Scalar curvature comparison of weakly convex rotationally symmetric sets

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Outline

Gauss-Bonnet theorem

Theorem Let (S, γ) be a surface with a metric γ, then

$$2\pi\chi(\Sigma) = \int_{\Sigma} K + \int_{\partial\Sigma} \kappa + \sum_i (\pi - \alpha_i)$$

where K is the Gauss curvature, κ is the geodesic curvature of $\partial\Sigma$ in Σ and α_i is the interior turning angles.

- Euler characteristic for surfaces: Triangulate the surface, then calculate $\chi(\Sigma) = V - E + F$; for surface, $\chi(\Sigma) = 2 - 2g - b$ where g is the genus and b is the number of the boundary components

Gauss-Bonnet theorem

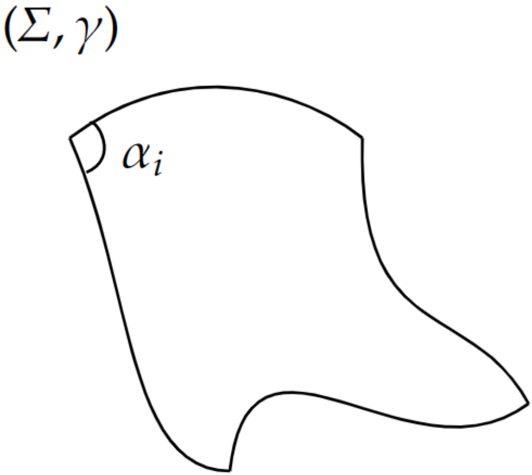

Figure: A surface with piecewise smooth boundary

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- On $(\mathbf{T}^2, g)$, there exists no metrics with $K_g \geq 0$ but $K \neq 0$;
  - There is no boundary term and angle term; so $2\pi\chi(\Sigma) = 0 = \int_{\mathbf{T}^2} K \geq 0$, which means that K has to vanish identically

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  - Has to be equalities

Scalar curvature: Generalizations of Gauss curvature

On an n-dimensional manifold (M, g), the volume of small geodesic ball at $p \in M$ satisfies

$$\mathrm{vol}(B_r(p)) = \omega_n r^n \left(1 - \frac{R_g(p)}{6(n+2)} r^2 + O(r^4)\right)$$

- The only place where I saw an application of this definition is where L. Guth re-proved the Gromov's systolic inequality.

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  - Llarull used spinors

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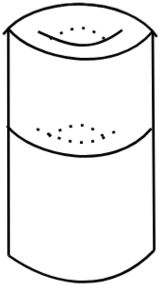
- Observation Each t slice is of mean curvature -2
- Theorem Let g be another metric on $M^3 = \mathbf{T}^2 \times [0, 1]$, if $H_0 \geq 2$, $H_1 \geq -2$ and $R_g \geq -6$, then g is hyperbolic. (Min Oo 95, Andersson-Cai-Galloway 08)

Figure of $\mathbf{T}^2 \times [0,1]$



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    - For $\mathbf{R}^3$ identify cubes and get the torus
    - For $\mathbf{H}^3$ identify cubes on horospheres and get a "band"

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- (Min-Oo 95) There is no deformation of metric fixing the induced metric on the boundary and $H_{\partial \mathbf{S}^3_+} \geq 0$ such that the scalar curvature is increased.
- Not true: Brendle-Marques 2010; true with $g \geq \bar{g}$

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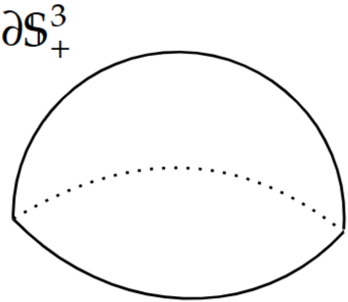

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Easier case with proof: non-rigidity

Schoen-Yau (1975) proof of nonexistence dimension 3

Stable minimal surfaces

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μ-bubble or prescribed mean curvature surface

Given a function $h \in C^\infty(M)$, we say that Σ is of prescribed mean curvature h if the mean curvature H agrees with h along Σ.

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- Equivalent to

$$\int_\Sigma (\nabla_\nu h + \mathrm{Ric}(\nu) + |A|^2)\phi^2 \leqslant \int_\Sigma |\nabla\phi|^2$$

  for all smooth ϕ.

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- It is also stable $\phi = \langle \partial_t, \nu \rangle$

Notes μ-bubble

I will only talk about the case with h being a constant. However, I would like to mention for general h, it is used in

- Nonexistence of positive scalar curvature on aspherical manifolds in $4, 5$ dimensions (Chodosh-Li 20, Gromov 20)

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- Classification of complete 3-manifolds with uniformly PSC (Jian Wang 22)
- Nonexistence of PSC metrics on $T^n \sharp M$ where M is not compact

By choosing suitable h.

Restatement of Geroch conjecture

Schoen-Yau 75 There exists no metrics on $(\mathbf{T}^3, g)$ with $R_g \geq 0$ but $R_{.} \neq 0$.

Proof of Geroch conjecture via minimal surface

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- So
$$\tfrac{1}{2}\int_\Sigma (R - R_\Sigma + |A|^2)\zeta^2 \leqslant \int_\Sigma |\nabla\zeta|^2$$

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- But by construction $\chi(\Sigma) = 0$

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  - $\mathrm{Ric}(\nu) + |A|^2 = \frac{1}{2}(R + 6 - 2K + |A^0|^2)$

Boundary version

What is the boundary version?

Result with a free boundary proof

- $\mathbf{T}^{n-1} \times [0,1]$?

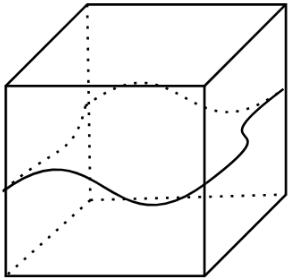
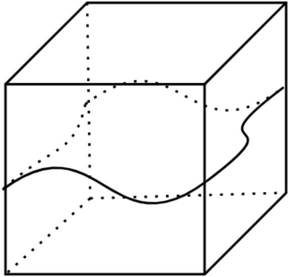


Figure: Finding a free boundary minimal surface in a cube

Result with a free boundary proof

- $\mathbf{T}^{n-1} \times [0,1]$?
- Cube $[0,1]^{n-k} \times \mathbf{T}^k$ (Li 2020)



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Results with capillary (constant angle) surface proof

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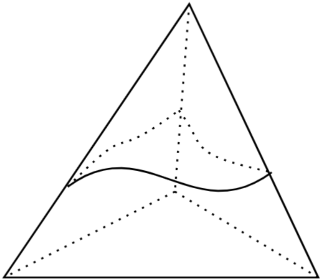

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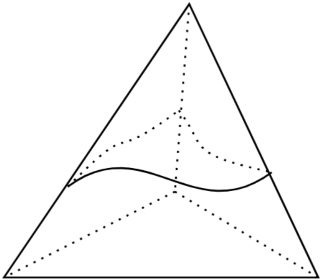
- hyperbolic tetrahedron (Chai-Wang 22)



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- The method is suggested by Gromov 19 in his *Four Lectures on Scalar Curvature*

More previous Results

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  .
- Lott 21: with extra assumption $g \geq \bar{g}$; used spinors in spirit of Llarull

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3. The set $\partial M \cap P_{\pm}$ contains only $p_{\pm}$ and ∂M is smooth at $p_{\pm}$.

Structure of vertex

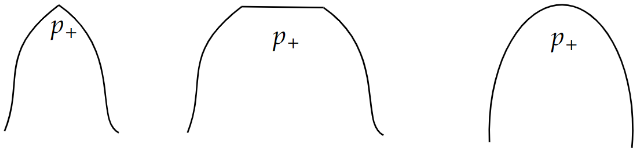

Figure: Structure at p_+

Theorem

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2. If ∂M is conical at $p_{\pm}$, we further assume that $\sigma = \bar{\sigma}$ at $p_{\pm}$.

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Finding a capillary minimal surface

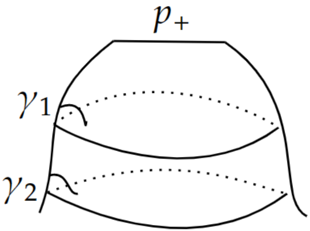

Figure: Finding a capillary surface with varying contact angle (Usually $\gamma_1 \neq \gamma_2$

Comment

- Assume that ∂M is isometric to the model, we can remove weak convexity

Notations

- Use bar to denote every geometric quantites of the model

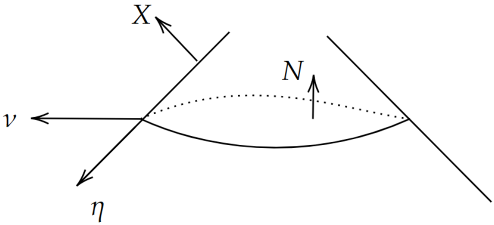

Figure: Labelling of various vectors

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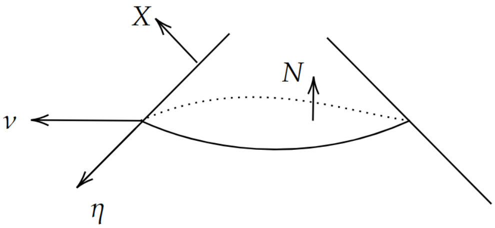
- See the following figure:



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- Each level set $\bar{M} \cap \{x^3 = s\}$ is minimal and meeting ∂M with angle $\bar{\gamma}$

Model

- The angles $\cos\bar{\gamma} = \langle \bar{X}, \frac{\partial}{\partial x^3} \rangle$
- Each level set $\bar{M} \cap \{x^3 = s\}$ is minimal and meeting ∂M with angle $\bar{\gamma}$
- It is stable

Variational problem

- Consider the following functional

$$F(E) := \mathcal{H}^2(\partial E \cap \operatorname{int} M) - \int_{\partial E \cap \partial M} \cos \bar{\gamma}$$

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- First variation of F: Letting $f = \langle Y, N \rangle$,

$$\mathcal{A}'(0) = \int_{\Sigma} Hf + \int_{\partial \Sigma} \langle Y, \nu - \eta \cos \bar{\gamma} \rangle$$

Second variation

Assume that Σ is minimal capillary, we have the second variation formula

$$\mathcal{A}''(0) = Q(f,f) := -\int_\Sigma (f\Delta f + (|A|^2 + \mathrm{Ric}(N))f^2) + \int_{\partial\Sigma} f(\frac{\partial f}{\partial \nu} - qf),$$

- where q is defined to be
$$q = \frac{1}{\sin\bar{\gamma}} A_{\partial M}(\eta,\eta) - \cot\bar{\gamma} A(\nu,\nu) + \frac{1}{\sin^2\bar{\gamma}} \partial_\eta \cos\bar{\gamma}.$$

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- Rewrite of q: Along the boundary $\partial\Sigma$,
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- Note the free boundary version: $A_{\partial M}(\eta,\eta) = H_{\partial M} - \kappa$

Using rewrites

Taking $f \equiv 1$ in the second variation and use the rewrites we obtain that we have that

$$\int_\Sigma K + \int_{\partial\Sigma} \kappa \geqslant \int_{\partial\Sigma} \frac{H_{\partial M}}{\sin\bar{\gamma}} + \frac{1}{\sin^2\bar{\gamma}} \partial_\eta \cos\bar{\gamma} + \frac{1}{2}\int_\Sigma R_g + |A|^2.$$

Using the bounds $R_g + |A|^2 \geqslant 0$, $H_{\partial M} \geqslant \bar{H}_{\partial M}$ and the Gauss-Bonnet theorem,

$$2\pi\chi(\Sigma) \geqslant \int_{\partial\Sigma} \left(\frac{H_{\partial M}}{\sin\bar{\gamma}} + \frac{1}{\sin^2\bar{\gamma}} \partial_\eta \cos\bar{\gamma} \right) \mathrm{d}\lambda.$$

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- Now we can trace back inequalities

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