# LLARULL TYPE THEOREMS FOR BANDS IN THREE AND FOUR DIMENSIONS

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The Llarull theorem asserts that the scalar curvature and the metric on the $n$-sphere cannot be bounded below at the same time by those of the standard $n$-sphere. Using the warped $\mu$-bubble method, we develop Llarull type theorems for three and four-dimensional bands with spectral scalar curvature bounds. Our method is flexible and also applies to the Perelman scalar curvature.

## 1. Introduction

The standard $n$-sphere has the Llarull type rigidity.

**Theorem 1.1** ([Lla98])**.** *Let $f : (M, g) \to (\mathbb{S}^n, g_{\mathbb{S}^n})$ be a distance non-increasing spin map such that $R_g \geqslant n(n-1)$, then $f$ is an isometry.*

Here, $R_g$ denotes the scalar curvature. In recent years, the theorem was established via different techniques: spinors [BBHW24], [LSW24], [WX25], spactime harmonic function [HKKZ25], $\mu$-bubble [Gro23], [HLS23], [CWXZ24]. See also [Lot21], [GS02] and [Lis10] for generalisations. We shall revisit [CWXZ24] and [Lis10] later.

Among many of the works listed here, a Llarull type theorem for warped products with a log-concave warping factor or *bands* was also established. Let

$$g_\xi = \mathrm{d}t^2 + \xi(t)^2 g_{\mathbb{S}^{n-1}} \tag{1.1}$$

on $\mathbb{S}^n_I := I \times \mathbb{S}^{n-1}$ where $I$ is some closed interval $[t_-, t_+]$ and $\xi : I \to \mathbb{R}$ is a positive function. Define

$$h_\xi(t) = (n-1)\tfrac{\xi'(t)}{\xi(t)}.$$

The log-concavity says that $(\log \xi)'' < 0$.

We now introduce the manifold with a band structure. A *band* is a manifold $M$ with at least two boundary components which are re-groupped into two non-empty groups $\partial_- M$ and $\partial_+ M$. We say that a band is *over-spherical* if there is a map $f : M \to S^n_I$ such that $f(\partial_\pm M) = \{t_\pm\} \times \mathbb{S}^{n-1}$. We also say that $f$ is over-spherical. For a hypersurface $\Sigma$ homologous to $\partial_- M$, we fix the direction of the unit normal of $\Sigma$ to point outside of the region bounded by $\Sigma$ and $\partial_- M$. In particular, In particular, we fix $\nu_-$ (*resp.* $\nu_+$) to be the unit normal of $\partial_- M$ (*resp.* $\partial_+ M$) to point inside (*resp.* outside) of $M$. And the second fundamental form $A$ and the mean curvature $H$ of a given hypsurface $\Sigma$ are computed using the choice of the unit normals, that is, $A = \nabla \nu$ and $H = \mathrm{tr}_\Sigma(\nabla \nu)$.

The Llarull type theorem for the warped products is given in the following:

**Theorem 1.2.** *Let $f$ be an over-spherical spin map from $(M, g)$ to $(\mathbb{S}^n_I, g_\xi)$ such that $R_g \geqslant f^* R_{g_\xi}$ in $M$, $H_{\partial_+ M} \geqslant h_\xi(t_+)$ along $\partial_+ M$ and $H_{\partial_- M} \leqslant h_\xi(t_-)$ along $\partial_- M$, then $f$ is an isometry.*

The $\mu$-bubble proof [Gro23], [HLS23] of Theorem 1.2 was previously known to work in three dimensions with the recent development of Cecchini-Wang-Xie-Zhu [CWXZ24] proof of Theorem 1.1 in dimension four using Listing's theorem [Lis10] and the Ricci flow. However, Theorem 1.2 remains open for higher dimensions for non-spin bands.

In this article, we settle the four-dimensional case of Theorem 1.2 in the non-spin setting, see Theorem 1.4. In fact, we can prove a more general theorem which involves the so-called *spectral scalar curvature.*

**Definition 1.3.** *Let $\gamma$ be a real number and $(M, g)$ be a Riemannian manifold and $u$ be a positive function, we call*

$$-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g \tag{1.2}$$

*the spectral scalar curvature. More generally, let $\sigma$ be a real number, and we consider*

$$\Lambda(g, u) := -\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g + \sigma\gamma u^{-2}|\nabla u|^2 \tag{1.3}$$

*which we refer as the spectral scalar curvature with a gradient term. We also use the convention $\Lambda_{g,u} = \Lambda(g, u)$. We omit the reference to $u$ when the context is clear.*

The earliest occurence of the spectral scalar curvature seems to be in the work of Schoen-Yau [SY83]. Also, a special case of (1.3) is the Perelman or weighted scalar curvature $P = R_g + 2\Delta f - |\nabla f|^2$ (with $\sigma = 1 - \gamma/2$, $f = -\gamma \ln u$), see [Wan25] and the references therein.

Define

$$u_\xi(t) = \xi^{\frac{1}{2(1-\sigma)-\gamma}},$$
$$m_\xi(t) = h_\xi(t) + \gamma\tfrac{u_\xi'(t)}{u_\xi(t)} = (n - 1 + \tfrac{\gamma}{2(1-\sigma)-\gamma})\tfrac{\xi'}{\xi}$$

for (1.1). We require that $m_\xi'(t) < 0$ on $[t_-, t_+]$, $\sigma < 1$; $\gamma < 4(1 - \sigma)$ when $n = 3$ and $\gamma < 3(1 - \sigma)$ when $n = 4$.

**Theorem 1.4.** *Let $n = 3$ or 4, $(M, g)$ be an $n$-dimensional oriented band, $f : (M, g) \to (S_I^n, g_\xi)$ be a smooth map of non-zero degree which does not increase the distance and $f(\partial_\pm M) \subset f(\partial_\pm S_\xi^n)$. In addition, there exists a positive function $u$ such that the following curvature bounds*

*(a) $\Lambda_{g,u} \geqslant f^*\Lambda(\xi, u_\xi)$ in $(M, g)$,*
*(b) $H_{\partial_+M} + \gamma u^{-1}\langle\nabla u, \nu_-\rangle \geqslant m_\xi(t_+)$ on $\partial_+M$ and $H_{\partial_-M} + \gamma u^{-1}\langle\nabla u, \nu_-\rangle \leqslant m_\xi(t_-)$ on $\partial_-M$,*

*hold, then $f$ is an isometry and $u$ is a constant multiple of $u_\xi$.*

**Remark 1.5.** *When $\gamma = 2(1 - \sigma)$, the definition of $u_\xi$ might have a minor issue which can be easily addressed by replacing $u_\xi$ by $u(t)$, $\xi$ by a constant and $m_\xi$ by $(2(n - 1)(1 - \sigma) - (n - 2)\gamma)u'/u$ in Theorem 1.4.*

**Remark 1.6.** *Note that also the warping factor in (1.1) is not necessarily log-concave, in stead the condition is $m_\xi' < 0$, and $\xi$ can be log-convex if $\gamma > 2(1 - \sigma)$.*

Warped $\mu$-bubble method is a generalization of the $\mu$-bubble method introduced by Gromov [Gro23], and warped $\mu$-bubbles recently were used in the resolution of several important problems together with the spectral version of various curvatures.

It is desirable to establish higher dimensional Llarull type theorems with lower bounds on (1.3), and in this article, we are able to show the following Llarull type theorem for bands in three and four dimensions.

A spectral Llarull theorem was established in [CPW24] with (1.2) bounded below by a positive constant via spinors and spacetime harmonic functions. There were also similar results which deals with bounds on (1.2), see [CS25], [HSS24], [HKKZ24], [ZZ25], [Cho25]. Motivated by the warped $\mu$-bubble proof of the band width estimates [CS25], in particular for over-torical bands, we are interested in a non-spin analog of Theorem 1.2 for over-spherical bands with spectral curvature bounds.

The case $\sigma \neq 0$ of Theorem 1.4 is new in the sense that it is actually not known via spinors or spacetime harmonic functions, in particular, it is desirable to show a higher dimensional (i.e., $n \geqslant 5$) version of Theorem 1.4. Our proof is based on a combination of the warped $\mu$-bubble method with the Gauss-Bonnet theorem when $n = 3$ and with the Listing's Theorem 1.7 when $n = 4$.

Listing's theorem stated below is stronger than Llarull's Theorem 1.1.

**Theorem 1.7** ([Lis10])**.** *If $f : (M^n, g) \to (\mathbb{S}^n, g_{\mathbb{S}^n})$ is a smooth spin map of non-zero degree such that*

$$R_g(p) \geqslant n(n-1)\|\mathrm{d}f_p\|^2 \text{ for all } p \in M,$$

*then there exists a constant $c > 0$ such that $f : (M, cg) \to (\mathbb{S}^n, g_{\mathbb{S}^n})$ is an isometry. In particular, when $n = 3$ (since all 3-manifolds are spin).*

Our use of Listing's theorem in dimension four is different from [CWXZ24], more specifically, we are able to avoid the Ricci flow by exploiting the full rigidity statement of Listing's Theorem 1.7, see Propositions 3.1 and 3.4. It should be quite straightforward to extend our method to some of the boundary cases, see [CW23], [KY24], [WWZ24]. We also showed a simple generalisation of Thoerem 1.7 in Theorem 4.1 which recovers a special case obtained for the Perelman scalar curvature by Zhou-Zhu [ZZ25] via spinor methods.

As Llarull's Theorem 1.1 is the most important case, we also need to develop a strategy. To this end, we need a new setup for the function $\xi$ appearing in the warped product (1.1). Now we assume that $t_- = 0$, $\xi$ is positive on $(t_-, t_+]$ and $\xi(t) = A(t - t_-) + O(|t - t_-|^2)$ near $t_-$ for some positive constant $A$. Topologically, $\{t_-\} \times \mathbb{S}^{n-1}$ is a conical point $p_\xi$ in $S^n_I$. Now we setup $M$ such that it has similar structures. We assume that $M$ is a compact metric space such that $M \backslash (\cup_i \{p_i\})$ is a smooth Riemannian manifold with the Riemannian metric $g$, and $M$ is the metric completion of $(M \backslash \cup_i \{p_i\}, g)$. Let the resulting metric be $d_g$ and we call $p_i$ a singular point of $M$. We assume that near $p_i$, the metric $g$ is of the form $\mathrm{d}t^2 + t^2 g_S + g_1$ for some closed Riemannian manifold $S$ with the Riemannian metric $g$ for some small (compared to $\mathrm{d}t^2 + t^2 g_S$) 2-form $g_1$, and that $\{t = 0\}$ corresponds to the singular point $p_i$. The limit of the spaces $\{(M, \lambda^{-1} d_g, p_i)\}_{\lambda > 0}$ as $\lambda \to 0$ in the pointed Gromov-Hausdorff distance is called the tangent cone of $M$ at $p_i$. It is clear that the tangent cone metric is given by $\mathrm{d}t^2 + t^2 g_S$. For the notion of tangent cones, see [BBI01, Chapter 8].

For simplicity, we assume that $f^{-1}(p_\xi)$ exhausts all the conical points of $(M, g)$. Note that $f^{-1}(p_\xi)$ can include smooth points of $M$, but such points still have the structure described above with $(S, g_S)$ being the standard $(n-1)$-sphere. Let $p \in f^{-1}(p_\xi)$, and we further assume that the tangent cones of $(M, g)$ at $p$ and of $(S^n_I, g_\xi)$ at $p_\xi$ are isometric. Let $\bar{u}(t) = t^{\frac{1}{2(1-\sigma)-\gamma}}$, now we assume that $\gamma < 2(1-\sigma)$, and we require that the tangent cone metric given by $\bar{g} = \mathrm{d}t^2 + A^2 t^2 g_{\mathbb{S}^{n-1}}$ satisfies

$$(n-2)(A^{-2} - 1) + \tfrac{2(n-1)(1-\sigma)}{2(1-\sigma)-\gamma} \geqslant 0. \tag{1.4}$$

**Theorem 1.8.** *Let $n = 3$ or 4, and let $f$, $(M, g)$ and $(S^n_I, g_\xi)$ be given as above, if there exists a positive function $u$ on $M \backslash f^{-1}(p_\xi)$ which satisfies*

*(a) $\Lambda_{g,u} \geqslant f^* \Lambda(g_\xi, u_\xi)$ in $(M, g)$,*
*(b) $H_{\partial_+ M} + \gamma u^{-1} \langle \nabla u, \nu_- \rangle \geqslant m_\xi(t_+)$ on $\partial_+ M$,*
*(c) $u = t^\alpha v(x) + O(t^{1+\alpha})$ near for some $\alpha \in \mathbb{R}$ near $t = 0$,*

*then $f$ is an isometry and $u$ is a constant multiple of $u_\xi$.*

**Remark 1.9.** *The 2-tensor $\mathcal{R}_{\bar{g}}$ defined in* (4.9) *of the metric $\bar{g}$ is non-negative under the assumption of* (1.4)*; note that $\bar{g} = \mathrm{d}t^2 + A^2 t^2 g_{\mathbb{S}^{n-1}}$ might not have a* *positive curvature operator since $A$ might be greater than 1. On the other hand,* (1.4) *is always satisfied for* $0 < A \leqslant 1$.

We believe the assumption of isometric tangent cones are removable, but at the oment it seems challenging to prove Theorem 1.8 without the isometric tangent cone assumption for general $\gamma \neq 0$ because of $u$. Actually, similar conditions were imposed in [CPW24]. For $\gamma = 0$, however, we can remove the assumption by making use of the distance comparison.

**Theorem 1.10.** *Let $n = 3$ or 4, and let $f$, $(M, g)$ and $(S^n_I, g_\xi)$ be given as above. Assume that $\xi = At + O(t^2)$ with $0 < A \leqslant 1$,*
*(a) $R_g \geqslant f^* R_{g_\xi}$ in $(M, g)$,*
*(b) and $H_{\partial_+ M} \geqslant h_\xi(t_+)$ on $\partial_+ M$,*
*then $f$ is an isometry.*

Our construction is local, and it also applies to the case when $\xi = A_+(t_+ - t) + O(|t - t_+|^2)$ for $0 < A_+ \leqslant 1$ near $t = t_+$. So this reproves Llarull's Theorem 1.1 in four dimensions by taking $\xi : [0, \pi] \to \mathbb{R}$ to be $\sin t$. The three dimensional version of Theorem 1.10 already appeared in [CW25, Theorem 1.6].

The article is organized as follows:

In Section 2, we introduce the warped $\mu$-bubble, compute the related first and second variation formulas, in particular, we relate the second variation with the curvature (1.3). We also calculate some model metrics.

In Section 3, we develop a foliation analysis to prove Theorem 1.4. In Section 4, we construction a foliation near the conical point with its leaf satisfying a certain mean curvature condition to show Theorem 1.8. In particular, we prove Theorem 1.10.

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## 2. Basics of warped $\mu$-bubbles

In this section, we introduce the warped $\mu$-bubble which is a critical point of an energy functional related to the weighted area. We compute the first and second variation of the energy functional, in particular, we relate the second variation with the curvature (1.3).

2.1. **The variational problem.** We assume that $M$ lies in a slightly larger manifold and we move $\partial_- M$ slightly outward and obtain a resulting band $M_1$. Let $\mathcal{C}_1$ to be the collection of all Cacciopolli sets $\Omega$ such that $\Omega$ contains a neighborhood of $\partial_- M_1$ and define the warped $\mu$-bubble functional

$$E(\Omega) = \int_{\partial^* \Omega \cap \operatorname{int} M_1} u^\gamma \mathrm{d}\mathcal{H}^{n-1} - \int_\Omega u^\gamma \mu \mathrm{d}\mathcal{H}^n, \tag{2.1}$$

where $\partial^* \Omega$ denotes the reduced boundary of $\Omega$. Let $\mathcal{C}$ to be the sub-collection of Cacciopolli sets $\Omega$ such that $\partial^* \Omega \backslash \partial_- M$ is a subset in the closure of $M$. We consider the minimisers of $E$ in the class $\mathcal{C}$. In Gromov's terminology [Gro23, Section 1.6.4], a minimiser of $E$ is called a *warped $\mu$-bubble.*

Let $\Omega_t$ be a smooth family of Cacciopolli sets,

$$\tfrac{\mathrm{d}}{\mathrm{d}t} E(\Omega_t) = \int_{\Sigma_t} (H + \gamma u^{-1} u_\nu - \mu) u^\gamma \phi \mathrm{d}\mathcal{H}^{n-1},$$

where $\Sigma_t = \partial\Omega_t \cap \operatorname{int} M$ and $H$ is the mean curvature of $\Sigma_t$. The quantity $H + \gamma u^{-1} u_\nu$ is called the *$u^\gamma$-weighted mean curvature*, and we often omit the dependence on $u^\gamma$.

Let $\Omega = \Omega_0$, if $\Sigma := \partial\Omega \cap \operatorname{int} M$ satisfies $H + \gamma u^{-1} u_\nu - \mu = 0$ along $\Sigma$, then the second variation of $E(\Omega_t)$ is given by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} E(\Omega_t) = \int_{\Sigma_t} \delta_Y (H + \gamma u^{-1} u_\nu - \mu) u^\gamma \phi \mathrm{d}\mathcal{H}^{n-1}. \tag{2.2}$$

**Definition 2.1.** *We say that $\Omega = \Omega_0$ is stable if*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} E(\Omega_t)|_{t=0} \geqslant 0. \tag{2.3}$$

*For convenience, we also say $\Sigma$ is stable if $\Sigma$ is of vanishing $H + \gamma u^{-1} u_\nu - \mu$ and the right-hand side of* (2.2) *is non-negative.*

By a direct calculation (see [AX24]), the first variation of $\eta = H + \gamma u^{-1} u_\nu - \mu$ is given by

$$\begin{aligned} (2.4) \qquad & \delta_{\phi\nu}\eta \\ = & -\Delta_\Sigma \phi - (\operatorname{Ric}(\nu) + |A|^2)\phi \\ & - \gamma w_\nu^2 \phi + \gamma u^{-1} \phi (\Delta_g u - \Delta_\Sigma u - H u_\nu) - \gamma \langle \nabla w, \nabla^\Sigma \phi \rangle - \mu_\nu \phi. \end{aligned}$$

We also call $\delta_{\phi\nu}\eta$ to be the linearisation of $\eta$.

2.2. **Rewrite of the second variation.** Now we write the linearisation of $\eta$ or the second variation of the warped $\mu$-bubble function in a way that is related to (1.3).

**Lemma 2.2.** *The first variation of $\eta = H + \gamma u^{-1} u_\nu - \mu$ is $\delta_{\phi\nu}\eta = \mathcal{L}_\mu \phi$, where given a function $f$ on $\Sigma$, the elliptic operator $\mathcal{L}_f$ is given by*

$$\mathcal{L}_f \phi = L\phi + \tfrac{1}{2} R_\Sigma \phi + Z_\mu \phi + W\phi,$$

*with $L$, $Z_f$ and $W$ given by*

$$L\phi = -\Delta_\Sigma \phi - \gamma u^{-1} \phi \Delta_\Sigma u - \gamma \langle \nabla w, \nabla^\Sigma \phi \rangle + \sigma\gamma |\nabla^\Sigma w|^2 \phi, \tag{2.5}$$

*and*

$$Z_f = -\tfrac{1}{n-1} \eta (nf - \gamma w_\nu) - \tfrac{n}{2(n-1)} f^2, \tag{2.6}$$

*and*

$$\begin{aligned} (2.7) \qquad W = & -\tfrac{1}{2}|A^0|^2 - \left( \frac{2n(1-\sigma) - (n-1)\gamma}{2(2(n-1)(1-\sigma) - (n-2)\gamma)} \mu^2 + \mu_\nu + \Lambda \right) \\ & - \tfrac{2(n-1)(1-\sigma)-(n-2)\gamma}{2(n-1)} \gamma (w_\nu - \tfrac{1}{2(n-1)(1-\sigma)-(n-2)\gamma} \mu)^2. \end{aligned}$$

*Proof.* It follows from Schoen-Yau's rewrite [SY79], $|A|^2 = |A^0|^2 + \frac{1}{n-1} H^2$ that

$$\begin{aligned} \operatorname{Ric}(\nu) + |A|^2 = & \tfrac{1}{2} R_g - \tfrac{1}{2} R_\Sigma + \tfrac{1}{2} H^2 + \tfrac{1}{2} |A|^2 \\ = & \tfrac{1}{2} R_g - \tfrac{1}{2} R_\Sigma + \tfrac{n}{2(n-1)} H^2 + \tfrac{1}{2} |A^0|^2. \end{aligned}$$

Here, $A^0$ is the traceless part of the second fundamental form. Since $H = \eta + \mu - \gamma w_\nu$, then

$$\begin{aligned} & \operatorname{Ric}(\nu) + |A|^2 \\ = & \tfrac{1}{2} R_g - \tfrac{1}{2} R_\Sigma + \tfrac{1}{2} |A^0|^2 + \tfrac{n}{2(n-1)} \eta^2 + \tfrac{n}{n-1} \eta (\mu - \gamma w_\nu) \\ & + \tfrac{n}{2(n-1)} \mu^2 - \tfrac{n}{n-1} \gamma \mu w_\nu + \tfrac{n}{2(n-1)} \gamma^2 w_\nu^2. \end{aligned}$$

Using the above and $|\nabla u|^2 = |\nabla^\Sigma u|^2 + u_\nu^2$ in (2.4), and after suitable regrouping, we obtain

$$\begin{aligned} & \delta_{\phi\nu}\eta \\ = & -\Delta_\Sigma\phi - \gamma u^{-1}\phi\Delta_\Sigma u - \gamma\langle\nabla w, \nabla^\Sigma\phi\rangle + \sigma\gamma|\nabla^\Sigma w|^2\phi \end{aligned}$$

$$-\tfrac{1}{2}|A^0|^2\phi - \tfrac{n}{n-1}\eta(\mu - \gamma w_\nu)\phi - \gamma\eta w_\nu - \tfrac{n}{2(n-1)}\eta^2\phi \tag{2.8}$$

$$-\tfrac{n}{2(n-1)}\mu^2\phi + (\tfrac{n}{n-1}\gamma - \gamma)\phi\mu w_\nu + w_\nu^2(-\tfrac{n}{2(n-1)}\gamma^2 + \gamma^2 - \gamma + \sigma\gamma)\phi \tag{2.9}$$

$$+\tfrac{1}{2}R_\Sigma\phi - (-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g + \sigma\gamma|\nabla^g w|^2)\phi - \mu_\nu\phi.$$

With some simplification of the terms on line (2.8), completing the square on the line (2.9) and using the definitions of $\Lambda$, $Z_\mu$ in (2.6) and $W$ in (2.7) finishes the proof. □

**Lemma 2.3.** *Let* $\psi = u^{\gamma/2}\phi$ *and* $L$ *be given in* (2.5), *then*

$$\int_\Sigma u^\gamma\phi L\phi = \tfrac{4-4\sigma}{4-\gamma-4\sigma}\int_\Sigma|\nabla_\Sigma\psi|^2 - \gamma(1-\sigma-\tfrac{\gamma}{4})\int_\Sigma|\psi\nabla_\Sigma w - \tfrac{1}{2(1-\sigma-\frac{\gamma}{4})}\nabla_\Sigma\psi|^2.$$

*Proof.* Inserting $\psi = u^{\gamma/2}\phi$ in the definition (2.5) of $L$,

$$\begin{aligned} u^\gamma\phi L\phi = & -u^{\gamma/2}\psi\Delta_\Sigma(u^{-\gamma/2}\psi) - \gamma u^{-1}\psi^2\Delta_\Sigma u \\ & -\gamma\langle\nabla w, \nabla^\Sigma(u^{-\gamma/2}\psi)\rangle u^{\gamma/2}\psi + \sigma\gamma|\nabla^\Sigma w|\psi^2. \end{aligned}$$

Take the integration on $\Sigma$, and by integration by parts on the first two terms and a direct calculation,

$$\int_\Sigma u^\gamma\phi L\phi = \int_\Sigma|\nabla_\Sigma\psi|^2 + \gamma\langle\nabla w, \nabla^\Sigma\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma + \sigma)|\nabla^\Sigma w|\psi^2. \tag{2.10}$$

The rest is a simple completing of the squares. □

The following is an immediate corollary from Lemmas 2.2 and 2.3.

**Corollary 2.4.** *For a warped h-hypersurface* $\Sigma$, *the second variation* (2.3) *can be rewritten as*

$$\begin{aligned} & \tfrac{\mathrm{d}^2}{\mathrm{d}t^2}E(\Omega_t)|_{t=0} \\ = & \tfrac{4-4\sigma}{4-\gamma-4\sigma}\int_\Sigma|\nabla_\Sigma\psi|^2 + \tfrac{1}{2}\int_\Sigma R_\Sigma\psi^2 \\ & -\gamma(1-\sigma-\tfrac{\gamma}{4})\int_\Sigma|\psi\nabla_\Sigma w - \tfrac{1}{2(1-\sigma-\frac{\gamma}{4})}\nabla_\Sigma\psi|^2 + \int_\Sigma W\psi^2, \end{aligned}$$

*where* $W$ *is given in* (2.7) *and* $u^{\gamma/2}\phi = \psi$.

2.3. **Model metrics for Theorem 1.4.** Now we discuss some model metrics where $\Lambda(g_\xi, u_\xi)$ is a constant. We set $\sigma = 0$. The model metrics are $g_\xi = \mathrm{d}t^2 + \xi(t)^2 g_{\mathbb{S}^{n-1}}$ with

$$\xi(t) = a\sin(bt),$$

where

$$a(\gamma, \Lambda) = \sqrt{\frac{\gamma(n-1)(n-2)(2n-(n-1)\gamma)}{2\Lambda(2(n-2)-(n-3)\gamma)}},$$

$$b(\gamma, \Lambda) = \frac{(2-\gamma)\sqrt{2\Lambda}}{\sqrt{\gamma(2n-(n-1)\gamma)(2(n-1)-(n-2)\gamma)}}.$$

The functions $u$, $h$ in $H + \gamma u^{-1}u' - h = 0$ by

$$u = \xi^{\frac{1}{2-\gamma}},\ h = \frac{2(n-1)+\gamma(2-n)}{2-\gamma}\xi'\xi^{-1}. \tag{2.11}$$

and $h$ satisfies the ODE

$$h' + \tfrac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)} h^2 + \Lambda = \tfrac{1}{2}(n-1)(n-2)\xi^{-2}. \tag{2.12}$$

This metric also satisfies $-\gamma u^{-1}\Delta_g u + \frac{1}{2}R_g = \Lambda$. Here, $\Lambda > 0$, the case already appeared in [CPW24]. Now for $\Lambda < 0$, then $\xi = a(\gamma, -\Lambda)\sinh(b(\gamma, -\Lambda)t)$, and $\xi$ satisfies (2.11) and (2.12). Now for $\Lambda = 0$, $\xi = a(\gamma, 1)b(\gamma, 1)t$ and $\xi$ satisfies (2.11) and (2.12).

## 3. Llarull type theorem for bands

In this section, we give the proof of Theorem 1.4. Our strategy is a classical foliation argument which makes the full use of the rigidity of Listing (Theorem 1.7).

3.1. **Construction of $\mu$.** Given a warped product metric $g_\xi = \mathrm{d}t^2 + \xi(t)^2 g_{\mathbb{S}^{n-1}}$ where $\xi(t) > 0$ for all $t \in [t_-, t_+]$ and $\gamma \geqslant 0$ on $S^n_I$, define

$$m_\xi(t) = (n-1)\xi'/\xi,\ h_\xi(t) = (n - 1 + \tfrac{\gamma}{2(1-\sigma)-\gamma})\tfrac{\xi'}{\xi},\ u_\xi(t) = \xi^{\frac{1}{2(1-\sigma)-\gamma}}.$$

Let $(M, g)$ be a band such that the map $f : (M, g) \to (S^n_I, g_\xi)$ is of non-zero degree, define

$$\mu := \mu_\xi = h_\xi \circ \mathrm{pr}_I \circ f : M \to \mathbb{R}, \tag{3.1}$$

and

$$f_\Sigma : \mathrm{pr}_{\mathbb{S}^{n-1}} \circ f \circ i_\Sigma : \Sigma \to \mathbb{S}^{n-1} \tag{3.2}$$

where $i_\Sigma : \Sigma \to M$ is the inclusion map of $\Sigma$ into $M$. Also define $\mathrm{pr}_I : \mathbb{S}^n_I := I \times \mathbb{S}^{n-1} \to I$ projection.

3.2. **Local rigidity.**

**Proposition 3.1.** *Assume that $(M, g)$ is given in Theorem 1.4, and that $\Sigma$ is a stable warped $\mu$-hypersurface in $(M, g)$ disjoint from $\partial_\pm M$ and such that $f_\Sigma$ is of non-zero degree, then $\Sigma$ is a level set of $\mathrm{pr}_I \circ f$, $(\Sigma, g_\Sigma)$ is isometric to $(S^{n-1}, (\xi \circ \mathrm{pr}_I \circ f)^2 g_{\mathbb{S}^{n-1}})$ via $f_\Sigma$.*

*Proof.* First, we estimate $\mu_\nu$ appeared in the definition of $W$ in (2.7). Recall that $\mu = \mu_\xi \circ \mathrm{pr}_I \circ f$, see (3.1). By the chain rule,

$$\mu_\nu = m'_\xi(\mathrm{pr}_I \circ f)\langle \nabla(\mathrm{pr}_I \circ f), \nu\rangle.$$

Since $f$ is of distance non-increasing, $|\nabla(\mathrm{pr}_I \circ f)| \leqslant 1$; and since $m'_\xi < 0$,

$$\mu_\nu \geqslant m'_\xi(\mathrm{pr}_I \circ f). \tag{3.3}$$

Now we can estimate the term $\frac{2n(1-\sigma)-(n-1)\gamma}{2(2(n-1)(1-\sigma)-(n-2)\gamma)}\mu^2 + \mu_\nu + \Lambda$ appeared in $W$, see (2.7). We set

$$\Gamma = \frac{2n(1-\sigma) - (n-1)\gamma}{2(2(n-1)(1-\sigma) - (n-2)\gamma)}$$

for convenience. By the comparison $\Lambda \geqslant f^*\Lambda_{g_\xi} = \Lambda_{g_\xi} \circ f$, $\mu = h_\xi \circ \mathrm{pr}_I \circ f$ and (3.3), we see

$$\Gamma\mu^2 + \mu_\nu + \Lambda \geqslant \Gamma m^2_\xi(\mathrm{pr}_I \circ f) + m'_\xi(\mathrm{pr}_I \circ f) + \Lambda_{g_\xi} \circ f.$$

It is easy to check that the right hand side of the above is $\frac{1}{2}(n-1)(n-2)\frac{1}{\xi^2(\mathrm{pr}_I \circ f)}$. Hence,

$$\Gamma\mu^2 + \mu_\nu + \Lambda \geqslant \tfrac{1}{2}(n-1)(n-2)\tfrac{1}{\xi^2(\mathrm{pr}_I \circ f)},$$

and evidently then

$$W \leqslant -\tfrac{1}{2}(n-1)(n-2)\tfrac{1}{\xi^2(\mathrm{pr}_I \circ f)}. \tag{3.4}$$

Using the rewrite of Corollary 2.4 on the stability of $\Sigma$, and then (3.4),

$$0 \leqslant \tfrac{4-4\sigma}{4-\gamma-4\sigma}\int_\Sigma |\nabla_\Sigma \psi|^2 + \tfrac{1}{2}\int_\Sigma R_\Sigma \psi^2 - \tfrac{1}{2}(n-1)(n-2)\int_\Sigma \tfrac{1}{\xi^2(\mathrm{pr}_I \circ f)}\psi^2 =: B_1(\psi,\psi) \tag{3.5}$$

for all $\psi \in C^2(\Sigma)$. Note that, we have dropped the integrals of the non-negative terms listed below (up to constant multiples)

$$|\psi\nabla_\Sigma w - \tfrac{1}{2(1-\sigma-\frac{\gamma}{4})}\nabla_\Sigma \psi|^2,\ (w_\nu - \tfrac{1}{2(n-1)(1-\sigma)-(n-2)\gamma}\mu)^2\psi^2,\ |A^0|^2\psi^2. \tag{3.6}$$

Set

$$\tilde{L} = -\tfrac{4-4\sigma}{4-\gamma-4\sigma}\Delta_\Sigma + \tfrac{1}{2}(R_\Sigma - (n-1)(n-2)\tfrac{1}{\xi^2(\mathrm{pr}_I \circ f)}). \tag{3.7}$$

Now we perform an analysis similar to [FS80] in three dimensions. Now $\tilde{L} = -\tfrac{4-4\sigma}{4-\gamma-4\sigma}\Delta_\Sigma + \tfrac{1}{2}(R_\Sigma - \tfrac{2}{\xi^2(\mathrm{pr}_I \circ f)})$. We set $\psi = 1$ and use the Gauss-Bonnet theorem in (3.5), and obtain

$$2\pi - \int_\Sigma \tfrac{1}{\xi^2(\mathrm{pr}_I \circ f)}\mathrm{d}A_g \geqslant B(1,1) = 2\pi\chi(\Sigma) - \int_\Sigma \tfrac{1}{\xi^2(\mathrm{pr}_I \circ f)}\mathrm{d}A_g \geqslant 0. \tag{3.8}$$

where $\mathrm{d}A_g$ denotes the area element of $\Sigma$ with respect to the metric $g$. Since $g \geqslant g_\xi$, $\mathrm{d}A_g \geqslant f^*(\xi^2 \mathrm{d}\mathcal{A}_{\mathbb{S}^2})$. So the left hand side of (3.8) is non-positive, which forces all inequalities of (3.8) have to be equalities, in particular, $B(1,1) = 0$.

It is easy to check $B_1(\psi,\psi) = \int_\Sigma \psi\tilde{L}\psi$. Hence by (3.5), the first eigenvalue $\lambda_1$ of $\tilde{L}$ is non-negative. In fact, since $B_1(1,1) = 0$, $\lambda_1 = 0$ and hence $\tilde{L}1 = 0$, that is, $R_\Sigma = \tfrac{2}{\xi^2(\mathrm{pr}_I \circ f)}$ along $\Sigma$. Tracing back the equalities in the full stability inequality (see Corollary 2.4), and noting that we have used the non-negativity of the terms in (3.6), we see

$$\nabla_\Sigma w = 0,\ w_\nu - \tfrac{1}{4(1-\sigma)-\gamma}\mu = 0,\ \text{and } |A^0| = 0 \text{ along } \Sigma \tag{3.9}$$

(note that $n = 3$). Moreover, $\nabla(\mathrm{pr}_I \circ f) = \nu$ and $\Lambda = f^*\Lambda_{g_\xi}$ along $\Sigma$.

Now we handle four dimensions. We use the estimate

$$\|(\mathrm{d}f_\Sigma)_p\|^2 \leqslant \tfrac{1}{\xi^2(\mathrm{pr}_I(f(p)))} \tag{3.10}$$

in the estimate (3.4) of $W$, and we obtain (using also $n = 4$)

$$0 \leqslant \tfrac{4-4\sigma}{4-\gamma-4\sigma}\int_\Sigma |\nabla_\Sigma \psi|^2 + \tfrac{1}{2}\int_\Sigma (R_\Sigma - 6\|(\mathrm{d}f_\Sigma)_p\|^2)\psi^2 =: B_2(\psi,\psi) \tag{3.11}$$

instead of (3.5). Let $\hat{L} = -\tfrac{4-4\sigma}{4-\gamma-4\sigma}\Delta_\Sigma + \tfrac{1}{2}(R_\Sigma - 6\|(\mathrm{d}f_\Sigma)_p\|^2)$, by the above, the first eigenvalue of $\hat{L}$ is non-negative. We choose the corresponding eigenfunction $v$ to be positive that is, $\hat{L}v = \lambda_1 v \geqslant 0$. Without loss of generality, we assume that $\sup_\Sigma v = 1$.

Let $\alpha = \tfrac{1-\sigma}{4-\gamma-4\sigma}$ and $g_\Sigma$ be the induced metric on $\Sigma$ from $g$. We consider the conformal metric $v^{4\alpha}g_\Sigma$, and the scalar curvature of $\Sigma$ with respect to $v^{4\alpha}g_\Sigma$ is

$$\begin{aligned}
R_\Sigma(v^{4\alpha}g_\Sigma) &= v^{-4\alpha}(R_\Sigma - 8\alpha v^{-1}\Delta_\Sigma v - 8\alpha(\alpha-1)|\nabla^\Sigma \log v|^2) \\
&= v^{-4\alpha}(2\lambda_1 + 6\|(\mathrm{d}f_\Sigma)_p\|^2 - 8\alpha(\alpha-1)|\nabla^\Sigma \log v|^2) \\
&= v^{-4\alpha}(2\lambda_1 - 8\alpha(\alpha-1)|\nabla^\Sigma \log v|^2) + 6\|(\mathrm{d}f_\Sigma)_p\|^2_{v^{4\alpha}g_\Sigma} \\
&\geqslant 6\|(\mathrm{d}f_\Sigma)_p\|^2_{v^{4\alpha}g_\Sigma}
\end{aligned}$$

by $\lambda_1 \geqslant 0$ and the fact $0 < \alpha < 1$ from the assumptions. By the rigidity statement of Theorem 1.7 in dimension $n - 1 = 3$, the above has to be an equality, and then

$\lambda_1 = 0$ and $v = 1$. In particular, $f_\Sigma : (\Sigma, \sigma) \to (S^3, c g_{\mathbb{S}^3})$ where the map $f_\Sigma$ is defined in (3.2) is an isometry for some constant $c > 0$. We will determine the constant $c$ shortly.

As in dimension 3, we can trace back the equalities: letting $\psi = 1$ in (3.11) first give $B_2(1, 1) = 0$ and then in Corollary 2.4, which forces

$$\nabla_\Sigma w = 0,\ w_\nu - \tfrac{1}{6(1-\sigma)-2\gamma}\mu = 0,\ \text{and } |A^0| = 0 \text{ along } \Sigma \tag{3.12}$$

similar to (3.9). Also, the equality of (3.10) is achieved, in particular, it implies $c = \xi \circ \mathrm{pr}_I \circ f$. As before, Moreover, $\nabla(\mathrm{pr}_I \circ f) = \nu$ and $\Lambda = f^* \Lambda_{g_\xi}$ along $\Sigma$. □

**Remark 3.2.** *We have also shown that the first eigenvalue of the operator* $\tilde{L}$ *defined in* (3.7) *is non-positive for any hypersurface* $\Sigma$ *such that* $f_\Sigma$ *is non-zero degree: assume otherwise, then it is a contradiction to the Gauss-Bonnet theorem in dimension* $n = 3$ *and to Theorem 1.7 due to Listing in dimension* $n = 4$.

### 3.3. **Analysis of the foliation.**

The most important property of $\Sigma$ in Lemma 3.1 is that fact that the linearisation of $H + \gamma u^{-1} u_\nu - \mu$ gives only $-\Delta_\Sigma$ which allows us to construct a foliation with the property that $H + \gamma u^{-1} u_\nu - \mu$ is constant along every leaf of the foliation.

**Proposition 3.3** (cf. [CS25, Lemma 3.4])**.** *Let* $\Sigma$ *be a hypsurface such that* $\delta_Y(H + \gamma u^{-1} u_\nu - \mu) = -\Delta_\Sigma \phi$, *then there exists a foliation* $\{\Sigma_t\}_{t \in (-\varepsilon, \varepsilon)}$ *such that* $\Sigma_0 = \Sigma$ *and that* $H + \gamma u^{-1} u_\nu - \mu$ *is constant along* $\Sigma_t$.

The following is our key estimate of the foliation.

**Proposition 3.4.** *Let* $\{\Sigma_t\}_{t \in (-\varepsilon, \varepsilon)}$ *be the foliation constructed in Proposition 3.3, let* $\eta(t) = H + \gamma u^{-1} u_\nu - \mu$, *then* $\eta(t) \leqslant 0$ *if* $t \in (0, \varepsilon)$ *and* $\eta(t) \geqslant 0$ *if* $t \in (-\varepsilon, 0)$.

*Proof.* Using the variational formula (2.4),

$$\eta'(t) = \mathcal{L}_{\mu,t} \phi_t = L_t \phi_t + \tfrac{1}{2} R_{\Sigma_t} \phi_t + Z_{\mu,t} \phi_t + W_t \phi_t$$

where $\phi_t = \langle \partial_t, \nu_t \rangle > 0$. We use the subscript $t$ to indicate that the quantities live in $\Sigma_t$. Note that $Z_{\mu,t} \leqslant -\frac{1}{n-1} \eta(t)(n\mu - \gamma w_{\nu_t}) =: -q_t \eta(t)$ by (2.6) and then by (3.4),

$$\eta'(t) \leqslant L_t \phi_t + \tfrac{1}{2}(R_{\Sigma_t} - \tfrac{2}{\xi^2(\mathrm{pr}_I(f(p)))}) \phi_t - q_t \eta(t) \phi_t.$$

Now we divide the above by $\phi_t$ and multiply by $\psi^2$, integrate on $\Sigma_t$ and obtain that

$$\eta'(t) \int_{\Sigma_t} \phi_t^{-1} \psi^2 \leqslant \int_{\Sigma_t} \psi^2 \phi_t^{-1} L_t \phi_t + \tfrac{1}{2}(R_{\Sigma_t} - \tfrac{2}{\xi^2(\mathrm{pr}_I(f(p)))}) \psi^2 - \eta(t) \int_{\Sigma_t} q_t \psi^2. \tag{3.13}$$

Let $\phi_t = u^{-\gamma/2} e^{\zeta_t}$, then by a direct calculation

$$\begin{aligned} \phi_t^{-1} L_t \phi_t &= -\phi_t \Delta_{\Sigma_t} \phi_t - \gamma u^{-1} \Delta_\Sigma u - \gamma \phi_t^{-1} \langle \nabla w, \nabla^{\Sigma_t} \phi_t \rangle + \sigma\gamma |\nabla^{\Sigma_t} w|^2 \\ &= -|\nabla^{\Sigma_t} \zeta_t|^2 - \Delta_{\Sigma_t} \zeta_t + (\tfrac{\gamma^2}{4} - \gamma) |\nabla^{\Sigma_t} w|^2 - \tfrac{\gamma}{2} \Delta_{\Sigma_t} w + \sigma\gamma |\nabla^{\Sigma_t} w|^2. \end{aligned}$$

Now we estimate $(-|\nabla^{\Sigma_t} \zeta_t|^2 - \Delta_{\Sigma_t} \zeta_t) \psi^2$ and $-\frac{\gamma}{2} \psi^2 \Delta_{\Sigma_t} w$ as

$$\begin{aligned} &(|\nabla^{\Sigma_t} \zeta_t|^2 + \Delta_{\Sigma_t} \zeta_t) \psi^2 \\ =&|\nabla^{\Sigma_t} \zeta_t|^2 \psi^2 - 2\langle \nabla^{\Sigma_t} \psi, \nabla^{\Sigma_t} \zeta_t \rangle + \mathrm{div}_{\Sigma_t}(\psi^2 \nabla^{\Sigma_t} \zeta_t) \\ \geqslant& -|\nabla^{\Sigma_t} \psi|^2 + \mathrm{div}_{\Sigma_t}(\psi^2 \nabla^{\Sigma_t} \zeta_t) \end{aligned}$$

by Cauchy-Schwarz inequality; and

$$\tfrac{\gamma}{2} \Delta_{\Sigma_t} w = \tfrac{\gamma}{2} \mathrm{div}_{\Sigma_t}(\psi^2 \nabla^{\Sigma_t} w) - \gamma \psi \langle \nabla^{\Sigma_t} \psi, \nabla^{\Sigma_t} w \rangle.$$

With these two estimates in (3.13) and using the divergence theorem, we see

$$\begin{aligned}&\eta'(t)\int_{\Sigma_t}\phi_t^{-1}\psi^2\\ \leqslant&\int_{\Sigma_t}(|\nabla^{\Sigma_t}\psi|^2+\gamma\psi\langle\nabla^{\Sigma_t}\psi,\nabla^{\Sigma_t}w\rangle+(\tfrac{\gamma^2}{4}-\gamma+\sigma\gamma)|\nabla^{\Sigma_t}w|^2\psi^2)\\ &+\tfrac{1}{2}\int_{\Sigma_t}(R_{\Sigma_t}-\tfrac{2}{\xi^2(\mathrm{pr}_I(f(p)))})\psi^2-\eta(t)\int_{\Sigma_t}q_t\psi^2.\end{aligned}$$

Note that this has the same form as (2.10). Applying Lemma 2.3, we obtain that

$$\eta'(t)\int_{\Sigma_t}\phi_t^{-1}\psi^2\leqslant\tfrac{4-4\sigma}{4-\gamma-4\sigma}\int_{\Sigma}(|\nabla_\Sigma\psi|^2+\tfrac{1}{2}(R_{\Sigma_t}-\tfrac{2}{\xi^2(\mathrm{pr}_I(f(p)))})\psi^2)-\eta(t)\int_{\Sigma_t}q_t\psi^2.$$

Recall that in the proof of Lemma 3.1, we have shown that the first eigenvalue of

$$\tilde{L}_t=-\tfrac{4-4\sigma}{4-\gamma-4\sigma}\Delta_{\Sigma_t}+\tfrac{1}{2}(R_{\Sigma_t}-(n-1)(n-2)\tfrac{1}{\xi^2(\mathrm{pr}_I\circ f)})$$

is non-positive, see Remark 3.2. Note that this is for every $t\in(-\varepsilon,\varepsilon)$. Let $\psi$ be replaced by the first eigenfunction $\psi_t$ of $\tilde{L}_t$, then

$$\eta'(t)\int_{\Sigma_t}\phi_t^{-1}\psi_t^2\leqslant-\eta(t)\int_{\Sigma_t}q_t\psi_t^2$$

which gives

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\eta(t)\exp(\int^t Q(s)\mathrm{d}s)\right)\leqslant 0$$

where $Q(t)=\frac{\int_{\Sigma_t}q_t\psi_t^2}{\int_{\Sigma_t}\phi_t^{-1}\psi_t^2}$. The above ODE and that $\eta(0)=0$ finish the proof of the proposition. □

Now we are ready to finish the proof of Theorem 1.4.

*Proof of Theorem 1.4.* By an argument of Chai-Sun [CS25, Appendix B] (replacing the $\mu$-hypersurface with the warped $\mu$-hypersurface), we can assume that there exists a non-trivial minimiser $\Omega$ to the functional (2.1). Let $\Sigma'=\partial\Omega\backslash\partial_-M$, then by [Rad23, Lemma 6.3], there exists a connected component $\Sigma$ of $\Sigma'$ such that $f_\Sigma:\Sigma\to\mathbb{S}^{n-1}$ has non-zero degree. Moreover, $\Sigma$ is a stable warped $\mu$-hypersurface. By Lemma 3.1, the linearisation of $H+\gamma u^{-1}u_\nu-\mu$ along $\Sigma$ is just $-\Delta_\Sigma$ and it leads to the existence of a foliation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ such that $\Sigma_0=\Sigma$ by Proposition 3.3. By Proposition 3.4, $\eta(t)\leqslant 0$ if $t\in(0,\varepsilon)$ and $\eta(t)\geqslant 0$ if $t\in(-\varepsilon,0)$.

For each non-zero $t\in(-\varepsilon,\varepsilon)$, define $\Omega_t$ as follows

$$\Omega_t=\Omega\cup\tilde{\Omega}_t\text{ if }t>0,\ \Omega\backslash\tilde{\Omega}_t\text{ if }t<0,$$

where $\tilde{\Omega}_t$ is the region bounded by $\Sigma_t$ and $\Sigma$.

We consider $t>0$ and the case $t<0$ is similar. By the first variation of $E$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}E(\Omega_t)=\int_{\Sigma_t}(H+\gamma u^{-1}u_\nu-\mu)u^\gamma\phi_t=\eta(t)\int_{\Sigma_t}u^\gamma\phi_t\leqslant 0$$

where $Y=\phi_t\nu$. So, each $\Omega_t$ is also a minimiser to the functional (2.1), and each $\Sigma_t$ satisfies the conclusions of Lemma 3.1. Hence, by connectedness, we find that $M$ is globally foliated by such hypersufaces. Since each level set is isometric to $(\xi\circ\mathrm{pr}_I\circ f)^2g_{\mathbb{S}^{n-1}}$ and $\nabla(\mathrm{pr}_I\circ f)=\nu$, we see the metric is $\mathrm{d}t^2+\xi^2(t)g_{\mathbb{S}^{n-1}}$, and the function $u$ is a contant multiple of $u^{\frac{1}{2(1-\sigma)-\gamma}}$ follows from (3.12). □

**Remark 3.5.** *The proof also works if we replace the comparisons of the mean curvatures by $H+\gamma u^{-1}u_\nu\geqslant\mu$ on $\partial_+M$ and $H+\gamma u^{-1}u_\nu\leqslant\mu$ on $\partial_-M$. In this case, we can obtain additionally that $\partial_\pm M$ are both level sets of the function $\mathrm{pr}_I\circ f$.*

## 4. Llarull's theorem with conical end points

In this section, we give the proof for Theorems 1.8 and 1.10.

4.1. **Tangent cone analysis.** Let $p \in f^{-1}(\bar{p})$ where $\bar{p}$ is the conical point of the metric $g_\xi$ at $t = t_-$. Without loss of generality, we assume that $t_- = 0$. In this subsection, we assume that the tangenet cone at $p$ of the metric $g$ and the tangent cone at $\bar{p}$ of the metric $g_\xi$ are isometric. In other words, near $p$, there exists a coordinat system such that the metric is of the form $\mathrm{d}t^2 + A^2t^2 g_{\mathbb{S}^{n-1}} + g_1$ with $A$ satisfying (1.4) and $g_1$ is a 2-form which is small compared to the metric $\mathrm{d}t^2 + A^2t^2 g_{\mathbb{S}^{n-1}}$.

First, we prove a spectral version of Theorem 1.7.

**Theorem 4.1.** *Let $0 \leqslant \gamma < \frac{2(n-1)(1-\sigma)}{n-2}$ and $f : (M^n, g) \to (\mathbb{S}^n, g_{\mathbb{S}^n})$ be a spin map and $u$ a positive smooth function on $M$ such that*

$$\Lambda_{g,u} \geqslant \tfrac{1}{2}n(n-1)\tfrac{(f^*\bar{g})(X,X)}{g(X,X)}$$

*at all points $p$ and for all non-zero vectors $X \in T_pM$, then there exists a positive constant $c$ such that $f : (M, g) \to (\mathbb{S}^n, cg_{\mathbb{S}^n})$ is an isometry and $u$ is a constant.*

*Proof.* The theorem follows from Listing's Theorem 1.7 by a conformal change. Let $f_1 = u^\beta$ where $4(n-1)/(n-2)\beta = 2\gamma$ and $\hat{g} = f_1^{\frac{4}{n-2}} g$. By the assumption, $1 - \sigma > \beta$. We use the well known formula of conformal change

$$R_{\hat{g}} = f_1^{-\frac{4}{n-2}}(R_g - \tfrac{4(n-1)}{n-2} f^{-1}\Delta f).$$

We have the estimate using that $1 - \sigma > \beta$,

$$\begin{aligned} R_{\hat{g}} =& f_1^{-\frac{4}{n-2}}(R_g - \tfrac{4(n-1)}{n-2}(\beta\tfrac{\Delta_g u}{u} + \beta(\beta-1)\tfrac{|\nabla u|^2}{u^2})) \\ =& f_1^{-\frac{4}{n-2}}(R_g - 2\gamma u^{-1}\Delta_g u - 2\gamma(\beta-1)\tfrac{|\nabla u|^2}{u^2}) \\ =& f_1^{-\frac{4}{n-2}}(2\Lambda_{g,u} - 2\gamma(\beta - 1 + \sigma)\tfrac{|\nabla u|^2}{u^2}) \\ \geqslant& f_1^{-\frac{4}{n-2}} n(n-1)\tfrac{(f^*\bar{g})(X,X)}{g(X,X)} \\ =& n(n-1)\tfrac{(f^*\bar{g})(X,X)}{\hat{g}(X,X)}. \end{aligned} \tag{4.1}$$

By Listing's Theorem 1.7 for the metric $\hat{g}$, $\hat{g}$ is isometric to some $cg_{\mathbb{S}^n}$ for some positive constant $c_1$. Then the inequality (4.1) must be an identity, so $u$ must be a constant and $f_1$ is also a constant. Hence, the theorem is proved. ☐

**Remark 4.2.** *We can apply the same trick and replace $\frac{(f^*\bar{g})(X,X)}{g(X,X)}$ with the square root of $\frac{(f^*\bar{g})(X\wedge Y, X\wedge Y)}{g(X\wedge Y, X\wedge Y)}$ where $X$ and $Y$ are vectors such that $X \wedge Y \neq 0$. Also, we can replace $(\mathbb{S}^n, g_{\mathbb{S}^n})$ with a manifold $(N, \bar{g})$ with positive curvature operator and $n(n-1)$ with $R_{\bar{g}} \circ f$. See* [Lis10, Theorem 1].

Denote

$$\lambda(g_\Sigma, v) = -\gamma v^{-1}\Delta_\Sigma v + \tfrac{1}{2}R_\Sigma + \sigma\gamma\tfrac{|\nabla^\Sigma v|^2}{v^2},$$

where $v$ is a function on $\Sigma$. We study the relation of $\Lambda_{\Sigma,v}$ and $\Lambda_{\bar{g},\bar{u}}$ where $\Sigma$ is the unit cross-section of the cone $(C_1 = (0,\infty) \times \Sigma, \bar{g} = \mathrm{d}t^2 + t^2 g_\Sigma)$ for some metric $g_\Sigma$ and $\bar{u} = t^\alpha v$.

Let $\Sigma_t = \{t\} \times \Sigma$, using the first variation formula (2.4) on $\Sigma_t$, we see

$$\begin{aligned} -\tfrac{n-1}{t^2} - \tfrac{\alpha\gamma}{t^2} =& \partial_t (H + \gamma u^{-1} \partial_t u) \\ =& -\gamma u^{-1} \Delta_g u + \tfrac{1}{2} R_g \\ & - (-\gamma u^{-1} \Delta_{\Sigma_t} u + \tfrac{1}{2} R_{\Sigma_r}) \\ & + \tfrac{1}{2} \tfrac{n}{n-1} H_{\Sigma_t}^2 + \gamma (\tfrac{\partial_t u}{u})^2 + \gamma H_{\Sigma_t} \tfrac{\partial_t u}{u}. \end{aligned}$$

Considering that $\frac{|\nabla^g u|^2}{u^2} = \frac{|\nabla^\Sigma v|^2}{v^2} + \alpha^2 t^{-2}$, $H_{\Sigma_t} = \frac{n-1}{t}$, the definitions of $\Lambda_{g,u}$ and $\lambda_{\Sigma,v}$, we have

$$\lambda(g_\Sigma, v) = \Lambda_{g,u} + \alpha\gamma(\alpha + n - 2 - \sigma\alpha) t^{-2} + \tfrac{1}{2}(n-1)(n-2) t^{-2}. \tag{4.2}$$

Now we show that the exponent in the asymptotics of $u$ is $\frac{1}{2(1-\sigma)-\gamma}$.

**Lemma 4.3.** *Let $u$ be given in Theorem 1.8, then $u(t) = t^{\frac{1}{2(1-\sigma)-\gamma}}(1 + O(t))$.*

*Proof.* First, if $\alpha < \bar{\alpha} := \frac{1}{2(1-\sigma)-\gamma}$, then $H_{\Sigma_t} = (n - 1 + \alpha\gamma)t^{-1} + o(t^{-1})$ which implies that $\Sigma_t$ is a strict barrier, but by Theorem 1.4, this is impossible, so $\alpha \geqslant \bar{\alpha}$. By the comparisons $\Lambda_{g,u} \geqslant \Lambda_{g_\xi, u_\xi} \circ f$, and that $f$ is distance non-increasing, then these properties were preserved by taking the tangent cones. Also, (4.2) holds up to some error term of order $O(t^{-1})$. By $\alpha \geqslant \bar{\alpha}$ and the comparison $\Lambda_{g,u} \geqslant \Lambda_{g_\xi, u_\xi} \circ f$ in the limit, it follows that

$$\lambda(g_\Sigma, v) \geqslant \lambda(g_\Sigma, 1).$$

By Theorem 4.1, $v$ is a positive constant, in particular, the equality holds in the above which implies $\alpha = \bar{\alpha}$ by (4.2). □

**Proposition 4.4.** *There exists a foliation $\{\Sigma_t\}_{t \in (0,\varepsilon)}$ such that each leaf has constant $\eta = H + \gamma u^{-1} u_\nu - h$.*

Given $t > 0$, we consider $\hat{g}_t(\tau, x) = t^{-2} g(t\tau, x) = (\mathrm{d}\tau^2 + A^2 \tau^2 g_{\mathbb{S}^{n-1}} + g_1 t^{-2})$ and $\hat{u}_t(\tau, x) = t^{-\frac{1}{2(1-\sigma)-\gamma}} u(t\tau, x)$ with $\tau \in (0, 1]$ and $x \in \mathbb{S}^{n-1}$.

Let $\hat{\nu}_\tau$ be the unit normal of the $\tau$-level set, $\hat{H}_\tau(x)$ denotes the mean curvature of $\tau$-level set $\hat{\Sigma}_\tau$ with respect to the metric $\hat{g}$. It is clear that $\hat{H}_\tau(x) = t H_{t\tau}(x)$ where $H_t(x)$ is the mean curvature of the $t$-level set with respect to the metric $g$.

Set

$$\hat{\eta}_{t,\tau}(x) = \hat{H}_\tau(x) + \gamma(\log \hat{u})_{\hat{\nu}} - \hat{h}. \tag{4.3}$$

Note that all of the above quantities depends on $t$ and we have made some dependence implicit. Now we consider only the $\tau$-level set $\hat{\Sigma}_1$ and the perturbations $\hat{\Sigma}_{\tau,v}$ defined by

$$(\tau + v(x), x)$$

where $v : \mathbb{S}^{n-1} \to \mathbb{R}$ is a function. Denote every geometric quantitiy on $\hat{\Sigma}_{\tau,v}$ with a subscript $t, \tau, v$. Using this setup, Proposition 4.4 is equivalent to the following.

**Proposition 4.5.** *For a sufficiently small $\varepsilon$, for each $t \in (0, \varepsilon)$ there exists a $v(x, t)$, $x \in \mathbb{S}^{n-1}$ such that $\hat{\Sigma}_{1,tv(\cdot,t)}$ has constant $\hat{\eta}$.*

*Proof.* By the first variation (2.4) and the Taylor expansion of $\hat{\eta}_{1,tv}$ with respect to $t$,

$$\begin{aligned} & (\hat{\eta}_{t,1,tv} - \hat{\eta}_{t,1,0}) t^{-1} \\ =& -\Delta_t v - |\hat{A}|_{\hat{g}}^2 v - \mathrm{Ric}_{\hat{g}}(\hat{\nu}_1, \hat{\nu}_1) v - \gamma \hat{u}^{-2} \hat{u}_{\hat{\nu}}^2 v \\ & + \gamma \hat{u}^{-1} (\Delta_{\hat{g}} \hat{u} - \Delta_t \hat{u} - \hat{H}_1 \hat{u}) v - \gamma \hat{u}^{-1} \langle \hat{\nabla} \hat{u}, \hat{\nabla} v \rangle \\ & - \langle \hat{\nabla} \hat{h}, \hat{\nu}_1 \rangle v + \hat{\partial}_\tau^\top \hat{\eta} + O(t). \end{aligned}$$

By the convergence of $\hat{g}$ to $\mathrm{d}\tau^2 + A^2\tau^2 g_{\mathbb{S}^{n-1}}$ and $\hat{u}$ to $\tau^{\overline{2(1-\sigma)-\gamma}^{1}}$,

$$(\hat{\eta}_{t,1,tv} - \hat{\eta}_{t,1,0})t^{-1} = -\bar{\Delta}_1 v + O(t). \tag{4.4}$$

By the convergence, $\hat{\eta}_{t,1,0} = O(t^2)$ and hence

$$\hat{\eta}_{t,1,tv}t^{-1} = -\bar{\Delta}_1 v + O(t).$$

Due to the convergence as $t \to 0$,

$$\hat{\eta}_{t,1,tv}t^{-1} = -\bar{\Delta}_1 v \tag{4.5}$$

Fix a sufficiently small positive number $\varepsilon$, and define the mapping

$$F(t,v) = \tfrac{\hat{\eta}_{t,1,tv}}{t} - |\hat{\Sigma}_1|^{-1}\int_{\hat{\Sigma}_1} \tfrac{\hat{\eta}_{t,1,tv}}{t},$$

where the integration is with respect to the metric $\mathrm{d}\tau^2 + A^2\tau^2 g_{\mathbb{S}^{n-1}}$ and $|\hat{\Sigma}_1|$ is the volume of $\hat{\Sigma}_1$ with respect to the metric $\mathrm{d}\tau^2 + A^2\tau^2 g_{\mathbb{S}^{n-1}}$. The space of $C_0^{k,\alpha}(\hat{\Sigma}_1)$ is the subspace of $C^{k,\alpha}(\hat{\Sigma}_1)$ with zero averages. We extend the definition of $F(t,v)$ to $t=0$ by taking limits and we see that $F(0,v) = -\bar{\Delta}_1 v$ from (4.5). Evidently,

$$DF_{(0,0)}(0,v) = \tfrac{\partial}{\partial s}F(0,sv)|_{s=0} = -\bar{\Delta}_1 v.$$

This says that $DF_{(0,0)}$ is an isomorphism restricted to $C_0^{0,\alpha}(\hat{\Sigma}_1)$. By the implicit function theorem, there exists a family of functions $v(\cdot,t)$ for all small $t$ that $F(t,v(\cdot,t)) = 0$, which gives that $\hat{\eta}_{t,1,tv(\cdot,t)}$ is a constant for such $v(\cdot,t)$. □

Now we need to determine the sign of $\hat{\eta}$.

**Proposition 4.6.** *Under the assumption of isometric tangent cones and the scalar curvature comparison, $\hat{\eta}(t) := \hat{\eta}_{t,1,tv(\cdot,t)} \leqslant 0$ for all $t \in (0,\varepsilon)$ where $v_t$ are the functions constructed above.*

Before proving this proposition, we discuss the previous strategy (cf. [CW25]): we integrate (4.4) on $\hat{\Sigma}_1$ with respect to the metric $\mathrm{d}\tau^2 + A^2\tau^2 g_{\mathbb{S}^{n-1}}$,

$$t^{-1}\hat{\eta}(t) = \int_{\hat{\Sigma}_1} \hat{\eta}_{t,1,0}t^{-1} + O(t). \tag{4.6}$$

The limit of $\hat{\eta}_{1,0}$ is zero and note that the term $\hat{\eta}_{1,0}t^{-1}$ is a first variation of (4.3) at $t=0$ for the metric $\hat{g}_t$ and the functions $\hat{u}_t$. As easily seen, we have a comparison of the metric by the distance non-increasing property, but there is no comparison of the functions $\hat{u}_t$. It turns out that we can remedy the issue by considering the weighted version of (4.6).

We introduce some notations. Let $g_t$ and $u_t$ be a family of metrics and functions index by $t \in (-\varepsilon,\varepsilon)$ such that $g_0 = g$ and $u_0 = u$. Denote $\delta g = \lim_{t\to 0} t^{-1}g_t$ and $\delta u = \lim_{t\to} t^{-1}u_t$ (derivatives at $t=0$ of $g_t$ and $u_t$). Let $G(g,u)$ be any geometric quantity which depends on $g$ and $u$ and $G(g_t,u_t)$ be the corresponding geometric quantity computed with respect to $g_t$ and $u_t$, then $\delta_u G$, the variation of $G$ with only $u$ varying is defined to be $\lim_{t\to 0} t^{-1}(G(g,u_t) - G(g,u))$. We can define similarly $\delta_g G$. Let $(g_t^{(i)}, u_t^{(i)})$, $i = 1,2$ be any two families of metrics and functions, we use $\delta^i$ to denote the variation is taken with respect to each family index by $i$.

**Proposition 4.7.** *Let $g_t$ and $u_t$ be a family of metrics and functions on a manifold $M$ with non-empty boundary $\partial M$, then*

$$\int_{\partial M} \delta_u(H + \gamma u^{-1}u_\nu)u^{2-2\sigma} + \int_M u^{2-2\sigma}\delta_u\Lambda_g = 0, \tag{4.7}$$

*and*

$$\int_{\partial M}\delta_g(H+\gamma u^{-1}u_\nu)u^{2-2\sigma}+\int_M u^{2-2\sigma}\delta_g\Lambda_g=-\tfrac{1}{2}\int_M u^{2-2\sigma}\mathcal{R}_g-\tfrac{1}{2}\int_{\partial M}u^{2-2\sigma}\mathcal{H}_g. \tag{4.8}$$

*Here, the integration is with respect to the metric $g$ and the tensors $\mathcal{R}$ and $\mathcal{H}$ are defined by*

$$\begin{aligned}\mathcal{R}_g=&\operatorname{Ric}_g-2\gamma\tfrac{\nabla^2 u}{u}+2\sigma\gamma\frac{\nabla u\otimes\nabla u}{u^2}+2\gamma\frac{\nabla(u^{1-2\sigma}\nabla u)}{u^{2-2\sigma}}\\&-\gamma\frac{\nabla_k(u^{1-2\sigma}\nabla^k u)}{u^{2-2\sigma}}g-\frac{\nabla^2 u^{2-2\sigma}}{u^{2-2\sigma}}+\frac{\Delta_g u^{2-2\sigma}}{u^{2-2\sigma}}g,\\ \mathcal{A}_g=&A_{\partial M}-(2(1-\sigma)-\gamma)u^{-1}u_\nu g_{\partial M}\end{aligned} \tag{4.9}$$

*where $\nabla u$ is understood as a 1-form $(\nabla u)(e_i)=\nabla_{e_i}u$.*

*Proof.* Since both $\delta_u H$ and $\delta_u R_g$ vanish, (4.7) is equivalent to

$$\int_{\partial M}\delta_u(u^{-1}u_\nu)u^{2-2\sigma}+\int_M\delta_u(-u^{-1}\Delta_g u+\sigma|\nabla u|^2u^{-2})u^{2-2\sigma}=0.$$

By direct calculation and integration by parts,

$$\begin{aligned}&\int_M\delta_u(-u^{-1}\Delta_g u+\sigma|\nabla u|^2u^{-2})u^{2-2\sigma}\\=&-\int_M(-u^{-2}\delta u\Delta_g u+u^{-1}\Delta_g(\delta u))u^{2-2\sigma}\\&+\sigma\int_M(-2u^{-3}\delta u|\nabla u|^2+2u^{-2}\langle\nabla u,\nabla(\delta u)\rangle)u^{2-2\sigma}\\=&\int_M(u^{-2\sigma}\delta u\Delta_g u-\delta u\Delta_g u^{1-2\sigma})\\&+\int_{\partial M}(-u^{1-2\sigma}(\delta u)_\nu+(u^{1-2\sigma})_\nu\delta u)\\&-2\sigma\int_M(u^{-1-2\sigma}\delta u|\nabla u|^2+\operatorname{div}_g(u^{-2\sigma}\nabla u)\delta u)\\&+2\sigma\int_{\partial M}u^{-2\sigma}\delta u u_\nu.\end{aligned}$$

The interior terms of the above vanish and the boundary term is precisely

$$-\int_M u^{2-2\sigma}\delta_u(u^{-1}u_\nu)=-\int_M u^{2-2\sigma}(-u^{-2}\delta u u_\nu+u^{-1}(\delta u)_\nu).$$

Hence, (4.7) is proved.

The calculation of (4.8) is more involved but direct. We start with the well known variation

$$\delta R_g=-\langle\operatorname{Ric},\delta g\rangle+\operatorname{div}_g^2(\delta g)-\Delta_g\operatorname{tr}_g(\delta g)$$

of $R_g$ and the variation

$$2\delta_g H_g=(\mathrm{d}(\operatorname{tr}_g(\delta g))-\operatorname{div}_g(\delta g))(\nu)-\operatorname{div}_{\partial M}(\delta g(\cdot,\nu))^\top-\langle A_{\partial M},\delta g\rangle$$

of the mean curvature $H_g$. For the variation $\delta_g(u^{-1}\Delta_g u)$, we need the variation of Christoffel symbols $\Gamma_{ij}^k$ of $g$, which is given by

$$\delta_g\Gamma_{ij}^k=\tfrac{1}{2}g^{kl}(\nabla_i(\delta g)_{jl}+\nabla_j(\delta g)_{il}-\nabla_l(\delta g)_{ij}),$$

and it yields

$$\begin{aligned}
&\delta_g(\Delta_g u) = \delta_g(g^{ij}\nabla_i\nabla_j u)\\
=&\delta_g(g^{ij}(\partial_i\partial_j u - \Gamma_{ij}^k\partial_k u))\\
=&-\delta_g g^{ij}(\partial_i\partial_j u - \Gamma_{ij}^k\partial_k u) - g^{ij}\delta_g\Gamma_{ij}^k\partial_k u\\
=&-\langle\delta g, \nabla^2 u\rangle - \tfrac{1}{2}\nabla^k u(2\nabla^i(\delta g)_{ik} - \nabla_k(\operatorname{tr}_g(\delta g))).
\end{aligned}$$

The variation $\delta_g(|\nabla u|^2) = -\langle\delta g, \nabla u\otimes\nabla u\rangle$. Hence,

$$\begin{aligned}
&\int_M u^{2-2\sigma}\delta_g\Lambda\\
=&\int_M \delta_g(-\gamma u^{-1}\Delta_g u + \tfrac{1}{2}R_g + \sigma\gamma|\nabla u|^2u^{-2})u^{2-2\sigma}\\
=&-\gamma\int_M u^{1-2\sigma}(-\langle\delta g, \nabla^2 u\rangle - \tfrac{1}{2}\nabla^k u(2\nabla^i(\delta g)_{ik} - \nabla_k(\operatorname{tr}_g(\delta g))))\\
&+\tfrac{1}{2}\int_M u^{2-2\sigma}(-\langle\mathrm{Ric}, \delta g\rangle + \operatorname{div}_g^2(\delta g) - \Delta_g\operatorname{tr}_g(\delta g))\\
&-\sigma\gamma\int_M u^{-2\sigma}\langle\delta g, \nabla u\otimes\nabla u\rangle\\
=&\gamma\int_M u^{1-2\sigma}\langle\delta g, \nabla^2 u\rangle - \sigma\gamma\int_M u^{-2\sigma}\langle\delta g, \nabla u\otimes\nabla u\rangle\\
&+\gamma\int_{\partial M} u^{1-2\sigma}(\delta g)(\nabla u, \nu) - \gamma\int_M(\delta g)_{ik}\nabla^i(u^{1-2\sigma}\nabla^k u)\\
&-\tfrac{\gamma}{2}\int_{\partial M} u^{1-2\sigma}u_\nu\operatorname{tr}_g(\delta g) + \tfrac{\gamma}{2}\int_M\nabla_k(u^{1-2\sigma}\nabla^k u)\operatorname{tr}_g(\delta g)\\
&-\tfrac{1}{2}\int_M u^{2-2\sigma}\langle\mathrm{Ric}, \delta g\rangle\\
&+\tfrac{1}{2}\int_{\partial M}(u^{2-2\sigma}\operatorname{div}_g(\delta g)(\nu) - (\delta g)(\nabla u^{2-2\sigma}, \nu)) + \tfrac{1}{2}\int_M\langle\nabla^2u^{2-2\sigma}, \delta g\rangle\\
&+\tfrac{1}{2}\int_{\partial M}(\nabla_\nu u^{2-2\sigma}\operatorname{tr}_g(\delta g) - u^{2-2\sigma}\nabla_\nu\operatorname{tr}_g(\delta g)) - \tfrac{1}{2}\int_M\operatorname{tr}_g(\delta g)\Delta_g u^{2-2\sigma}.
\end{aligned}$$

In the last equality, we have used integration by parts so that the interior terms do not contains the derivatives of $\delta g$.

Now we turn to $\delta_g(H + \gamma u^{-1}u_\nu)$. Recall that

$$2\delta_g H = (\mathrm{d}(\operatorname{tr}_g(\delta g)) - \operatorname{div}_g(\delta g))(\nu) - \operatorname{div}_{\partial M}(\delta g(\nu, \cdot))^\top - \langle\delta g, A_{\partial M}\rangle,$$

and $\delta_g(\nu^j) = -((\delta g(\nu, \cdot))^\top)^j - \tfrac{1}{2}\delta g(\nu, \nu)\nu^j$ (see [MP22]). Here, $\delta g(\nu, \cdot)^\top$ is understood as the dual vector field of the tangential component of the 1-form $\delta g(\nu, \cdot)$. So

$$\begin{aligned}
&\int_{\partial M} u^{2-2\sigma}\delta_g(H + \gamma u^{-1}u_\nu)\\
=&\tfrac{1}{2}\int_{\partial M} u^{2-2\sigma}((\mathrm{d}\operatorname{tr}_g(\delta g) - \operatorname{div}_g(\delta g))(\nu) - \operatorname{div}_{\partial M}(\delta g(\nu, \cdot))^\top - \langle\delta g, A_{\partial M}\rangle)\\
&-\gamma\int_{\partial M} u^{1-2\sigma}\delta g(\nu, \nabla^{\partial M}u) - \tfrac{\gamma}{2}\int_{\partial M} u^{1-2\sigma}\delta g(\nu, \nu)u_\nu\\
=&\tfrac{1}{2}\int_{\partial M} u^{2-2\sigma}((\mathrm{d}\operatorname{tr}_g(\delta g) - \operatorname{div}_g(\delta g))(\nu) - \langle\delta g, A_{\partial M}\rangle) + \tfrac{1}{2}\int_{\partial M}\delta g(\nu, \nabla^{\partial M}u^{2-2\sigma})\\
&-\gamma\int_{\partial M} u^{1-2\sigma}\delta g(\nu, \nabla^{\partial M}u) - \tfrac{\gamma}{2}\int_{\partial M} u^{1-2\sigma}\delta g(\nu, \nu)u_\nu.
\end{aligned}$$

In the last equality, we have used integration by parts on $\partial M$. By simply collecting $\int_M u^{2-2\sigma}\delta_g\Lambda$ and $\int_{\partial M}\delta_g(H+\gamma u^{-1}u_\nu)$,

$$\int_{\partial M}\delta_g(H+\gamma u^{-1}u_\nu)u^{2-2\sigma}+\int_M u^{2-2\sigma}\delta_g\Lambda_g=\tfrac{1}{2}\int_M u^{2-2\sigma}\mathcal{R}_g+\tfrac{1}{2}\int_{\partial M}u^{2-2\sigma}\mathcal{A}_1$$

with $\mathcal{A}_1$ given by

$$\begin{aligned}\mathcal{A}_1={}&A_{\partial M}-2\gamma u^{-1}\nabla u\otimes\nu+\gamma u^{-1}u_\nu g+\tfrac{\nabla u^{2-2\sigma}}{u^{2-2\sigma}}\otimes\nu\\&-\frac{\nabla_\nu u^{2-2\sigma}}{u^{2-2\sigma}}g-\frac{\nabla^{\partial M}u^{2-2\sigma}}{u^{2-2\sigma}}\otimes\nu+2\gamma u^{-1}\nu\otimes\nabla^{\partial M}u+\gamma u^{-1}\nu\otimes\nu.\end{aligned}$$

Checking that $\mathcal{A}_1=\mathcal{A}_g$ finishes the proof the proposition. ☐

By taking the difference of two pairs $(g_t^{(1)},u_t^{(1)})$ and $(g_t^{(2)},u_t^{(2)})$ which tend to the same limit, we easily obtain the following.

**Corollary 4.8.** *Let $(M,g)$ be a smooth manifold with non-empty boundary and $u$ be a smooth function on $M$ such that $\mathcal{R}_g\geqslant 0$ and $\mathcal{A}_g\geqslant 0$. Assume that the pair $(g_t^{(1)},u_t^{(1)})$ and $(g_t^{(2)},u_t^{(2)})$ which tends to the same limit $(g,u)$ as $t\to 0$, and $g_t^{(1)}\geqslant g_t^{(2)}$ for all $t$ small, then*

$$\int_{\partial M}(\delta_u^{(2)}(H+\gamma u^{-1}u_\nu)-\delta_u^{(1)}(H+\gamma u^{-1}u_\nu))u^{2-2\sigma}+\int_M u^{2-2\sigma}(\delta_u^{(2)}\Lambda_g-\delta_u^{(1)}\Lambda_g)=0,$$

*and*

$$\int_{\partial M}(\delta_g^{(2)}(H+\gamma u^{-1}u_\nu)-\delta_g^{(1)}(H+\gamma u^{-1}u_\nu))u^{2-2\sigma}u^{2-2\sigma}+\int_M u^{2-2\sigma}(\delta_g^{(2)}\Lambda_g-\delta_g^{(1)}\Lambda_g)\geqslant 0.$$

Now we compute the $\mathcal{R}_{\bar g}$ and $\mathcal{A}_{\bar g}$ for the tangent cone metric $\bar g=\mathrm{d}t^2+A^2t^2g_{\mathbb{S}^{n-1}}$ and the function $\bar u$.

**Lemma 4.9.** *Let $0<\gamma<2(1-\sigma)$, $\bar g=\mathrm{d}t^2+A^2t^2g_{\mathbb{S}^{n-1}}$ and $\bar u(t)=t^{\frac{1}{2(1-\sigma)-\gamma}}$, then $\mathcal{R}_{\bar g}\geqslant 0$ and $\mathcal{A}_{\bar g}\geqslant 0$.*

*Proof.* Let $e_i$ be a vector orthogonal to $\partial_t$ which has length with respect to the metric $\bar g$ and $\bar\nabla$ be the connection with respect to the metric $\bar g$. The Ricci tensor of $\bar g$ is given by $\mathrm{Ric}_{\bar g}(\partial_t,\partial_t)=0$ and $\mathrm{Ric}_{\bar g}(e_i,e_i)=(n-2)(A^{-2}-1)t^{-2}$. Let $\alpha=\frac{1}{2(1-\sigma)-\gamma}$, then $\bar u(t)=t^\alpha$. Then $\bar\nabla^2\bar u(\partial_t,\partial_t)=\alpha(\alpha-1)t^{\alpha-2}$, $\bar\nabla^2u(e_i,e_i)=\alpha t^{\alpha-2}$ and $\Delta_{\bar g}u=\alpha(\alpha+n-2)t^{\alpha-2}$. Using these in the expression of $\mathcal{R}_{\bar g}$ and then $\alpha=\frac{1}{2(1-\sigma)-\gamma}$, we find that

$$\begin{aligned}t^2\mathcal{R}_{\bar g}(\partial_t,\partial_t)&=\frac{2(n-1)(1-\sigma)-(n-2)\gamma}{2(1-\sigma)-\gamma},\\t^2\mathcal{R}_{\bar g}(e_i,e_i)&=(n-2)(A^{-2}-1)+\tfrac{2(n-1)(1-\sigma)}{2(1-\sigma)-\gamma}.\end{aligned}$$

By the assumption of Theorem 1.8, see (1.4), $\mathcal{R}_{\bar g}\geqslant 0$. And it is not difficult to see that $\mathcal{A}_{\bar g}=0$. ☐

**Remark 4.10.** *The authors have used a computer algebra system to assist the computation.*

With the help of this corollary, we give the proof of Theorem 1.8.

*Proof of Theorem 1.8.* Now let $(g_t^{(1)},u_t^{(1)})=(\hat g_t,\hat u_t)$ and $(g_t^{(2)},u_t^{(2)})=(\hat g_{\xi,t},\hat u_{\xi,t})$. We multiply (4.4) by $\bar u^{2-2\sigma}$, and by an application of the divergence theorem,

$$\hat\eta(t)t^{-1}\bar u^{2-2\sigma}|\hat\Sigma_1|=\int_{\hat\Sigma_1}\bar u^{2-2\sigma}\hat\eta_{t,1,0}t^{-1}+O(t),$$

where we have used that $\bar{u}$ is constant on $\hat{\Sigma}_1$. Now $t^{-1}\hat{\eta}_{t,1,0}$ can be interpretated as

$$\delta_g^{(2)}(H+\gamma u^{-1}u_\nu)-\delta_g^{(1)}(H+\gamma u^{-1}u_\nu)+(\delta_u^{(2)}(H+\gamma u^{-1}u_\nu)-\delta_u^{(1)}(H+\gamma u^{-1}u_\nu))$$

along $\hat{\Sigma}_1$. By Corollary 4.8 and the assumptions on comparisons $g \geqslant g_\xi$ and $\Lambda_{g,u} \geqslant \Lambda(g_\xi, u_\xi)$, we have that $\hat{\eta}(t) \leqslant O(t)$, which implies that $\lim_{t\to 0}\hat{\eta}(t) \leqslant 0$.

By the argument of Proposition 3.4, $\hat{\eta}(t) \leqslant 0$ for all small $t > 0$. By rescaling back, we obtain that each leaf of the foliation $\{\Sigma_t\}_{t\in(0,\varepsilon)}$ in Proposition 4.4 satisfies $\eta(t) \leqslant 0$. Each choice of $\Sigma_t$ gives a lower barrier. If there are two conical points, we may apply this construction twice and obtain both the upper barrier and the lower barrier. In either case, Theorem 1.8 are reduced to Theorem 1.4, see Remark 3.5. By choosing $t$ smaller, we obtain the global rigidity. □

### 4.2. **Llarull's theorem.** Now we prove Theorem 1.10.

*Proof of Theorem 1.10.* We briefly skecth an argument of Chai-Wang (cf. [CW25, Section 3.3]) which reduces the theorem to Theorem 1.4 with $\gamma = 0$.

Let $N$ be the north pole of $S^n_I$, and $U_t$ be the neigborhood of $N$ such that the $I$-direction coordinate is less than $t$. We consider $f^{-}(U\backslash\{N\})$. The metric of $g$ in $f^{-1}(U\backslash\{N\})$ takes the form of $\mathrm{d}r^2 + r^2 g_S$ where $S$ is some manifold of dimension $n-1$ and $g_S$ be the inducd metric, and by the metric comparison, $r \geqslant t + o(t)$. (Note that $t = O(r)$.) We can see that the mean curvature of the $r$-level set $\Sigma_r$ is $\frac{n-1}{r} + o(r^{-1})$, by the relation $r \geqslant t + o(r)$, $\Sigma_r$ is an approximate (non-strict) barrier in the sense that $\frac{n-1}{r} \leqslant \frac{n-1}{t} + o(r^{-1})$. In fact, $\Sigma_r$ can be perturbed into a strict barrier if $\lim_{r\to 0^+}\frac{r}{t} \gneqq 1$. Hence, $\lim_{r\to 0}\frac{r}{t} = 1$. In that case, taking the tangent cones $C_1$ and $C_2$ at $N$ and $p$, we see there exists a map from the unit cross-section of $C_1$ to the unit cross-section of $C_2$, which is of non-zero degree and of distance non-increasing. Since the scalar curvature comparison is preserved by taking the tangent cones and the comparison descents to the unit cross-section. Hence the lower dimensional Llarull's Theorem 1.1 shows that the cross-sections are isometric. Hence, we showed that $(M, g)$ and $(S^n_I, g_\xi)$ have isometric tangent cones at $p \in f^{-1}(N)$ and $N$. Using Theorem 1.4 with $u$ being constant finishes the proof. □

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