2020 考古

1. $f(x) = \begin{cases} 0 & , -\pi < x < 0 \\ 1 & , 0 < x < \pi \end{cases}$ (週期 2π)

A: $a_0 = \frac{1}{2\pi} \int_0^{\pi} 1 \, dx = \frac{1}{2}$

$a_n = \frac{1}{\pi} \int_0^{\pi} \cos nx \, dx = \frac{1}{n\pi} \cdot \sin nx \Big|_0^{\pi} = 0$

$b_n = \frac{1}{\pi} \int_0^{\pi} \sin nx \, dx = \frac{-1}{n\pi} \cos nx \Big|_0^{\pi} = \frac{1}{n\pi} (1 - \cos n\pi)$

$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{n\pi} (1 - \cos n\pi) \cdot \sin nx \right)$

$= \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n} (1 - \cos n\pi) \cdot \sin nx \right)$

2. $f(x) = 3 - 2x \quad , -\pi < x < \pi$ 週期 2π

A: $a_0 = 3$

$a_n = 0$

$b_n = \frac{1}{\pi} \int_{-2}^{2} -2x \cdot \sin nx \, dx = \frac{-4}{\pi} \int_0^2 x \sin nx \, dx$

$= \frac{4}{\pi} \left(-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right) \Big|_0^{\pi} = \frac{4}{n} \cos n\pi$

$f(x) = 3 + \sum_{n=1}^{\infty} \frac{4}{n} \cos n\pi \sin nx$

3. Ans: 2

4. $f(x) = \begin{cases} x+1 & -1 < x < 0 \\ x-1 & 0 < x < 1 \end{cases}$ 週期 $=2$ ，求傅立葉級數

$$\Rightarrow \sum_{n=1}^{\infty} \frac{-2}{n\pi} \sin n\pi x$$

$$a_0 = \frac{1}{2}\left(\int_{-1}^{0}(x+1)dx + \int_{0}^{1}(x-1)dx\right) = \frac{1}{2}\left(\frac{1}{2}x^2 + x\Big|_{-1}^{0} + \frac{1}{2}x^2 - x\Big|_{0}^{1}\right) = 0$$

$$a_n = \int_{-1}^{0}(x+1)\cdot \cos n\pi x\, dx + \int_{0}^{1}(x-1)\cdot \cos n\pi x\, dx$$

$$= \int_{-1}^{1} x \cdot \cos n\pi x\, dx + \int_{-1}^{0} \cos n\pi x\, dx + \int_{0}^{1} \cos \pi \pi x\, dx$$

$$= \frac{1}{n\pi}\sin n\pi x\Big|_{-1}^{0} + \frac{1}{n\pi}\sin n\pi x\Big|_{0}^{1} = \frac{-1}{n\pi}\sin n\pi x + \frac{1}{n\pi}\sin n\pi x = 0$$

$$b_n = \int_{-1}^{1} x \sin n\pi x\, dx + \int_{-1}^{0} \sin n\pi x\, dx - \int_{0}^{1} \sin n\pi x\, dx$$

$$= 2\left(\frac{-x}{n\pi}\cos n\pi x + \frac{1}{(n\pi)^2}\sin n\pi x\right)\Big|_{0}^{1} + \frac{-1}{n\pi}\sin n\pi x\Big|_{-1}^{0} + \frac{1}{n\pi}\sin n\pi x\Big|_{0}^{1}$$

$$= 2\left(\frac{-1}{n\pi}\cos n\pi x + \frac{1}{(n\pi)^2}\sin n\pi x\right) + \frac{-2}{n\pi}$$

5. $f(x)=x^2$, $0<x<L$ $f(x)$ 的奇函數半幅展開式

$$\Rightarrow \frac{2}{L}\int_0^2 x^2 \sin\frac{n\pi}{L}x\,dx$$

$$\Rightarrow \frac{2}{L}\left(x^2\frac{-L}{n\pi}\cdot\cos\frac{n\pi}{L}x+2x\left(\frac{L}{n\pi}\right)^2\sin\frac{n\pi}{L}x+2\left(\frac{L}{n\pi}\right)^3\cos\frac{n\pi}{L}x\right)\Big|_0^2$$

$$\Rightarrow \frac{2}{L}\left(-\frac{L^3}{n\pi}(-1)^x+2\left(\frac{L}{n\pi}\right)^3(-1)^n-2\cdot\left(\frac{L}{n\pi}\right)^3\right)$$

$$=\frac{2L^2}{\pi}\left(\frac{(-1)^{n+1}}{n}+\frac{2}{n^3\pi^2}((-1)^n-1)\right)$$

$$f(x)=\frac{2L^2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n}+\frac{2}{n^3\pi^2}((-1)^3-1)\right)\cdot\sin\frac{n\pi}{L}x$$

6. $f(x)=x^2$, $0<x<2\pi$ 的傅立葉級數（週期 2π）

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}x^2\,dx=\frac{1}{2\pi}\frac{1}{3}x^3\Big|_0^{2\pi}=\frac{4}{3}\pi^2$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nx\,dx=\frac{1}{\pi}\frac{4\pi}{n^2}=\frac{4}{n^2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x^2\sin nx\,dx=\frac{1}{\pi}\left(\frac{-x^2}{n}\cos nx+\frac{2}{n^3}\cos nx\right)\Big|_0^{2n}$$

$$=\frac{1}{\pi}\left(\frac{-4\pi^2}{n}+\frac{2}{n^3}-\frac{2}{n^3}\right)=\frac{-4\pi}{n}$$

$$f(x)=\frac{4}{3}\pi^2+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx\right)$$

7. $f(x)=x^2$, $0<x<L$ 的傅立葉級數 (週期為 L)

$$a_0=\frac{1}{L}\int_0^L x^2dx=\frac{1}{L}\cdot\frac{L^3}{3}=\frac{L^2}{3}$$

$$a_n=\frac{2}{L}\int_0^L x^2\cos\frac{2n\pi x}{L}dx$$

$$=\frac{2}{L}\cdot\left(\frac{x^2L}{2n\pi}\cdot\sin\frac{2n\pi x}{L}+2x\frac{L^2}{4n^2\pi^2}\cos\frac{2n\pi x}{L}-\frac{L^3}{4n^3\pi^3}\frac{2n\pi x}{L}\right)$$

$$=\frac{2}{L}\left(2\cdot\frac{L^3}{4n^2\pi^2}\right)=\frac{L^2}{n^2\pi^2}$$

$$b_n=\frac{2}{L}\int_0^L x^2\sin\frac{2n\pi x}{L}dx$$

$$=\frac{2}{L}\left(\frac{-x^2L}{2n\pi}\cdot\cos\frac{2n\pi x}{L}+\frac{L^3}{4n^2\pi^2}\cos\frac{2n\pi x}{L}\right)\Big|_0^2$$

$$=\frac{2}{L}\left(\frac{-L^3}{2n\pi}+\frac{L^3}{2n\pi}\right)=\frac{-L^2}{n\pi}$$

$$f(x)=\frac{L^2}{3}+\frac{L^2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2\pi}\cos\frac{2n\pi}{L}-\frac{1}{n}\sin\frac{2n\pi}{L}$$