

On Constructing Secure and Hardware-Efficient Invertible Mappings

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Abstract—Our society becomes increasingly dependent on wireless communications. The tremendous growth in the number and type of wirelessly connected devices in a combination with the dropping cost for performing cyberattacks create new challenges for assuring security of services and applications provided by the next generation of wireless communication networks. The situation is complicated even further by the fact that many end-point Internet of Things (IoT) devices have very limited resources for implementing security functionality. This paper addresses one of the aspects of this important, many-faceted problem - the design of hardware-efficient cryptographic primitives suitable for the protection of resource-constrained IoT devices. We focus on cryptographic primitives based on the invertible mappings of type $\{0, 1, \dots, 2^n - 1\} \rightarrow \{0, 1, \dots, 2^n - 1\}$. In order to check if a given mapping is invertible or not, we generally need an exponential in n number of steps. In this paper, we derive a sufficient condition for invertibility which can be checked in $O(n^2N)$ time, where N is the size of representation of the largest function in the mapping. Our results can be used for constructing cryptographically secure invertible mappings which can be efficiently implemented in hardware.

Index Terms—Invertible mapping, permutation, Boolean function, NLFSR

I. INTRODUCTION

This paper addresses the problem of constructing hardware-efficient cryptographic primitives suitable for assuring trustworthiness of resource-constrained devices used in services and applications provided by the next generation of wireless communication networks.

The importance of improving security of wireless networks for our society is hard to overestimate. In 2014, the annual loss to the global economy from cybercrimes was more than \$400 billion [1]. This number can quickly grow larger with the rapid growth of *Internet-of-Things* (IoT) applications. In coming years “things” such as household appliances, meters, sensors, vehicles, etc. are expected to be accessible and controlled via local networks or the Internet, opening an entirely new range of services appealing to users. The ideas of self-driving cars, health-tracking wearables, and remote surgeries do not sound like science-fiction any longer. The number of wirelessly connected devices is expected to grow to a few tens of billions in the next five years [2].

Unfortunately, the new technologies are appealing to the attackers as well. As processing power and connectivity become cheaper, the cost of performing a cyberattack drops, making it easier for adversaries of all types to penetrate networks. Attacks are becoming more frequent, more sophisticated, and more widespread. A connected household appliance becomes a target for all hackers around the globe unless appropriate security mechanisms are implemented. Household appliances typically do not have the same level of protection as computer systems. A compromised device can potentially be used as an entry point for a cyberattack on other devices connected to the network. The first proven cyberattack involving “smart” household appliances has been already reported in [3]. In this attack, more than 750.000 malicious emails targeting enterprises and individuals worldwide were sent from more than 100.000 consumer devices such

as home-networking routers, multi-media centers, TVs, refrigerators, etc. No more than 10 emails were initiated from any single IP-address, making the attack difficult to block based on location. The attack surface of future IoT with billions of connected devices will be enormous.

In addition to a larger attack surface, the return value for performing a cyberattack grows. The assets accessible via tomorrow’s networks (hardware, software, information, and revenue streams) are expected to be much greater than the ones available today, increasing incentive for cyber criminals and underground economies [4]. A growing black market for breached data serves as a further encouragement. The damage caused by an individual actor may not be limited to a business or reputation, but could have a severe impact on public safety, national economy, and/or national security.

The tremendous growth in the number and type of wirelessly connected devices, as well as the increased incentive for performing attacks change the threat landscape and create new challenges for security. The situation is complicated even further by the fact that many end-point IoT devices require utmost efficiency in the use of communication, computing, storage and energy resources. A typical IoT device spends most of its “life” in a sleep mode. It gets activated at periodic intervals, transmits a small amount of data and then shuts down again. To satisfy extreme limitations of resource-constrained IoT devices, very efficient cryptographic primitives for implementing encryption, data integrity protection, authentication, etc. are required.

Invertible mappings are among the most frequently used primitives in cryptography [5]. For example, the round function of a block cipher [6], [7] has to be invertible in order to result in unique decryption. Stream ciphers [8], [9] and hash functions [10], [11] use invertible state mappings to prevent incremental reduction of the entropy of the state.

This paper presents a sufficient condition for invertibility of mappings $x \rightarrow f(x)$ of type $\{0, 1, \dots, 2^n - 1\} \rightarrow \{0, 1, \dots, 2^n - 1\}$. Such a single-variable 2^n -valued mapping can be interpreted as an n -variable Boolean mapping $\{0, 1\}^n \rightarrow \{0, 1\}^n$ in which the Boolean variable $x_i \in \{0, 1\}$ represents the bit number i of the input x and the Boolean function $f_i : \{0, 1\}^n \rightarrow \{0, 1\}$ represents the bit number i of the output $f(x)$, i.e.:

$$\begin{pmatrix} x_0 \\ \dots \\ x_{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} f_0(x_0, \dots, x_{n-1}) \\ \dots \\ f_{n-1}(x_0, \dots, x_{n-1}) \end{pmatrix} \quad (1)$$

for $i \in \{0, 1, \dots, n-1\}$. For example, the 4-variable Boolean mapping

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \oplus 1 \\ x_1 \oplus x_0 \\ x_2 \oplus x_0 x_1 \\ x_3 \oplus x_0 x_1 x_2 \end{pmatrix}$$

corresponds to the single-variable 16-valued mapping

$$x \rightarrow x + 1 \pmod{16},$$

where “+” is addition modulo 16. Note that the corresponding single-variable 2^n -valued mapping may not have a closed form.

In order to check if a mapping of type (1) is invertible or not, we generally need an exponential in n number of steps. The condition derived in this paper can be checked in $O(n^2N)$ time, where N is the size of representation of the largest Boolean function in the mapping. So, it can be used for constructing secure invertible mappings with a small hardware implementation cost. For example, we show how the presented results can be used in stream cipher design.

The paper is organized as follows. Section II summarizes basic notations used in the sequel. In Section III, we describe previous work and show that previous methods cannot explain invertibility of some mappings which can be handled by the presented approach. Section IV presents the main result. Section V estimates the complexity of checking the presented condition. Section VI shows how the presented results can be used in stream cipher design. Section VII concludes the paper.

II. PRELIMINARIES

Throughout the paper, we use “ \oplus ” to denote the Boolean XOR, “ \cdot ” to denote the Boolean AND and \bar{x} to denote the Boolean complement of x , defined by $\bar{x} = 1 \oplus x$.

The Algebraic Normal Form (ANF) [12] of a Boolean function $f: \{0,1\}^n \rightarrow \{0,1\}$ (also called positive polarity *Reed-Muller canonical form* [13]) is a polynomial in the Galois Field of order 2, $GF(2)$, of type

$$f(x_0, x_1, \dots, x_{n-1}) = \sum_{i=0}^{2^n-1} c_i \cdot x_0^{i_0} \cdot x_1^{i_1} \cdot \dots \cdot x_{n-1}^{i_{n-1}},$$

where $c_i \in \{0,1\}$ are constants, “ \cdot ” is the Boolean AND, “ \sum ” is the Boolean XOR, the vector $(i_0 i_1 \dots i_{n-1})$ is the binary expansion of i , and $x_j^{i_j}$ denotes the i_j th power of x_j defined by $x_j^{i_j} = x_j$ for $i_j = 1$ and $x_j^{i_j} = 1$ otherwise, for $j \in \{0, 1, \dots, n-1\}$.

The *dependence set* (or *support set* [14]) of a Boolean function $f: \{0,1\}^n \rightarrow \{0,1\}$ is defined by

$$\text{dep}(f) = \{j \mid f|_{x_j=0} \neq f|_{x_j=1}\},$$

where $f|_{x_j=k} = f(x_0, \dots, x_{j-1}, k, x_{j+1}, \dots, x_{n-1})$ for $k \in \{0,1\}$.

A mapping $x \rightarrow f(x)$ on a finite set is called *invertible* if $f(x) = f(y)$ if, and only if, $x = y$. Invertible mappings are also called *permutations*.

III. PREVIOUS WORK

In this section, we describe previous work on invertible mappings and show that previous methods cannot explain invertibility of some mappings which can be handled by the presented approach.

Many methods for constructing different classes of invertible mappings are known. The simplest one is to compose simple invertible operations. A Substitution-Permutation Network (SPN) is a typical example. An SPN consists of S-boxes, which permute input bits locally, and P-boxes, which diffuse input bits globally. This method of construction cannot explain the invertibility of, for example, the following 4-variable mapping

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \oplus x_1 x_2 \\ x_0 \oplus x_3 \end{pmatrix} \quad (2)$$

since a non-invertible operation Boolean AND is used.

Feistel [15] proposed a powerful technique which makes possible constructing invertible mappings from non-invertible basic operations. It is used in many block ciphers, including DES [16]. The

basic Feistel construction maps two inputs, l and r , into two outputs as follows:

$$\begin{pmatrix} l \\ r \end{pmatrix} \rightarrow \begin{pmatrix} r \\ l \oplus f(r) \end{pmatrix}$$

where f is any single-variable function. The full Feistel construction iterates the above mapping any number of times. This method was extended in several directions, including *unbalanced*, *homogeneous*, *heterogeneous*, *incomplete*, and *inconsistent* Feistel networks [17]. However, the Feistel construction requires at least two variables. It cannot explain the invertibility of mappings $x \rightarrow f(x)$ of type $\{0, 1, \dots, 2^n - 1\} \rightarrow \{0, 1, \dots, 2^n - 1\}$. The presented method can explain it by looking into the structure of Boolean functions f_i representing the bits of the output $f(x)$.

Shamir [18] introduced an interesting construction based on the fact that the mapping of type (1) is invertible for almost all inputs if each f_i has the form:

$$f_i(x_0, \dots, x_{n-1}) = h_i(x_0, \dots, x_{i-1})x_i + g_i(x_0, \dots, x_{i-1}) \pmod{N}$$

where g_i and h_i are arbitrary non-zero i -variable polynomials modulo a large RSA modulus N . The “triangular” nature of functions f_i makes it possible to perform the inversion process similarly to the way we do Gaussian elimination to solve a system of linear equations. This approach can also handle g_i ’s and h_i ’s which mix arithmetic and Boolean operations [19], [20]. The approach presented in this paper is similar to the approach of Shamir in that it also uses triangulation to prove invertibility. However, our functions f_i are of the type

$$f_i(x_0, \dots, x_{n-1}) = x_j \oplus g_i(x_0, \dots, x_{n-1}), \quad (3)$$

where $j \in \{0, 1, \dots, n-1\}$ and g_i does not depend on x_j . Thus, in our case, the i th output may depend on the input j such that $j > i$. For example, in the mapping defined by equations (2), f_2 depends on x_1, x_2 and x_3 .

Shamir’s construction was further extended by Klimov and Shamir [19], [20] to a class on invertible mappings based on T-functions. A single-variable function $f(x)$ of type $\{0, 1, \dots, 2^n - 1\} \rightarrow \{0, 1, \dots, 2^n - 1\}$ is defined to be a T-function if each of its output bits $f_{i-1}(x)$ depends only on the first i input bits x_0, x_1, \dots, x_{i-1} :

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \dots \\ x_{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} f_0(x_0) \\ f_1(x_0, x_1) \\ f_2(x_0, x_1, x_2) \\ \dots \\ f_{n-1}(x_0, x_1, \dots, x_{n-1}) \end{pmatrix}.$$

A T-function is invertible if, and only if, each output bit f_i can be represented as

$$f_i(x_0, \dots, x_i) = x_i \oplus g_i(x_0, \dots, x_{i-1}).$$

This fundamental result inspired the construction which we present in this paper. The reader will easily notice that, in our construction (3), functions g_i ’s are T-functions while functions f_i ’s are not if $j > i$. Another difference is that Klimov and Shamir targeted software implementation and therefore focused on mappings whose expression can be evaluated by a program with the minimal number of instructions. We target the hardware implementation. Therefore, we are interested in minimizing the number of Boolean operations in the expressions of Boolean functions representing output bits. The construction method which we present can be used as a starting point for constructing nonlinear invertible mappings which have an efficient hardware implementation.

Another group of construction methods consider *permutation polynomials* which involve only the arithmetic operation of addition,

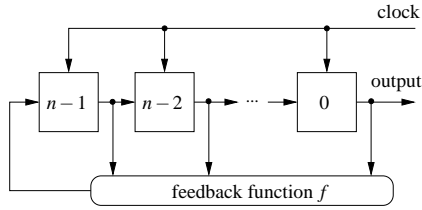


Fig. 1: The structure of an n -bit Non-Linear Feedback Shift Register.

subtraction, and multiplication. Permutation polynomials are well-studied in mathematics. Hermite [21] made a substantial progress in characterizing univariate permutation polynomials modulo a prime p . Dickson [22] described all univariate polynomials with degrees smaller than 5. However, the problem remains unsolved for high degree polynomials modulo a large prime p .

The problem appears simpler for the ring of integers modulo 2^n . Rivest [23] provided a complete characterization of all univariate permutation polynomials modulo 2^n . He proved that a polynomial

$$p(x) = a_0 + a_1x + \dots + a_dx^d$$

with integral coefficients is a permutation polynomial for $n > 2$ if, and only if, a_1 is odd, $(a_2 + a_4 + \dots)$ is even and $(a_3 + a_5 + \dots)$ is even. His powerful algebraic proof technique was further generalized by Klimov and Shamir [19], [20] to polynomials of type

$$p(x) = a_0 \circ a_1x \circ \dots \circ a_dx^d,$$

where $\circ \in \{+, -, \oplus\}$, which mix arithmetic and Boolean operations. However, since the resulting polynomials are T-functions, they do not cover the construction method presented in this paper for the case when i th output depends on the input j such that $j > i$.

Finally, we would like to discuss the relation between the mappings of type (1) and the state mappings generated by *Non-Linear Feedback Shift Registers* (NLFSRs) [24]. An n -bit NLFSR consists of n binary stages, each capable of storing 1 bit of information, a nonlinear Boolean function, called *feedback function*, and a clock (see Figure 1). At each clock cycle, the stage $n-1$ is updated to the value computed by the feedback function. The rest of the stages shift the content of the previous stage. Thus, an n -bit NLFSR with the feedback functions f implements the state mapping of type

$$\begin{pmatrix} x_0 \\ x_1 \\ \dots \\ x_{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ f(x_0, \dots, x_{n-1}) \end{pmatrix}.$$

where the variable x_i represents the value of the stage i , for $i \in \{0, 1, \dots, n-1\}$.

It is well-known [25] that an n -bit NLFSR is invertible if, and only if, its feedback function is of type

$$f(x_0, \dots, x_{n-1}) = x_0 \oplus g(x_1, x_2, \dots, x_{n-1}).$$

The mappings considered in this paper can be implemented by a more general type of non-linear state machines, shown in Figure 2. Since the content of stages is not any longer shifted from one stage to the next, but rather it is updated by some arbitrary functions, such registers are not called shift registers any longer. Instead, they are called *binary machines* [25] or *registers with non-linear update* [26]. Binary machines are typically smaller and faster than NLFSRs generating the same sequence [27], [28]. For example, the 4-bit NLFSR with the feedback function $f(x_0, x_1, x_2, x_3) = x_0 \oplus x_3 \oplus x_1x_2 \oplus x_2x_3$

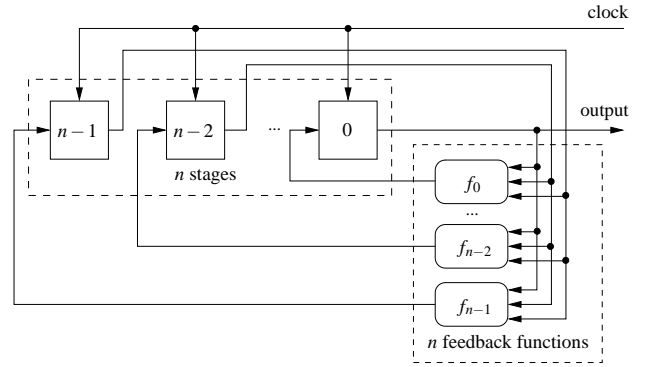


Fig. 2: The structure of an n -bit binary machine.

generates the same set of sequences as the 4-bit binary machine implementing the 4-variable mapping defined by equations (2). We can see that the binary machine uses 3 binary Boolean operations, while the NLFSR uses 5 binary Boolean operations. Furthermore, the depth of feedback functions of the binary machine is smaller than the depth of the feedback function of the NLFSR. Thus, the binary machine has a smaller propagation delay than the NLFSR.

IV. CONDITIONS FOR INVERTIBILITY

As we mentioned in Section III, it is well-known how to construct invertible NLFSRs [25]. An n -bit NLFSR is invertible if, and only if, its feedback function is of type:

$$f(x_0, x_1, \dots, x_{n-1}) = x_0 \oplus g(x_1, x_2, \dots, x_{n-1}). \quad (4)$$

The proof is simple, because every two consecutive states of an NLFSR overlap in $n-1$ positions. This implies that each state can have only two possible predecessors and two possible successors. If f is in the form (4), then the NLFSR states which correspond to the binary n -tuples $x = (x_0, x_1, \dots, x_{n-1})$ and $y = (\bar{x}_0, x_1, \dots, x_{n-1})$ always have different successors. The values of $f(x)$ and $f(y)$ depend on the value of $g(x_1, \dots, x_{n-1})$ and on the value of x_0 . The value of $g(x_1, \dots, x_{n-1})$ is the same for x and y . The value of x_0 is different for x and y . Thus, $f(x) \neq f(y)$. It is also easy to see that, if both x and y have the same successor, f cannot have the form (4).

In the general case of mappings $\{0, 1\}^n \rightarrow \{0, 1\}^n$, any binary n -tuple can have 2^n possible predecessors and 2^n possible successors. Therefore, to guarantee that a mapping $\{0, 1\}^n \rightarrow \{0, 1\}^n$ is invertible, we have to check that, for all $x, y \in \{0, 1\}^n$, $x \neq y$ implies that $f_i(x) \neq f_i(y)$, for some $i \in \{0, 1, \dots, n-1\}$. This clearly requires the number of steps which is exponential in n . The main contribution of this paper is a more restricted sufficient condition which can be checked in $O(n^2N)$ steps. To formulate this condition, we first introduce the notion of a free variable of a function.

Definition 1 (Free variable): A variable $x_i \in \text{dep}(f)$ is called a free variable if f can be decomposed as

$$f(x_0, x_1, \dots, x_{n-1}) = x_i \oplus g(x_0, x_1, \dots, x_{n-1}).$$

where $x_i \notin \text{dep}(g_i)$.

Definition 2 (Set of free variables): The set of free variables of f , $\Phi(f)$, contains all free variables of f

$$\Phi(f) = \{x_i \mid x_i \text{ is a free variable of } f\}.$$

Now we are ready to formulate the following sufficient condition for invertibility.

Theorem 1: A mapping of type (1) is invertible if the following two conditions hold:

- 1) For each $i \in \{0, 1, \dots, n-1\}$, f_i is of type

$$f_i(x_0, x_1, \dots, x_{n-1}) = x_{j_i} \oplus g_i(x_0, x_1, \dots, x_{n-1}), \quad (5)$$

where $x_{j_i} \in \Phi(f_i)$.

- 2) Functions f_0, f_1, \dots, f_{n-1} can be re-ordered as $f_{i_0}, f_{i_1}, \dots, f_{i_{n-1}}$ to satisfy the property

$$\text{dep}(g_{i_k}) \subseteq \begin{cases} \emptyset, & \text{for } k = 0 \\ \bigcup_{j=0}^{k-1} \text{dep}(f_{i_j}), & \text{for } 1 \leq k \leq n-1 \end{cases}$$

where “ \cup ” stands for the union.

Proof: By contradiction.

Suppose that there exist a non-invertible mapping of type (1) for which the conditions of the theorem hold. Then, this mapping is of type

$$\begin{pmatrix} x_{i_0} \\ x_{i_1} \\ x_{i_2} \\ \dots \\ x_{i_{n-1}} \end{pmatrix} \rightarrow \begin{pmatrix} x_{j_0} \oplus g_{i_0} \\ x_{j_1} \oplus g_{i_1}(x_{j_0}) \\ x_{j_2} \oplus g_{i_2}(x_{j_0}, x_{j_1}) \\ \dots \\ x_{j_{n-1}} \oplus g_{i_{n-1}}(x_{j_0}, x_{j_1}, \dots, x_{j_{n-2}}) \end{pmatrix} \quad (6)$$

where $x_{j_k} \in \Phi(f_{i_k})$, for $k \in \{0, 1, \dots, n-1\}$.

Since the mapping (6) is non-invertible, there exist at least two input assignments $a = (a_{i_0}, a_{i_1}, \dots, a_{i_{n-1}}) \in \{0, 1\}^n$ and $a' = (a'_{i_0}, a'_{i_1}, \dots, a'_{i_{n-1}}) \in \{0, 1\}^n$, $a \neq a'$, which are mapped into the same output. We analyzing (6) row by row, we can conclude that:

- $a_{i_1} \rightarrow a_{j_1} \oplus g_{i_0}$ and $a'_{i_1} \rightarrow a'_{j_1} \oplus g_{i_0}$ and $a_{j_0} \oplus g_{i_0} = a'_{j_0} \oplus g_{i_0}$ implies $a_{j_0} = a'_{j_0}$
- $a_{i_1} \rightarrow a_{j_1} \oplus g_{i_1}(a_{j_0})$ and $a'_{i_1} \rightarrow a'_{j_1} \oplus g_{i_1}(a'_{j_0})$ and $a_{j_1} \oplus g_{i_1}(a_{j_0}) = a'_{j_1} \oplus g_{i_1}(a'_{j_0})$ and $a_{j_0} = a'_{j_0}$ implies $a_{j_1} = a'_{j_1}$
- ...
- $a_{i_{n-1}} \rightarrow a_{j_{n-1}} \oplus g_{i_{n-1}}(a_{j_0}, a_{j_1}, \dots, a_{j_{n-2}})$ and $a'_{i_{n-1}} \rightarrow a'_{j_{n-1}} \oplus g_{i_{n-1}}(a'_{j_0}, a'_{j_1}, \dots, a'_{j_{n-2}})$ and $a_{j_{n-1}} \oplus g_{i_{n-1}}(a_{j_0}, a_{j_1}, \dots, a_{j_{n-2}}) = a'_{j_{n-1}} \oplus g_{i_{n-1}}(a'_{j_0}, a'_{j_1}, \dots, a'_{j_{n-2}})$ and $a_{j_0} = a'_{j_0}$, $a_{j_1} = a'_{j_1}$, ... $a_{j_{n-2}} = a'_{j_{n-2}}$ implies $a_{j_{n-1}} = a'_{j_{n-1}}$.

Therefore, $a = a'$. We reached a contradiction. Thus, the assumption that there exist a non-invertible mapping of type (1) for which the conditions of the theorem hold is not true. \square

An example of mapping which satisfies the conditions of Theorem 1 is:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \dots \\ x_{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \oplus g_0 \\ x_2 \oplus g_1(x_1) \\ x_3 \oplus g_2(x_1, x_2) \\ \dots \\ x_0 \oplus g_{n-1}(x_1, x_2, \dots, x_{n-1}) \end{pmatrix} \quad (7)$$

The reader may notice that mappings defined by Theorem 1 can be obtained by re-labeling variables in a T-function. The re-labeling is given by the ordering of free variables. So, the mapping (5) can be viewed as a composition of a bit permutation and a T-function. Since there are $n!$ bit permutations, the class of invertible mappings considered in this paper is by a factor of $n!$ larger than the class of invertible mappings based on T-functions.

Clearly, the re-labeling of variables does not change the implementation cost of a mapping. However, it might drastically change the cycle structure of the underlying state transition graph, as illustrated

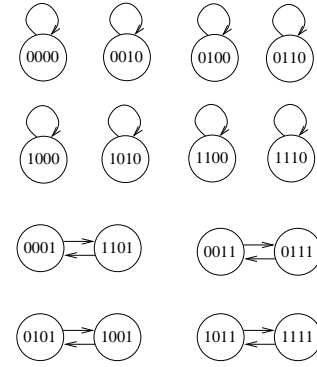


Fig. 3: The state transition graph of the mapping (8).

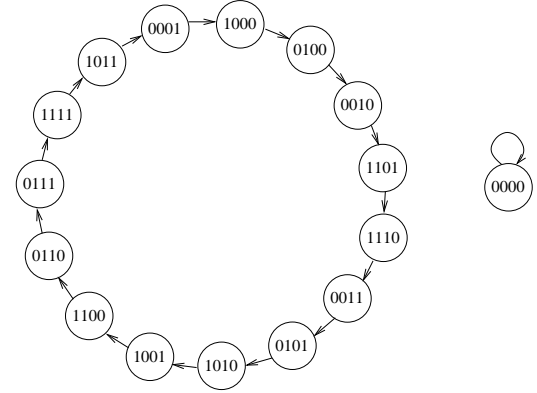


Fig. 4: The state transition graph of the mapping (9).

by the following example. Consider the following 4-variable mapping based on a T-function:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ x_1 \\ x_2 \oplus x_0 \\ x_3 \oplus x_0 \oplus x_0 x_1 \end{pmatrix}. \quad (8)$$

This mapping has a quite uninteresting state transition graph shown in Figure 3. Let us re-label the variables as $(0, 1, 2, 3) \rightarrow (1, 2, 3, 0)$. We get the mapping:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \oplus x_1 \\ x_0 \oplus x_1 \oplus x_1 x_2 \end{pmatrix}. \quad (9)$$

which has the state transition graph shown in Figure 4. All states, except the all-0 state, are included in one cycle. If we implement this mapping by the binary machine shown in Figure 2 and initialize the binary machine to any non-zero state, then its output generates a pseudo-random sequence with period 15 which satisfies the first two randomness postulates of Golomb [25], namely the balance property and the run property. No sequence generated by a nonlinear Boolean function satisfy the third randomness postulate (two-level autocorrelation property). Therefore, the mapping we obtained by re-labeling has much more interesting statistical properties than the original mapping.

The cycle structure of a mapping is important for some cryptographic applications, e.g. stream ciphers [29]. Obviously, if we iterate a mapping a large number of times, we do not want the sequence of

generated states to be trapped in a short cycle. It is worth noting that formal verification tools based on reachability analysis can be adopted for finding short cycles in a state transition graph. There are dedicated tools to compute the number and length of cycles in state transition graphs of synchronous sequential networks, e.g. BNS [30] which is based on SAT-based bounded model checking and BooleNet [31] which is based on Binary Decision Diagrams.

The mappings defined by Theorem 1 are invertible for any choice of Boolean functions $g_i(x_0, x_2, \dots, x_{i-1})$. The smaller is the number of Boolean operations in the expressions of g_i 's, the smaller is its hardware implementation cost. Note, that we are not restricted to represent g_i 's in ANF. Any Boolean expression combining AND, OR, NOT and XOR can be used. Multi-level logic optimization tools, such as UC Berkeley tool ABC [32] can be applied to transform the ANF or other Boolean expression representing the function into an optimized multi-level expression. Clearly, the choice of g_i 's will be guided not only by the hardware cost, but also by other criteria determining the cryptographic strength of the resulting function, e.g. nonlinearity, correlation immunity, algebraic degree, etc. (see [12] for requirements on cryptographically strong functions).

Finally, we would like to point out that the conditions of Theorem 1 are sufficient, but not necessary conditions for invertibility. For example, the following 4-variable mapping is invertible, but it does not satisfy them:

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \rightarrow \begin{pmatrix} x_1 \oplus x_0 \oplus x_0 x_2 \\ x_2 \oplus x_1 \oplus x_3 \\ x_3 \oplus x_0 x_2 \\ x_0 \end{pmatrix}.$$

V. CONDITION CHECKING

Next, let us estimate the number of steps required to check the conditions of Theorem 1. Suppose that all Boolean functions of the mapping are represented in ANF. Let N be the ANF size of the largest function in the mapping. The *size* of an ANF is defined as the total number of variables appearing in the ANF. For example, the ANF $x_1 \oplus x_1 x_2$ is of size 3.

Consider the pseudocode shown as Algorithm 1. Φ is a set which keeps track of variables which are identified as free for some f_i . If this set is implemented as a hash table of size n , then adding a variable to the set or checking if a variable belongs to the set can be done in constant time.

In the first **for**-loop, we check if each f_i is of type $f_i = x_j$ or $f_i = x_j \oplus 1$, for some $j \in \{0, 1, \dots, n-1\}$. If yes, we add x_j to the set Φ and mark f_i . Since the steps 4, 5, 6 and 7 can be done in $O(1)$ time, the complexity of the first **for**-loop is $O(n)$.

If none of the functions f_i are marked during the first **for**-loop, the algorithm terminates with the conclusion that the conditions of Theorem 1 are not satisfied.

In the second **for**-loop, for each non-marked f_i , we check if it is of type (5) and if every $x_k \in \text{dep}(g_i)$ belongs to Φ . If yes, add x_j to Φ , mark f_i , and return to step 13. The steps 15, 17 and 18 can be done in $O(1)$ time. The step 16 requires $O(N)$ time. Thus, the complexity of the second **for**-loop is $O(nN)$. Since we return to the step 13 at most n times, the overall complexity of steps 13-22 is $O(n^2 N)$.

In the third **for**-loop, we check if all f_i are marked. If yes, the mapping is invertible. Otherwise, the algorithm returns "conditions are not satisfied". Since the conditions of Theorem 1 are sufficient, but not necessary conditions for invertibility, in this case the mapping may or may not be invertible. The complexity of the third **for**-loop is $O(n)$. We can conclude that overall complexity of Algorithm 1 is $O(n^2 N)$.

Algorithm 1 Checks conditions of Theorem 1 for a mapping of type (1) in which all Boolean functions are represented in ANF

```

1:  $\Phi = \emptyset$ ;
2: flag = 0;
3: for every  $i$  from 0 to  $n-1$  do
4:   if  $f_i = x_j$  or  $f_i = x_j \oplus 1$ , for some  $j \in \{0, 1, \dots, n-1\}$  then
5:     add  $x_j$  to  $\Phi$ 
6:     mark  $f_i$ 
7:     flag = 1;
8:   end if
9: end for
10: if flag = 0 then
11:   Return "Conditions are not satisfied"
12: end if
13: loop:
14: for every  $i$  from 0 to  $n-1$  do
15:   if  $f_i$  is not marked then
16:     if  $f_i$  is of type (5) and  $x_k \in \Phi \ \forall x_k \in \text{dep}(g_i)$  then
17:       add  $x_j$  to  $\Phi$ 
18:       mark  $f_i$ 
19:       go to loop
20:     end if
21:   end if
22: end for
23: for every  $i$  from 0 to  $n-1$  do
24:   if  $f_i$  is marked then
25:     continue
26:   else
27:     Return "Conditions are not satisfied"
28:   end if
29: end for
30: Return "Mapping is invertible"

```

VI. APPLICATIONS

In this section we show how the presented results can be used in stream cipher design.

A possible way to construct a key stream generator for a stream cipher is to run several FSRs in parallel and to combine their outputs with a nonlinear Boolean function [12]. The resulting structure is called a *combination generator*. If the periods of FSRs are pairwise co-prime, then the period of the resulting key stream generator is equal to the product of periods of FRSs [33]. Examples of stream ciphers based on combination generators are VEST [34], Achterbahn-128/80 [35], and the cipher [36]. VEST uses 16 10-bit and 16 11-bit NLFSRs. Achterbahn-128/80 uses 13 NLFSRs of size from 21 to 33 bits. The cipher from [36] uses 10 NLFSRs of size 22-29, 31 and 32 bits.

At present it is not known how to construct large NLFSRs with a guaranteed long period. Existing algorithms cover special cases only, e.g. [37], [38]. Small NLFSRs which are used in combination generators are computed by a random search. Lists of n -bit NLFSRs of size < 25 bits with the period $2^n - 1$ whose feedback functions contain up to 6 binary Boolean operations in their ANFs are available in [39]. It is known that, for example, there are no 20-bit NLFSRs with the period $2^{20} - 1$ whose feedback function contains only four binary Boolean operations in its ANF (i.e. implementable with four 2-input gates).

Using Algorithm 1 to bound random search, in 12 hours we were able to find 63 20-variable nonlinear invertible mappings with the

period $2^{20} - 1$ whose functions f_i contain no more than 4 binary Boolean operations in their ANFs in total. Two representatives are:

- 1) $f_{16} = x_{17} \oplus x_4 x_{11}$, $f_{13} = x_{14} \oplus x_{13}$, $f_4 = x_5 \oplus x_1$.
- 2) $f_{19} = x_0 \oplus x_{16}$, $f_{15} = x_{16} \oplus x_3 \oplus x_1 x_{15}$.

The omitted functions are of type $f_i = x_{(i+1) \bmod 20}$, for $i \in \{0, 1, \dots, 19\}$. This shows that Algorithm 1 is useful for finding nonlinear invertible mappings with a small hardware implementation cost. Other important properties, such as nonlinearity, correlation immunity, algebraic degree, etc., can then be used to further guide the search.

VII. CONCLUSION

We derived a sufficient condition for invertibility of mappings of type $\{0, 1, \dots, 2^n - 1\} \rightarrow \{0, 1, \dots, 2^n - 1\}$ which can be checked in $O(n^2 N)$ steps, where N is the ANF size of the largest Boolean function in the mapping. The presented method can be used as a starting point for constructing nonlinear invertible mappings which can be efficiently implemented in hardware.

Future work remains on constructing new cryptographic primitives based on the presented class of invertible mappings and evaluating their security and hardware cost. We also plan to investigate the usability our results in reversible computing.

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