# RESEARCH STATEMENT

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## 1. Introduction

My primary research interest is geometric analysis with a focus on minimal surface theory, especially questions arising from general relativity. Recently, I developed an interest in min-max construction of minimal surfaces.

The relationship between minimal surface and the geometry of scalar curvature is revealed by the seminal work of Schoen and Yau [SY79] in the 1970s. Their theorem states that for an asymptotically flat manifold with non-negative scalar curvature, a geometric invariant called ADM mass [ADM60] is non-negative. The idea of the proof is that if the mass is negative, then there exists an area-minimizing surface lying between two fixed coordinate planes. These planes serve as barriers for the existence of area-minimizing surfaces in the sense that surfaces outside these planes has strictly larger area. Moreover this surface is strongly stable. However the stability of the surface requires a topological condition contradicting the Gauss-Bonnet theorem with boundary.

In a recent article [ABdL16], a positive mass theorem is proved for an asymptotically flat manifold with a non-compact boundary using minimal surface techniques. The ADM mass in this case involves an integration on the boundary. Their proof follows Schoen and Yau and uses the non-compact boundary and a coordinate plane as barriers after deforming the metric. We restate an asymptotically flat manifold with a noncompact boundary by [ABdL16].

**Definition 1.** *(asymptotically flat, abbreviated as AF) We say that $(M, g)$ is asymptotically flat with decay rate $\tau > 0$ if there exists a compact subset $K \subset M$ and a diffeomorphism $\Psi : M\backslash K \to \mathbb{R}^n_+\backslash \bar{B}^+_1(0)$ such that the following asymptotics holds as $r \to +\infty$:*

$$|g_{ij}(x) - \delta_{ij}| + r|g_{ij,k}| + r^2|g_{ij,kl}| = o(r^{-\tau})$$

*where $\tau > \frac{n-2}{2}$. We identify $\mathbb{R}^n_+$ as $\{x_1 \geqslant 0\} \cap \mathbb{R}^n$.*

Note that the diffeomorphism only maps to half space $\mathbb{R}^n_+$. They define an *ADM mass* [ABdL16, Definition 1.1] (cf. [ADM60]) for $M$ as well given by

$$m_{\text{ADM}} = \lim_{r\to+\infty} \left\{ \int_{S^{n-1}_{r,+}} (g_{ij,j} - g_{jj,i})\mu^i dS^{n-1}_{r,+} + \int_{S^{n-2}_r} g_{\alpha 1}\vartheta^\alpha dS^{n-2}_r \right\} \tag{1}$$

where the comma denotes partial differentiation, $S^{n-1}_{r,+} \subset M$ is a large coordinate hemisphere of radius $r$ with outward unit normal $\mu$, and $\vartheta$ is the outward pointing unit co-normal to $S^{n-2}_r = \partial S^{n-1}_{r,+}$, viewed as the boundary of the bounded region $\Sigma_r \subset \Sigma$. We use the Einstein convention on repeated indices. The index $i, j$ ranges from 1 to $n$ and $\alpha$ ranges from 2 to $n$. See the following figure for an illustration.

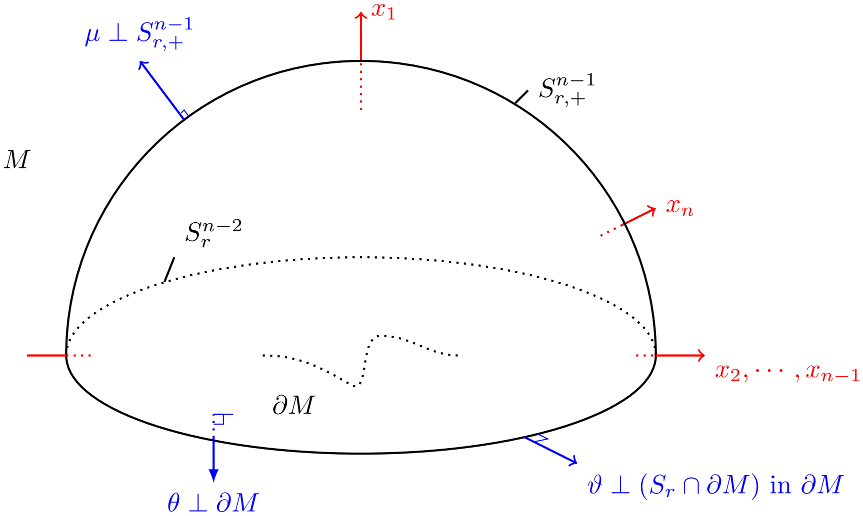


FIGURE 1. A region in $M$ enclosed by a coordinate half-sphere $S^{n-1}_{r,+}$.

**Theorem 1.** *(Positive mass theorem with a non-compact boundary) If an asymptotically flat manifold $M$ with a non-compact boundary has a metric of nonnegative scalar curvature and mean convex boundary, then the ADM mass is nonnegative. The equality happens if and only if $M$ is isometric to $\mathbb{R}^n_+$ where $3 \leqslant n \leqslant 7$.*

Similar statements appeared first in Escobar's paper [Esc92] in terms of expansion of a Green function addressing the Yamabe problem with boundary. Explicit expression of energy is due to [ABdL16].

## 2. Alternate Proofs of Theorem 1

I give two alternate proofs in [Cha18a] using free boundary minimal surface techniques in . Recall the definition of a free boundary minimal hypersurface.

**Definition 2.** *(Free boundary minimal hypersurface) Given a manifold $M$ of dimension $n$ with boundary $\partial M$, a hypersurface $\Sigma$ is called a free boundary minimal hypersurface (often abbreviated as FBMH) if $\Sigma$ meets $\partial M$ orthogonally along $\partial\Sigma$ and has vanishing mean curvature.*

A free boundary minimal hypersurface is just a critical point of the area functional with respect to variations tangent to the boundary of ambient manifold. The first variation of the area functional $\mathcal{H}^n$ at $\Sigma$ along a vector field $X$ is given by

$$\delta\Sigma(X) = \int_\Sigma H\langle X, \nu\rangle + \int_{\partial\Sigma} \langle \eta, X\rangle$$

where $\nu$ is normal to $\Sigma$ in $M$, $\eta$ is normal to $\partial\Sigma$ in $\Sigma$. Choosing a compactly supported function $f$ on $\Sigma$, the *second variation* along the vector field $f\nu$ is given by

$$\delta^2\Sigma(X) = \int_\Sigma |\nabla f|^2 - (\mathrm{Ric}(\nu,\nu) + |B|^2) f^2 - \int_{\partial\Sigma} h(\nu,\nu) f^2. \tag{2}$$

The inequality $\delta^2\Sigma(f\nu) \geqslant 0$ is called the *stability inequality*. This inequality is the key ingredient of the proof.

2.1. **First alternate approach to Theorem 1.** Our first proof is adapted from Schoen and Yau [SY79, Sch89]. Assume that the mass $m < 0$, by [ABdL16, Proposition 4.1], we can assume that the metric is conformally flat at infinity, i.e. $g = h^{\frac{4}{n-2}}\delta$ outside a compact set and $R_g, H_g > 0$. Let $\eta = h^{-\frac{2}{n-2}}\partial_n$, choose $a_0 > 0$ such that for $x^n \geqslant a_0$, $\operatorname{div}_g \eta > 0$ and $\operatorname{div}_g(-\eta) < 0$ for $x^n \leqslant -a_0$. This is because of the negativity of the mass.

Take $\sigma$ large, let

$$\Gamma_{\sigma,a} = \{x = (\bar{x}, x^n) : x^n = a, x^1 \geqslant 0, |\bar{x}| = \sigma\}.$$

We solve a Plateau problem with fixed boundary $\Gamma_{\sigma,a}$ and partially free boundary on $\partial M$. Using geometric measure theory, the solution is smooth except at where the free boundary and fixed boundary meet for $3 \leqslant n \leqslant 7$. We can also find an $a = a(\sigma)$ such that $\Gamma_{\sigma,a(\sigma)}$ has least area among all $\{\Gamma_{\sigma,a}\}_{a\in(-\infty,\infty)}$. This is possible since the translation along $\partial_n$ direction decreases the volume for large $a$. This is the notion of *strong stability*. We take a subsequence $\sigma_i \to \infty$, we obtain a limiting area-minimizing hypersurface $\Sigma \subset M$ with a noncompact boundary. Strong stability of $\Sigma$ allows us to take $f = 1$ in (2) and obtain

$$\int_\Sigma (-\operatorname{Ric}(\nu) - |B|^2) - \int_{\partial\Sigma} A(\nu,\nu) \geqslant 0. \tag{3}$$

Using decomposition of the mean curvature $H_{\partial M}$, Gauss equation, $R_g, H_g > 0$ and minimality of $\Sigma$, the inequality becomes

$$\int_\Sigma \frac{1}{2}R_\Sigma + \int_{\partial\Sigma} H_{\partial\Sigma} > 0. \tag{4}$$

The contradiction will follow from Gauss-Bonnet theorem if $n = 3$. We can also follow [SY79]. The rest of the proof follows from a dimension reduction argument.

2.2. **Second alternate approach to Theorem 1.** We use Lohkamp's approach. Lohkamp's method of concentrating all geometry into a compact region carries over to our setting. We can obtain a manifold $M$ which is Euclidean outside a compact set and in this compact set $R_g, H_g \geqslant 0$, but $M$ is not Euclidean if the mass is not negative. We can then identify the edges $\{x_i = a\}$ and $\{x_i = -a\}$ for $i \in \{2, \ldots, n\}$, cut off the region $x_1 > a$. The result is a connected sum $(\mathbb{T}^{n-1} \times [0,1]) \# M_o$ which has two boundary components. Now the dimension 3 case follows directly from the Gauss-Bonnet theorem with boundary except that we do not have to deal with limits like the first approach. The dimension reduction of this approach uses geometric measure theory [Fed96] and does not have to deal with asymptotics.

## 3. Extension to asymptotically hyperbolic settings

We wish to have an analog of the ADM type mass (1) in the settings of asymptotically hyperbolic manifolds. Since for the *standard* asymptotically hyperbolic settings we have a *mass functional* $H_\Phi$ [CH03] instead of a single quantity, it turns out only possible for this particular $V = \cosh r$.

Let $\mathbb{H}^n$ be the hyperbolic $n$-space with constant sectional curvature -1. Take an arbitrary point $o \in \mathbb{H}^n$ as the origin and let $r(x) = \mathrm{dist}_{\mathbb{H}^n}(o, x)$ be the geodesic distance from a point $x$ to the origin and $V = \cosh r$.

As a model for the standard $n$-space $\mathbb{H}^n$ we may also take $\mathbb{H}^{n-1} \times \mathbb{R}$. Any point $x \in \mathbb{H}^n$ has the coordinate $x = (x\prime, s)$. We assume that $o = (o\prime, 0) \in \mathbb{H}^{n-1} \times \mathbb{R}$ where we take $o\prime$ as the origin in $\mathbb{H}^{n-1}$. Let $U(x\prime) = \cosh(\mathrm{dist}_{\mathbb{H}^{n-1}}(o\prime, x\prime))$, now the metric $b$ takes the form

$$b := \bar{h} + U^2 ds \otimes ds \tag{5}$$

where $\bar{h}$ is the metric for $\mathbb{H}^{n-1}$. Note that $U = V$ when $s = 0$. Now define

$$\mathbb{H}^n_+ = \mathbb{H}^{n-1} \times \{s \in \mathbb{R} : s \geqslant 0\} \tag{6}$$

and $B^n_+ = \{x \in \mathbb{H}^n_+ : \mathrm{dist}(x, o) \leqslant 1\}$. Now let $\bar{\nabla}$ and $\bar{D}$ be respectively the standard connection on $\mathbb{H}^n$ and $\mathbb{H}^{n-1} = \mathbb{H}^{n-1} \times \{0\}$. Denote the Christoffel symbols of $\mathbb{H}^{n-1}$ by $(\Gamma)$.

Motivated by the notion of an asymptotically flat manifold with a noncompact boundary [ABdL16], a Riemannian manifold $(M^n, g)$ is called *asymptotically hyperbolic with a noncompact boundary* of decay order $\tau > \frac{n}{2}$ if there exist a compact set $K$ and a diffeomorphism $\Psi : M \backslash K \to \mathbb{H}^n_+ \backslash B^n_+$ such that $(\Psi^{-1})^* g$ is uniformly equivalent with $b$ and

$$\|e\|_b + \|\bar{\nabla} e\|_b + \|\bar{\nabla}\bar{\nabla} e\|_b = O(e^{-\tau r}) \tag{7}$$

where $e := (\Psi^{-1})^* g - b$. For the *standard asymptotically hyperbolic* manifolds, one could just drop the positive signs in $\mathbb{H}^n_+ \backslash B^n_+$.

**Definition 3.** *(Chai* [Cha18b]*) If $(M^n, g)$ is an asymptotically hyperbolic manifold with decay order $\tau > \frac{n}{2}$, $V(R + n(n-1))$ is integrable and $VH \in L^1(\partial M)$, then*

$$m(g) = \lim_{r \to \infty} \int_{S^{n-1}_{r,+}} (V \bar{\nabla}_l e_{jk} - e_{jk} \bar{\nabla}_l V) P^{ijkl} \nu_i + \int_{S^{n-2}_r} e_{\alpha n} \theta^\alpha \tag{8}$$

*is a finite number and is called mass integral.*

Here, $S^{n-1}_{r,+}$ and $S^{n-2}_r$ are defined similarly to those in (1). We define the *modified Einstein tensor*

$$\tilde{G} := G - \frac{1}{2}(n-1)(n-2) g.$$

**Theorem 2.** *We have*

$$m(g) = -\frac{2}{n-2} \lim_{r \to \infty} \left[ \int_{S^{n-1}_{r,+}} \tilde{G}(X, \nu) + \int_{S^{n-2}_r} (A - Hh)(X, \theta) \right]. \tag{9}$$

## 4. Quasi-local mass with boundary

4.1. **Quasi-local mass with boundary.** The advantage of the above approaches compared to [ABdL16] is that we have found an important role of free boundary minimal surface techniques in dealing with AF manifolds with a noncompact boundary. This leads us to consider *quasi-local masses with boundary.* One of the notable quasi-local masses is derived by Hawking [Haw68] which is evaluated on closed 2-surfaces. I have an analog definition for a 2-surface $\Sigma$ with boundary which meets $\partial M$ orthogonally,

**Definition 4.** *(Hawking mass evaluated on 2-surfaces with boundary)*

$$m_H(\Sigma) = \sqrt{\frac{\mathrm{Area}(\Sigma)}{8\pi}} \left( \chi(\Sigma) - \frac{1}{8\pi} \int_\Sigma H^2 \right) \tag{10}$$

*where* $\chi(\Sigma)$ *is the Euler characteristic of* $\Sigma$*.*

The above Hawking mass usually appears in the settings of asymptotically flat manifolds. In the settings of asymptotically hyperbolic manifolds, one uses the following from [Cha18b],

$$m_H(\Sigma) = \sqrt{\frac{\mathrm{Area}(\Sigma)}{8\pi}} \left( \chi(\Sigma) - \frac{1}{8\pi} \int_\Sigma (H^2 - 4) \right). \tag{11}$$

Another quasi-local mass I successfully defined is the isoperimetric mass proposed firstly by G. Huisken, my definition is the following

**Definition 5.** *(Isoperimetric mass evaluated on surfaces with boundary,*[Cha18c]*)*

$$m_{\mathrm{iso}}(\Omega) = \frac{2}{\mathrm{Area}(\partial\Omega \cap \partial M)} (\mathrm{Vol}(\Omega) - \frac{\sqrt{2}\,\mathrm{Area}^{3/2}(\partial\Omega \cap \partial M)}{6\sqrt{\pi}\,\mathrm{Vol}(\Omega)})$$

*where* $\Omega \subset M$ *is an open set with smooth boundary. One can relax the regularity assumption on* $\Omega$ *and assume that* $\Omega$ *is an open set with finite perimeter.*

*Remark* 1. This definition also appears in Volkman [Vol14] but in a more restricted setting. In fact, we showed in [Cha18c] that it is a special case of ours.

## 5. Other results

I proved in [Cha18d] the following based a monotonicity formula. Actually, these results generalizes easily to Riemannian manifolds with upper sectional curvature bounds.

**Theorem 3.** *Given any closed 2-surface* $\Sigma$ *in* $\mathbb{H}^n$ *and a real number* $\rho > 0$*, let* $(r, x)$ *be the geodesic polar coordinates centered at* $o \in \Sigma$*, then*

$$4\pi + |\Sigma_\rho| \leqslant \frac{1}{w(\rho)} \int_{\Sigma_\rho} V + \frac{1}{4} \int_\Sigma |\mathbf{H}|^2, \tag{12}$$

*where* $\Sigma_\rho = \{x \in \Sigma : \mathrm{dist}_{\mathbb{H}^n}(o, x) < \rho\}$*,* $V = \cosh r$*,* $w(r) = \int_0^r \sinh t dt$ *and* $\mathbf{H}$ *is the mean curvature vector of* $\Sigma$*.*

It directly implies a Willmore type inequality in hyperbolic space when taking limits $\rho \to \infty$. I also a give alternate proof of the equality case. Also, a similar version holds for $\Sigma^2 \subset \mathbb{S}^n$.

**Theorem 4.** *Given any closed 2-surface* $\Sigma$ *in* $\mathbb{S}^n$ *and a real number* $0 < \rho < \pi$*, let* $(r, x)$ *be the geodesic polar coordinates centered at* $o \in \Sigma$*, then*

$$4\pi - |\Sigma_\rho| \leqslant \frac{1}{w(\rho)} \int_{\Sigma_\rho} V + \frac{1}{4} \int_\Sigma |\mathbf{H}|^2, \tag{13}$$

*where* $\Sigma_\rho = \{x \in \Sigma : \mathrm{dist}_{\mathbb{S}^n}(o, x) < \rho\}$*,* $V = \cos r$*,* $w(r) = \int_0^r \sin t dt$ *and* $\mathbf{H}$ *is the mean curvature vector of* $\Sigma$*.*

## 6. Research Prospects

**6.1. Positive mass theorem for asymptotically hyperbolic manifolds with a noncompact boundary.** We propose the following conjecture.

**Conjecture 1.** *Given an asymptotically hyperbolic manifold $(M^3, g)$with a noncompact boundary, if $R_g \geqslant -6$ and $R_g \in L^1(M)$, and $H_g \geqslant 0$ along $\partial M$ and $H_g \in L^1(\partial M)$, then $m(g) \geqslant 0$ is well defined and nonnegative.*

I have only proven a special case of this conjecture in [Cha18b]. It is direct to see that spinors techniques of both [CH03] and [Wan01] do not work any more in the presence of a noncompact boundary. The only existing approach left is by Anderson, Cai and Galloway [ACG08].

**6.2. Generalizing mass with $k$-th mean curvature.** By our computation in both the asymptotically flat and asymptotically hyperbolic manifolds, it is promising to define a higher order mass using the $k$-th mean curvature. We only discuss here the asymptotically flat graphs similar to [Lam10].

Let the shape operator of the graph $M$ be $h^i_j$, then one has

$$\sigma_1(h) = H, \sigma_2(h) = \frac{1}{2} R_M. \tag{14}$$

We have proven that under suitable decay of $u$, $\sigma_1(h)$ define a mass and coincides with (1). $\sigma_2(h) = R/2$ is the scalar curvature and Lam [Lam10] prove that the mass coincides with the ADM mass (i.e. (1) without the boundary term). For $m_{2k}$, we note the works of Ge, Wang and Wu [GWW14] which generalizes Lam's work.

It remains to check other odd values of $k$ for the mass. One might consider a Chern-Simons type boundary term as explained using differential forms by Meyers [MEY63].

**6.3. Higher dimensional positive mass theorem.** Higher dimensional positive mass theorem was open until recently when Schoen and Yau [SY17] develop a slicing theory for minimal surface and Lohkamp's work [Loh06, Loh16] and use it to extend positive mass theorem to arbitrary higher dimensions. I am interested to extend these theories especially the minimal slicing theory of Schoen and Yau to the settings of free boundary as well.

My recent progress on this direction is an analog functional of [SY17, (2.1)] which at least gives us a proof of positive mass theorem for $3 \leqslant n \leqslant 7$. This is closely related to the following problem:

**Problem:** Given a manifold $M = (\mathbb{T}^{n-1} \times [0, 1]) \# M_0$, does there exist a metric of positive scalar curvature which the boundary $\partial M$ has nonnegative mean curvature?

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