수학으로부터 인류를 자유롭게 하라

Free Humankind from Mathematics

Basic Algebra

다항식

Chap.4 Polynomial Expressions/Equations

1. linear
2. quadratic
3. cubic

4.1 Linear Equations

Linear Equations

linear expression

$$\underbrace{ax+b}=0$$

x : variable

a, b : constants

The equation is a condition

True가 될수도,
false가 될수도

"LINEAR"

1. $ax+b$
2. D.B
3. homogeneity
additivity

$f(ax+by)=af(x)+bf(y)$

ex.1) $a=0,\ b=5 \longrightarrow 5=0$

ex.2) $a=1,\ b=0 \longrightarrow x=0$

ex.3) $a=1,\ b=5 \longrightarrow x+5=0$

ex.4) $a=3,\ b=0 \longrightarrow 3x=0$

ex.5) $a=2,\ b=5 \longrightarrow 2x+5=0$

ex.6) $a=-3,\ b=-1 \longrightarrow -3x-1=0$

Solutions of Linear Equations

Object

주어진 linear equation을 참으로 만드는 x값을 구하기 위해

- additive inverse
- multiplicative inverse

를 이용하여 방정식을 $x = \alpha$으로 만들기

$$ax + b = 0 \longrightarrow x = \alpha$$

Solutions of Linear Equations

Case.1 $ax + b = 0$ $\rightarrow$ $x = \alpha$

$$ax + b = 0 \xrightarrow[\text{additive inverse}]{} ax + b + (-b) = -b \longrightarrow ax = -b$$

$$\xrightarrow[\substack{\text{multiplicative inverse} \\ a \neq 0}]{} ax \cdot \left(\frac{1}{a}\right) = -b \cdot \left(\frac{1}{a}\right) \longrightarrow x = -\frac{b}{a}$$

if $a = 0$

$ax + b = 0 \longrightarrow b = 0$

if $b = 0 \longrightarrow S = \{x \mid x \in \mathbb{R}\}$

if $b \neq 0 \longrightarrow S = \varnothing$

예) $2 = 0$ $\rightarrow$ 항상 false

Solutions of Linear Equations

Case.2 $ax + b = cx + d$

$$ax + b = cx + d \xrightarrow[\text{additive inverse}(cx)]{} ax + b + (-cx) = cx + d + (-cx) \longrightarrow ax - cx + b = d$$

$$\xrightarrow[\text{additive inverse}(b)]{} ax - cx + b + (-b) = d + (-b) \longrightarrow ax - cx = d - b$$

$$\xrightarrow[\text{distributivity}]{} ax - cx = d - b \longrightarrow (a - c) \cdot x = d - b$$

$$\xrightarrow[\substack{\text{multiplicative inverse} \\ a \neq c}]{} (a - c) \cdot x = d - b \longrightarrow x = \frac{d - b}{a - c}$$

if $\boxed{a = c}$

$ax + b = cx + d \longrightarrow b = d$

if $b = d \longrightarrow S = \{x \mid x \in \mathbb{R}\}$ $\mathbb{R}$

if $b \neq d \longrightarrow S = \varnothing$

empty set

$\boxed{ax+b=0}$

Case 1) $S = \mathbb{R}$

Case 2) $S = \varnothing$

Case 3) $S = \{\alpha\}$

Solutions of Linear Equations

ex.1) $x + 2 = 0 \longrightarrow S = \{-2\}$

ex.2) $3x = 6 \longrightarrow S = \{2\}$

ex.3) $2x + 3 = 0 \longrightarrow S = \{-3/2\}$

ex.4) $3x - 2 = 0 \longrightarrow S = \{2/3\}$

ex.5) $2x + 3 = x - 2 \longrightarrow S = \{-5\}$

ex.6) $2x + 3 = 2x + 3 \longrightarrow S = \{x \mid x \in \mathbb{R}\}$

ex.7) $2x + 3 = 2x - 4 \longrightarrow S = \varnothing$

Linear Inequalities

$$ax + b \le 0 \qquad ax + b \ge 0$$
$$ax + b < 0 \qquad ax + b > 0$$

ex.1) $x \le 0, \quad x \ge 0$
$x < 0, \quad x > 0$

ex.2) $x + 3 \le 0, \quad x + 3 \ge 0$
$x + 3 < 0, \quad x + 3 > 0$

ex.3) $2x + 3 \le 0, \quad 2x + 3 \ge 0$
$2x + 3 < 0, \quad 2x + 3 > 0$

Solutions of Linear Inequalities

$$(\text{LHS}) \quad \begin{matrix} \leq & \quad & \geq \\ < & \quad & > \end{matrix} \quad (\text{RHS})$$

Additions and Inequalities

$$(\text{LHS}) < (\text{RHS}) \longrightarrow (\text{LHS}) + \alpha < (\text{RHS}) + \alpha$$

Multiplications and Inequalities

$$2 < 3 \longrightarrow \begin{cases} 2 \cdot 3 < 3 \cdot 3 \\ 2 \cdot (-3) > 3 \cdot (-3) \end{cases}$$

$$(\text{LHS}) < (\text{RHS}) \longrightarrow (\text{LHS}) \cdot \alpha < (\text{RHS}) \cdot \alpha, \text{ if } \alpha > 0$$

$$(\text{LHS}) < (\text{RHS}) \longrightarrow (\text{LHS}) \cdot \alpha > (\text{RHS}) \cdot \alpha, \text{ if } \alpha < 0$$

음수를 곱하면

부호 바꾸기.

Solutions of Linear Inequalities

Case.1 $ax+b<0$

$$ax+b<0 \xrightarrow[\text{additive inverse}]{} ax+b+(-b)<-b \longrightarrow ax<-b$$

$$\xrightarrow[\substack{\text{(1) multiplicative inverse} \\ a>0}]{} ax\cdot\left(\frac{1}{a}\right)<-b\cdot\left(\frac{1}{a}\right) \longrightarrow x<-\frac{b}{a}$$

$$\xrightarrow[\substack{\text{(2) multiplicative inverse} \\ a<0}]{} ax\cdot\left(\frac{1}{a}\right)>-b\cdot\left(\frac{1}{a}\right) \longrightarrow x>-\frac{b}{a}$$

if $a=0$

$ax+b<0 \longrightarrow b<0$

if $b<0 \longrightarrow S=\{x \mid x\in\mathbb{R}\}$

if $b\geq 0 \longrightarrow S=\varnothing$

Solutions of Linear Inequalities

Case.2 $ax + b < cx + d$

$$ax + b < cx + d \xrightarrow[\text{additive inverse}(cx)]{} ax + b + (-cx) < cx + d + (-cx) \longrightarrow ax - cx + b < d$$

$$\xrightarrow[\text{additive inverse}(b)]{} ax - cx + b + (-b) < d + (-b) \longrightarrow ax - cx < d - b$$

$$\xrightarrow[\text{distributivity}]{} ax - cx < d - b \longrightarrow (a-c)\cdot x < d - b$$

$$\xrightarrow[\substack{\text{(1)multiplicative inverse} \\ a > c}]{} (a-c)\cdot x < d - b \longrightarrow x < \frac{d-b}{a-c}$$

$$\xrightarrow[\substack{\text{(2)multiplicative inverse} \\ a < c}]{} (a-c)\cdot x < d - b \longrightarrow x > \frac{d-b}{a-c}$$

if $a = c$

$ax + b < cx + d \longrightarrow b < d$

if $b < d \longrightarrow S = \{x \mid x \in \mathbb{R}\}$

if $b \geq d \longrightarrow S = \varnothing$

Solutions of Linear Inequalities

ex.1) $2x + 4 > 0 \longrightarrow S = \{x \mid x > -2\}$

$2x + 4 \geq 0 \longrightarrow S = \{x \mid x \geq -2\}$

$2x + 4 < 0 \longrightarrow S = \{x \mid x < -2\}$

$2x + 4 \leq 0 \longrightarrow S = \{x \mid x \leq -2\}$

ex.2) $-2x + 4 > 0 \longrightarrow S = \{x \mid x < 2\}$

$-2x + 4 \geq 0 \longrightarrow S = \{x \mid x \leq 2\}$

$-2x + 4 < 0 \longrightarrow S = \{x \mid x > 2\}$

$-2x + 4 \leq 0 \longrightarrow S = \{x \mid x \geq 2\}$

$\overset{2}{\cancel{4}} > \cancel{2}x$

$2 > x$

$x < 2$

$\cancel{4} \leq \cancel{2}x$

$2 \leq x$

$x \geq 2$

Powers

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ times}}$$

x^a

exponent $\longrightarrow$ constant

base $\longrightarrow$ variable

ex.1) $2^2 = 4, \quad 3^2 = 9, \quad 4^2 = 16, \quad 5^2 = 25$

ex.2) $2^3 = 8, \quad 3^3 = 27, \quad 4^3 = 64, \quad 5^3 = 125,$

ex.3) $2^4 = 16, \quad 3^4 = 81, \quad 4^4 = 256, \quad 5^4 = 625$

Decomposing Numbers

$$10^0 = 1, \quad 10^1 = 10, \quad 10^2 = 100, \quad 10^3 = 1000,$$

$$\begin{aligned} a_3a_2a_1a_0 &= a_3 \cdot 1000 + a_2 \cdot 100 + a_1 \cdot 10 + a_0 \cdot 1 \\ &= a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10^1 + a_0 \cdot 10^0 \end{aligned}$$

ex.1) $123 = 10^2 + 2 \cdot 10^1 + 3 \cdot 10^0$

ex.2) $3002 = 3 \cdot 10^3 + 2 \cdot 10^0$

$2345 = 2000 + 300 + 40 + 5$

$2 \times 10^3 + 3 \times 10^2 + 4 \times 10^1 + 5 \times 10^0$

Special Exponents and Bases

Zero Exponents

$$x^0 = 1,\ x \neq 0$$

$0^0 = 1$ in C.S.

(1) $1^0 = 1$ (2) $2^0 = 1$ (3) $100^0 = 1$ (4) $(-2)^0 = 1$ (5) $(-100)^0 = 1$

Unit Exponents

$$x^1 = x$$

(1) $1^1 = 1$ (2) $2^1 = 2$ (3) $100^1 = 100$ (4) $(-2)^1 = -2$ (5) $(-100)^1 = -100$

Special Exponents and Bases

부호를 바꿔주는 연산

-1 Bases

짝수 홀

$$(-1)^{2n} = 1, \quad (-1)^{2n-1} = -1$$

(1) $(-1)^0 = 1$ (2) $(-1)^1 = -1$ (3) $(-1)^2 = 1$ (4) $(-1)^3 = -1$ (5) $(-1)^4 = 1$

0 Bases

$$0^n = 0$$

(1) $0^1 = 0$ (2) $0^2 = 0$ (3) $0^3 = 0$ (4) $0^4 = 0$

Property of Powers

$$(x \cdot y)^n = \underbrace{(x \cdot y) \cdot (x \cdot y) \cdot \ldots \cdot (x \cdot y)}_{n \text{ times}}$$

$$= \underbrace{(x \cdot x \cdot \ldots \cdot x)}_{n \text{ times}} \cdot \underbrace{(y \cdot y \cdot \ldots \cdot y)}_{n \text{ times}} \quad \text{by commutativity}$$

$$= x^n \cdot y^n$$

ex.1) $6^3 = 216$
$= (2 \cdot 3)^3 = 2^3 \cdot 3^3 = 8 \cdot 27 = 216$

ex.2) $20^4 = (2 \cdot 10)^4 = 2^4 \cdot 10^4$
$= 16 \cdot 10000 = 160000$

ex.3) $(-2)^2 = (-1 \cdot 2)^2 = (-1)^2 \cdot 2^2 = 4$
$(-2)^3 = (-1 \cdot 2)^3 = (-1)^3 \cdot 2^3 = -8$
$(-2)^4 = (-1 \cdot 2)^4 = (-1)^4 \cdot 2^4 = 16$
$(-2)^5 = (-1 \cdot 2)^5 = (-1)^5 \cdot 2^5 = -32$

Quadratic Expressions

$$ax^2 + bx + c$$

x : variable

a, b, c : constants

$(-x^2) = (-1)\cdot(x)^2$

coefficients(계수): 변수에 대한 곱셈인자

a : x^2 의 계수　b : x의 계수　c : 상수항

ex) $2x^2 + x - 1$에서 계수와 상수항을 구하고, x가 $-2,\ -1,\ 0,\ 1,\ 2$일때의 값을 구하세요.

coeff. of x^2 / x : 2 / 1,　constant: -1

(1) $x = -2 \longrightarrow 2\cdot(-2)^2 - 2 - 1 = 5$

(2) $x = -1 \longrightarrow 2\cdot(-1)^2 - 1 - 1 = 0$

(3) $x = 0 \longrightarrow 2\cdot(0)^2 + 0 - 1 = -1$

(4) $x = 1 \longrightarrow 2\cdot(1)^2 + 1 - 1 = 2$

(5) $x = 2 \longrightarrow 2\cdot(2)^2 + 2 - 1 = -9$

Multiplication Rules of Quadratic Expressions

- $(a+b)^2 = a^2 + 2ab + b^2$

$$\begin{aligned} \underbrace{(a+b)}_{\alpha} \cdot (a+b) &= \alpha \cdot (a+b) \\ &= \alpha \cdot a + \alpha \cdot b \quad \text{by distributivity} \\ &= (a+b) \cdot a + (a+b) \cdot b \\ &= a^2 + ba + ab + b^2 \quad \text{by distributivity} \\ &= a^2 + ab + ab + b^2 \quad \text{by commutativity} \\ &= a^2 + 2ab + b^2 \end{aligned}$$

$$(a+b)(a-b)$$

- $(a-b)^2 = a^2 - 2ab + b^2$

Multiplication Rules of Quadratic Expressions

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$

- $(a+b)(a-b) = a^2 - b^2$

  $(a+b)(a-b) = a^2 - \cancel{ab} + \cancel{ba} - b^2$

variable

- $(x + a)^2 = x^2 + 2ax + a^2$
- $(x+a)(x+b) = x^2 + (a+b)x + ab$

  $(x+a)(x+b) = x^2 + xb + ax + ab$

  $= x^2 + bx + ax + ab$ by commutativity

  $= x^2 + (a+b)x + ab$ by distributivity

Multiplication Rules of Quadratic Expressions

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$
- $(a+b)(a-b) = a^2 - b^2$
- $(x+a)^2 = x^2 + 2ax + a^2$
- $(x+a)(x+b) = x^2 + (a+b)x + ab$
- $(ax+b)(cx+d) = acx + (ad+bc)x + bd$

$$
\begin{aligned}
(ax+b)(cx+d) &= a \cdot \left(x + \frac{b}{a}\right) \cdot c \cdot \left(x + \frac{d}{c}\right) \\
&= ac \cdot \left(x + \frac{b}{a}\right) \cdot \left(x + \frac{d}{c}\right) \\
&= ac \cdot \left(x^2 + \left(\frac{b}{a} + \frac{d}{c}\right)x + \frac{bd}{ac}\right) \\
&= ac \cdot x^2 + (bc + ad) \cdot x + bd
\end{aligned}
$$

Multiplication Rules of Quadratic Expressions

- $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

$$(a+b+c)(a+b+c)$$

$$\begin{aligned}(a+b+c)(a+b+c) &= a(a+b+c)+b(a+b+c)+c(a+b+c)\\ &= (a^2+ab+ac)+(ba+b^2+bc)+(ca+cb+c^2)\\ &= a^2+b^2+c^2+2ab+2bc+2ca\end{aligned}$$

Multiplication Rules of Quadratic Expressions

Review

- $(a+b)^2 = a^2 + 2ab + b^2$
- $(a-b)^2 = a^2 - 2ab + b^2$

- $(a+b)(a-b) = a^2 - b^2$

- $(x+a)^2 = x^2 + 2ax + a^2$
- $(x+a)(x+b) = x^2 + (a+b)x + ab$
- $(ax+b)(cx+d) = acx + (ad+bc)x + bd$

- $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

Multiplication Rules of Quadratic Expressions

Examples

ex.1) $a(a^2b - ab + 3) = a^3b - a^2b + 3a$

ex.2) $ab(a^2 - ab + 4) = a^3b - a^2b^2 + 4ab$

ex.3) $(x+1)(x+2) = x^2 + 3x + 2$

ex.4) $(x+4)(x-2) = x^2 + 2x - 8$

ex.5) $(x+10)(x-5) = x^2 + 5x - 50$

ex.6) $(x+a)\left(x - \frac{1}{a}\right) = x^2 + \left(a - \frac{1}{a}\right)x - 1$

ex.7) $(x+4)(x-4) = x^2 - 16$

ex.8) $(x+1)(x-1) = x^2 - 1$

Factorizations of Quadratic Expressions

- $x^2 + 2ax + a^2 = (x+a)^2$
- $x^2 - 2ax + a^2 = (x-a)^2$

ex.1) $x^2 + 2x + 1 = (x+1)^2$

ex.2) $x^2 - 2x + 1 = (x-1)^2$

ex.3) $x^2 + 4x + 4 = (x+2)^2$

ex.4) $x^2 - 4x + 4 = (x-2)^2$

- $x^2 - a^2 = (x+a)(x-a)$

ex.1) $x^2 - 1 = (x+1)(x-1)$

ex.2) $x^2 - 9 = (x+3)(x-3)$

ex.3) $x^2 - 25 = (x+5)(x-5)$

Factorizations of Quadratic Expressions

- $x^2 + (a+b)x + ab = (x+a)(x+b)$

ex.1) $x^2 + 3x + 2 = (x+1)(x+2)$

ex.2) $x^2 + 7x + 12 = (x+3)(x+4)$

ex.3) $x^2 + 8x + 15 = (x+3)(x+5)$

ex.4) $2x^2 + 6x + 4 = 2(x^2 + 3x + 2) = 2(x+1)(x+2)$

ex.5) $x^2 + 2x - 3 = (x+3)(x-1)$

ex.6) $x^2 - 2x - 3 = (x+1)(x-3)$

ex.7) $x^2 + x - 12 = (x+4)(x-3)$

ex.8) $x^2 + 2x - 15 = (x+5)(x-3)$

ex.9) $x^2 - 4x + 3 = (x-1)(x-3)$

ex.10) $x^2 - 8x + 15 = (x-3)(x-5)$

Factorizations of Quadratic Expressions

- $x^2 + 2ax + a^2 = (x + a)^2$
- $x^2 - 2ax + a^2 = (x - a)^2$

- $x^2 - a^2 = (x + a)(x - a)$

- $x^2 + (a + b)x + ab = (x + a)(x + b)$

Two Representations of Quadratic Expressions

General Form(일반형)

$$ax^2 + bx + c$$

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x\right)+c \\ &= a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c \\ &= a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right)+c-\frac{b^2}{4a} \\ &= a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a} \\ &= a\left(x-\left(-\frac{b}{2a}\right)\right)^2+c-\frac{b^2}{4a} \end{aligned}$$

Standard Form(표준형) =Vertex Form

vertex: 꼭지점

$$\alpha(x-p)^2+q$$

if $a=1 \longrightarrow x^2+bx+c$

$$\begin{aligned} x^2+bx+c &= x^2+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c \\ &= \left(x+\frac{b}{2}\right)^2+c-\frac{b^2}{4} \end{aligned}$$

Two Representations of Quadratic Expressions

Examples 다음의 quadratic expression들을 vertex form으로 바꾸고, x가 $-2,\ 0,\ 2$일 때의 값을 구하세요.

ex.1) x^2+4x+9

$$\begin{aligned} x^2+4x+9 &= x^2+4x+4-4+9 \\ &= (x+2)^2+5 \end{aligned}$$

(1) $x=-2 \longrightarrow 0^2+5=5$

(2) $x=0 \longrightarrow 2^2+5=9$

(3) $x=2 \longrightarrow 4^2+5=21$

ex.2) x^2-4x+9

$$\begin{aligned} x^2-4x+9 &= x^2-4x+4-4+9 \\ &= (x-2)^2+5 \end{aligned}$$

(1) $x=-2 \longrightarrow (-4)^2+5=21$

(2) $x=0 \longrightarrow (-2)^2+5=9$

(3) $x=2 \longrightarrow 0^2+5=5$

ex.3) $2x^2+4x+9$

$$\begin{aligned} 2x^2+4x+9 &= 2(x^2+2x)+9 \\ &= 2(x^2+2x+1)+7 \\ &= 2(x+1)^2+7 \end{aligned}$$

(1) $x=-2 \longrightarrow 2\cdot(-1)^2+7=9$

(2) $x=0 \longrightarrow 2\cdot 1^2+7=9$

(3) $x=2 \longrightarrow 2\cdot 3^2+7=25$

ex.4) $2x^2-4x+9$

$$\begin{aligned} 2x^2-4x+9 &= 2(x^2-2x)+9 \\ &= 2(x^2-2x+1)+7 \\ &= 2(x-1)^2+7 \end{aligned}$$

(1) $x=-2 \longrightarrow 2\cdot(-3)^2+7=25$

(2) $x=0 \longrightarrow 2\cdot(-1)^2+7=9$

(3) $x=2 \longrightarrow 2\cdot 1^2+7=9$

ex.5) x^2-4x

$$\begin{aligned} x^2-4x &= x^2-4x+4-4 \\ &= (x-2)^2-4 \end{aligned}$$

(1) $x=-2 \longrightarrow (-4)^2-4=12$

(2) $x=0 \longrightarrow (-2)^2-4=0$

(3) $x=2 \longrightarrow 0^2-4=-4$

Cubic Expressions

$$ax^3 + bx^2 + cx + d$$

x : variable
a, b, c, d : constants

ex) $-x^3 + x^2 - x + 1$에서 계수와 상수항을 구하고, x가 -1, 0, 1일때의 값을 구하세요.

coeff. of x^3 / x^2 / x : -1 / 1 / -1 constant: 1

(1) $x = -1 \longrightarrow 1 + 1 + 1 + 1 = 4$

(2) $x = 0 \longrightarrow 0 + 0 + 0 + 1 = 1$

(3) $x = 1 \longrightarrow -1 + 1 - 1 + 1 = 0$

Multiplication Rules of Cubic Expressions

- $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\begin{aligned}(a+b)(a+b)(a+b) &= (a+b)(a^2+2ab+b^2)\\ &= (a^3+2a^2b+ab^2)+(a^2b+2ab^2+b^3)\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

- $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$
- $(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc$

ex.1) $(x+4)^3 = x^3 + 12x^2 + 48x + 64$

ex.2) $(x-4)^3 = x^3 - 12x^2 + 48x - 64$

Multiplication Rules of Cubic Expressions

w.r.t Coefficients

$$(a+b)(a^2+3ab+b^2) \quad a^3 \quad a^2b$$

$$(a+b)(a^2+3ab+b^2) \quad ab^2 \quad b^3$$

Polynomials

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$

$$3x+1, \quad x^2+2x-1, \quad 3x^3+3x^2-2x+1$$

x : variable

$a_0, a_1, \ \ldots \ , a_n$: constants

a_i : coeff. of x^i

degree of polynomials(차수): 최고차항의 차수

$ax+b$: linear expressions / first degree polynomials / 1차 다항식

ax^2+bx+c : quadratic expressions / second degree polynomials / 2차 다항식

ax^3+bx^2+cx+d : cubic expressions / third degree polynomials / 3차 다항식

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0:$$

n-th degree polynomials / n차 다항식

Polynomials

Polynomial Sets

$$P_1 = \{a_1x + a_0 \mid a_1,\, a_0 \in \mathbb{R}\}$$

$$P_2 = \{a_2x^2 + a_1x + a_0 \mid a_2,\, a_1,\, a_0 \in \mathbb{R}\}$$

$$\vdots$$

$$P_n = \{a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R},\, 0 \le i \in \mathbb{Z} \le n\}$$

$$P_1 \subset P_2 \subset \ldots \subset P_n$$

P_i's are increasing sequences of sets

Additions / Subtractions of Polynomials

Additions of Polynomials

$$\left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0\right) + \left(b_n x^n + b_{n-1}x^{n-1} + \ldots + b_2 x^2 + b_1 x + b_0\right)$$
$$= (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \ldots + (a_1 + b_1)x + (a_0 + b_0)$$

Subtractions of Polynomials

$$\left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0\right) - \left(b_n x^n + b_{n-1}x^{n-1} + \ldots + b_2 x^2 + b_1 x + b_0\right)$$
$$= (a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \ldots + (a_1 - b_1)x + (a_0 - b_0)$$

Additions / Subtractions of Polynomials

ex.1) $(2x^2-6x+10)+(-x^2+2x+4)=(2-1)x^2+(-6+2)x+(10+4)$
$=x^2-4x+14$

ex.2) $(2x^2-6x+10)-(-x^2+2x+4)=(2+1)x^2+(-6-2)x+(10-4)$
$=3x^2-8x+6$

ex.3) $(x^2+4x-2)+(-2x+4)=x^2+(4-2)x+(-2+4)$
$=x^2+2x+2$

ex.4) $(3x^3+2x-4)+(-2x^3+3x^2-2x+8)=(3-2)x^3+3x^2+(2-2)x+(-4+8)$
$=x^3+3x^2+4$

Multiplication of Polynomials

Algebraic Properties of Polynomials Multiplication

polynomials A, B, C

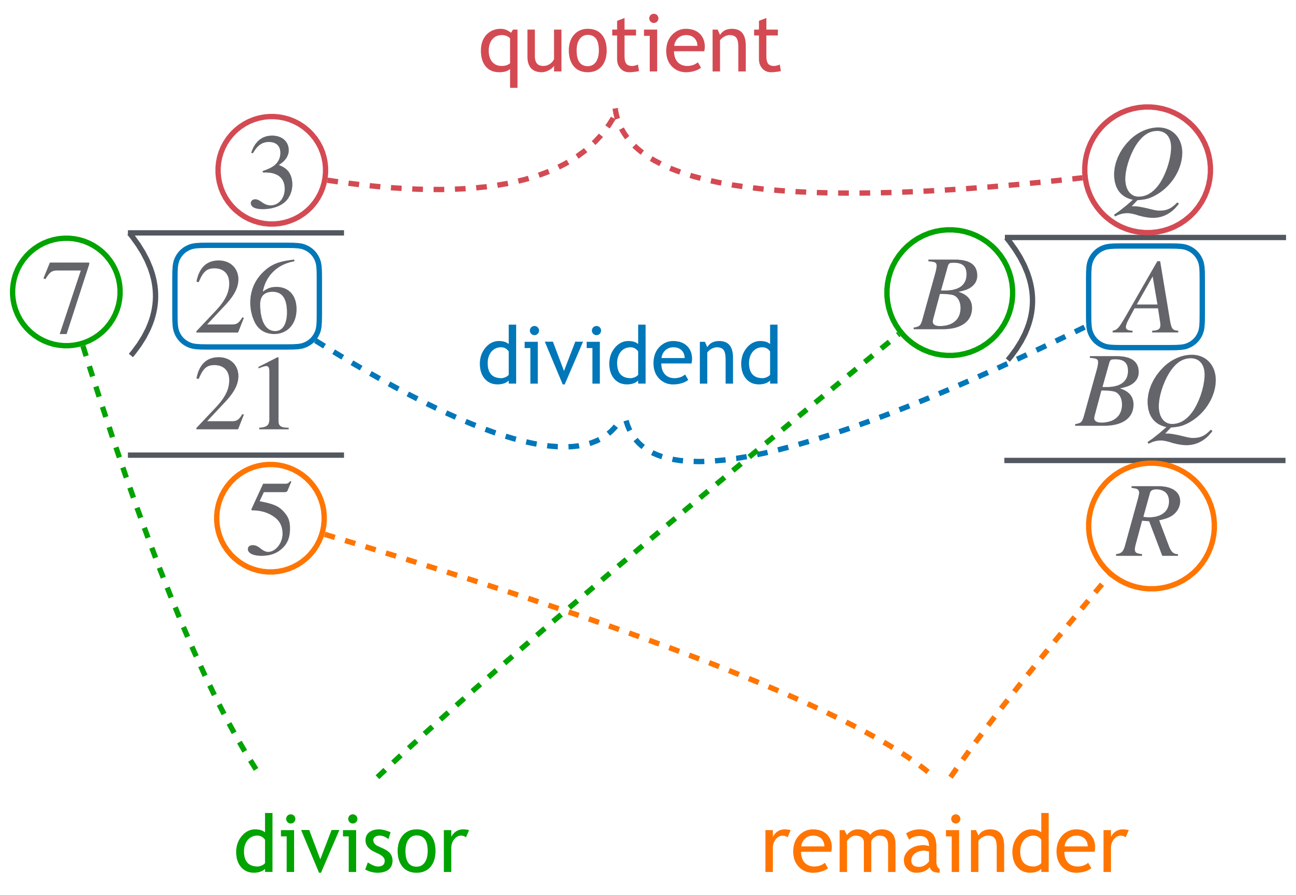
commutativity $AB = BA$

associativity $(AB)C = A(BC)$

distributivity
$$A(B+C) = AB + AC$$
$$(A+B)C = AC + BC$$
$$(A+B)(C+D) = (AC+AD) + (BC+BD)$$

Division of Polynomials

Terminologies



$$26 = 7 \cdot 3 + 5 \qquad A = B \cdot Q + R$$

Division of Polynomials

ex.1) $(3x^2 - 4x + 7) \;/\; (x + 1)$

$$
\begin{array}{r|l}
 & 3x \quad -7 \\
\hline
x+1 & 3x^2 - 4x + 7 \\
 & 3x^2 + 3x \\
\hline
 & \quad -7x + 7 \\
 & \quad -7x - 7 \\
\hline
 & \qquad\quad 14
\end{array}
$$

$$
\begin{aligned}
&3x^2 - 4x + 7 \\
&\quad = (x+1)(3x-7) + 14
\end{aligned}
$$

ex.2) $(4x^4 + 3x^3 - 4x) \;/\; (-x^2 + x - 1)$

$$
\begin{array}{r|l}
 & -4x^2 \quad -7x \quad -3 \\
\hline
-x^2 + x - 1 & 4x^4 + 3x^3 \qquad\qquad -4x \\
 & 4x^4 - 4x^3 + 4x^2 \\
\hline
 & \qquad 7x^3 - 4x^2 - 4x \\
 & \qquad 7x^3 - 7x^2 + 7x \\
\hline
 & \qquad\qquad 3x^2 - 11x \\
 & \qquad\qquad 3x^2 - 3x + 3 \\
\hline
 & \qquad\qquad\quad -8x - 3
\end{array}
$$

$$
\begin{aligned}
&4x^4 + 3x^3 - 4x \\
&\quad = (-x^2 + x - 1)(-4x^2 - 7x - 3) + (-8x - 3)
\end{aligned}
$$

remainder의 차수는 quotient의 차수보다 낮다.

Factorization with Division

$$ax^3 + bx^2 + cx + d \longrightarrow a(x+\alpha)(x+\beta)(x+\gamma)$$

$x+\alpha,\ \ x+\beta,\ \ x+\gamma$로 나눴을 때, remainder가 0이 된다.

ex) $x^3 + 9x^2 + 26x + 24$

-2를 대입했을 때 0이 된다. $\longrightarrow$ $(x+2)$로 나눴을 때 remainder는 0

$$\begin{array}{r|l}
 & x^2 \quad +7x \quad +12 \\
\hline
x+2 & x^3 + 9x^2 + 26x + 24 \\
 & x^3 + 2x^2 \\
\hline
 & \qquad 7x^2 + 26x \\
 & \qquad 7x^2 + 14x \\
\hline
 & \qquad\qquad 12x + 24 \\
 & \qquad\qquad 12x + 24 \\
\hline
 & \qquad\qquad\qquad 0
\end{array}$$

$$\begin{aligned} x^3 + 9x^2 + 26x + 24 &= (x+2)(x^2+7x+12) \\ &= (x+2)(x+3)(x+4) \end{aligned}$$

Tip. 24의 약수로 찾기

Quadratic Equations

$$ax^2 + bx + c = 0 \longrightarrow a(x - \alpha_1)(x - \alpha_2) = 0$$
$$\longrightarrow S = \{\alpha_1, \alpha_2\}$$

ex.1) $x^2 + 2x + 1 = 0 \quad \longrightarrow (x+1)^2 = 0 \longrightarrow S = \{-1\}$

ex.2) $x^2 - 4x + 4 = 0 \quad \longrightarrow (x-2)^2 = 0 \longrightarrow S = \{2\}$

ex.3) $x^2 + 3x + 2 = 0 \quad \longrightarrow (x+1)(x+2) = 0 \longrightarrow S = \{-1, -2\}$

ex.4) $x^2 + 7x + 12 = 0 \quad \longrightarrow (x+3)(x+4) = 0 \longrightarrow S = \{-3, -4\}$

ex.5) $x^2 + 8x + 15 = 0 \quad \longrightarrow (x+3)(x+5) = 0 \longrightarrow S = \{-3, -5\}$

ex.6) $x^2 - 1 = 0 \quad \longrightarrow (x+1)(x-1) = 0 \longrightarrow S = \{1, -1\}$

ex.7) $x^2 - 9 = 0 \quad \longrightarrow (x+3)(x-3) = 0 \longrightarrow S = \{3, -3\}$

ex.8) $x^2 - 25 = 0 \quad \longrightarrow (x+5)(x-5) = 0 \longrightarrow S = \{5, -5\}$

Quadratic Equations

Quadratic Formula

$$ax^2 + bx + c = 0 \longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

ex.1) $x^2 + 2x + 1 = 0$

$\longrightarrow (x+1)^2 = 0 \longrightarrow S = \{-1\}$

$$x = \frac{-2 \pm \sqrt{4-4}}{2} = -1$$

ex.2) $x^2 - 4x + 4 = 0$

$\longrightarrow (x-2)^2 = 0 \longrightarrow S = \{2\}$

$$x = \frac{4 \pm \sqrt{16-16}}{2} = 2$$

ex.3) $x^2 + 3x + 2 = 0$

$\longrightarrow (x+1)(x+2) = 0 \longrightarrow S = \{-1, -2\}$

$$x = \frac{-3 \pm \sqrt{9-8}}{2} = -1 \text{ or } -2$$

ex.4) $x^2 + 7x + 12 = 0$

$\longrightarrow (x+3)(x+4) = 0 \longrightarrow S = \{-3, -4\}$

$$x = \frac{-7 \pm \sqrt{49-48}}{2} = -3 \text{ or } -4$$

Quadratic Equations

Complex Solutions

$\sqrt{a},\ a < 0$는 실수로 정의되지 않는다. ⇒ complex numbers

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac > 0 \longrightarrow \pm\sqrt{b^2 - 4ac} \in \mathbb{R} \longrightarrow$ **TWO** real sol.s

$b^2 - 4ac = 0 \longrightarrow \sqrt{b^2 - 4ac 0} \longrightarrow$ **ONE** real sol(중근)

$b^2 - 4ac < 0 \longrightarrow \pm\sqrt{b^2 - 4ac} \notin \mathbb{R} \longrightarrow$ **TWO** complex sol.s

complex solution은 0, 2개로만 존재한다.

Quadratic Equations

Discriminants

$$D = b^2 - 4ac$$

$D > 0 \longrightarrow$ **TWO** real sol.s

$D = 0 \longrightarrow$ **ONE** real sol(중근)

$D < 0 \longrightarrow$ **TWO** complex sol.s

ex.1) $x^2 + 2x + 1 = 0$

$D = 4 - 4 = 0 \longrightarrow$ **ONE** real sol

ex.2) $x^2 + 3x + 2 = 0$

$D = 9 - 8 = 1 \longrightarrow$ **TWO** real sol.s

ex.3) $x^2 + 2x + 2 = 0$

$D = 4 - 8 < 0 \longrightarrow$ **TWO** complex sol.s

Cubic Equations

$$ax^3 + bx^2 + cx + d = 0 \longrightarrow a(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) = 0$$
$$\longrightarrow S = \{\alpha_1, \alpha_2, \alpha_3\}$$

The Types of Solutions

$$ax^3 + bx^2 + cx + d = 0 \longrightarrow a(x - \alpha_1)(x^2 + \alpha x + \beta) = 0$$

$D > 0 \longrightarrow$ **TWO** real sol.s + **ONE** real sol = **THREE** real sol.s

$D = 0 \longrightarrow$ **ONE** real sol + **ONE** real sol

$\alpha_1 = \alpha_2 = \alpha_3 \longrightarrow$ **ONE** real sol

$\alpha_1 = \alpha_2 \neq \alpha_3 \longrightarrow$ **TWO** real sol.s

$D < 0 \longrightarrow$ **TWO** complex sol.s + **ONE** real sol = **ONE** real sol

$$S = S_R \cup S_C$$

Cubic Equations

ex.1) $(x+1)(x-1)(x+3)=0$

$S = S_R = \{-1,\ 1,\ 3\} \longrightarrow$ **THREE** real sol

ex.2) $(x+1)^3=0$

$S = S_R = \{-1\} \longrightarrow$ **ONE** real sol

ex.3) $(x+1)(x-2)^2=0$

$S = S_R = \{-1,\ 2\} \longrightarrow$ **TWO** real sol

ex.4) $(x+1)(x^2+2x+2)=0$

$S_R = \{-1\}, \quad S_C = \{-1+j\cdot 2,\ -1-j\cdot 2\} \longrightarrow$ **ONE** real sol

$S = S_R \cup S_C = \{-1,\ -1+j\cdot 2,\ -1-j\cdot 2\}$

Polynomial Equations

Solutions of Polynomial Equations

$$a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = 0$$
$$\longrightarrow a(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n) = 0$$
$$\longrightarrow S = \{\alpha_1, \alpha_2, \ \ldots \ , \alpha_n\} \quad |S| = n$$

real solution의 개수는 n보다 작을 수 있다.

Solution Sets

$$|S| = n, \quad |S_C| = 2m, m \in \mathbb{W}$$
$$\longrightarrow |S_R| = n - 2m$$

$$S = S_R \cup S_C$$

Equal Polynomials

$P(x) = 0$

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

모든 x값에 대해 $P(x) = 0$을 만족시킬 조건

Case.1 Quadratic Equations

$$ax + b = 0 \xrightarrow[x=0]{} b = 0 \longrightarrow ax = 0 \xrightarrow[x=1]{} a = 0$$

Case.2 Cubic Equations

$$ax^2 + bx + c = 0 \xrightarrow[x=0]{} c = 0 \longrightarrow ax^2 + bx = 0 \xrightarrow[\div x]{} ax + b = 0 \longrightarrow ax = 0 \xrightarrow[\div x]{} a = 0$$

Case.3 Polynomial Equations

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$$

$$\longrightarrow \forall a_i,\ 0 \le i \in \mathbb{W} \le n$$

Equal Polynomials

$$\left(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0\right) = \left(b_nx^n + b_{n-1}x^{n-1} + \ldots + b_2x^2 + b_1x + b_0\right)$$

$$\longrightarrow \left(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0\right) - \left(b_nx^n + b_{n-1}x^{n-1} + \ldots + b_2x^2 + b_1x + b_0\right) = 0$$

$$\longrightarrow (a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \ldots + (a_1 - b_1)x + (a_0 - b_0) = 0$$

$$\longrightarrow \forall(a_i - b_i),\ 0 \le i \in \mathbb{W} \le n$$

ex.1) $a_1x + a_0 = 3x + 2 \quad \longrightarrow a_1 = 3,\ a_0 = 2$

ex.2) $a_2x^2 + a_1x + a_0 = 3x^2 - 2x + 1 \ \longrightarrow a_2 = 3,\ a_1 = -2,\ a_0 = 1$

ex.3) $a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 4x^4 - 2x$

$\longrightarrow a_4 = 4,\ a_1 = -2,\ a_3 = a_2 = a_0 = 0$

CLOSING

Basic Algebra

Chap.4 Polynomial Expressions/Equations