SOME RIGIDITY THEOREMS WITH SPECTRAL CURVATURE BOUNDS

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Abstract. By systematically making use of the warped μ-bubble method, we investigate the geometric implications of a spectral Ricci or scalar curvature lower bound. In particular, we prove several spectral band width estimates, and some splitting theorems such as the spectral version of the Geroch conjecture and the Milnor conjecture.

1. Introduction

A groundbreaking result in scalar curvature geometry is the resolution of Geroch conjecture due to Schoen-Yau [SY79a]. The conjecture states that an n-dimensional torus does not admit a metric of non-negative scalar curvature. Schoen-Yau used a minimal hypersurface approach and it has become a major tool, and there are other approaches by spinors [GL83] and the technique of harmonic functions due to Stern [Ste22].

A weaker version of curvatures, called the *spectral curvatures*, which are defined as the first eigenvalue of an elliptic operator involving with the Laplace-Beltrami operator and the curvature, recently has found its place in several important problems. For example, the spectral curvature is useful in the stable Bernstein conjecture [CL23], [CLMS24], [Maz24] and the aspherical conjecture [CL24].

We will use the following definition of the spectral curvatures using a positive function.

Definition 1.1. *Let (M, g) be a Riemannian manifold and u be a positive function, we call*

$$-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g \tag{1.1}$$

the spectral scalar curvature and

$$-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g > 0 \tag{1.2}$$

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the spectral Ricci curvature where

$$\mathrm{Ric}_g := \inf_{e\in T_xM, |e|_g=1} \mathrm{Ric}_g(e,e)$$

is the least Ricci curvature at a given point $x \in M$. We will say spectral scalar (resp. Ricci) curvature modified by γ and u if the dependence on γ and u needs to be explicit.

As easily checked, in a closed manifold, a lower bound on (1.1) would imply the same lower bound on the first eigenvalue of $-\gamma\Delta_g + \frac{1}{2}R_g$. Similar implications works for (1.2).

Here we fix the convention of Ric_g: if it is given no argument it means the least Ricci curvature; two arguments means the usual Ricci curvature. We also find it convenient to use

$$-\gamma u^{-1}\Delta_g u + 2\,\mathrm{Ric}_g\,.$$

1.1. **Weighted minimal hypersurfaces.** A suitable generalization of Schoen-Yau's technique of minimal hypersurfaces in the study of spectral curvatures, such as the spectral Ricci curvature and the spectral scalar curvature, is the notion of a weighted minimal hypersurface, which arises as a critical point of the weighted area functional.

Definition 1.2. *We say that Σ is a u^γ-weighted minimal hypersurface if it is a critical point of the u^γ-weighted area functional*

$$\mathcal{A}_u(S) = \int_S u^\gamma \mathrm{d}\mathcal{H}^{n-1} \tag{1.3}$$

defined for all oriented hypersurfaces. The references to u^γ would be omitted if the dependence on γ and u is clear.

Given a smooth family of hypersurfaces $\{\Sigma_t\}$ such that $\Sigma_0 = \Sigma$, the first variation of the weighted area functional is

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_u(\Sigma_t)|_{t=0} = \int_\Sigma (H + \gamma u^{-1}u_\nu)u^\gamma \langle Y, \nu\rangle \mathrm{d}\mathcal{H}^{n-1},$$

where ν is a chosen unit normal to Σ, H is the mean curvature defined as $\mathrm{div}_S\,\nu$ and Y is the variational vector field.

Definition 1.3. *We call $H+\gamma u^{-1}u_\nu$ the u^γ-weighted mean curvature. If the second variation is non-negative for a weighted minimal hypersurface, we call Σ stable, and we call Σ u^γ-weighted area-minimizing if Σ is a minimiser to the functional* (1.3)*. Again, the references to u^γ would be omitted if the dependence on γ and u is clear.*

We will introduce generalizations of the weighted minimal hypersurfaces in Section 2, in particular, warped h-bubbles.

We now state our first result using the weighted minimal hypersurface, which is a classification result of stable weighted surfaces in a 3-manifold of non-negative spectral curvature. The result is an analog of [FCS80, Theorem 3], [CG00] which classifies stable minimal surfaces in a 3-manifold of non-negative scalar curvature.

Theorem 1.4. *Let (M^3, g) be a 3-dimensional complete manifold with spectral nonnegative scalar curvature and Σ be a stable, complete, oriented weighted minimal surface with weight u^γ in (M, g). Then there are two possibilities:*

(a) Σ is compact, then Σ is a sphere or a torus, in the case of a torus, Σ is flat, if further Σ is weighted area-minimizing, then (M, g) locally splits;
(b) Σ is non-compact, then Σ is conformally equivalent to the complex plane $\mathbb{C}$ or a cylinder $\mathbb{A}$; Σ is flat if it is conformally equivalent to a cylinder.

The strategy of proving the compact case of Theorem 1.4 is as follows: we use the stability and the spectral curvature condition (with the Gauss-Bonnet theorem) to show that Σ is infinitesimally rigid (cf. [FCS80]); second, construct a foliation $\{\Sigma_t\}$ with Σ be a leaf using Theorem 2.3; third, determine the sign of weighted mean curvature of each leaf using the curvature condition again; fourth, show that every nearby Σ_t is also minimizing using the first variation (cf. [BBN10]). We can show a global splitting if we assume an additional topological condition, see Theorem 1.21 (cf. [CG00]).

This strategy in proving the rigidity will used in Subsection 1.2 and the band width estimates in Subsection 1.3 with suitable adaptations. To avoid repetition, we unify and streamline these proofs by introducing the variables Z and W in Subsection 2.3. The key differences left are estimating the mean curvature of a leaf of the foliation and calculating the rigid metrics.

The case with Σ being non-compact is more subtle in Theorem 1.4 and subsequent Theorem 1.5 (cf. [FCS80, Theorem 3]). It is due to a question posed in [FCS80, Remark 1] regarding the inverse spectral properties of the elliptic operator $-\Delta_\Sigma + aK$ where $a \in \mathbb{R}$ and K is the Gaussian curvature of the surface Σ. Here, we have made use of [BC14].

With a slightly different proof, we have the following.

Theorem 1.5. *Theorem 1.4 holds if the spectral non-negative scalar curvature is replaced by the condition that $-\gamma u^{-1}\Delta_g u + 2\,\mathrm{Ric}_g \geqslant 0$, $0 \leqslant \gamma < 4$.*

For the Ricci curvature $-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g$, the splitting result was obtained by [APX24, CMMR25] which works in any dimensions.

1.2. **Constant weighted mean curvature.** The concept of a (Riemannian) *band* is introduced by Gromov in his studies of metric inequalities [Gro18]. It is a Riemannian manifold with at least two boundaries. We can group the boundaries into two groups $\partial_- M$ and $\partial_+ M$ allowing the *band width* to be defined as the distance from $\partial_- M$ to $\partial_+ M$.

We fix the notation of band, let Σ be an oriented hypersurface homologous to $\partial_- M$ we choose the direction of the unit normal of Σ such that it points outward of the region bounded by Σ and $\partial_- M$. In particular, we choose the direction of the unit normal ν_- of $\partial_- M$ pointing inside of M and the unit normal ν_+ of $\partial_+ M$ outside of M.

By using hypersurfaces of constant weighted mean curvature, we can show some rigidity theorem for n-dimensional *torical bands* which are simply $T^{n-1} \times [-1, 1]$ with a smooth metric g.

Below is a spectral analog of [ACG08, Theorem 1.1].

Theorem 1.6. *Let* $0 \leqslant \gamma < \frac{2(n-1)}{n-2}$*,* $\Gamma = \frac{2n-(n-1)\gamma}{4(n-1)-2(n-2)\gamma}$*, and* Λ *be a constant such that either* $\Gamma\Lambda < 0$ *or* $\Gamma = \Lambda = 0$*. If* $\Gamma\Lambda < 0$*, set* $\eta = \eta_{\Gamma,\Lambda} := \sqrt{-\Lambda/\Gamma}$*; and if* $\Gamma = \Lambda = 0$*, set* $\eta = 0$*. Set* $\alpha = \frac{1}{2(n-1)-(n-2)\gamma}\eta$ *and* $\beta = \frac{2-\gamma}{2(n-1)-(n-2)\gamma}\eta$*. If* (M, g) *be an* n*-dimensional torical band and* $u > 0$ *such that*

$$-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g \geqslant \Lambda$$

in M *and*

$$H_{\partial_+ M} + \gamma u^{-1} u_{\nu_+} \geqslant \eta \text{ along } \partial_+ M = \{1\} \times T^{n-1},$$

and

$$H_{\partial_- M} + \gamma u^{-1} u_{\nu_-} \leqslant \eta \text{ along } \partial_- M = \{-1\} \times T^{n-1}.$$

Then (M, g) *must be isometric to* $([t_-, t_+] \times T^{n-1}, \mathrm{d}t^2 + e^{2\alpha t} g_{\mathbb{T}^{n-1}})$ *for some* $t_- < t_+$ *with* u *given by a constant multiple of* $e^{\beta t}$*.*

As for the spectral Ricci curvatures, we have the following.

Theorem 1.7. *Let* $0 \leqslant \gamma < 3 + \frac{1}{n-2}$*,* $\Lambda \leqslant 0$*,* $\eta = \sqrt{-\Lambda}/\sqrt{1-\gamma/4}$*,* (M, g) *be an* n*-dimensional torical band and* $u > 0$ *such that*

$$-\gamma u^{-1}\Delta_g u + (n-1)\operatorname{Ric}_g \geqslant \Lambda, \tag{1.4}$$

in M *and*

$$H_{\partial_+ M} + \gamma u^{-1} u_{\nu_+} \geqslant \eta \text{ along } \partial_+ M = \{1\} \times T^{n-1},$$

and

$$H_{\partial_- M} + \gamma u^{-1} u_{\nu_-} \leqslant \eta \text{ along } \partial_- M = \{-1\} \times T^{n-1}.$$

Then (M, g) *is foliated by flat* $(n-1)$*-torus with* $H + \gamma u^{-1} u_\nu = \eta$ *and the equality in* (1.4) *is achieved everywhere. In particular, if* $n = 3$*, then* (M, g) *is isometric to*

$$([t_-, t_+] \times T^2, \mathrm{d}t^2 + e^{\eta(1-\gamma/2)t} g_{\mathbb{T}^2})$$

for some $t_- < t_+$ with u given by a constant multiple of $e^{\eta t/2}$.

Remark 1.8. *For $n \geqslant 4$, $u = e^{\eta t/2}$ and the warped product metric $g = \mathrm{d}t^2 + e^{2(1-\gamma/2)/(n-1)t} g_{\mathbb{T}^{n-1}}$ satisfy the conditions in Theorem 1.7, however, we do not know whether there are other metrics.*

Theorem 1.9. *Let $0 \leqslant \gamma < \frac{n-1}{n-2}$, $\Gamma = \frac{4-(n-1)\gamma}{4(n-1-(n-2)\gamma)}$, Λ be a number such that $\Gamma\Lambda < 0$ or $\Gamma = \Lambda = 0$. Let η be the constant $\sqrt{-\Lambda/\Gamma}$ when $\Gamma\Lambda < 0$ and 0 when $\Gamma = \Lambda = 0$. Let $\alpha = -\frac{n-3}{2(n-1-(n-2)\gamma)}\eta$ and $\beta = \frac{2-\gamma}{2(n-1-(n-2)\gamma)}\eta$. If (M, g) is a band with*

(a) $-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g \geqslant \Lambda$ in M,

(b) $H_{\partial_+ M} + \gamma u^{-1} u_{\nu_+} \geqslant \eta$ along $\partial_+ M$, and $H_{\partial_- M} + \gamma u^{-1} u_{\nu_-} \leqslant \eta$ along $\partial_- M$,

then (M, g) is isometric to $([t_-, t_+] \times S, \mathrm{d}t^2 + e^{2\alpha t} g_S)$ for some $t_- < t_+$ and (S, g_S) with $\mathrm{Ric}_{g_S} \geqslant 0$ and u is a constant multiple of $e^{\beta t}$.

1.3. **Band width estimates.** One of the basic result of the band width is that the band width of a torical band is bounded from above due to the effect of the positive scalar curvature, see [Gro18]. Gromov's approach is by considering hypersurfaces of prescribed mean curvature h where h is a Lipschitz function related to the band width. The proper generalization of Gromov's h-bubble is the notion of a *warped h-bubble* which we will use to show several band width estimates under spectral curvature bounds.

To facilitate the description of the band width estimates, we introduce some notations. Let Γ and Λ be two constants and we are concerned with the ODE

$$\Gamma\eta^2 + \eta' + \Lambda = 0 \tag{1.5}$$

such that the solution η satisfies $\eta' < 0$. To ensure $\eta' < 0$, at least one of Γ and Λ should be positive. Indeed, the solution to (1.5) is given by the following

$$\eta(t) := \eta_{\Lambda,\Gamma}(t) := \begin{cases} \sqrt{-\Lambda/\Gamma}\coth(\sqrt{-\Lambda\Gamma}t), & \Gamma > 0,\ \Lambda < 0, \\ \frac{1}{\Gamma t}, & \Gamma > 0,\ \Lambda = 0, \\ \sqrt{\Lambda/\Gamma}\cot(\sqrt{\Lambda\Gamma}t), & \Gamma > 0,\ \Lambda > 0, \\ -\Lambda t, & \Gamma = 0,\ \Lambda > 0, \\ -\sqrt{-\Lambda/\Gamma} + \frac{2\sqrt{-\Lambda/\Gamma}}{1+\exp(2\sqrt{-\Lambda\Gamma}t)} & \Gamma < 0,\ \Lambda > 0. \end{cases}$$

Evidently, $\eta_{\Lambda,\Gamma}$ is only well defined on the interval $I_{\Lambda,\Gamma}$ given by

$$I_{\Lambda,\Gamma} := \begin{cases} (0, \infty), & \Gamma > 0,\ \Lambda \leqslant 0; \\ (0, \pi/\sqrt{\Lambda\Gamma}), & \Gamma > 0,\ \Lambda > 0, \\ (-\infty, +\infty), & \Gamma \leqslant 0,\ \Lambda > 0. \end{cases}$$

Now we give a slight generalization of the band width estimate under a positive spectral scalar curvature bound in [CS25] by allowing negative and zero spectral scalar curvature bounds.

Theorem 1.10. *Let Λ be a constant, $0 \leqslant \gamma < \frac{2(n-1)}{n-2}$, $\Gamma = \frac{2n-(n-1)\gamma}{4(n-1)-2(n-2)\gamma}$. Assume that at least one of Γ and Λ is positive, and let $t_- < t_+$ be two numbers such that $[t_-, t_+] \subset I_{\Lambda,\Gamma}$. Let (M^n, g) be a torical band such that*

(a) there exits a positive function u with $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g \geqslant \Lambda$,
(b) and $H_{\partial_+ M} + \gamma u^{-1} u_{\nu_+} \geqslant \eta(t_+)$ on $\partial_+ M$, $H_{\partial_- M} + \gamma u^{-1} u_{\nu_-} \leqslant \eta(t_-)$ on $\partial_- M$,

then

$$\mathrm{width}(M, g) \leqslant t_+ - t_-.$$

Equality occurs if and only if (M, g) is isometric to the model

$$([t_-, t_+] \times T^{n-1}, \mathrm{d}t^2 + \phi(t)^2 g_{\mathbb{T}^{n-1}})$$

where $\phi(t) = \exp(\frac{2-\gamma}{2(n-1)-(n-2)\gamma} \int^t \eta)$ and $g_{\mathbb{T}^{n-1}}$ is some flat metric on T^{n-1} and u is a constant multiple of $\exp(\frac{1}{2(n-1)-(n-2)\gamma} \int^t \eta)$.

Remark 1.11. *A new feature of this band width estimate is the possible negative or zero value of Γ in the ODE* (1.5)*, which is not present in the non-spectral case (i.e. $\gamma = 0$).*

Now let's turn to the Ricci curvature. The Bonnet-Myers theorem is a fundamental result regarding the control of diameter by a Ricci curvature bound. More specifically, it states that if a complete manifold with $\mathrm{Ric}_g \geqslant n-1$, then its diameter must be less than π. This can be interpreted as a band type estimate: any band with $\mathrm{Ric}_g \geqslant n-1$ must have its width less than π. Indeed, for a closed manifold, we can remove a pair of points which realizes the diametr, and the manifold become a band in some sense. This interpretation was achieved by [HKKZ25] via spacetime harmonic function techniques.

Now, we apply the same interpretation for the Bonnet-Myers theorem of spectral Ricci curvature which was shown by Antonelli-Xu [AX24]. Their theorem states that if a complete manifold has positive spectral Ricci curvature with $0 \leqslant \gamma < 4/(n-1)$, then the diameter of the manifold is bounded. This result has been further extended to the spectral Bakry-Emery curvatures [CH24, Yeu25, Wu25]. With the alternative interpretation, we can show the band with estimates with different curvature bounds and a rigidity statement.

Theorem 1.12. *Let Λ be a constant, $0 \leqslant \gamma < \frac{n-1}{n-2}$ and $\Gamma = \frac{4-(n-1)\gamma}{4(n-1-(n-2)\gamma)}$. Assume that at least one of Γ and Λ is positive, and $t_- < t_+$ be two numbers such that $[t_-, t_+] \subset I_{\Lambda,\Gamma}$. Let (M, g) be a band such that* *(a) there exists a positive function u with*

$$-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g \geqslant \Lambda;$$

(b) $H_{\partial_+ M} + \gamma u^{-1} u_{\nu_+} \geqslant \eta(t_+)$ on $\partial_+ M$, $H_{\partial_- M} + \gamma u^{-1} u_{\nu_-} \leqslant \eta(t_-)$ on $\partial_- M$,

then

$$\mathrm{width}(M, g) \leqslant t_+ - t_-.$$

Equality occurs if and only if (M, g) is isometric to the model

$$([t_-, t_+] \times S, \mathrm{d}t^2 + \phi^2(t) g_S) \tag{1.6}$$

where $\phi(t) = \exp(\frac{2-\gamma}{2(n-1-(n-2)\gamma)} \int^t \eta)$ for some closed manifold (S, g_S) with its Ricci curvature

$$\mathrm{Ric}_{g_S} \geqslant -(n-2) \max_{t \in [t_-, t_+]} \phi^2 (\tfrac{\phi'}{\phi})',$$

and u is a constant multiple of $\exp(-\frac{n-3}{2(n-1-(n-2)\gamma)} \int^t \eta)$.

Remark 1.13. *It should be viable to prove via minimizing weighted geodesics from $\partial_- M$ to $\partial_+ M$ (see* [CMMR25, HW25]*). Our approach is based on warped μ-bubbles and to be consistent with other parts of the article.*

Remark 1.14. *The equality case of Theorem 1.12 suggests that the diameter estimate proved in* [AX24] *cannot achieve an equality for $\gamma > 0$, because the completeness requires that S is an $(n-1)$-dimensional sphere and $\alpha = 1$ (from which $\gamma = 0$ for $n > 3$).*

Theorem 1.15. *Let $0 \leqslant \gamma < 3 + \frac{1}{n-2}$, $\Gamma = 1 - \gamma/4$ (note that $\Gamma > 0$), and $[t_-, t_+] \subset I_{\Lambda,\Gamma}$. If a torical band $M = [-1, 1] \times T^{n-1}$ with a metric g and some positive function u satisfy*

(a) $-\gamma u^{-1}\Delta_g u + (n-1)\,\mathrm{Ric} \geqslant \Lambda$,
(b) $H_{\partial_+ M} + \gamma u^{-1} u_{\nu_+} \geqslant \eta(t_+)$ on $\partial_+ M$ and $H_{\partial_- M} + \gamma u^{-1} u_{\nu} \leqslant \eta(t_-)$ on $\partial_- M$,

then

$$\mathrm{width}(M, g) \leqslant t_+ - t_-. \tag{1.7}$$

The rigidity of the band width estimate in Theorem 1.15 is interesting, and we are able to show the following result. Note that the exponent γ now has a smaller range and the dimension is 3.

Theorem 1.16. *Let $0 \leqslant \gamma \leqslant 2$ and (M, g) be a 3-dimensional torical band which satisfies the assumptions of Theorem 1.15, the equality in* (1.7) *is achieved if and only if the torical band is isometric to the model*

$$\bar{g} = \mathrm{d}t^2 + \phi(t)^2 \mathrm{d}s_1^2 + \varphi(t)^2 \mathrm{d}s_2^2,$$

where ϕ and φ are given by the following: If $\Lambda = 0$,

$$\phi'/\phi + \varphi'/\varphi = \tfrac{1-\gamma/2}{1-\gamma/4} t^{-1}, \phi'/\phi - \varphi'/\varphi = F_0 t^{-\frac{1-\gamma/2}{1-\gamma/4}}$$

with $F_0 \in \mathbb{R}$, $t \in [t_-, t_+]$ satisfying $F_0^2 t^{\frac{2\gamma}{4-\gamma}} \leqslant \frac{1-\gamma/2}{1-\gamma/4}$.

If $\Lambda < 0$,

$$\phi'/\phi + \varphi'/\varphi = (1-\gamma/2)\sqrt{\tfrac{-\Lambda}{1-\gamma/4}} \coth(\sqrt{-\Lambda(1-\gamma/4)}t),$$
$$\phi'/\phi - \varphi'/\varphi = F_0 \sinh^{-\frac{1-\gamma/2}{1-\gamma/4}}(\sqrt{-\Lambda(1-\gamma/4)}t),$$

with $F_0 \in \mathbb{R}$, $t \in [t_-, t_+]$ satisfying

$$F_0^2 \sinh^{2\gamma/(4-\gamma)}(\sqrt{-\Lambda(1-\gamma/4)}t) + \Lambda(1-\gamma/2) \leqslant 0.$$

If $\Lambda > 0$,

$$\phi'/\phi + \varphi'/\varphi = (1-\gamma/2)\sqrt{\tfrac{\Lambda}{1-\gamma/4}} \cot(\sqrt{\Lambda(1-\gamma/4)}t),$$
$$\phi'/\phi - \varphi'/\varphi = F_0 \sin^{-\frac{1-\gamma/2}{1-\gamma/4}}(\sqrt{\Lambda(1-\gamma/4)}t),$$

with $F_0 \in \mathbb{R}$, $t \in [t_-, t_+]$ satisfying

$$F_0^2 \sin^{2\gamma/(4-\gamma)}(\sqrt{-\Lambda(1-\gamma/4)}t) - \Lambda(1-\gamma/2) \leqslant 0.$$

The conditions on F_0 and $t \in [t_-, t_+]$ are to ensure that the Ricci curvature normal to the ∂_t direction is greater than or equal to $\mathrm{Ric}(\partial_t, \partial_t)$.

Remark 1.17. *It is interesting to note that the rigid band for Theorem 1.15 is a doubly warped product when $0 \leqslant \gamma < 2$; when $\gamma = 2$, the rigid band is a warped product; but when $2 < \gamma < 4$, there is no rigid band which realizes the width.*

Most of the proof of Theorem 1.15 was laid out in [CS25, Theorems 1.1-1.2]. We only need an additional argument to deal with the boundary which is very similar to that of Theorem 1.12. We omit the proof of Theorem 1.15, see also Remark 5.1.

1.4. **Applications and extensions.** We have found some applications of the band width estimate to the splitting theorems and Geroch conjecture for manifolds with positive spectral scalar curvature with arbitrary ends.

Theorem 1.18. *Let $(M^n, g) = (T^{n-1} \times \mathbb{R}, g)$ and u a positive function on M, if*

$$-\gamma\Delta u + \tfrac{1}{2} R_g u \geq 0$$

for some $0 < \gamma < \frac{2n}{n-1}$ and $3 \leq n \leq 7$, then (M^n, g) is isometric to $(T^{n-1} \times \mathbb{R}, g_{\mathbb{T}^{n-1}} + \mathrm{d}t^2)$ where $g_{\mathbb{T}^{n-1}}$ is a flat metric on T^n and $\mathrm{d}t^2$ is the metric on $\mathbb{R}$.

Below is a generalization of [CL24, Theorem 3] to the spectral setting.

Theorem 1.19. *For any n-manifold ($3 \leqslant n \leqslant 7$) X, the connected sum $M = T^n \# X$ does not admit a complete metric of spectral positive scalar curvature with $0 \leqslant \gamma < \frac{2n}{n-1}$.*

In [HSY25, Corollary 2.7], He-Yu-Shi proved the manifold $N\#X$ does not admit a complete metric of spectral positive scalar curvature with $0 \leq \gamma < 2$, where N is an enlargeable manifold. Theorem 1.19 suggests that closely related results such as [CCZ24] can be generalized to the settings of spectral scalar curvature.

Remark 1.20. *For X closed, this theorem could be easily proved via a conformal change $u^{\frac{n-2}{2(n-1)}\gamma} g$, and we only have to require that $\gamma < \frac{2(n-1)}{n-2}$. We can conclude that g is flat and u is a positive constant.*

In dimension 3, we have the following which is an extension of [CEM19] to the weighted setting.

Theorem 1.21. *Let (M^3, g) be a connected, orientable, complete Riemannian manifold with $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g \geqslant 0$ for some positive function u. Assume that (M, g) contains a properly embedded surface $\Sigma \subset M$ that is both homeomorphic to the cylinder and absolutely weighted area-minimising. Then (M, g) is flat and u is a constant.*

Theorem 1.21 can be easily extended to the case with $-\gamma u^{-1}\Delta_g u + 2\operatorname{Ric}_g$, $0 \leqslant \gamma < 4$. We left it to the reader.

Theorem 1.22. *Theorem 1.21 holds if the non-negativity of spectral scalar curvature is replaced by the condition $-\gamma u^{-1}\Delta_g u + 2\operatorname{Ric}_g \geqslant 0$, $0 \leqslant \gamma < 7/2$.*

Interestingly, we find an application of Theorem 1.22 to the spectral version of the Milnor conjecture and establish the following.

Theorem 1.23. *Let (M^3, g) be a complete oriented, non-compact 3-dimensional manifold with $-\gamma\Delta_M u + \operatorname{Ric}_M u \geqslant 0$ and $-\gamma\Delta_M u + 2\operatorname{Ric}_M u \geqslant 0$ for some $0 \leqslant \gamma < 7/2$, $u > 0$. Then either M is diffeomorphic to $\mathbb{R}^3$ or the universal cover $\tilde{M}^3$ of M^3 is isometric to the product $\bar{M}^2 \times \mathbb{R}$ where $\bar{M}^2$ is a complete 2-manifold with nonnegative Ricci curvature.*

Finally, we would like to remark that there is some freedom to consider the spectral curvature condition with an extra gradient term $c|\nabla_g u|^2/u^2$, $c \in \mathbb{R}$ in most of the results obtained in this article, for example,

$$\mathcal{R}_c = -\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g + c\gamma u^{-2}|\nabla_g u|^2.$$

One can also consider $-\gamma u^{-1}\Delta_g u +$ (*resp.* 2) $\operatorname{Ric}_g + c\gamma u^{-2}|\nabla u|^2$. The reason is the observation that

$$u^{-2}|\nabla_g u|^2 = |\nabla_g w|^2 = |\nabla_\Sigma w|^2 + w_\nu^2,$$

and we can run the same procedures as done in Subsection 2.3. As a consequence, we can generalize the results which are for $c = 0$ to the case $c \neq 0$ with suitable range of c and γ. Now setting $c = 1 - \gamma/2$, $f = -\gamma \ln u$ gives the Perelman scalar curvature

$$\begin{aligned} P &= R_g + 2\Delta f - |\nabla f|^2 \\ &= R_g - 2\gamma u^{-1}\Delta_g u + (2\gamma - \gamma^2)u^{-2}|\nabla u|^2 \end{aligned}$$

which was introduced by Perelman in his gradient flow formulation of the Ricci flow. Recently, the Perelman scalar curvature has sparkled some interests, see [CZ23] and the references therein.

The article is organized as follows:

In Section 2, we introduce basics of warped μ-bubbles including the first, second variation formulas.

In Section 3, we make use of the weighted minimal hypersurfaces, in particular, we prove Theorems 1.4 and 1.5.

In Section 4, we develop the rigidity results Theorems 1.6, 1.9 and 1.7 by using hypersurfaces of constant weighted mean curvature.

In Section 5, by selecting suitable μ we prove several band width estimates which include a band width interpretation of the Bonnet-Myers theorem (Theorem 1.12).

In the final Section 6, we show some applications of the band width estimates and extend some of the results in the earlier sections to the non-compact setting.

In Appendix A, we record some curvature computation for a warped product and a doubly warped product.

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2. Basics of warped μ-bubbles

In this section, we introduce our main technical tool, warped μ-bubbles including the first, second variation formulas and the rewrites which relate the second variation to the spectral curvature condition. The warped μ-bubbles includes weighted minimal hypersurfaces and hypersurfaces of constant weighted mean curvature as special cases.

2.1. **Warped μ-bubble in bands.** Let Ω be a Cacciopolli set which contains a neighborhood of $\partial_- M$ and disjoint from $\partial_+ M$, a positive function u and a Lipschitz function $h \in C^{0,1}(\bar{M})$, we define

$$E(\Omega) = \int_{\partial\Omega\cap \mathrm{int} M} u^\gamma \mathrm{d}\mathcal{H}^{n-1} - \int_\Omega h u^\gamma \mathrm{d}\mathcal{H}^n.$$

Let Ω be a Cacciopolli set and Σ be a connected component of $\partial\Omega \cap \mathrm{int} M$. Let Σ_t be a variation of Σ with the variation vector field given by Y, and Ω_t be the Caccioppolli set enclosed by $\partial\Omega \setminus \Sigma$ and Σ_t, then

$$\tfrac{\mathrm{d}}{\mathrm{d}t} E(\Omega_t)|_{t=0} = \int_\Sigma (H + \gamma u^{-1} u_\nu - h)\langle Y, \nu\rangle u^\gamma \mathrm{d}\mathcal{H}^{n-1}. \tag{2.1}$$

We say that Ω is a *warped h-bubble* if Ω is a critical point of E, in particular (2.1) vanishes for all Y. If Ω is a minimiser of E, we call Ω a minimising h-bubble. We say Σ a *warped h-hypersurface* if $H + \gamma u^{-1} u_\nu = 0$ along Σ. If $h = 0$ along Σ, we say that Σ is a *warped or weighted minimal hypersurface*. The terminology of warped h-hypersurface and warped minimal hypersurface is just for convenience.

Given a warped h-hypersurface Σ, we can calculate the first variation (or linearisation) of $H + \gamma u^{-1} u_\nu - h$, and we obtain

$$\begin{aligned}
&\delta(H + \gamma u^{-1} u_\nu - h)\\
=& - \Delta_\Sigma \phi + (-|A|^2 - \mathrm{Ric}(\nu,\nu) - \gamma u^{-2} u_\nu^2 + \gamma u^{-1} \nabla^2_{\nu\nu} u - h_\nu)\phi - \gamma u^{-1}\langle \nabla_\Sigma \phi, \nabla_\Sigma u\rangle\\
=& - \Delta_\Sigma \phi + (-|A|^2 - \mathrm{Ric}(\nu,\nu) - \gamma u^{-2} u_\nu^2 + \gamma u^{-1} (\Delta_g u - H u_\nu - \Delta_\Sigma u) - h_\nu)\phi\\
& - \gamma u^{-1}\langle \nabla_\Sigma \phi, \nabla_\Sigma u\rangle := L_\Sigma \phi.
\end{aligned} \tag{2.2}$$

In the above, we have used that $\nabla^2_{\nu\nu} = \Delta_g u - H u_\nu - \Delta_\Sigma u$. It follows immediately that given a critical point of E, we have the second variation

$$\tfrac{\mathrm{d}^2}{\mathrm{d}t^2} E(\Omega_t)|_{t=0} = \int_\Sigma u^\gamma \phi L_\Sigma \phi,$$

where Σ, Σ_t, Ω_t and L_Σ are given as above, see [AX24, CS25].

2.2. **Variation of a warped h-hypersurface.**

Definition 2.1. *We say that Σ a warped h-hypersurface is stable there exists a positive function ϕ such that $L\phi \geqslant 0$.*

The eigenvalue with least real part is called the *principal eigenvalue*. By Krein-Rutman theorem (see [AMS05]), the principal eigenvalue of L_Σ is real and the corresponding eigenfunction has a sign which we choose here the positive. Equivalently, if Σ is stable, then the principal eigenvalue is non-negative.

Lemma 2.2. *If Σ is a non-stable warped h-hypersurface, then there exits a hypersurface Σ_- which lies in the side of Σ which ν points into and $H_{\Sigma_-}+\gamma u^{-1}\langle\nabla u,\nu_{\Sigma_-}\rangle - h<0$.*

Proof. Let $\phi>0$ be the principal eigen-function of L_Σ. Since Σ is non-stable, $L\phi<0$. Let Y be a vector field defined in an open neighborhood of Σ and such that $Y=\phi\nu$ along Σ and Φ_t be the local flow of Y, set $\Sigma_t=\Phi_t(\Sigma)$. By the Taylor expansion,

$$H_{\Sigma_t}+\gamma u^{-1}\langle\nabla u,\nu_{\Sigma_t}\rangle - h = tL_\Sigma\phi + O(t^2)<0$$

for all $t>0$ sufficiently small. Taking $\Sigma_-=\Sigma_t$ for a fixed small t finishes the proof. □

Lemma 2.3. *If Σ is a warped h-hypersurface such that $\delta(H+\gamma u^{-1}u_\nu-h)=-\Delta_\Sigma\phi$, then there exists a foliation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ such that $H_{\Sigma_t}+\gamma u^{-1}u_{\nu_t}-h$ is constant along each Σ_t (i.e., depending only on t).*

For the proof, see [CS25, Lemma 3.4] where actually only the facts that $H+\gamma u^{-1}u_\nu-h=0$ and that $\delta(H+\gamma u^{-1}u_\nu-h)=-\Delta_\Sigma\phi$ were needed. Specially, such a foliation exists if Σ is infinitesimally rigid. Infinitesimal rigidity of a warped h-hypersurface Σ is a condition stronger than the condition that Σ satisfies that $\delta(H+\gamma u^{-1}u_\nu-h)=-\Delta_\Sigma\phi$. However, the infinitesimal rigidity is a condition which depends on the context, and to save the bother of stating the condition of infinitesimal rigidity for every case, we will refer to Lemma 2.3.

We find it useful to have the following (cf. [AM09, Lemma 5.2]).

Lemma 2.4. *If Σ satisfies $H+\gamma u^{-1}u_\nu-h\lneqq 0$, there exists a hypersurface Σ_- near Σ lying to the side which ν points into such that $H_{\Sigma_-}+\gamma u^{-1}u_{\nu_-}-h<0$.*

Proof. We run the mean curvature flow

$$\partial_t x=-(H+\gamma u^{-1}u_\nu-h)\nu,\ x\in S$$

starting from Σ. Here S is a manifold diffeomorphic to Σ. Let $\Sigma_t=x(t,S)$. By writing the equation as a graph of a function u over Σ, we see only $-H\nu$ contains second order derivatives of u, hence the flow is a quasi-linear parabolic equation. By standard theory, the flow exists in a short time interval $[0,t_0)$. We have the evolution equation for $\tilde{H}$ that

$$\begin{aligned}
&\partial_t \tilde{H} \\
=&\Delta_{\Sigma_t}\tilde{H} - (-|A|^2 - \mathrm{Ric}(\nu,\nu) - \gamma u^{-2}u_\nu^2 + \gamma u^{-1}\nabla^2_{\nu\nu}u - h_\nu)\tilde{H} + \gamma u^{-1}\langle \nabla_{\Sigma_t}\tilde{H}, \nabla_{\Sigma_t}u\rangle \\
=:&\Delta_{\Sigma_t}\tilde{H} - Q(x,t)\tilde{H} + \gamma u^{-1}\langle \nabla_{\Sigma_t}\tilde{H}, \nabla_{\Sigma_t}u\rangle
\end{aligned}$$

using the short hand $\tilde{H} = H + \gamma u^{-1}u_\nu - h$ and the first variation of $\tilde{H}$ (2.2). Here, $x \in \Sigma_t$. By existence of short time, we can assume that Q is bounded on $[0, t_0/2]$. We take

$$K > \max_{x\in\Sigma_t, t\in[0,t_0]} |Q(x,t)|.$$

Then

$$(\partial_t - \Delta_{\Sigma_t})(e^{-Kt}\tilde{H}) = \gamma u^{-1}\langle \nabla_{\Sigma_t}(e^{-Kt}\tilde{H}), \nabla_{\Sigma_t}u\rangle - (Q+K)(e^{-Kt}\tilde{H}),$$

and the coefficient of the zeroth term is negative. Hence, by the strong maximum principle of parabolic equations, $e^{-Kt}\tilde{H} < 0$ for all $t \in (0, t_0/2]$. Take any Σ_t, $t \in [0, t_0/2)$ as Σ_- would suffice. □

2.3. **Rewrite of second variation.**

Lemma 2.5. *The second variation* (2.1) *of the functional* $E(\Omega)$ *can be rewritten as*

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}E(\Omega_t)|_{t=0} = &\frac{4}{4-\gamma}\int_\Sigma |\nabla_\Sigma \psi|^2 - \int_\Sigma (1-\tfrac{\gamma}{4})\gamma \left|\psi\nabla_\Sigma w - \frac{1}{2(1-\gamma/4)}\nabla_\Sigma\psi\right|^2 \\
&+ \int_\Sigma (\gamma u^{-1}\Delta_g u - (|A|^2 + \mathrm{Ric}(\nu,\nu)))\psi^2 \\
&- \int_\Sigma (\gamma H w_\nu + h_\nu + \gamma w_\nu^2)\psi^2,
\end{aligned}$$

where $\psi = \phi u^{\gamma/2}$ *and* $w = \log u$.

Proof. From [CS25, Lemma 2.4], we see

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}t^2}E(\Omega_t)|_{t=0} = &\int_\Sigma |\nabla_\Sigma \psi|^2 + \int_\Sigma (\gamma\psi\langle \nabla_\Sigma w, \nabla_\Sigma \psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_\Sigma w|^2) \\
&+ \int_\Sigma (\gamma u^{-1}\Delta_g u - (|A|^2 + \mathrm{Ric}(\nu,\nu)))\psi^2 \\
&- \int_\Sigma (\gamma H w_\nu + h_\nu + \gamma w_\nu^2)\psi^2.
\end{aligned}$$

The following identity

$$\begin{aligned}&(\tfrac{\gamma^2}{4}-\gamma)\psi^2|\nabla_\Sigma w|^2+\gamma\psi\langle\nabla_\Sigma w,\nabla_\Sigma\psi\rangle\\=&\tfrac{1}{4}(1-\tfrac{\gamma}{4})^{-1}\gamma|\nabla_\Sigma\psi|^2-(1-\tfrac{\gamma}{4})\gamma\left|\psi\nabla_\Sigma w+\tfrac{1}{2(\gamma/4-1)}\nabla_\Sigma\psi\right|^2\end{aligned}$$

finishes the proof. □

Now we set $w=\log u$ and

$$Z=-\gamma u^{-1}\Delta_g u+(|A|^2+\mathrm{Ric}(\nu,\nu))+\gamma Hw_\nu+h_\nu+\gamma w_\nu^2. \tag{2.3}$$

First, in proving various band width estimates, the function h will be chosen as $\eta\circ\rho$ where η is a non-increasing function and ρ a Lipschitz function with $|\nabla\rho|\leqslant 1$; we can choose η constant in other cases. So

$$h_\nu=\eta'\circ\rho\langle\nabla\rho,\nu\rangle=\eta'\circ\rho(\langle\nabla\rho,\nu\rangle-1)+\eta'\circ\rho.$$

For different spectral curvature conditions, we rewrite Z in different ways where $H+\gamma w_\nu=h$.

I. Case $-\gamma u^{-1}\Delta_g u+\mathrm{Ric}$ (e.g., for Theorem 1.12) with $0\leqslant\gamma<\frac{4}{n-1}$: we use $|A|^2=(|A|^2-\frac{1}{n-1}H^2)+\frac{1}{n-1}H^2$, $H+\gamma w_\nu=h$ in (2.3), and with suitable rearrangement, we obtain that

(2.4)

$$\begin{aligned}W:=&(-\gamma u^{-1}\Delta_g u+\mathrm{Ric}(\nu))+(|A|^2-\tfrac{1}{n-1}H^2)\\&+\tfrac{1}{n-1}(h-\gamma w_\nu)^2+\gamma(h-\gamma w_\nu)w_\nu+h_\nu+\gamma w_\nu^2\\=&(-\gamma u^{-1}\Delta_g u+\mathrm{Ric}(\nu))+(|A|^2-\tfrac{1}{n-1}H^2)\\&+\tfrac{1}{n-1}h^2+\tfrac{n-3}{n-1}\gamma hw_\nu+\gamma(1-\tfrac{n-2}{n-1}\gamma)w_\nu^2+h_\nu\\=&\Big(\tfrac{4-(n-1)\gamma}{4(n-1-(n-2)\gamma)}(\eta\circ\rho)^2+\eta'\circ\rho+(-\gamma u^{-1}\Delta_g u+\mathrm{Ric})\Big)+(\mathrm{Ric}(\nu,\nu)-\mathrm{Ric})\\&+(|A|^2-\tfrac{1}{n-1}H^2)\\&+(1-\tfrac{n-2}{n-1}\gamma)\gamma(w_\nu+\tfrac{n-3}{2(n-1-(n-2)\gamma)}h)^2\\&+\eta'\circ\rho(\langle\nabla\rho,\nu\rangle-1),\end{aligned}\tag{2.5}$$

and Z, W are related by

$$Z=W+\tfrac{1}{n-1}\tilde{H}^2+\tilde{H}(\tfrac{2}{n-1}(h-\gamma w_\nu)+\gamma w_\nu).$$

II. Case $-\gamma u^{-1}\Delta_g u+\frac{1}{2}R_g$ (e.g., for Theorem 1.10): Using the Schoen-Yau's rewrite [SY79b]

$$\begin{aligned}2(|A|^2+\mathrm{Ric}(\nu))=&H^2+|A|^2+R_g-R_\Sigma\\=&(|A|^2-\tfrac{1}{n-1}H^2)+\tfrac{n}{n-1}H^2+R_g-R_\Sigma\end{aligned}\tag{2.6}$$

of $\mathrm{Ric}(\nu)$ in (2.3). And with similar argument as in the previous case, we obtain that

$$W := -\tfrac{1}{2}R_\Sigma + \Big(\tfrac{2n-(n-1)\gamma}{4(n-1)-2(n-2)\gamma}(\eta\circ\rho)^2 + \eta'\circ\rho + (-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g)\Big) + \tfrac{1}{2}(|A|^2 - \tfrac{1}{n-1}H^2)$$
$$+ (1 - \tfrac{n-2}{2(n-1)}\gamma)(w_\nu - \tfrac{1}{(2(n-1)-(n-2)\gamma)}h)^2$$
$$+ \eta'\circ\rho(\langle\nabla\rho,\nu\rangle - 1).$$

And Z and W are related by

$$Z = W + \tfrac{n}{2(n-1)}\tilde{H}^2 + \tfrac{1}{n-1}\tilde{H}(nh - \gamma w_\nu). \tag{2.7}$$

III. Case $-\gamma u^{-1}\Delta_g u + (n-1)\,\mathrm{Ric}$ (e.g., for Theorem 1.15): let $\{e_i\}_{1\leqslant i\leqslant n-1}$ be an orthonormal frame along Σ, and using

$$\mathrm{Ric}_g(\nu,\nu) + |A|^2 = \sum_{i\leqslant n-1}\mathrm{Ric}_g(e_i,e_i) + H^2 - R_\Sigma,$$

(as easily seen by the definition of the scalar curvature, the above is equivalent to (2.6); see also [Zhu21, (5.2)]) and following arguments in the previous cases, we obtain

$$\begin{aligned} W = &- R_\Sigma + ((1 - \tfrac{1}{4}\gamma)(\eta\circ\rho)^2 + \eta'\circ\rho + (-\gamma u^{-1}\Delta_g u + (n-1)\,\mathrm{Ric})) \\ &+ (\sum_{i\leqslant n-1}\mathrm{Ric}_g(e_i) - (n-1)\,\mathrm{Ric}_g) + \gamma(w_\nu - \tfrac{1}{2}h)^2 \\ &+ \eta'\circ\rho(\langle\nabla\rho,\nu\rangle - 1), \end{aligned}$$

and Z, W are related by

$$Z = W + \tilde{H}^2 + \tilde{H}(2h - \gamma w_\nu).$$

Remark 2.6. *Note that $Z = W$ when $H + \gamma w_\nu = h$ along Σ in every case.*

3. Classification of stable weighted minimal surfaces

In this section, we show some immediate applications of weighted minimal surfaces, which are the simplest case of the warped h-bubbles by taking h to be identically zero in dimension 3. In particular, we prove Theorem 1.4.

3.1. **Classification of stable surfaces.**

Proof of Theorem 1.4. For a compact, stable weighted minimal surface Σ, the second variation (2.1) is non-negative, so the rewrite (Lemma 2.5 and case **II** in Subsection 2.3) yields

(3.1)

$$0 \leqslant \tfrac{4}{4-\gamma}\int_\Sigma |\nabla_\Sigma\psi|^2 - \int_\Sigma (1-\tfrac{\gamma}{4})\gamma \left|\psi\nabla_\Sigma w - \tfrac{1}{2(1-\gamma/4)}\nabla_\Sigma\psi\right|^2 + \tfrac{1}{2}\int_\Sigma R_\Sigma\psi^2 - \int_\Sigma (Z+\tfrac{1}{2}R_\Sigma)\psi^2.$$

We note that $Z+\frac{1}{2}R_\Sigma \geqslant 0$ by the assumptions, by taking $\psi \equiv 1$ and using the Gauss-Bonnet theorem, $2\pi\chi(\Sigma) \geqslant 0$. Hence, Σ can only be a sphere or a torus.

In the case of a torus, then $\nabla_\Sigma w = 0$ and $Z + \frac{1}{2}R_\Sigma = 0$ (which forces $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g = 0$, $A = \frac{1}{n-1}H$ and $w_\nu = 0$). Let $\tilde{L}_\Sigma = -\frac{4}{4-\gamma}\Delta_\Sigma + \frac{1}{2}R_\Sigma$. Since taking $\psi = 1$ in (3.1) implies that the right hand side of (3.1) must vanish, hence the eigenvalue of $\tilde{L}_\Sigma$ is zero and the corresponding eigenfunction is 1, that is, $0 = \tilde{L}_\Sigma\psi = \frac{1}{2}R_\Sigma$. Hence Σ is flat. We conclude that Σ satisfies the assumptions of Lemma 2.3 with $h = 0$.

Now we show that (M, g) locally splits if Σ is weighted area-minimising. Using Lemma 2.3, there exits a foliation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ of constant $H + \gamma u^{-1}u_\nu$ near Σ and $\Sigma_0 = \Sigma$. By (2.2),

$$\phi_t^{-1}\tilde{H}'(t) = -\phi_t^{-1}\Delta_{\Sigma_t}\phi_t - \gamma\phi_t^{-1}\langle\nabla_{\Sigma_t}w, \nabla_{\Sigma_t}\phi_t\rangle - \gamma u^{-1}\Delta_{\Sigma_t}u - Z.$$

Setting ξ_t to be $\phi_t = u^{-\gamma/2}e^{\xi_t}$ and using (2.7), we see

$$\begin{aligned}
&\phi_t^{-1}\tilde{H}'(t)\\
=&-|\nabla_{\Sigma_t}\xi_t|^2 - \Delta_{\Sigma_t}\xi_t + (\tfrac{\gamma^2}{4}-\gamma)|\nabla_{\Sigma_t}w|^2 - \tfrac{\gamma}{2}\Delta_{\Sigma_t}w + \tfrac{1}{2}R_{\Sigma_t} - (W + \tfrac{1}{2}R_{\Sigma_t})\\
&-\tfrac{3}{4}\tilde{H}^2 - \tfrac{1}{2}\tilde{H}(3h - \gamma w_\nu).
\end{aligned}$$

By integration on both sides, the divergence theorem and the Gauss-Bonnet theorem,

$$\begin{aligned}
&\tilde{H}'(t)\int_{\Sigma_t}\phi_t^{-1}\\
\leqslant&-\int_{\Sigma_t}(|\nabla_{\Sigma_t}\xi_t|^2 + (\tfrac{\gamma^2}{4}-\gamma)|\nabla_{\Sigma_t}w|^2) + 2\pi\chi(\Sigma_t) - \int_{\Sigma_t}(W+\tfrac{1}{2}R_{\Sigma_t}) - \tfrac{1}{2}\tilde{H}\int_{\Sigma_t}(3h-\gamma w_\nu)\\
\leqslant&2\pi\chi(\Sigma_t) - \tfrac{1}{2}\tilde{H}\int_{\Sigma_t}(3h-\gamma w_\nu)\\
=&-\tfrac{1}{2}\tilde{H}\int_{\Sigma_t}(3h-\gamma w_\nu)
\end{aligned}$$

where we have used $0 \leqslant \gamma < 3$ and that $W + \frac{1}{2}R_{\Sigma_t} \geqslant 0$. Hence,

$$\tilde{H}'(t) \leqslant -\tfrac{1}{2}\tilde{H}\frac{\int_{\Sigma_t}(3h - \gamma w_\nu)}{\int_{\Sigma_t}\phi_t^{-1}},$$

which by solving we conclude that $\tilde{H} \leqslant 0$ for all $t \in [0, \varepsilon)$ and $\tilde{H} \geqslant 0$ for all $t \in (-\varepsilon, 0]$. By (2.1), Σ_t is also weighted area-minimising. Hence, the argument works for Σ also works for Σ_t.

Since $H + \gamma u^{-1}u_\nu = 0$ and $w_\nu = 0$, Σ_t is minimal. From $A - \frac{1}{n-1}H = 0$, we see Σ_t is totally geodesic. So (M, g) locally splits and hence u is a positive constant.

In the case that Σ is non-compact, (3.1) holds for all $\psi \in C_c^\infty(\Sigma)$ which implies that the operator

$$L = -\tfrac{4}{4-\gamma}\Delta_\Sigma + \tfrac{1}{2}R_\Sigma - (Z + \tfrac{1}{2}R_\Sigma)$$

in the condition that $0 \leqslant \gamma < 3$, is non-negative by (2.5). By [BC14, Theorems 1.1-1.3], Σ is conformally equivalent to either the complex plane $\mathbb{C}$ or the cylinder $\mathbb{A}$; in the case of a cylinder, Σ is flat and $Z + \frac{1}{2}R_\Sigma = 0$ which implies $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g = 0$, $A = \frac{1}{n-1}H$ and $w_\nu = 0$. By taking a simple cutoff ψ approximating 1 in (3.1), we see $\nabla_\Sigma w = 0$. □

With almost the same proof, we can give a proof of Theorem 1.5.

Proof of Theorem 1.5. It suffices to replace the rewrites of Z and W by case **III** of Subsection 2.3. The proof is almost verbatim except deriving the consequences of $\nabla_\Sigma w = 0$, $R_\Sigma = 0$ and $Z + R_\Sigma = 0$. The condition $Z + R_\Sigma = 0$ implies that $\mathrm{Ric}_g = \mathrm{Ric}_g(e_i, e_i)$, $-\gamma u^{-1}\Delta_g u + 2\,\mathrm{Ric}_g = 0$ and $w_\nu = 0$, where e_i is any tangent vector of Σ. Using [CS25, Subsection 3.4], (M, g) is locally a doubly warped product, say $\mathrm{d}t^2 + \phi(t)^2\mathrm{d}s_1^2 + \varphi(t)^2\mathrm{d}s_2^2$ which satisfies (A.1) and (A.2). Denote t-level set by Σ_t. First,

$$H = \phi'/\phi + \varphi'/\varphi = -\gamma w_\nu = 0.$$

By (A.2), $\phi'/\phi = \varphi'/\varphi$. Hence, $\phi'/\phi = \varphi'/\varphi = 0$ giving that both ϕ and φ are constants. □

4. Hypersurfaces of constant weighted mean curvature

In this section, we prove Theorems 1.6, 1.9 and 1.7 by choosing h to be a constant in the definitions of warped h-bubble.

Note that now we have the structure of a band, that is, two boundary components. And, along the boundaries, the barrier condition is not strict. We need to address the existence of a warped h-bubble first.

4.1. **Spectral scalar curvature case.**

Theorem 4.1. *Let γ, Λ, α, β and η be as in Theorem 1.6. There does not exist a torical band (M, g) such that $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g \geqslant \Lambda$, $H + \gamma u^{-1}u_\nu > (n-1)\alpha + \gamma\beta$ along $\partial_+ M$ and $H + \gamma u^{-1}u_\nu < (n-1)\alpha + \gamma\beta$.*

Proof. We assume that there exists such a torical band (M, g). Since $\eta = (n-1)\alpha + \beta\gamma$, using the strict barrier condition, we have a minimising warped η-bubble Ω. Let Σ be one of the connected component of $\partial\Omega\backslash\partial_- M$.

Using the stability inequality with the rewrite **II** of Subsection 2.3, we obtain

$$0 \leqslant \int_\Sigma \left(\tfrac{4}{4-\gamma}|\nabla_\Sigma \psi|^2 + \tfrac{1}{2}R_\Sigma \psi^2\right) - \int_\Sigma (1-\tfrac{\gamma}{4})\gamma\psi^2 \left|\nabla_\Sigma w - \tfrac{1}{2(1-\gamma/4)} \tfrac{\nabla_\Sigma \psi}{\psi}\right|^2 - \int_\Sigma (Z + \tfrac{1}{2}R_\Sigma)\psi^2. \tag{4.1}$$

For $n = 3$, we use an argument similar to Theorme 1.4. We deal with $n \geqslant 4$. Let

$$L = -\tfrac{4}{4-\gamma}\Delta_\Sigma + \tfrac{1}{2}R_\Sigma - (Z + \tfrac{1}{2}R_\Sigma).$$

Let λ_1 be the first eigenvalue of L and v be the corresponding eigenfunction. Note that $\lambda_1 \geqslant 0$ by (4.1).

We define a constant κ by $4(n-2)\kappa/(n-3) = 8/(4-\gamma)$. By the range of γ, $\kappa \in (0, 1)$. Let $\hat{g} = (v^\kappa)^{4/(n-3)} g|_\Sigma$ be the conformal metric. Then the scalar curvature of Σ with respect to $\hat{g}$ is

$$\begin{aligned}
(v^\kappa)^{\frac{n+1}{n-3}} R_\Sigma(\hat{g}) &= v^\kappa R_\Sigma - \tfrac{4(n-2)}{n-3}\Delta_\Sigma v^\kappa \\
&= v^\kappa (R_\Sigma - \tfrac{4(n-2)}{n-3}\alpha v^{-1}\Delta_\Sigma v - \tfrac{4(n-2)}{n-3}\kappa(\kappa-1)v^{-2}|\nabla_\Sigma v|^2) \\
&\geqslant v^\kappa (\tfrac{2Lv}{v} + (Z + \tfrac{1}{2}R_\Sigma) - \tfrac{4(n-2)}{n-3}\kappa(\kappa-1)v^{-2}|\nabla_\Sigma v|^2) \\
&= v^\kappa \left(2\lambda_1 + (Z + \tfrac{1}{2}R_\Sigma) - \tfrac{4(n-2)}{n-3}\kappa(\kappa-1)v^{-2}|\nabla_\Sigma v|^2\right) \\
&\geqslant 0.
\end{aligned}$$

Using the assumptions, it is direct to check that $Z + \frac{1}{2}R_\Sigma \geqslant 0$ along Σ. So $R_\Sigma(\hat{g}) \geqslant 0$. By the resolution of the Geroch conjecture and that $0 < \kappa < 1$, $R_\Sigma(\hat{g})$ has to vanish which implies that $\lambda_1 = 0$, $\nabla_\Sigma v = 0$, $Z + \frac{1}{2}R_\Sigma = 0$ (i.e., $A - \frac{1}{n-1}H = 0$, $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g = \Lambda$, $w_\nu = \frac{1}{2(n-1)-(n-2)\gamma}\eta$) and $R_\Sigma = 0$. Now with $\psi = v = 1$ in (4.1) whose right hand side vanishes, we have $\nabla_\Sigma w = 0$. So we have show that Σ is infinitesimally rigid. It follows from Theorem 2.3 that we have a foliation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ near Σ.

Claim: $\tilde{H}_t \leqslant 0$ for $t \in (0, \varepsilon)$ and $\tilde{H} \geqslant 0$ for $t \in (-\varepsilon, 0)$.

To show this claim, we repeat an argument in [CS25, Lemma 4.4]. By (2.2),

$$\phi_t^{-1}\tilde{H}'(t) = -\phi_t^{-1}\Delta_{\Sigma_t}\phi_t - \gamma\phi_t^{-1}\langle\nabla_{\Sigma_t} w, \nabla_{\Sigma_t}\phi_t\rangle - \gamma u^{-1}\Delta_{\Sigma_t} u - Z.$$

Setting ξ_t to be $\phi_t = u^{-\gamma/2}e^{\xi_t}$ and using (2.7), we see

$$\begin{aligned}&\phi_t^{-1}\tilde{H}'(t)\\ =&-|\nabla_{\Sigma_t}\xi_t|^2-\Delta_{\Sigma_t}\xi_t+(\tfrac{\gamma^2}{4}-\gamma)|\nabla_{\Sigma_t}w|^2-\tfrac{\gamma}{2}\Delta_{\Sigma_t}w+\tfrac{1}{2}R_{\Sigma_t}-(W+\tfrac{1}{2}R_{\Sigma_t})\\ &-\tfrac{n}{2(n-1)}\tilde{H}^2-\tfrac{1}{n-1}\tilde{H}(nh-\gamma w_\nu),\end{aligned}$$

which leads to

$$\tilde{H}'+q_t\tilde{H}\leqslant\phi_t(-|\nabla_{\Sigma_t}\xi_t|^2-\Delta_{\Sigma_t}\xi_t+(\tfrac{\gamma^2}{4}-\gamma)|\nabla_{\Sigma_t}w|^2-\tfrac{\gamma}{2}\Delta_{\Sigma_t}w+\tfrac{1}{2}R_{\Sigma_t}).$$

Here, $q_t := -\frac{1}{n-1}(nh-\gamma w_\nu)\phi_t$. For each t, so for any positive function $\varphi\in C^2(\Sigma_t)$,

$$\tilde{H}'\int_{\Sigma_t}\varphi+\tilde{H}\int_{\Sigma_t}q_t\varphi\leqslant\int_{\Sigma_t}\varphi\phi_t(-|\nabla_{\Sigma_t}\xi_t|^2-\Delta_{\Sigma_t}\xi_t+(\tfrac{\gamma^2}{4}-\gamma)|\nabla_{\Sigma_t}w|^2-\tfrac{\gamma}{2}\Delta_{\Sigma_t}w+\tfrac{1}{2}R_{\Sigma_t}).\tag{4.2}$$

It suffices to show that there exists a positive function φ such that the right hand side is non-positive. Assume the contrary, and without loss of generality, we can replace $\varphi\phi_t$ by φ^2. First,

$$\begin{aligned}&(|\nabla_{\Sigma_t}\xi_t|^2+\Delta_{\Sigma_t}\xi_t)\varphi^2\\ =&|\nabla_{\Sigma_t}\xi_t|^2\psi^2-2\langle\nabla_{\Sigma_t}\varphi,\varphi\nabla_{\Sigma_t}\xi_t\rangle+\mathrm{div}_{\Sigma_t}(\varphi^2\nabla_{\Sigma_t}\xi_t)\\ \geqslant&-|\nabla_{\Sigma_t}\varphi|^2+\mathrm{div}_{\Sigma_t}(\varphi^2\nabla_{\Sigma_t}\xi_t).\end{aligned}$$

It follows from integration by parts that

$$-\int(|\nabla_{\Sigma_t}\xi_t|^2+\Delta_{\Sigma_t}\xi_t)\varphi^2\leqslant\int_{\Sigma_t}|\nabla_{\Sigma_t}\varphi|^2.$$

And also

$$\tfrac{\gamma}{2}\int_{\Sigma_t}\varphi^2\Delta_{\Sigma_t}w=-\int_{\Sigma_t}\gamma\varphi\langle\nabla_{\Sigma_t}w,\nabla_{\Sigma_t}\varphi\rangle.$$

So

$$\begin{aligned}&\int_{\Sigma_t}(-|\nabla_{\Sigma_t}\xi_t|^2-\Delta_{\Sigma_t}\xi_t+(\tfrac{\gamma^2}{4}-\gamma)|\nabla_{\Sigma_t}w|^2-\tfrac{\gamma}{2}\Delta_{\Sigma_t}w+\tfrac{1}{2}R_{\Sigma_t})\varphi^2\\ =&\int_{\Sigma_t}(|\nabla_{\Sigma_t}\varphi|^2+\gamma\varphi\langle\nabla_{\Sigma_t}w,\nabla_{\Sigma_t}\varphi\rangle+(\tfrac{\gamma^2}{4}-\gamma)|\nabla_{\Sigma_t}w|^2\varphi^2+\tfrac{1}{2}R_{\Sigma_t}\varphi^2)\\ =&\int_{\Sigma_t}(\tfrac{4}{4-\gamma}|\nabla_{\Sigma_t}\varphi|^2-(1-\tfrac{\gamma}{4})\gamma\left|\varphi\nabla_{\Sigma_t}w-\tfrac{1}{2(1-\gamma/4)}\nabla_{\Sigma_t}\varphi\right|^2+\tfrac{1}{2}R_{\Sigma_t}\varphi^2)\\ >&0.\end{aligned}$$

Let $L=-\frac{4}{4-\gamma}\Delta_{\Sigma_t}+\frac{1}{2}R_{\Sigma_t}$, λ_1 be the first eigenvalue of L and $v>0$ be the first eigenfunction. By the above inequality, $Lv=\lambda_1 v>0$. Let $\hat{g}_t=(v^\kappa)^{4/(n-3)}g|_{\Sigma_t}$ be the conformal metric. Then the scalar curvature of Σ_t with respect to $\hat{g}$ is

$$\begin{aligned}(v^\kappa)^{\frac{n+1}{n-3}} R_{\Sigma_t}(\hat{g}_t) &= v^\kappa R_{\Sigma_t} - \tfrac{4(n-2)}{n-3}\Delta_{\Sigma_t} v^\kappa \\ &= v^\kappa (R_{\Sigma_t} - \tfrac{4(n-2)}{n-3}\kappa v^{-1}\Delta_{\Sigma_t} v - \tfrac{4(n-2)}{n-3}\kappa(\kappa-1)v^{-2}|\nabla_{\Sigma_t} v|^2) \\ &\geqslant v^\kappa (\tfrac{2Lv}{v} - \tfrac{4(n-2)}{n-3}\kappa(\kappa-1)v^{-2}|\nabla_{\Sigma_t} v|^2) \\ &= v^\kappa \left(2\lambda_1 - \tfrac{4(n-2)}{n-3}\kappa(\kappa-1)v^{-2}|\nabla_{\Sigma_t} v|^2\right) \\ &> 0.\end{aligned}$$

This is in contradiction with the resolution of the Geroch conjecture. Hence, for each t, there exists some φ such that the right hand side of (4.2) is non-negative. Choosing such φ for each t, and solving (4.2), we finish the proof of the claim.

By the first variation (2.1), every Σ_t gives rise to a minimiser to the warped functional. And the rigidity extend to all M, which means that there exists a leaf would meet $\partial_- M$ tangentially. However, the strong maximum principle implies that $\partial_- M$ satisfies $H + \gamma u^{-1} u_\nu = \eta$. This is a contradiction to the assumption of the existence of (M, g). □

Now we are ready to finish the proof of Theorem 1.6.

Proof of Theorem 1.6. First, for $\partial_- M$, we claim that either there exists a hypersurface Σ_- near $\partial_- M$ such that $H_{\Sigma_-} + \gamma u^{-1}\langle \nabla u, \nu_{\Sigma_-}\rangle < \eta$ or there exits a *maximal* foliation $\{\Sigma_t^-\}_{t\in[0,t_1]}$ such that $\Sigma_0^- = \partial_- M$ and every leaf is of vanishing $H + \gamma u^{-1} u_\nu - \eta$. The maximality means that either the foliation foliates all of M or if the foliation were to extend beyond $\Sigma_{t_1}^-$ (since $\Sigma_{t_1}^-$ is stable by Definition 2.1) which will give a leaf Σ_- with $H + \gamma u^{-1} u_\nu - \eta < 0$ in the extended foliation.

Indeed, if $H + \gamma u^{-1} u_\nu - \eta \lneqq 0$ along $\partial_- M$, we use Lemma 2.4; If $H + \gamma u^{-1} u_\nu - \eta = 0$ but not stable, then we use Lemma 2.2. In both cases, we obtain a hypersurface which satisfies the claim.

If $\partial_- M$ is stable, then by Lemma 2.3, we obtain a foliation $\{\Sigma_t^-\}_{t\in[0,\varepsilon]}$. By the proof of Theorem 4.1, $H + \gamma u^{-1} u_\nu - \eta \leqslant 0$ for every $\{\Sigma_t^-\}_{t\in[0,\varepsilon]}$. Either there exists some $t \in (0, \varepsilon]$ such that $H + \gamma u^{-1} u_\nu - \eta < 0$ or $H + \gamma u^{-1} u_\nu - \eta = 0$ for all Σ_t^-, $t \in [0, \varepsilon]$. (It is worth noting that $\varepsilon = 0$ is also possible.) In the latter case, all leaves are stable by Definition 2.1. Hence, the foliation can be extended beyond Σ_ε^-. In light of this, we can assume that starting from $\partial_- M$, there is a maximal foliation $\{\Sigma_t^-\}_{t\in[0,t_1]}$. If the union of the leaves are the closure of M, then we are done. If not, by maximality, we have a hypersurface Σ_- with $H + \gamma u^{-1} u_\nu - \eta < 0$ constructed as a leaf of the foliation started from $\Sigma_{t_1}^-$ by Lemma 2.3. Hence, the claim is proved.

We can argue similarly for $\partial_+ M$ to show that there exists a hypersurface Σ_+ near $\partial_- M$ such that $H_{\Sigma_+} + \gamma u^{-1}\langle \nabla u, \nu_{\Sigma_+}\rangle > \eta$ or there exits a maximal foliation $\{\Sigma_t^+\}_{t\in[0,t_2]}$ such that $\Sigma_0^- = \partial_- M$ and every leaf is of vanishing $H + \gamma u^{-1} u_\nu - \eta$. (For this, we have to reverse the signs of the mean curvature, ν and η.)

Note that the two leafs from $\{\Sigma_t^-\}$ and $\{\Sigma_t^+\}$ can only touch which by the strong maximum principle are the same leaf. It then implies that the two foliations are the same one and foliate all of M. If this happens, we can finish the rigidity of M. If not, by maximality, we can extend the foliations and obtain the hypersurfaces Σ_- and Σ_+ which satisfies the barrier condition strictly.

To summarize, either the rigidity holds for M or there exists two hypersurfaces Σ_- and Σ_+ which satisfies the barrier condition strictly. However, the latter case is ruled out by Theorem 4.1. Now we find the metric g and u based on the foliation. The foliation gives (as in Theorem 4.1)

$$\nabla_{\Sigma_t} w = 0, \ \ -\gamma u^{-1}\Delta_g u + \tfrac{1}{2} R_g = \Lambda, \ A - \tfrac{1}{n-1} H = 0, \ w_\nu = \tfrac{1}{2(n-1)-(n-2)\gamma}\eta$$

and that each Σ_t is a flat torus.

The equation $A - \frac{1}{n-1}H = 0$ gives that (M, g) is isometric to some warped product $\mathrm{d}t^2 + \phi(t)^2 g_{\mathbb{T}^{n-1}}$, and we can assume that the foliation is given by the level set of the coordinate t. Note that $\nabla_{\Sigma_t} w = 0$ gives that u is constant along each Σ_t, so $w_\nu = \frac{1}{2(n-1)-(n-2)\gamma}\eta$ leads to $u = e^{\eta t/(2(n-1)-(n-2)\gamma)}$ (up to a constant). Now the equation

$$H + \gamma u^{-1} u_\nu = \eta = (n-1)\phi'/\phi + \gamma u^{-1} u_\nu$$

reduces to an ODE for the warping factor ϕ, and (up to a constant) $\phi = e^{\alpha t}$. □

The proof of Theorem 1.7 differs only by the calculation of rigid metrics.

Proof of Theorem 1.7. It suffices to follow Theorem 1.6 and to replace the rewrites of Z and W by case **III** of Subsection 2.3. For dimensions $n \geqslant 4$, we need to establish a version of Theorem 4.1. We proceed the proof to the place where the range $0 \leqslant \gamma < 3 + \frac{1}{n-2}$ is needed and omit the rest. By the existence result, we have a stable hypersurface Σ which is a torus and with vanishing $H + \gamma u^{-1} u_\nu - \eta$. The stability gives

$$0 \leqslant \int_\Sigma \left(\tfrac{4}{4-\gamma}|\nabla_\Sigma \psi|^2 + R_\Sigma \psi^2\right) - \int_\Sigma (1-\tfrac{\gamma}{4})\gamma\psi^2 \left|\nabla_\Sigma w - \tfrac{1}{2(1-\gamma/4)}\tfrac{\nabla_\Sigma \psi}{\psi}\right|^2 - \int_\Sigma (Z + R_\Sigma)\psi^2. \tag{4.3}$$

Let

$$L = -\tfrac{4}{4-\gamma}\Delta_\Sigma + R_\Sigma - (Z + R_\Sigma).$$

Let λ_1 be the first eigenvalue of L and v be the corresponding eigenfunction. Note that $\lambda_1 \geqslant 0$ by (4.3).

We define a constant κ by $4(n-2)\kappa/(n-3) = 4/(4-\gamma)$. By the range of γ, $\kappa \in (0,1)$. Let $\hat{g} = (v^\kappa)^{4/(n-3)} g|_\Sigma$ be the conformal metric. Then the scalar curvature of Σ with respect to $\hat{g}$ is

$$\begin{aligned}(v^\kappa)^{\frac{n+1}{n-3}} R_\Sigma(\hat{g}) &= v^\kappa R_\Sigma - \tfrac{4(n-2)}{n-3}\Delta_\Sigma v^\kappa \\ &= v^\kappa (R_\Sigma - \tfrac{4(n-2)}{n-3}\kappa v^{-1}\Delta_\Sigma v - \tfrac{4(n-2)}{n-3}\kappa(\kappa-1)v^{-2}|\nabla_\Sigma v|^2) \\ &\geqslant v^\kappa (\tfrac{Lv}{v} + (Z + R_\Sigma) - \tfrac{4(n-2)}{n-3}\kappa(\kappa-1)v^{-2}|\nabla_\Sigma v|^2) \\ &= v^\kappa \left(\lambda_1 + (Z+R_\Sigma) - \tfrac{4(n-2)}{n-3}\kappa(\kappa-1)v^{-2}|\nabla_\Sigma v|^2\right) \\ &\geqslant 0.\end{aligned}$$

In order to obtain $\nabla_\Sigma v = 0$, $\lambda_1 = 0$ and $Z + R_\Sigma = 0$, we need $\kappa < 1$ which is equivalent to $\gamma < 3 + \frac{1}{n-2}$. We omit the rest of the proof which is almost the same with Theorm 1.6 except that we obtain $H + \gamma u^{-1} u_\nu = \eta$, $\nabla_\Sigma w = 0$, $R_\Sigma = 0$ and $Z + R_\Sigma = 0$ for the foliation. The condition $Z + R_\Sigma = 0$ implies that $\mathrm{Ric}_g = \mathrm{Ric}_g(e_i, e_i)$, $-\gamma u^{-1}\Delta_g u + 2\,\mathrm{Ric}_g = \Lambda$ and $w_\nu = \frac{1}{2}\eta$, where e_i is any tangent vector of Σ.

In dimension 3, using [CS25, Subsection 3.4], (M, g) is locally a doubly warped product, say $\mathrm{d}t^2 + \phi(t)^2 \mathrm{d}s_1^2 + \varphi(t)^2 \mathrm{d}s_2^2$. Denote t-level set by Σ_t. First, $w_\nu = \frac{1}{2}\eta$ gives $u = e^{\eta t/2}$, and

$$H = \phi'/\phi + \varphi'/\varphi = \eta - \gamma w_\nu = (1 - \tfrac{\gamma}{2})\eta. \tag{4.4}$$

By the Ricci curvatures of a doubly warped product metric in Appendix A.2, $\mathrm{Ric}(\partial_t, \partial_t) \geqslant \mathrm{Ric}(e_i, e_i)$ gives

$$0 \geqslant (\phi'/\phi + \varphi'/\varphi')' + (\phi'/\phi - \varphi'/\varphi)^2 = (\phi'/\phi - \varphi'/\varphi)^2$$

see (A.2). So $\phi'/\phi = \varphi'/\varphi = \frac{1}{2}(1 - \frac{\gamma}{2})\eta$ by (4.4). Then up to a factor we can choose $\phi = \varphi = e^{(1-\gamma/2)\eta/2}$. □

Remark 4.2. *In dimensions $n \geqslant 4$, we can not find a metric structure like a doubly warped product.*

4.2. **Spectral Ricci curvature case.** Before we prove Theorem 1.9, we need an analogous Theorem 4.1.

Theorem 4.3. *Let γ, Λ, α and β be as in Theorem 1.9. There does not exist a band (M, g) such that $-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g \geqslant \Lambda$, $H + \gamma u^{-1} u_\nu > (n-1)\alpha + \gamma\beta$ along $\partial_+ M$ and $H + \gamma u^{-1} u_\nu < (n-1)\alpha + \gamma\beta$.*

Proof. Let $\Sigma = \partial\Omega \cap \mathrm{int} M$, then Σ is a stable warped h-hypersurface. Then using $-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g \geqslant \Lambda$ and the condition on η, the stability (2.1) gives

$$0 \leqslant \tfrac{4}{4-\gamma}\int_\Sigma |\nabla_\Sigma \psi|^2 - \int_\Sigma (1-\tfrac{\gamma}{4})\left|\psi\nabla_\Sigma w - \tfrac{1}{2(1-\gamma/4)}\nabla_\Sigma\psi\right|^2 - \int_\Sigma Z\psi^2,$$

where Z is defined in (2.3). When Σ is an h-hypersurface, $Z = W \geqslant 0$. Hence taking $\psi = 1$, we find that $Z = W = 0$ and $\nabla_\Sigma w = 0$, and hence by (2.2), Σ is infinitesimally rigid. By Lemma 2.3, we can construct a foliation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ for some $\varepsilon > 0$.

Claim: for each $t \in (-\varepsilon, \varepsilon)$, $\tilde{H}(t) := H + \gamma u^{-1}u_\nu - h$ on $\{\Sigma_t\}$ satisfies

$$\tilde{H}(t) \leqslant 0 \text{ for } t \in (0,\varepsilon) \text{ and } \tilde{H}(t) \geqslant 0 \text{ for } t \in (-\varepsilon, 0).$$

Let Y be the variational vector field of the foliation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$, let $\phi_t = \langle Y, \nu_t\rangle$. Using the first variation (2.2) of $\tilde{H}(t)$, we see that

$$\phi_t^{-1}\tilde{H}' = -\phi_t^{-1}\Delta_{\Sigma_t}\phi_t - u^{-1}\Delta_{\Sigma_t}u - \gamma\phi_t^{-1}\langle\nabla_{\Sigma_t}w, \nabla_{\Sigma_t}\phi_t\rangle - Z_t$$

where $w = \log u$ and Z_t is defined as in (2.3) for Σ_t. Let W be defined as (2.4) for Σ_t. By the rewrite of Z_t when $\tilde{H}$ vanishes, $Z_t = W_t$ where W_t is given in (2.4). When $\tilde{H}$ might not vanish, Z_t and W_t are related by

$$Z_t = W_t + \tfrac{1}{n-1}\tilde{H}^2 + \tilde{H}q_t$$

where $q_t := \frac{2(h-\gamma w_{\nu_t})}{n-1} + \gamma w_{\nu_t}$. We set $\phi_t = u^{-\gamma/2}e^{\xi_t}$ and using the above relation, and after a tedious calculation, we get

$$\begin{aligned}
\phi_t^{-1}\tilde{H}' =& -|\nabla_{\Sigma_t}\xi_t|^2 - \Delta_{\Sigma_t}\xi_t + (\tfrac{\gamma^2}{4}-\gamma)|\nabla_{\Sigma_t}w|^2 - \tfrac{\gamma}{2}\Delta_{\Sigma_t}w - W_t - \tfrac{1}{n-1}\tilde{H}^2 - \tilde{H}q_t.\\
=:& L_t - W_t - \tfrac{1}{n-1}\tilde{H}^2 - \tilde{H}q_t\\
\leqslant& L_t - \tilde{H}q_t,
\end{aligned}$$

where we have used $W_t \geqslant 0$ using the assumptions. We integrate $\phi_t^{-1}\tilde{H}' \leqslant L_t - \tilde{H}q_t$ over Σ_t, and we obtain

$$\tilde{H}'\int_{\Sigma_t}\phi_t^{-1} + \tilde{H}\int_{\Sigma_t}q_t \leqslant \int_{\Sigma_t}L_t \tag{4.5}$$

By the range $0 \leqslant \gamma < \frac{4}{n-1}$ and the divergence theorem,

$$\int_{\Sigma_t}L_t = -\int_{\Sigma_t}(|\nabla_{\Sigma_t}\xi_t|^2 + (\gamma - \tfrac{\gamma^2}{4})|\nabla_{\Sigma_t}w|^2) \leqslant 0.$$

By noting that $\tilde{H}(0) = 0$, and solving the inequality (4.5), we finish the proof. □

Now we are ready to prove Theorem 1.9.

Proof of Theorem 1.9. Using Theorem 4.3 and similar arguments as in Theorem 1.6, we can show that M is foliated by $\{\Sigma_t\}_{t\in[t_-,t_+]}$ hypersurfaces of vanishing $H+\gamma u^{-1}u_\nu-\eta=0$ for some $t_-<t_+$. It remains to calculate the rigid metric. To this end, we observe from Theorem 4.3, every leaf must satisfy the identities

$$\begin{aligned}\nabla_{\Sigma_t} w &= 0,\\ |A|^2-\tfrac{1}{n-1}H^2 &= 0,\\ -\gamma u^{-1}\Delta_g u+\mathrm{Ric} &= \Lambda,\\ \mathrm{Ric} &= \mathrm{Ric}(\nu),\\ \langle w, v_{\Sigma_t}\rangle+\tfrac{n-3}{2(n-1-(n-2)\gamma)}\eta &= 0. \end{aligned}\tag{4.6}$$

(Other than $\nabla_{\Sigma_t}w=0$, the rest are implied by $Z=0$ along Σ_t.) The condition $|A|^2-\frac{1}{n-1}H^2=0$ implies that Σ_t is umbilic, hence a warped product $g=\mathrm{d}t^2+\phi(t)^2 g_S$ for some closed manifold (S,g_S). We can assume that the t parametrizes the foliation as well. By $\nabla_{\Sigma_t}w$ and $w_\nu+\frac{n-3}{2(n-1-(n-2)\gamma)}\eta=0$, u (since $w=\log u$) only depends on t. The equation (4.6) is then an ODE for u which we can solve, we obtain that $u=e^{\beta t}$. Now we solve

$$H+\gamma u^{-1}u_\nu=\eta=(n-1)\phi'/\phi+\gamma u^{-1}u_\nu$$

to get that $\phi=e^{\alpha t}$. The extra condition $\mathrm{Ric}_{g_S}\geqslant 0$ is the requirement that $\mathrm{Ric}(\partial_t,\partial_t)$ is the least Ricci curvature required by $Z=W=0$, see (2.4) and Appendix A. □

5. Band width estimates with spectral curvature bounds

In this section, by selecting the suitable h to be the composition of a decreasing function η and a distance function ρ, we prove band width estimates (Theorems 1.10, 1.12 and 1.15).

5.1. **Bonnet-Myers type band width estimate.**

Proof of Theorem 1.12. We show directly that if $\mathrm{width}(M,g)\geqslant t_+-t_-$, then the width must be t_+-t_- and rigidity would follow. In particular, it would imply that $\mathrm{width}(M,g)\leqslant t_+-t_-$.

We set

$$\rho(x)=\min\{\mathrm{dist}_g(x,\partial_-M)+t_+-t_-,t_+\},$$

since the width is greater than t_+-t_-, $\rho(x)=t_+$ for all $x\in\partial_+M$. Also, $|\nabla\rho|\leqslant 1$. We set $h=\eta\circ\rho$. Then $H_{\partial_+M}+\gamma u^{-1}u_{\nu_+}\geqslant h$ on ∂_+M, $H_{\partial_-M}+\gamma u^{-1}u_{\nu_-}\leqslant h$ on ∂_-M. Also, we can easily check that $W\geqslant 0$ along every hypersurface M from the estimate

$$h_\nu\geqslant\eta'\circ\rho\langle\nabla\rho,\nu\rangle\geqslant\eta'\circ\rho.$$

With these conditions, we can prove as Theorem 1.9 that M is foliated by hypsurfaces $\{\Sigma_t\}$ such that $H + \gamma u^{-1} u_\nu - \eta \circ \rho = 0$ along Σ_t, we see that

$$\nabla_{\Sigma_t} w = 0, \text{ and } Z_t = 0 \text{ along } \Sigma_t.$$

more specifically, we have the following identities,

$$\begin{aligned} \nabla_{\Sigma_t} w &= 0, \\ |A|^2 - \tfrac{1}{n-1} H^2 &= 0, \\ \langle \nabla \rho, \nu_{\Sigma_t} \rangle &= 1, \\ -\gamma u^{-1} \Delta_g u + \mathrm{Ric} &= \Lambda, \\ \mathrm{Ric} &= \mathrm{Ric}(\nu), \end{aligned} \tag{5.1}$$

$$\langle \nabla w, \nu_{\Sigma_t} \rangle + \tfrac{n-3}{2(n-1-(n-2)\gamma)} \eta \circ \rho = 0 \tag{5.2}$$

along Σ_t since all of the summands of Z_{Σ_t} are non-negative. The condition $|A|^2 - \frac{1}{n-1} H^2 = 0$ implies that Σ_t is umbilic, hence a warped product $g = \mathrm{d}t^2 + \phi(t)^2 g_S$ for some closed manifold (S, g_S). With $\langle \nabla \rho, \nu_t \rangle = 1$, the level sets of ρ agrees with the t-level set, and ρ and t differs by only a constant. By $\nabla_{\Sigma_t} w$ and $w_\nu + \frac{n-3}{2(n-1-(n-2)\gamma)} \eta \circ \rho = 0$, u (since $w = \log u$) only depends on t. The equation (5.2) is then an ODE for u which we can solve, we obtain the expression of u. With this, (5.1) is an ODE for ϕ, which we can solve and it gives the model (1.6). The extra condition is the requirement that $\mathrm{Ric}(\partial_t, \partial_t)$ is the least Ricci curvature in all directions forced by $Z = W = 0$, see (2.4) and Appendix A. □

5.2. **Band width estimate with spectral scalar curvature.** Now we briefly provide a proof of Theorem 1.10 regarding the band width estimates under spectral scalar curvature bounds. Most of the proof was already in [CS25, Theorem 1.4].

Proof of Theorem 1.10. We assume that $\mathrm{width}(M, g) \geqslant t_+ - t_-$, and set

$$\rho(x) = \min\{\mathrm{dist}_g(x, \partial_- M) + t_+ - t_-, t_+\},$$

since the width is greater than $t_+ - t_-$, $\rho(x) = t_+$ for all $x \in \partial_+ M$. Also, $|\nabla \rho| \leqslant 1$. We set $h = \eta \circ \phi$. As in the proof of Theorem 1.12, M is foliates by hypersurfaces with vanishing $\tilde{H}$. We can use the rigidity analysis of [CS25, Theorem 1.4] to conclude the proof. □

Remark 5.1. *Similarly, we can reuse the proof of Theorem* [CS25, Theorem 1.2] *to give the proof of Theorem 1.15, so we omit the proof of Theorem 1.15.*

6. Spectral splitting and non-compact settings

In this section, we show some applications of the band width estimates for the splitting theorems of the spectral curvatures.

6.1. **Proof of splitting result under spectral scalar curvature bound.**

Lemma 6.1 ([Zhu20, Lemma 2.1])**.** *There is a proper and surjective smooth function $\phi : M \to \mathbb{R}$ such that $\phi^{-1}(0) = \Sigma$ and $\mathrm{Lip}\phi < 1$. Where $\Sigma \subset M$ is an orientable closed hypersurface associated with a signed distance function.*

For the function ϕ, let

$$\Omega_0 = \{x \in M | \phi(x) < 0\}.$$

Given any smooth function $h : (-T, T) \to \mathbb{R}$, we introduce the following functional

$$\mathcal{B}^h(\Omega) = \int_{\partial^*\Omega} u^\gamma d\mathcal{H}^{n-1} - \int_M (\chi_\Omega - \chi_{\Omega_0}) u^\gamma h \circ \phi \, d\mathcal{H}^n,$$

on

$$\mathcal{C}_T = \{\text{Caccioppoli set } \Omega \subset M : \Omega \Delta \Omega_0 \Subset \phi^{-1}((-T, T))\}.$$

For the minimizing problem of the functional $\mathcal{B}^h$ on $\mathcal{C}_T$, we have the following existence result.

Lemma 6.2 ([Zhu20, Lemma 2.2])**.** *Assume that $\pm T$ are regular values of ϕ and also the function h satisfies*

$$\lim_{t \to -T} h(t) = +\infty \quad \text{and} \quad \lim_{t \to T} h(t) = -\infty,$$

then there exists a smooth minimizer $\hat{\Omega}$ in $\mathcal{C}_T$ for $\mathcal{B}^h$.

Lemma 6.3 ([Zhu20, Lemma 2.3])**.** *For any $\epsilon \in (0, 1)$, there is a function*

$$h_\epsilon : \left(-\frac{1}{n\epsilon}, \frac{1}{n\epsilon}\right) \to \mathbb{R}$$

such that

(a) h_ϵ satisfies

$$\frac{2n - \gamma n + \gamma}{2(n-1) + \gamma(2-n)} h_\epsilon^2 + 2h_\epsilon' = \frac{2(n-1) + \gamma(2-n)}{2n + \gamma - \gamma n} n^2 \epsilon^2$$

on $\left(-\frac{1}{n\epsilon}, -\frac{1}{2n}\right] \cup \left[\frac{1}{2n}, \frac{1}{n\epsilon}\right)$ and there is a universal constant C so that

$$\sup_{-\frac{1}{2n} \le t \le \frac{1}{2n}} \left| \frac{2n - \gamma n + \gamma}{2(n-1) + \gamma(2-n)} h_\epsilon^2 + 2h_\epsilon' \right| \le C\epsilon.$$

(b) $h_\epsilon' < 0$ and

$$\lim_{t \to \mp \frac{1}{n\epsilon}} h_\epsilon(t) = \pm\infty.$$

(c) As $\epsilon \to 0$, h_ϵ converge smoothly to 0 on any closed interval.
(d) $h_\epsilon < 0$ on $\left[\frac{1}{2n}, \frac{1}{n\epsilon}\right)$.

Let

$$\mathcal{B}_\epsilon(\Omega) = \int_{\partial^*\Omega} u^\gamma d\mathcal{H}^{n-1} - \int_M (\chi_\Omega - \chi_{\Omega_0}) u^\gamma h_\epsilon \circ \phi \, d\mathcal{H}^n,$$

on

$$\mathcal{C}_\epsilon = \{\text{Caccioppoli set } \Omega \subset M : \Omega \Delta \Omega_0 \Subset \phi^{-1}\left(\left(-\frac{1}{n\epsilon}, \frac{1}{n\epsilon}\right)\right)\}.$$

Proposition 6.4. *For almost every $\epsilon \in (0,1)$, there is a smooth minimizer $\hat{\Omega}_\epsilon$ in $\mathcal{C}_\epsilon$ for the functional $\mathcal{B}^\epsilon$.*

Similar to Section 4.2 in [CS25], we can obtain a closed stable minimal surface Σ by letting $\epsilon \to 0$. The rigidity is also similar to Section 4.2 in [CS25].

6.2. **Geroch conjecture with arbitrary ends under spectral scalar curvature condition.**

Proof of Theorem 1.19. We follow the strategy of [CL24]. Because of the positivity of the spectral scalar curvature, there is some room so that we can select an h. The level sets of h where it takes the value of $\pm\infty$ serves as barriers for the existence of warped h-bubbles. The only differences are the construction of h and the use of warped μ-bubbles.

Fix $\varepsilon > 0$ small, we define

$$\Xi := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x - k| > \varepsilon, k \in \mathbb{Z}^n\}/\sim$$

where $(x_1, \ldots, x_n) \sim (x_1 + k_1, \ldots, x_n + k_n)$ for $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. By assumption, there is a map $\Psi : \Xi \to M$ so that Ψ is a diffeomorphism onto its image. By scaling, we can assume that $\Lambda > 1$ on $\Psi(\Xi)$.

Define $\gamma_1 = \sqrt{\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}}$.

The ρ_0 and ρ_1 are defined the same way as in [CL24], we assume that $\mathrm{Lip}(\rho_1) < L$ and L is taken so we can assume that $\frac{\pi}{2}\gamma_1^{-1}L = J + \frac{3}{4}$. On $\hat{M}_0 \cap \left\{|\rho_1| < \frac{\pi L}{2\gamma_1}\right\}$, define

$$h(p) = -\tfrac{1}{\gamma_1}\tan(\tfrac{\gamma_1}{L}\rho_1(p)).$$

For $0 \leqslant k \leqslant J$ and

$$p \in X_k^0 \cap \{\rho_1 < k + \tfrac{1}{2} + \frac{L}{\gamma_1 \tan(\frac{\gamma_1}{L}(k + \frac{1}{2}))}\},$$

or $-J \leqslant k < 0$ and

$$p \in X_k^0 \cap \{\rho_1 > k + \tfrac{1}{2} + \frac{L}{\gamma_1 \tan(\frac{\gamma_1}{L}(k + \frac{1}{2}))}\},$$

we set
$$h(p)=\frac{L}{\gamma_1^2\left(\rho_1-(k+\frac{1}{2})-\frac{L}{\gamma_1\tan(\frac{\gamma_1}{L}(k+\frac{1}{2}))}\right)}.$$
We can easily check as in [CL24, Lemma 22] that
$$\gamma_1^2h^2-|\nabla h|+\Lambda\geqslant\gamma_1^2h^2-|\nabla h|+1>0 \tag{6.1}$$
on $\{|h|<\infty\}$. We can smooth h slightly, so that the above is still satisfied. We still denote by h.

We fix
$$\Omega_0:=\left(\hat{\Psi}(\hat{\Xi}\cap\{x_n<-\tfrac{1}{2}\}\cup(\cup_{k<0}X_k^0))\right)\cap\{|h|<\infty\}.$$
we minimize
$$E(\Omega)=\int_\Omega u^\gamma+\int_M(\chi_\Omega-\chi_{\Omega_0})hu^\gamma$$
for all Caccioppoli sets Ω in M with $\Omega\Delta\Omega_0\Subset M$. Denote by Ω the connected component of the minimiser containing $\{\rho_1=-J\}$. By the stability [CS25, (4.5)] and (6.1)
$$\tfrac{4}{4-\gamma}\int_{\partial\Omega}|\nabla_{\partial\Omega}\psi|^2+\tfrac{1}{2}\int_{\partial\Omega}R_{\partial\Omega}\psi^2>\int_{\partial\Omega}[\gamma_1^2h^2-|\nabla h|+\Lambda]\psi^2>0$$
for all $\psi\in C^\infty(\Sigma)$. The rest argument is the same as [CL24]. □

We now prove Theorem 1.21 by adapting the argument of [CEM19].

Proof of Theorem 1.21. It follows from Theorem 1.4 that Σ is a flat, by scaling if needed, we can assume that Σ is isometric to the standard cylinder $\mathbb{S}^1\times\mathbb{R}$. If Σ is separating, then $M\backslash S$ has two components from which we choose one of them and denote by $(\hat{M},\hat{g})$. If Σ is non-separating, then we cut along Σ, we obtain a new manifold which we denote also by $(\hat{M},\hat{g})$; $(\hat{M},\hat{g})$ has two boundary components from which we choose one. We set $\mathbb{S}^1\times\{0\}\subset\Sigma$ to be ℓ and Σ_h to be $\mathbb{S}^1\times[-h,h]\subset\Sigma$.

Fix a unit speed geodesic $c:[0,\varepsilon)\to\hat{M}$ with $c(0)\in\ell$ and the tangent vector at $c(0)$ is normal to Σ. We can construct a family of positive functions $u_{r,t}$ (see [HW25, Lemma 3.3], cf. [APX24, Lemma 2.2]) with the following conditions:

(a) $u_{r,t}\to u$ in C^3 as $t,r\to0$;
(b) $u_{r,t}\to u$ smoothly as $t\to0$ for $r\in(0,\varepsilon)$ fixed;
(c) $u_{r,t}=u$ on $\{x\in\hat{M}:\ \mathrm{dist}_{\hat{g}}(x,c(2r))\geqslant3r\}$;
(d) $u_{r,t}<u$ on $\{x\in\hat{M}:\ \mathrm{dist}_{\hat{g}}(x,c(2r))<3r\}$;
(e) $-\gamma u_{r,t}^{-1}\Delta_g u_{r,t}+\frac{1}{2}R_g>0$ on $\{x\in\hat{M}:\ r<\mathrm{dist}_{\hat{g}}(x,c(2r))<3r\}$;

(f) $H_\Sigma + \gamma u_{r,t}^{-1}\partial_\nu u_{r,t} \geqslant 0$ (which we call weakly $u_{r,t}^\gamma$-weighted mean-convex with respect to the weight $u_{r,t}^\gamma$).

Fix $h > 1$. Let B_h denote a precompact open set with smooth boundary in $\hat{M}$ and such that $\{x \in \hat{M} : \ \mathrm{dist}_{\hat{g}}(x, \Sigma_h) < 2h\} \subset B_h$. We modify $u_{r,t}$ further near the boundary of B_h to $u_{r,t,h}$ so B_h is weakly weighted mean-convex with respect to the weight $u_{r,t,h}^\gamma$ and

$$(1-\delta)u_{r,t} \leqslant u_{r,t,h} \leqslant (1+\delta)u_{r,t} \tag{6.2}$$

where $\delta \in (0,1)$ is chosen to satisfy the relation (6.9). Among all compact, oriented surfaces in B_h with respect to $\partial\Sigma_h$ that bound an open subset of $\hat{M}$, there is one whose weighted area with respect to $u_{r,t,h}$ is least. Choose one such weight area-minimising surface and denote it by $\Sigma_{r,t,h}$.

We claim that $\Sigma_{r,t,h}$ intersects $\{x \in \hat{M} : \ \mathrm{dist}_{\hat{g}}(x, c(2r)) < 3r\}$. For if not, we have that $u_{r,t,h} = u$ along $\Sigma_{r,t,h}$, and

$$\begin{aligned}
0 <& \mathcal{A}_u(\Sigma_h) - \mathcal{A}_{u_{r,t}}(\Sigma_h) && (6.3)\\
\leqslant& \mathcal{A}_u(\Sigma_{r,t,h}) - \mathcal{A}_{u_{r,t}}(\Sigma_h) && (6.4)\\
=& \mathcal{A}_{u_{r,t}}(\Sigma_{r,t,h}) - \mathcal{A}_{u_{r,t}}(\Sigma_h) && (6.5)\\
\leqslant& \tfrac{1}{1-\delta}\mathcal{A}_{u_{r,t,h}}(\Sigma_{r,t,h}) - \mathcal{A}_{u_{r,t}}(\Sigma_h) && (6.6)\\
\leqslant& \tfrac{1}{1-\delta}\mathcal{A}_{u_{r,t,h}}(\Sigma_h) - \mathcal{A}_{u_{r,t}}(\Sigma_h) && (6.7)\\
\leqslant& \tfrac{1+\delta}{1-\delta}\mathcal{A}_{u_{r,t}}(\Sigma_h) - \mathcal{A}_{u_{r,t}}(\Sigma_h) && (6.8)\\
=& \tfrac{2\delta}{1-\delta}\mathcal{A}_{u_{r,t}}(\Sigma_h).
\end{aligned}$$

We explain these relations: (6.3) follows since $\{\mathrm{dist}_{\hat{g}}(x, c(2r)) < 3r\} \cap \Sigma_h$ is non-empty and $u_{r,t} < u$ on this set; (6.4) follows since Σ_h minimises the u^γ-weighted area; (6.5) follows since $\Sigma_{r,t,h}$ does not intersect $\{\mathrm{dist}_{\hat{g}}(x, 2r) < 3r\}$ on which $u_{r,t}$ and u differ; (6.6) follows from (6.2); (6.7) holds since $\Sigma_{r,t,h}$ minimises $u_{r,t,h}^\gamma$-weighted area; again, (6.8) is due to (6.2).

This is impossible if we choose $\delta = \delta(r,t,h) > 0$ with

$$\tfrac{2\delta}{1-\delta} < \tfrac{\mathcal{A}_u(\Sigma_h) - \mathcal{A}_{u_{r,t}}(\Sigma_h)}{\mathcal{A}_{u_{r,t}}(\Sigma_h)}. \tag{6.9}$$

Now we can take a subsequential limit as $h \to \infty$ and obtain a properly embedded surface $\Sigma_{r,t}$. From the construction, $\Sigma_{r,t}$ is a boundary and homologically* $u_{r,t}^\gamma$-weighted area-minimising.

Following [CEM19] with Theorem 1.4 in place of [CEM19, Lemmas 2.1-2.4] as we take limit as $t \to 0$, we obtain a family of surfaces Σ_r which converges to Σ. If Σ_r is a torus, we again use Theorem 1.4 to show that $\hat{M}$ is isometric to either standard $\mathbb{S}^1 \times \mathbb{R} \times [0, \infty)$ or $\mathbb{S}^1 \times \mathbb{R} \times [0, a]$ for some $a > 0$. We may assume that Σ_r

is cylindrical. By the proof of Theorem 1.4, the gradient of u vanishes along Σ_r. Assume that u take different values on Σ and Σ_{r_0} for some r_0 sufficiently small, by connecting two points $x \in \Sigma$ and $y \in \Sigma_{r_0}$ by a segment ℓ. There exists a point $z \in \ell$ not equal to x nor y such that u has non-vanishing gradient. Take a small geodesic ball $B_{r_1}(z)$ centered at z such that ∇u is nowhere vanishing, the family $\{\Sigma_r\}$ cannot intresect $B_{r_1}(z)$ from which we obtain a contradiction with that $\{\Sigma_r\}$ converges to Σ. Hence, u is constant along all Σ_r from which we are reduced to the case u is constant, the case handled [CEM19] and we finish the proof. □

Remark 6.5. *It is direct to defined the notions of absolutely u^γ-weighted area-minimising, homologically u^γ-weighted area-minimising and homologically* u^γ-weighted area minimising by adapting* [CEM19, Appendix] *to the weighted case.*

6.3. **Proof of Theorem 1.23.** We only need to prove u is a constant on M. Then we can use the result of Liu [Liu13].

First, by the result of Antonelli-Pozzetta-Xu[APX24, Theorem 1.1] and its proof, we may assume $\pi_2(M) = 0$ and M is not diffeomorphic to $\mathbb{R}^3$. Next, by passing to a suitable covering, we can further assume $\pi_1(M) = \mathbb{Z}$ and M is orientable. Let Γ represent the generator of the fundamental group of M. We may assume Γ is a smooth closed curve. Consider an exhaustion of M by Ω_i with smooth boundary $\partial\Omega_i$, where we can assume Γ lies in each Ω_i. By Poincaré duality for manifolds with boundary, there exists an oriented surface $\Sigma_i \subset \Omega_i$ such that $\partial\Sigma_i \subset \Omega_i$ and the oriented intersection number of Σ_i with Γ equals 1. We then consider minimizing the weighted area $\int_\Sigma u^\gamma$ over all surfaces that belong to the same homology class as Σ_i and have the same boundary as Σ_i. We can perturb the metric near $\partial\Omega_i$ such that $H + u_\nu u^{-1} > 0$ on the boundary $\partial\Omega_i$. For each i, there exists a weighted minimizing surface, which we still denote as Σ_i, and the intersection of Σ_i with Γ is nonempty. By the curvature estimate[ZZ20, Theorem 3.6], a subsequence of Σ_i converges to an oriented stable weighted minimal surface Σ in M. By Lemma 2.5, for $h = 0$ and $H + u_\nu u^{-1} = 0$, we have

$$\begin{aligned}\int_\Sigma \left[\frac{4}{4-\gamma}|\nabla_\Sigma\psi|^2 + R_\Sigma\psi^2\right] \geq& \int_\Sigma (1-\tfrac{\gamma}{4})\gamma\left|\psi\nabla_\Sigma w - \tfrac{1}{2(1-\gamma/4)}\nabla_\Sigma\psi\right|^2 + \int_\Sigma w_\nu^2\psi^2 \\ &+ \int_\Sigma \left(-\gamma u^{-1}\Delta_g u + 2\operatorname{Ric}_g\right)\psi^2,\end{aligned}$$

where $w = \log u$ and $\psi \in C_c^\infty(\Sigma)$.

If $-\gamma u^{-1}\Delta_g u + \operatorname{Ric}_g > 0$ and $-\gamma u^{-1}\Delta_g u + 2\operatorname{Ric}_g > 0$, then Σ is either compact and diffeomorphic to $\mathbb{S}^2$(which contradicts $\pi_2(M) = 0$) or Σ is non-compact. When Σ is non-compact, there are two subcases: (i) Σ is conformal to the cylinder, which contradicts (b) in Theorem 1.4; (ii) Σ is conformal to the complex plane.

In Case (ii), we apply Lemma 2.5 again with $h = 0$ and $H + u_\nu u^{-1} = 0$, yielding

$$\begin{aligned}\tfrac{4}{4-\gamma}\int_\Sigma |\nabla_\Sigma \psi|^2 \geq & \int_\Sigma (1-\tfrac{\gamma}{4})\gamma \left|\psi\nabla_\Sigma w - \tfrac{1}{2(1-\gamma/4)}\nabla_\Sigma\psi\right|^2 \\ & + \int_\Sigma \left(\left(-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g\right) + |A|^2\right)\psi^2 + \int_\Sigma w_\nu^2\psi^2,\end{aligned}$$

where $w = \log u$ and $\psi \in C_c^\infty(\Sigma)$. Recall that we assume that $-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g \geq 0$ and $0 < \gamma < 7/2$, by Theorem 1.1 in [BC14], Σ has at most quadratic volume growth. We can then apply the logarithmic cut-off trick (see Proposition 1.37[**?**]) to deduce that $-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g = 0$, $(\log u)_v = 0$ and $\nabla_\Sigma \log u = 0$. This is a contradiction.

The remaining case is $-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g \geq 0$ or $-\gamma u^{-1}\Delta_g u + 2\,\mathrm{Ric}_g \geq 0$. Following Liu[Liu13], we handle this case as follows. Fixed point $p \in M$ such that $p \notin \Gamma$; we aim to deform the metric g so that the spectral Ricci curvature is strictly positive in an annulus region around p. Let $g_t = e^{2tf}g$ and $|v|_g = 1$, we then have

$$\begin{aligned}& -\gamma\frac{\Delta_{g_t}u}{u} + \mathrm{Ric}_{g_t} \\ =\ & e^{-2tf}\left[-\frac{\Delta_g u}{u} + \mathrm{Ric}_g\right] \\ & + e^{-2tf}\left[\frac{t}{u}\,g(\nabla u, \nabla f) - t\nabla^2 f(v,v) - t\Delta_g f + t^2((v(f))^2 - |\nabla f|^2)\right]\end{aligned}$$

where we have used the identity

$$\Delta_{g_t}u = e^{-2tf}[\Delta_g u + t\langle \nabla u, \nabla f\rangle_g].$$

Let r denote the distance function to p. For a sufficiently small $R > 0$, consider the function $\rho = R - r$ defined for $\frac{R}{2} < r < R$; we then extend ρ to be a positive smooth function for $0 \leq r < \frac{R}{2}$. With this ρ, define $f = -\rho^5$. For $\frac{R}{2} < r < R$, we obtain

$$\begin{aligned}& -\gamma\frac{\Delta_{g_t}u}{u} + \mathrm{Ric}_{g_t} \\ \geq\ & e^{-2tf}\left[-\frac{\Delta_g u}{u} + \mathrm{Ric}_g\right] \\ & + e^{-2tf}\left[5t\rho^4 u^{-1}g(\nabla u, \nabla\rho) + 20t\rho^3 + 5t\rho^4(\Delta_g\rho + \nabla^2\rho(v,v)) - 25t^3\rho^8\right].\end{aligned}$$

By the almost Euclidean propery of M near p, for small R, we have

$$|\Delta_g\rho + \nabla^2\rho(v,v)| \leq \frac{9}{8(R-\rho)}.$$

For all small t, if r sufficiently close to R, then ρ is close to 0. In this case,

$$20t\rho^3 - 5t\rho^4 u^{-1} g(\nabla u, \nabla \rho) + 5t\rho^4(\Delta_g \rho + \nabla^2\rho(v,v)) - 25t^3\rho^8 > 0$$

since the leading term is ρ^3 when ρ is small. When t small enough, we can also ensure $-\gamma\frac{\Delta_{g_t} u}{u} + 2\,\mathrm{Ric}_{g_t} > 0$ at the same time. Note that the metric g_t remains unchanged outside $B_p(R)$, and this metric deformation(i.e., g_t) is C^4 continuous with respect to the metric g and C^∞-continuous with respect to t.

Since Γ is closed, we can apply this perturbation finitely many times such that the spectral Ricci curvature is positive on Γ (each time we perturb the metric a little bit around a point) and nonnegative except for a small neighborhood of p. We then minimize the weighted area functional as we did earlier, which yields a complete stable weighted minimal surface Σ.

We now claim that Σ must pass through this small neighborhood of p. If this were not the case, the spectral Ricci curvature on Σ would be nonnegative, with strictly positive at some point on Γ. This leads to a contradiction as before.

Let t denote the deformation parameter. We shrink the size of the neighborhood of p where the spectral Ricci curvature might be negative; this allows us to construct a sequence of metrics on M, each admitting a stable weighted minimal surface passing through the small neighborhood of p. We can choose $t \to 0$ sufficiently rapidly such that these metrics converge to the original metric in a C^4 sense.

By passing to a subsequence of these complete weighted minimal surfaces and taking the limit, we obtain a completely oriented stable weighted minimal surface Σ passing through p with respect to the original metric. This holds for any $p \in M \setminus \Gamma$.

By Lemma 2.5, for $h = 0$ and $H + u_\nu u^{-1} = 0$, we have

$$\begin{aligned}\int_\Sigma \left[\frac{4}{4-\gamma}|\nabla_\Sigma \psi|^2 + R_\Sigma \psi^2\right] \geq & \int_\Sigma (1-\tfrac{\gamma}{4})\gamma \left|\psi \nabla_\Sigma w - \tfrac{1}{2(1-\gamma/4)}\nabla_\Sigma \psi\right|^2 + \int_\Sigma w_\nu^2 \psi^2 \\ & + \int_\Sigma \left(-\gamma u^{-1}\Delta_g u + 2\,\mathrm{Ric}_g\right)\psi^2,\end{aligned}$$

and

$$\begin{aligned}\tfrac{4}{4-\gamma}\int_\Sigma |\nabla_\Sigma \psi|^2 \geq & \int_\Sigma (1-\tfrac{\gamma}{4})\gamma \left|\psi \nabla_\Sigma w - \tfrac{1}{2(1-\gamma/4)}\nabla_\Sigma \psi\right|^2 \\ & + \int_\Sigma \left(\left(-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g\right) + |A|^2\right)\psi^2 + \int_\Sigma w_\nu^2 \psi^2,\end{aligned}$$

where $w = \log u$ and $\psi \in C_c^\infty(\Sigma)$.

If u is not constant on Σ, then using the same argument as before (given our assumptions that $-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g \geq 0$, $-\gamma u^{-1}\Delta_g u + 2\,\mathrm{Ric}_g \geq 0$ and $0 < \gamma < 7/2$), we can deduce that $-\gamma u^{-1}\Delta_g u + \mathrm{Ric}_g = 0$, $A = 0$, $(\log u)_v = 0$ and $\nabla_\Sigma \log u = 0$, i.e., u is a constant on Σ. This is a contradiction.

This implies that Ric_g is non-negative on $p \in M \setminus \Gamma$ and that u is constant on $M \setminus \Gamma$. By the continuity of Ric_g, we conclude that Ric_g is non-negative on the entire manifold M. Then we can use the result of Liu[Liu13].

As an application, we consider an asymptotically flat manifold (M, g) with one end. Let Ric_- denote the negative part. Assume that $\int_M |\mathrm{Ric}_-|^{\frac{n}{2}} \leq \epsilon$ for some small ϵ. Then M is diffeomorphisc to $\mathbb{R}^3$.

The reason is that, by Proposition 2.10[Sch17], there exists a function $u > 0$ such that

$$\begin{cases} -\Delta_g u + \mathrm{Ric}_g u = 0 \text{ on } M; \\ u \to 1 \text{ as } |x| \to \infty. \end{cases}$$

Appendix A. Warped product curvatures

In this appendix, we calculate the frequently used curvatures and relations of a warped product and a doubly warped product.

A.1. **Warped product.** Let $g = \mathrm{d}t^2 + \phi(t)\bar{g}$ where g is a metric on the manifold M. We calculate $\mathrm{Ric}(e_i, e_i)$ where e_i is tangential to M of unit g-length. For convenience, we set $a_i = \overline{\mathrm{Ric}}(\phi e_i, \phi e_i)$ where $\overline{\mathrm{Ric}}$ is the Ricci curvature of $(M, \bar{g})$. Note that ϕe_i is of unit $\bar{g}$-length and $\{a_i\}$ are the Ricci curvature of the metric $\bar{g}$.

By the Gauss equation,

$$\begin{aligned} R_{ii} &= \sum_j R_{ijji} + R_{i\nu\nu i} \\ &= \sum_j (\bar{R}_{ijji} - h_{jj}h_{ii} + h_{ij}h_{ij}) + R_{i\nu\nu i} \\ &= \bar{R}_{ii} - Hh_{ii} + h_{ii}^2 + R_{i\nu\nu i} \\ &= a_i\phi^{-2} - (n-2)(\phi'/\phi)^2 + R_{i\nu\nu i} \end{aligned}$$

where we used that $H = (n-1)\frac{\phi'}{\phi}$ and $h_{ii} = \phi'/\phi$, it remains to calculate $\bar{R}_{i\nu\nu i}$. We first compute the necessary Christofel symbols Γ^i_{nn} and Γ^j_{ni} of g:

$$\Gamma^i_{nn} = \tfrac{1}{2}\sum_{l \leqslant n} g^{il}(2g_{nl,n} - g_{nn,l}) = 0, \Gamma^n_{nn} = 0,$$

and

$$\Gamma_{in}^j = \tfrac{1}{2}\sum_{l\leqslant n} g^{jl}(g_{li,n}+g_{ni,l}-g_{in,l}) = \tfrac{1}{2}\sum_{l\leqslant n-1}\phi^{-2}\bar{g}^{jl}(\phi^2\bar{g}_{li})_n = \phi'\delta_i^j/\phi.$$

And the component

$$\begin{aligned} R_{i\nu\nu i} =& \partial_i\Gamma_{nn}^i - \partial_n\Gamma_{in}^i + \sum_{s\leqslant n}(\Gamma_{nn}^s\Gamma_{is}^i - \Gamma_{in}^s\Gamma_{ns}^i) \\ =& -(\phi'/\phi)' - \sum_{s\leqslant n-1}\Gamma_{in}^s\Gamma_{ns}^i = -(\phi'/\phi)' - (\phi'/\phi)^2 \end{aligned}$$

follows easily. Hence,

$$R_{ii} = a_i\phi^{-2} - (n-1)(\phi'/\phi)^2 - (\phi'/\phi)'.$$

As for $\mathrm{Ric}(\nu,\nu)$, we use the first variation of the mean curvature of the t-level set. We get

$$H' = -\mathrm{Ric}(\nu,\nu) - |A|^2.$$

Hence,

$$\mathrm{Ric}(\nu,\nu) = -(n-1)(\phi'/\phi)' - (n-1)(\phi'/\phi)^2.$$

The scalar curvature of the metric $\bar{g}$ is

$$R_{\bar{g}} = -2(n-1)\frac{\phi''}{\phi} - (n-1)(n-2)\frac{(\phi')^2}{\phi^2},$$

which can be found via Schoen-Yau's rewrite (2.6). The Laplace of the metric is

$$\Delta_g = \partial_t^2 + H_t\partial_t + \frac{1}{\phi^2}\Delta_{g_{\mathbb{T}^{n-1}}}.$$

A.2. **Doubly warped product.** Let $g = \mathrm{d}t^2 + \phi(t)^2\mathrm{d}s_1^2 + \varphi(t)^2\mathrm{d}s_2^2$. The non-zero components of its Ricci curvatures are given by

$$\begin{aligned} \mathrm{Ric}(\partial_t,\partial_t) &= -(\phi\varphi)^{-1}(\phi\varphi''+\varphi''\phi), \\ \mathrm{Ric}(e_1,e_1) &= -(\phi\varphi)^{-1}(\varphi\phi''+\varphi'\phi'), \\ \mathrm{Ric}(e_2,e_2) &= -(\phi\varphi)^{-1}(\varphi''\phi+\varphi'\phi'), \end{aligned}$$

where $e_1 = \phi^{-1}\partial_{s_1}$ and $e_2 = \varphi^{-1}\partial_{s_2}$. It is convenient to introduce $G = \phi'/\phi + \varphi'/\varphi$ and $F = \phi'/\phi - \varphi'/\varphi$. We often need to impose that $\mathrm{Ric}(e_1,e_1) = \mathrm{Ric}(e_2,e_2)$ and $\mathrm{Ric}(\partial_t,\partial_t) \geqslant \mathrm{Ric}(e_i,e_i)$. The former gives $\phi''/\phi = \varphi''/\varphi$ and then

$$F' = \tfrac{\phi''}{\phi} - (\tfrac{\phi'}{\phi})^2 - \tfrac{\varphi''}{\varphi} + (\tfrac{\varphi'}{\varphi})^2 = -FG.$$

That is,

$$(\phi'/\phi - \varphi'/\varphi)' = -(\phi'/\phi - \varphi'/\varphi)(\phi'/\phi + \varphi'/\varphi). \tag{A.1}$$

With $\phi''/\phi = \varphi'/\varphi$, the latter gives

$$\begin{aligned} 0 \geqslant & \frac{\phi''}{\phi} - \frac{\phi'\varphi'}{\phi\varphi} \\ = & \frac{1}{2}\left(\frac{\phi''}{\phi} + \frac{\varphi''}{\varphi}\right) - \frac{\phi'\varphi'}{\phi\varphi} \\ = & \frac{1}{2}\left((\phi'/\phi + \varphi'/\varphi)' + \left(\frac{\phi'}{\phi}\right)^2 + \left(\frac{\varphi'}{\varphi}\right)^2\right) - \frac{\phi'\varphi'}{\phi\varphi} \\ = & \frac{1}{2}\left(G' + \frac{1}{2}(F^2 + G^2)\right) - \frac{1}{4}(G^2 - F^2) \\ = & \frac{1}{2}(G' + F^2), \end{aligned}$$

which is

$$(\phi'/\phi + \varphi'/\varphi)' + (\phi'/\phi - \varphi'/\varphi)^2 \leqslant 0 \tag{A.2}$$

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