Willmore type inequalities in geodesic balls of hyperbolic space

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Outline

Mean curvature

Mean curvature type flow

Free boundary surface

First result: Convergence of the flow

Second result: Monotonicity of a Willmore type quantity

Proof: Monotonicity of a Willmore type quantity

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- Let Σ_t be a differentiable family of hypersurfaces deforming $\Sigma_0 = \Sigma$, let X be the variational vector field (we choose compactly supported variations), then

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- The above formula is also known as the first variation of area

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- The second fundamental form is defined to be $\bar{\nabla}\nu$, in components, $h_{ij} = \langle \bar{\nabla}_{e_i}\nu, e_j \rangle$
- The mean curvature is just the trace of h: $H = \sum_i \langle \nabla_{e_i}\nu, e_i \rangle$

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3. Stahl 96 mean curvature flow with free boundary

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3. Neves 07, Wang-Hung 14 starting from convex hypersurface the IMCF does not converge. Converges in stronger conditions: principal curvatures $\kappa_i > 1$. The Penrose inequality asymptotically hyperbolic is still open; locally hyperbolic case is settled by Lee-Neves 13.

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- If the mean curvature of Σ is zero, we say that Σ is a free boundary minimal surface

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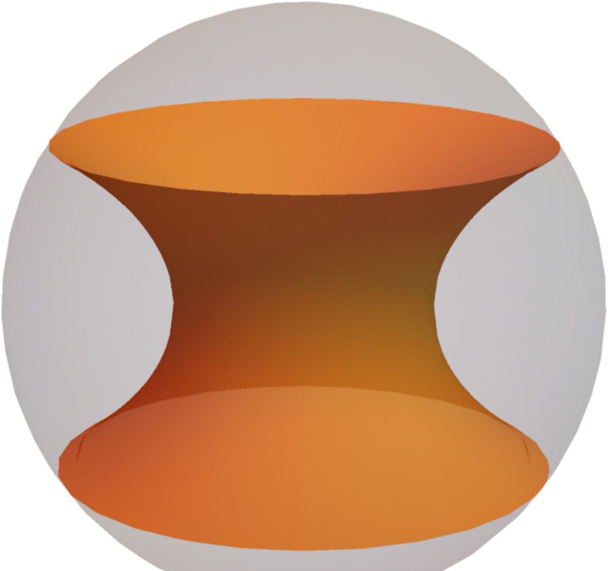
- The piece inside the sphere is called critical catenoid.

Critical catenoid



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3. Scheuer-Wang-Xia 22 Alexandrov-Fenchel type inequality which involves higher order mean curvature; Wang-Weng-Xia 22 capillary.

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- We are going to generalize Lambert-Scheuer 16, 17 result to M

Theorem: Convergence of the flow

Theorem Let M be a strictly convex, free boundary hypersurface in the ball B, then the inverse mean curvature flow M_t converges to a totally geodesic disk in finite time.

Monotonicity of a Willmore type quantity: Theorem

- Theorem (C. 22) Suppose that M is strictly convex, then $m_H(M)$ given by

$$m_H(M) = |M|^{\frac{2-n}{n}} \int_M (H^2 - n^2) + \Lambda|\partial M|$$

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  - $\Lambda = 2\coth \rho_0 \lambda^{(2-n)/n}$

Lower bound of the Willmore type quantity

- Theorem (C. 22) Assuming that the flow preserves strict convexity, and converges to a totally geodesic disk, then

$$|M|^{\frac{2-n}{n}} \int_M (H^2 - n^2) + \Lambda |\partial M| \geqslant -n^2 \lambda^{\frac{2}{n}} + \Lambda \omega_{n-1} \sinh^{n-1} \rho_0 .$$

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Key lemma from convex geometry in spheres

(Makowski-Scheuer 15) If ∂M_T is weakly convex in $\mathbb{S}^n$, then it lies in an open hemisphere or it is the equator.

Idea

- (easy) If ∂M_T is some equator, then using mean convexity of M_T, free boundary condition, maximum principle, M_T has to be a totally geodesic disk. (T being $\bar{T}$ or T^* is OK)

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  4. Which means at maximal existence time, the flow M_t converges to a totally geodesic disk.

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4. $\partial_t h_{ij} = -\bar{R}(\partial_i, \nu, \nu, \partial_j)\frac{1}{H} + \frac{h_i^k h_{jk}}{H} - \nabla_i \nabla_j \frac{1}{H}$

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5. $\partial_t H = -\Delta \frac{1}{H} - (\mathrm{Ric}(\nu) + |A|^2)\frac{1}{H}$

Evolution of coordinates

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- By Hessian decomposition

$$\Delta z^\alpha = n z^\alpha - H\nu^\alpha$$

Boundary derivative of mean curvature

- (Stahl 96)

$$\nabla_\eta \frac{1}{H} = \langle \nabla_\eta (\tfrac{1}{H}\nu), \nu\rangle \tag{1}$$
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- Boundary derivtives and then by maximum principle.

Height estimate

Theorem Let M be a strictly convex hypersurface, then $z^1 \geqslant \delta > 0$ for all M where the constant δ depends only on $\sup_M |A|$ and the distance of ∂M to some equator of $\mathbb{S}^n(\rho_0)$.

Comments The proof is long; and uses hyperbolic trignogeometry and convex geometry.

Lower bound of the mean curvature

Proposition Let M_t be the solution to the mean curvature flow with free boundary, if $\partial M_{\bar{T}}$ is positive distance from the equator, then

$$\sup_{M_t, t \in [0, \bar{T})} \frac{1}{H} \leq c$$

where c depends only on M_0 and the distance of $\partial M_{\bar{T}}$ to the equator.

Proof of the lower bound

1. $f(q) = -\log(\Lambda - q)$, $q = \lambda z^1 + z^0$,
   $F(z) = -\log(\Lambda - (\lambda z^1 + z^0))$ with $0 < \Lambda < \frac{1}{\cosh \rho_0}$ and
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$$-\frac{n\Lambda}{(n-q)H^2} + 2f' \frac{\partial q}{\partial \zeta^\alpha} \nu^\alpha \frac{1}{H} + |A|^2/H^2 \geqslant 0$$

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   - the equality implies an bound on $1/H$

Maximal existence time is also maximal convexity time

Theorem The strict convexity is preserved up to T^*.
Observation If $\bar{T} < T^*$, then the smallest principal curvature κ_1 reaches zero when the flow M_t accross the time $\bar{T}$

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However, this cannot always be true; just pick a big α.

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  However, this cannot always be true; just pick a big α.
- Then for large α, the supremum of ϕ would be decreasing which implies a bound on $\tilde{H}$ contradicting the definition of $\bar{T}$

Monotonicity of a Willmore type quantity: Theorem

- Theorem (C. 22) Suppose that M is strictly convex, then $m_H(M)$ given by

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is monotone under IMCF with free boundary (assume for now strict convexity is preserved and that $|M_t| \leq \lambda$).

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- A comment: The definition of $m_H(M)$ depends the supporting hypersurface of the boundary. If ∂M lies on the horosphere, or an equidistant hypersurface, then m_H is different.

Proof of monotonicity

- Use the evolution equation

$$\partial_t \int_{M_t} (H^2 - n^2)\sqrt{g}$$

$$= \int_{M_t} (H^2 - n^2)\sqrt{g} + 2\int_{M_t} H(-\Delta \tfrac{1}{H} - (-n + |A|^2)\tfrac{1}{H})\sqrt{g}.$$

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$$\begin{aligned}&\partial_t \int_{M_t} (H^2 - n^2)\sqrt{g}\\ &= \int_{M_t} (H^2 - n^2)\sqrt{g} + 2\int_{M_t} H(-\Delta\tfrac{1}{H} - (-n + |A|^2)\tfrac{1}{H})\sqrt{g}.\end{aligned}$$

- Integration on the Laplacian term

$$\begin{aligned}&-\int_{M_t} H\Delta\tfrac{1}{H}\\ &= -2\int_{\partial M_t} H\langle\nabla\tfrac{1}{H}, \eta\rangle + 2\int_{M_t}\langle\nabla H, \nabla\tfrac{1}{H}\rangle\\ &\leqslant -2\int_{\partial M_t} H\langle\nabla\tfrac{1}{H}, \eta\rangle\end{aligned}$$

Proof of monotonicity

- Use the evolution equation

$$\partial_t \int_{M_t} (H^2 - n^2)\sqrt{g}$$
$$= \int_{M_t} (H^2 - n^2)\sqrt{g} + 2\int_{M_t} H(-\Delta \tfrac{1}{H} - (-n + |A|^2)\tfrac{1}{H})\sqrt{g}.$$

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$$\leqslant -2\int_{\partial M_t} H\langle \nabla \tfrac{1}{H}, \eta\rangle$$

  - $\langle \nabla \frac{1}{H}, \eta\rangle = \coth \rho_0 / H$

Proof of monotonicity

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- Integration on the Laplacian term

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\end{aligned}
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  - $\langle\nabla\frac{1}{H}, \eta\rangle = \coth\rho_0/H$
  - $|A|^2 \geq \frac{H^2}{n}$ (by diagonalize A)

Proof of monotonicity: adding a correction term

- So

$$\partial_t \int_{M_t} (H^2 - n^2) \leqslant \tfrac{n-2}{n} \int_{M_t} (H^2 - n^2) - 2 \coth \rho_0 |\partial M_t|$$

Proof of monotonicity: adding a correction term

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- Let $q(t) = |M_t|^{\frac{2-n}{n}} \int_{M_t} (H^2 - n^2) + \Lambda |\partial M_t|$

Change rate of $q(t)$

- $$q'(t) \leq \Lambda \partial_t |\partial M_t| - 2 \coth \rho_0 |M|^{(2-n)/n} |\partial M_t|$$

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  - $H_{M,B} = H_{\partial M, \partial B} + A_M(\eta, \eta)$
- $|M| \leq \lambda$
- So $q' \leq 0$

Thank you

Thank you!