/c\, \partial_t \vec{E} = \frac{4\pi}{c}\vec{j} \Rightarrow \vec{j}$ is pure vector

That's all about tensors under O_3 and SO_3 groups!

1. 微分变换:

- 如果 $x \to x' = \omega x$, $\partial \to \partial' = \omega\partial$ （即在O_3 群中）
- 若 $x' = Lx$, 其中$L \neq 0$ 时:
  - $\partial/\partial x'_i = (\partial x_k/\partial x'_i) \cdot \partial/\partial x_k$
  - $x' = Lx \Rightarrow x = L^{-1}x'$，即 $x_k = L^{-1}_{ik}x'_i$
  - 因此 $\partial x'_i/\partial x_k = L^{-1}_{ik}$
  - 最终得到：当$x' = Lx$时，$\partial' = L^{-T}\partial$

2. O_3 群特性($L=\omega$):

- $\omega^T = \omega^{-1} \Rightarrow \partial' = \omega\partial$
- ∂是一个矢量（因为$\partial/\partial(-x) = -\partial/\partial x$，所以是纯矢量）
- 重要推论：所有用∂构造的量都是张量

3. Maxwell方程组中的张量性质:

- ρ是纯标量（物理假设）
- A和D是纯矢量
- Maxwell方程组的张量分析:
  - $\mathrm{div}\, E = 4\pi\rho$，$E_{i,i} = 4\pi\rho \Rightarrow E$是纯矢量
  - $\mathrm{rot}\, E + 1/c \cdot \partial B/\partial t = 0 \Rightarrow B$是赝矢量
  - $\mathrm{rot}\, B - 1/c \cdot \partial E/\partial t = 4\pi j \Rightarrow j$是纯矢量

这些内容揭示了:

1. 微分算符在坐标变换下的行为规律
2. O_3 群（三维正交群）中的特殊性质
3. 电磁场各物理量（E、B、j等）在坐标变换下的张量特性

这些知识对理解电磁场理论中的协变性和张量特性非常重要，它们确保了Maxwell方程组在不同参考系下的形式不变性。特别是区分了纯矢量和赝矢量的概念，这在电磁学中是非常基础的概念。

纯矢量和赝矢量的区别体现在它们在坐标变换（特别是反射变换）下的行为不同，这对理解电磁场的本质特性很重要。比如电场E是纯矢量，而磁场B是赝矢量，这反映了它们在物理本质上的区别。