A TILTED SPACETIME POSITIVE MASS THEOREM

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Abstract. We show a spacetime positive mass theorem for asymptotically flat initial data sets with a noncompact boundary. We develop a mass type invariant and a boundary dominant energy condition. Our proof is based on spinors.

1. Introduction

The first positive mass theorem was proven by Schoen and Yau in their seminal work [SY79] using a minimal surface technique. It says that if a complete manifold which is asymptotically flat and with nonnegative scalar curvature, an quantity called the ADM mass defined at infinity is nonnegative. The ADM mass is a characterization of scalar curvature at infinity. There are various works on the positive mass theorem: [Wit81], [EHLS16], [ACG08], [Wan01], [CH03], [Sak21]. Here the list is by no means exhaustive.

The study of the positive mass type theorems of the asymptotically flat manifold with a noncompact boundary were started in [ABdL16]. As a result, the effect of the mean curvature was included to the infinity and a boundary term was added to the ADM mass. See [AdL20], [AdLM19], [Cha18], [Cha21] and [AdL22] for some developments to the spacetime and hyperbolic settings.

We revisit the asymptotically flat initial data set with a noncompact boundary in this paper, we introduce the boundary dominant condition (1.2) and we prove two spacetime positive mass theorems (Theorems 1.4, 1.5). We use the spinorial argument of Witten [Wit81] (see also [PT82]) which greatly simplified the proof of the spacetime positive mass theorems when the initial data set is spin. The spin condition is automatically satisfied in the dimension 3 which is of more relevance to physics.

An initial data set (M^n, g, p) is an n-dimensional manifold which arises as a spacelike hypersurface of a Lorentzian manifold $(\mathcal{S}^{n,1}, \tilde{g})$ with p being the second fundamental form. The components T_{00} and T_{0i} of the Einstein tensor (or the energy-momentum tensor) T are respectively called the *energy density* μ and the *current density* J. Let e_0 be the future directed unit normal of M to $\mathcal{S}$, e_i be an orthonormal basis of the tangent space of M and we use the convention on p that $p_{ij} = \tilde{g}(\tilde{\nabla}_{e_i} e_0, e_j)$.

The energy density by the Gauss equation is

$$2\mu = R_g + (\operatorname{tr}_g p)^2 - |p|_g^2$$

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and the current density by the Gauss-Codazzi equation is

$$J = \operatorname{div} p - g\mathrm{d}(\operatorname{tr}_g p).$$

Definition 1.1. *We say that (M, g, p) satisfies the interior dominant energy condition if*

$$\mu \geqslant |J|. \tag{1.1}$$

If $\partial M \neq \emptyset$, let η be the outward normal of ∂M in M, $H_{\partial M} = \operatorname{div}_{\partial M} \eta$. We say that (M, g, p) satisfies the tilted boundary dominant energy condition if

$$H_{\partial M} \pm \cos\theta \operatorname{tr}_{\partial M} p \geqslant \sin\theta |p(\eta, \cdot)^{\top}| \text{ on } \partial M, \tag{1.2}$$

where $\theta \in [0, \frac{\pi}{2}]$ is a constant angle and $p(\eta, \cdot)^{\top}$ denotes the component of the 1-form $p(\eta, \cdot)$ tangential to ∂M.

The tilted boundary dominant energy condition (1.2) generalizes the tangential ($\theta = \pm\frac{\pi}{2}$) and normal boundary dominant energy conditions ($\theta = 0$) in [AdLM19]. Now we recall the definition of an asymptotically flat initial data set with a noncompact boundary and its ADM energy and linear momentum of [AdLM19].

Definition 1.2. *We say that an initial data set (M, g, p) is asymptotically flat with a noncompact boundary if there exists a compact set K such that M is diffeomorphic to the Euclidean half-space $\mathbb{R}^n_+ \backslash B_1$ and*

$$|g - \delta| + |x||\partial g| + |x|^2|\partial^2 g| + |x||p| + |x|^2|p| = o(r^{-\frac{n-2}{2}}), \tag{1.3}$$

where B_1 is a standard Euclidean ball.

Definition 1.3. *The quantities defined as*

$$E = \lim_{r\to\infty} \left[\int_{S^{n-1,r}_+} (g_{ij,j} - g_{jj,i})\nu^i - \int_{S^{n-2,r}} e_{\alpha n}\vartheta^\alpha \right],$$

and

$$P_i = 2\int_{S^{n-1,r}_+} \pi_{ij}\nu^j.$$

are respectively called the ADM energy and ADM linear momentum. Here, ν is unit normal to $S^{n-1,r}_+$, ϑ is normal to $S^{n-2,r}$ in ∂M and $\pi = p - g\operatorname{tr}_g p$. Denote $\hat{P} = (P_1, \ldots, P_{n-1})$, $S^{n-1,r}_+$ is the upper half of the coordinate sphere of radius r and $S^{n-2,r} = \partial S^{n-1,r}_+$.

Note that we include P_n in the ADM linear-momentum. This is a difference compared to [AdLM19]. We have the following two spacetime positive mass theorems.

Theorem 1.4. *If (M, g) is spin and (M, g, k) satisfies the interior dominant energy condition (1.1) and the tilted boundary dominant energy condition (1.2) for some nonzero $\theta \in (0, \frac{\pi}{2}]$, then*

$$E \pm \cos\theta P_n \geqslant \sin\theta |\hat{P}|. \tag{1.4}$$

The special case $\theta = \frac{\pi}{2}$ of the theorem is due to [AdLM19]. As we shall see later, (1.4) is related to an energy-momentum vector (2.1), and the proof of Theorem 1.4 already implies the particular case below.

Theorem 1.5. *If M is spin, $\mu \geqslant |J|$ and $H \pm \mathrm{tr}_{\partial M} p \geqslant 0$, then*

$$E \pm P_n \geqslant 0. \tag{1.5}$$

The time-symmetric case $p = 0$ of the theorem first appeared in [ABdL16] where a minimal surface proof was also given. The rough idea is: assume that the energy (mass) E is negative, then the boundary ∂M and a plane asymptotically parallel to ∂M serve as the barriers and we can find an area-minimizing minimal surface without boundary in between. Then the Gauss-Bonnet theorem applied on the stable minimal plane contradicts the nonnegativity of scalar curvature and mean curvature. An alternative proof was given by the author [Cha18]. Instead, the free boundary minimal surface was used. Observing the two works, one should be able to conclude that the two proofs using the minimal surface are actually proofs of two special cases when p vanishes: (I) $\theta = \pi/2$ in [Cha18]; (II) or $\theta = 0$ in [ABdL16]. This suggests that there is a proof via stable minimal surface with capillary boundary conditions for the case $p \equiv 0$ as well.

The capillarity also naturally arises in Gromov dihedral rigidity conjecture [Gro14]. The Euclidean version of Gromov dihedral rigidity conjecture says that if a Riemannian polyhedron has nonnnegative scalar curvature, mean convex faces and its dihedral angles are less than its Euclidean model, then it must be flat. The original motivation of Gromov dihedral rigidity is a characterization of nonnegative scalar curvature in the weak sense, it is also a localization of the positive mass theorems. Li [Li20] confirmed this conjecture in some special cases and the method he used is precisely minimal surface with a capillary bounday condition. It is reasonable to establish directly Theorem 1.4 implementing [EHLS16] using the capillary marginally outer trapped surface [ALY20].

The article is organized as follows:

In Section 2, we propose the mass related Theorems 1.4 and 1.5 and show the invariance. In Section 3, we collect basics of the chirality operator (3.2) and the hypersurface Dirac operator including the most important Schodinger-Lichnerowicz formula. In Section 4, we give the proofs of Theorems 1.4 and 1.5. We also include some partial results on the rigidity, that is, when equalities are achieved in (1.4) and (1.5). See Propositions 4.3 and 4.6.

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2. The invariance of mass

In this section, we introduce the energy-momentum vector (E^θ, P_i^θ) in (2.1) based on the Hamiltonian analysis performed in [HH96] and point out that the tilted dominant energy condition (1.2) appears in selecting a suitable lapse function and the shift vector.

2.1. **Hamiltonian formulation and mass invariance.** Assume at present that M is compact, we infinitesimally deform the initial data set (M, g, p) in $\mathcal{S}^{n,1}$ in the direction of a future directed timelike vector field T. Let ϕ_s be the local flow of T, $M_s = \phi_s(M)$. We assume that the unit normal e_0 to M_s is always tangential to the timelike hypersurface foliated by ∂M_s. Let $T = Ne_0 + X$, where N is called the *lapse* function and the vector field X tangent to M is called the *shift vector*, then the Hamiltonian along M is given by (see [HH96])

$$\mathcal{H}(N, X) = \int_M [N\mu + 2J(X)] + 2\int_{\partial M} [NH - p(X, \eta) + \operatorname{tr}_g p\langle X, \eta\rangle].$$

The tilted boundary dominant energy condition (1.2) now comes from selecting $N = 1$ and $X = \cos\theta\eta + \sin\theta\tau$ where τ is tangent to ∂M in the boundary term of the Hamiltonian. Indeed,

$$\begin{aligned} & NH - p(X, \eta) + \operatorname{tr}_g p\langle X, \eta\rangle \\ = & H - \cos\theta p(\eta, \eta) - \sin\theta p(\tau, \eta) + \cos\theta \operatorname{tr}_g p \\ = & H + \cos\theta \operatorname{tr}_{\partial M} p - \sin\theta p(\tau, \eta) \\ \geqslant & 0 \end{aligned}$$

if (1.2) holds.

Now let (M, g, p) be the backgroud $(\mathbb{R}^n_+, \delta, 0)$, we take N to be a constant and X be a translational Killing vector field of $\mathbb{R}^n_+$. We consider the Hamiltonian $\mathcal{H}_\varepsilon(N, X)$ on $(M, \delta + \varepsilon g, \varepsilon p)$ with (g, p) satisfying (1.3). We do the Taylor expansion of $\mathcal{H}_\varepsilon$ with respect to ε, due to the fact that M is noncompact, usually the first order terms do not vanish. These terms evaluated at infinity are precisely those given in Definition 1.3. For a more complete account of these facts, we refer the readers to [HH96], [Mic11] and [AdLM19].

We define the *charge density* which is a 1-form,

$$\begin{aligned} & \mathbb{U}_{(g,k)}(N, X) \\ = & N(\operatorname{div}_\delta g - \mathrm{d}(\operatorname{tr}_\delta g)) - (g - \delta)(\nabla^\delta N, \cdot) \\ & \quad + \operatorname{tr}_\delta(g - \delta)\mathrm{d}N + 2(p(X, \cdot) - \operatorname{tr}_\delta p\langle \cdot, X\rangle_\delta). \end{aligned}$$

Let $\mathcal{T}$ be the space of translational Killing vector fields of Minkowski spacetime denoted by $\mathbb{R}^{1,n}$. It is easy to see that $\mathcal{T}$ is identified with $\mathbb{R} \oplus W$ with $\mathbb{R}$ factor representing the translation in a chosen timelike direction ∂_0 and W being the linear space spanned by all translational Killing vector fields of $(\mathbb{R}^n, \delta)$ orthogonal to ∂_0. Each $T \in \mathcal{T}$ can be uniquely written in the form $T = N\partial_0 + X^i\partial_i$ where $N \in \mathbb{R}$ and $X^i \in \mathbb{R}$. We define the energy-momentum functional as follows:

$$\mathcal{M}(T) = \lim_{r\to\infty} \left[\int_{S^{n-1,r}_+} \mathbb{U}_{(g,k)}(N, X) + \int_{S^{n-2,r}} Ng(\bar{\eta}, \bar{\vartheta})\right].$$

It was shown in [AdLM19, Proposition 3.3] that the energy-momentum functional $\mathcal{M}(T)$ does not depend on the asymptotic coordinates (fixing ∂_0) chosen at infinity.

For any $\theta \in (0, \frac{\pi}{2}]$, we define

$$E^\theta = \mathcal{M}(\tfrac{1}{\sin\theta}\partial_0 + \tfrac{\cos\theta}{\sin\theta}\partial_n), P^\theta_i = \mathcal{M}(\partial_i) \text{ for any } i \neq n. \tag{2.1}$$

It is easy to check that $E = \mathcal{M}(\partial_0)$, $P_i = \mathcal{M}(\partial_i)$ where (E, P) is as defined in Definition 1.3, so $E^\theta = \frac{1}{\sin\theta}E + \frac{\cos\theta}{\sin\theta}P_n$. We have the following.

Theorem 2.1. *Given any asymptotically flat initial data set (M, g, k), for any $\theta \in (0, \frac{\pi}{2}]$, the vector $(E^\theta, P^\theta) \in \mathbb{R}^{1,n-1}$ is well defined (up to composition with an element of $SO_{1,n-1}$). In particular,*

$$-(E^\theta)^2 + \sum_{i \neq n} (P_i^\theta)^2$$

and the causal character $(E^\theta, P^\theta) \in \mathbb{R}^{1,n-1}$ do not depend on the chart at infinity to compute (E^θ, P^θ).

Proof. Let $\tilde{\partial}_0 = \frac{1}{\sin\theta}(\partial_0 + \cos\theta \partial_n)$, $\tilde{\partial}_n = \frac{1}{\sin\theta}(\cos\theta\partial_0 + \partial_n)$ and $\tilde{\partial}_i = \partial_i$. There is a Lorentz boost from $(\partial_0, \partial_1, \ldots, \partial_n)$ to $(\tilde{\partial}_0, \tilde{\partial}_1, \ldots, \tilde{\partial}_n)$ such that

$$\begin{pmatrix} \tilde{\partial}_0 \\ \tilde{\partial}_n \end{pmatrix} = \begin{pmatrix} \cosh\rho & \sinh\rho \\ \sinh\rho & \cosh\rho \end{pmatrix} \begin{pmatrix} \partial_0 \\ \partial_n \end{pmatrix},$$

on the plane spanned by $\{\partial_0, \partial_n\}$ with ρ defined by $\cosh\rho = \frac{1}{\sin\theta}$. So $(\tilde{\partial}_0, \tilde{\partial}_1, \ldots, \tilde{\partial}_n)$ gives a new coordinate system for the Minkowski spacetime $\mathbb{R}^{1,n}$. Let $(\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{R}^{1,n}$ where $\tilde{x}$ is expressed in the new coordinates. Obviously,

$$-(\tilde{x}_0)^2 + \sum_{i \neq n} (\tilde{x}_n)^2$$

is invariant under linear Lorentz transformations of $\mathbb{R}^{1,n}$ which fixes $\tilde{\partial}_n$. These transformations as a subgroup of the special Lorentz group $SO_{1,n}$ is isomorphic to $SO_{1,n-1}$. The discussion applies to

$$(\mathcal{M}(\tilde{\partial}_0), \mathcal{M}(\tilde{\partial_1}), \ldots, \mathcal{M}(\tilde{\partial}_{n-1}), \mathcal{M}(\tilde{\partial}_n)),$$

and this is our theorem. □

For the case $\theta = 0$, it is simpler.

Theorem 2.2. *Given any asymptotically flat initial data set (M, g, k), the quantity $E \pm P_n$ is a numerical invariant under isometries of $\mathbb{R}^n_+$ which includes rotations and translations of the $(n-1)$-dimensional hyperplane $\partial\mathbb{R}^n_+$.*

Proof. Note that E and P_n are invariant under rotations and translations of the hyperplane $\{\partial_1, \ldots, \partial_{n-1}\}$, see [AdLM19, Proposition 3.3]. □

3. Hypersurface Dirac operator

In this section, we recall the hypersurface Dirac spinors and the related Schrodinger-Lichnerowicz formula (3.1). We review the chirality operator (3.2) and we relate the boundary condition (3.3) to the geometric quantities along the boundary ∂M in Lemma 3.6.

3.1. **Hypersurface Dirac operator.** The standard reference of spin geometry is [LM89], we also refer to [PT82], [HZ03]. Denote by $\mathbb{S}$ the local spinor bundle of $\mathcal{S}$, since M is spin, $\mathbb{S}$ exists globally over M. This spinor bundle $\mathbb{S}$ is called the hypersurface spinor bundle of M. Let $\tilde{\nabla}$ and ∇ denote respectively the Levi-Civita connections of $\tilde{g}$ and g, we use the same symbols to denote the lifts of the connections to the hypersurface spinor bundle.

There exists a Hermitian inner product $(\cdot,\cdot)$ on $\mathbb{S}$ over M which is compatible with the spin connection $\tilde{\nabla}$. For any 1-form ω of $\mathcal{S}$ and the hypersurface spinors ϕ, ψ, we have

$$(\omega\cdot\phi,\psi)=(\phi,\omega\cdot\psi)$$

where the dot $\cdot$ denotes the Clifford multiplication. This inner product is not positive definite. However, there exists on $\mathbb{S}$ over M a positive definite Hermitian inner product defined by

$$\langle\phi,\psi\rangle=(e^0\cdot\phi,\psi)$$

where e^0 is the future-directed unit timelike normal to M. We see that

$$\langle e^0\cdot\phi,\psi\rangle=\langle\phi,e^0\cdot\psi\rangle,\ \langle e^i\cdot\phi,\psi\rangle=-\langle\phi,e^i\cdot\psi\rangle,$$

where $\{e^i\}$ are the dual frame of an orthonormal basis $\{e_i\}$ over M. Then the spinor connection $\tilde{\nabla}$ over $\mathbb{S}$ is related to ∇ by

$$\tilde{\nabla}_i=\nabla_i-\tfrac{1}{2}p_{ij}e^j\cdot e^0\cdot.$$

This is essentially the spinorial Gauss equation. Moreover, the connection ∇ is compatible with $\langle\cdot,\cdot\rangle$ and $\nabla_i(e^0\cdot\phi)=e^0\cdot\nabla_i\phi$.

The hypersurface Dirac (or Dirac-Witten) operator is then defined by

$$\tilde{D}=e^i\cdot\tilde{\nabla}_i=D+\tfrac{1}{2}\operatorname{tr}_g pe^0\cdot,$$

where D is the standard Dirac operator. We call a spinor ϕ satisfying $\tilde{D}\phi=0$ a *(spacetime) harmonic spinor.*

When our spacetime $\mathcal{S}$ is of dimension $3+1$, the local spinor bundle $\mathbb{S}$ have a simpler algebraic description by the representation theory of the special linear group $\mathrm{SL}(2,\mathbb{C})$. In this case, the theory is easier to understand, see [PT82, Section 2].

We now collect relevant facts regarding $\tilde{\nabla}$ and $\tilde{D}$.

Lemma 3.1 ([PT82])**.** *The adjoint of $\tilde{\nabla}$ is given by*

$$\tilde{\nabla}_i^*\psi=-\nabla_i\psi-\tfrac{1}{2}p_{ij}e^j\cdot e^0\psi,$$
$$\mathrm{d}(\langle\phi,\psi\rangle * e^i)=[\langle\tilde{\nabla}_i\phi,\psi\rangle-\langle\phi,\tilde{\nabla}_i^*\psi\rangle]*1.$$

The Dirac operator $\tilde{D}$ is self-adjoint with

$$\mathrm{d}(\langle e^i\cdot\phi,\psi\rangle * e^i)=(\langle\tilde{D}\phi,\psi\rangle-\langle\phi,\tilde{D}\psi\rangle)*1.$$

The Schrodinger-Lichnerowicz formula is given by

$$\tilde{D}^2-\tilde{\nabla}^*\tilde{\nabla}=\tfrac{1}{2}(\mu-J\cdot e^0\cdot).$$

The integration form of the Schrodinger-Lichnerowicz formula is a direct corollary of Lemma 3.1.

Theorem 3.2 ([PT82])**.** *Let Ω be a compact manifold with boundary, we have for any smooth spinor ϕ that*

$$\begin{aligned}&\int_\Omega|\tilde{D}\phi|^2-\int_\Omega|\tilde{\nabla}\phi|^2+\int_{\partial\Omega}[\langle e^i\cdot\tilde{D}\phi,\phi\rangle+\langle\phi,\tilde{\nabla}_i\phi\rangle]*e^i\\ =&\tfrac{1}{2}\int_\Omega\langle(\mu-J\cdot e^0\cdot)\phi,\phi\rangle.\end{aligned}\tag{3.1}$$

3.2. **Boundary chirality operator.** We fix the convention. We use the Greek letters α, β, γ to indicate the indices which are not n in the rest of the paper. The vector e_n is used to denote the outer normal of ∂M in M and h denotes the the second fundamental form of ∂M given by $h_{\alpha\beta} = \langle \nabla_{e_\alpha} e_n, e_\beta \rangle$, then $H := H_{\partial M} = \sum_\alpha h_{\alpha\alpha}$.

The following chirality operator Q was introduced by [AdL22, Definition 3.3] where $e^0 \cdot e^n \cdot$ is replaced by the Clifford multiplication of the complex volume element.

Definition 3.3. *Let $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, define Q by*

$$Q\phi = \cos\theta e^0 \cdot e^n \cdot \phi + \sqrt{-1} \sin\theta e^n \cdot \phi. \tag{3.2}$$

In (3.2), we require that $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, but in (1.2) we require that $\theta \in [0, \frac{\pi}{2}]$. The reason is that we need a choice on the sign of θ later in the proof of Theorem 1.4. We collect a few commutative and anti-commutative properties of Q below.

Lemma 3.4. *For the operator Q, we have*

(1) $Q^2 = 1$ *and Q is self-adjoint;*
(2) $e^n \cdot Q + Q \cdot e^n = -2\sqrt{-1} \sin\theta$;
(3) $e^\alpha \cdot e^\beta \cdot e^n \cdot Q + Q e^\alpha \cdot e^\beta \cdot e^n \cdot = -2\sqrt{-1} \sin\theta e^\alpha \cdot e^\beta \cdot$;
(4) $e^0 \cdot Q + Q \cdot e^0 = 0$;
(5) $e^\alpha \cdot e^\beta \cdot e^0 \cdot Q + Q e^\alpha \cdot e^\beta \cdot e^0 \cdot = 0$;
(6) $e^\alpha \cdot Q\phi - Q(e_\alpha \cdot \phi) = 2\sqrt{-1} \sin\theta e^\alpha \cdot e^n \cdot \phi$;
(7) $e^\alpha \cdot Q\phi + Q(e_\alpha \cdot \phi) = 2\cos\theta e^\alpha \cdot e^0 \cdot e^n \cdot \phi$
(8) $e^\alpha \cdot e^n \cdot Q + Q \cdot e^\alpha \cdot e^n = 0$;
(9) $e^n \cdot e^0 \cdot Q + Q e^n \cdot e^0 = -2\cos\theta$;
(10) $e^\alpha \cdot e^\beta \cdot e^0 \cdot e^n \cdot Q + Q e^\alpha \cdot e^\beta \cdot e^0 \cdot e^n \cdot = 2\cos\theta e^\alpha \cdot e^\beta \cdot$.
(11) $e^\alpha \cdot e^0 \cdot Q + Q e^\alpha \cdot e^0 = 2\sqrt{-1} \sin\theta e^\alpha \cdot e^0 \cdot e^n$;
(12) *For* $\alpha \neq \beta$, $e^\alpha \cdot e^\beta \cdot Q = Q e^\alpha \cdot e^\beta \cdot$;
(13) $e^\alpha \cdot e^n \cdot e^0 \cdot Q = Q \cdot e^\alpha \cdot e^n \cdot e^0$.

Proof. All the items follows from direct calculation starting from the definition of Q in (3.2). As an example, we only show the last item. By (3.2) and anti-commutative property of the Clifford multiplication,

$$\begin{aligned} e^\alpha \cdot e^n \cdot e^0 \cdot Q\phi &= \cos\theta e^\alpha \cdot e^n \cdot e^0 \cdot e^0 \cdot e^n \cdot \phi + \sqrt{-1} \sin\theta e^\alpha \cdot e^n \cdot e^0 \cdot e^n \cdot \phi \\ &= -\cos\theta e^\alpha \cdot \phi + \sqrt{-1} \sin\theta e^\alpha \cdot e^0 \cdot \phi, \\ Q(e_\alpha \cdot e^n \cdot e^0 \cdot \phi) &= \cos\theta e^0 \cdot e^n \cdot e^\alpha \cdot e^n \cdot e^0 \cdot \phi + \sqrt{-1} \sin\theta e^n \cdot e^\alpha \cdot e^n \cdot e^0 \cdot \phi \\ &= -\cos\theta e^\alpha \cdot \phi + \sqrt{-1} \sin\theta e^\alpha \cdot e^0 \cdot \phi. \end{aligned}$$

So we know that the last item holds true. $\square$

3.3. **Boundary terms in Schrodinger-Lichnerowicz formula.** We calculate

$$[\langle e^i \cdot \tilde{D}\phi, \phi \rangle + \langle \phi, \tilde{\nabla}_i \phi \rangle] * e^i$$

along ∂M when

$$Q\phi = \pm\phi. \tag{3.3}$$

First, we compute a few inner products of spinors satisfying (3.3).

Lemma 3.5. *If a spinor ϕ satisfies* (3.3) *along ∂M, then*

$$\langle\sqrt{-1}e^n\cdot\phi,\phi\rangle=\pm\sin\theta|\phi|^2, \tag{3.4}$$

$$\langle e^n\cdot e^0\cdot\phi,\phi\rangle=\mp\cos\theta|\phi|^2, \tag{3.5}$$

$$\langle e^\alpha\cdot e^0\cdot\phi,\phi\rangle=\mp\langle\sqrt{-1}\sin\theta e^\alpha\cdot e^n\cdot e^0\cdot\phi,\phi\rangle. \tag{3.6}$$

Proof. The first term (3.4) already appeared in [AdL22, Proposition 3.11]. From Lemma 3.4, we have

$$\langle\sqrt{-1}e^n\cdot Q\phi,\phi\rangle+\langle Q\cdot\sqrt{-1}e^n\cdot\phi,\phi\rangle=2\sin\theta|\phi|^2.$$

Since Q is self-adjoint, so

$$\langle\sqrt{-1}e^n\cdot Q\phi,\phi\rangle+\langle\sqrt{-1}e^n\cdot\phi,Q\phi\rangle=2\sin\theta|\phi|^2.$$

Because $Q\phi=\pm\phi$, we have

$$\pm2\langle\sqrt{-1}e^n\cdot\phi,\phi\rangle=2\sin\theta|\phi|^2,$$

which is the first item. The rest follow similarly from corresponding relations from Lemma 3.4. □

The following lemma relates the boundary term in the integration form of Schrodinger-Lichnerowicz formula (3.1) with the mean curvature H, $\operatorname{tr}_{\partial M}p$, p_{nj} along the boundary, and in particular, the tilted boundary dominant energy condition (1.2).

Lemma 3.6. *If a spinor ϕ satisfies* (3.3) *along ∂M, then*

$$\begin{aligned}&\langle\tilde\nabla_{e^n}\phi+e^n\cdot\tilde D\phi,\phi\rangle\\ =&\langle D^{\partial M}\phi,\phi\rangle-\tfrac12H|\phi|^2\mp\tfrac12\cos\theta\operatorname{tr}_{\partial M}p|\phi|^2\\ &\pm\tfrac12\sin\theta\langle\sqrt{-1}p_{n\gamma}e^\gamma\cdot e^n\cdot e^0\cdot\phi,\phi\rangle.\end{aligned}$$

Proof. Let $D^{\partial M}$ be a boundary Dirac operatoror defined by

$$D^{\partial M}=e^n\cdot e^\alpha\cdot\nabla^{\partial M}_\alpha.$$

Here, $\nabla^{\partial M}$ is the spin connection intrinsic to ∂M explicitly defined on spinor fields on M restricted to ∂M as

$$\nabla^{\partial M}_\alpha=\nabla_\alpha-\tfrac12h_{\alpha\beta}e^n\cdot e^\beta\cdot.$$

We calculate $D^{\partial M}\phi$ with ϕ satisfying (3.3) and we see

$$\begin{aligned}&D^{\partial M}\phi\\ =&e^n\cdot e^\alpha\cdot(\nabla_\alpha\phi-\tfrac12h_{\alpha\beta}e^n\cdot e^\beta\cdot\phi)\\ =&e^n\cdot(D\phi-e^n\cdot\nabla_{e^n}\phi)+\tfrac12H\phi\\ =&e^n\cdot D\phi+\nabla_{e^n}\phi+\tfrac12H\phi\\ =&e^n\cdot(\tilde D\phi-\tfrac12\operatorname{tr}_g pe^0\cdot\phi)+(\tilde\nabla_{e^n}\phi+\tfrac12p_{nj}e^j\cdot e^0\cdot\phi)+\tfrac12H\phi\end{aligned}$$

So we have that

$$\begin{aligned}&\langle\tilde\nabla_{e^n}\phi+e^n\cdot\tilde D\phi,\phi\rangle\\ =&\langle D^{\partial M}\phi,\phi\rangle+\langle\tfrac12\operatorname{tr}_g pe^n\cdot e^0\cdot\phi-\tfrac12p_{nj}e^j\cdot e^0\cdot\phi,\phi\rangle-\tfrac12H|\phi|^2.\end{aligned}$$

From (3.5), we have

$$
\begin{aligned}
&\left\langle \tfrac{1}{2}\operatorname{tr}_g p e^n \cdot e^0 \cdot \phi - \tfrac{1}{2} p_{nj} e^j \cdot e^0 \cdot \phi, \phi \right\rangle \\
=&\left\langle \tfrac{1}{2}(\operatorname{tr}_{\partial M} p e^n + p_{nn} e^n) \cdot e^0 \cdot \phi - \tfrac{1}{2}(p_{n\alpha} e^\alpha + p_{nn} e^n) \cdot e^0 \cdot \phi, \phi \right\rangle \\
=&\tfrac{1}{2}\operatorname{tr}_{\partial M} p \langle e^n \cdot e^0 \cdot \phi, \phi\rangle - \tfrac{1}{2}\langle p_{n\alpha} e^\alpha \cdot e^0 \cdot \phi, \phi\rangle \\
=&\mp \tfrac{1}{2}\cos\theta \operatorname{tr}_{\partial M} p |\phi|^2 - \tfrac{1}{2}\langle p_{n\alpha} e^\alpha \cdot e^0 \cdot \phi, \phi\rangle \\
=&\mp \tfrac{1}{2}\cos\theta \operatorname{tr}_{\partial M} p |\phi|^2 \mp \tfrac{1}{2}\sin\theta \langle \sqrt{-1} p_{n\alpha} e^\alpha \cdot e^0 \cdot e^n \cdot \phi, \phi\rangle
\end{aligned}
$$

which follows from (3.5) and (3.6). □

4. THE POSITIVE MASS THEOREM

With the help of results from the previous sections, in this section, we finish the proofs of Theorems 1.4 and 1.5. We give some consequences of vanishing mass, that is,

$$E \pm \cos\theta P_n = \sin\theta |\hat{P}|. \tag{4.1}$$

See Propositions 4.3 and 4.6.

4.1. **Existence of a spacetime harmonic spinor.**
When the initial data set (M, g, p) is flat and totally geodesic, i.e. (M, g, p) is $(\mathbb{R}^n_+, \delta, 0)$, we define

$$\bar{Q}\phi = \cos\theta \mathrm{d}x^0 \cdot \mathrm{d}x^n \cdot \phi + \sqrt{-1}\sin\theta \mathrm{d}x^n \cdot \phi.$$

Note that $\bar{Q}^2 = I$, the eigenvalues of $\bar{Q}$ are ± 1. The standard hypersurface spinor bundle $\bar{\mathbb{S}}$ over $(\mathbb{R}^n_+, \delta, 0)$ splits into two eigen subbundles and the spinor ϕ satisfying

$$\bar{Q}\phi = \pm\phi \tag{4.2}$$

is closely related to our problem.

We recall the following existence of a spacetime harmonic spinor ϕ which is asymptotic to a constant spinor ϕ_0 satisfying (4.2), and we extract the mass from the boundary integral in (3.1). By [AdLM19, Proposition 5.3] and the discussions that followed, we have the following.

Theorem 4.1. *Assume that (M, g, k) satisfies the dominant energy conditions* (1.1) *and* (1.2)*, given any nonzero constant spinor ϕ_0 satisfying* (4.2) *, there exists a spinor ϕ which is asymptotic to ϕ_0 and satisfies*

$$
\begin{aligned}
&\tilde{D}\phi = 0 \text{ in } M, \\
&Q\phi = \pm\phi \text{ on } \partial M.
\end{aligned}
$$

4.2. **Proof of positive mass theorems.**
Using the ϕ of Theorem 4.1 in (3.1), we can give a proof of Theorem 1.2 and the proof works equally well for Theorem 1.5.

Proof of Theorem 1.4. Let M_r be the compact region bounded by ∂M and $S_r^{n-1,+}$. By the integral form of Schrodinger-Lichnerowicz formula (3.1), we have for any spinor ϕ, we have

$$
\begin{aligned}
&\int_{M_r} |\tilde{D}\phi|^2 - \int_{M_r} |\tilde{\nabla}\phi|^2 + \int_{\partial M_r} [\langle e^i \cdot \tilde{D}\phi, \phi\rangle + \langle \phi, \tilde{\nabla}_i \phi\rangle] * e^i \\
=&\tfrac{1}{2}\int_{M_r} \langle (\mu - J \cdot e^0 \cdot)\phi, \phi\rangle.
\end{aligned}
$$

Note that ∂M_r are made of two portions: one lies in the interior of M and the other lies on ∂M. We require that $Q\phi = \pm\phi$ along ∂M, so by Lemma 3.6,

$$\begin{aligned}
&\int_{M_r} |\tilde{D}\phi|^2 - \int_{M_r} |\tilde{\nabla}\phi|^2 + \int_{\partial M_r \cap \operatorname{int} M} [\langle e^i \cdot \tilde{D}\phi, \phi\rangle + \langle \phi, \tilde{\nabla}_i \phi\rangle] * e^i \\
&+ \int_{\partial M_r \cap \partial M} \langle D^{\partial M}\phi, \phi\rangle - \tfrac{1}{2} H |\phi|^2 \mp \tfrac{1}{2}\cos\theta \operatorname{tr}_{\partial M} p |\phi|^2 \\
&\pm \int_{\partial M_r \cap \partial M} \tfrac{1}{2}\sin\theta \langle \sqrt{-1} p_{n\gamma} e^\gamma \cdot e^n \cdot e^0 \cdot \phi, \phi\rangle \\
=&\tfrac{1}{2}\int_{M_r} \langle (\mu - J \cdot e^0 \cdot)\phi, \phi\rangle.
\end{aligned}$$

It follows that $\langle D^{\partial M}\phi, \phi\rangle = 0$ from [CH03, (4.27)] (with ε there replaced by Q). We have

$$\begin{aligned}
&\int_{\partial M_r \cap \operatorname{int} M} \\
\to& \tfrac{1}{4}(E \pm \cos\theta P_n)|\phi|^2 \pm \sin\theta P_\gamma \langle \sqrt{-1} \mathrm{d}x^\gamma \cdot \mathrm{d}x^n \cdot \mathrm{d}x^0 \cdot \phi_0, \phi_0\rangle_\delta
\end{aligned}$$

as $r \to \infty$. We calculate

$$\begin{aligned}
&[\langle e^i \cdot \tilde{D}\phi, \phi\rangle + \langle \phi, \tilde{\nabla}_i \phi\rangle] * e^i \\
=&\langle \nu^\flat \cdot \tilde{D}\phi, \phi\rangle + \langle \phi, \tilde{\nabla}_\nu \phi\rangle \\
=&\langle \nu^\flat \cdot D\phi, \phi\rangle + \langle \phi, \nabla_\nu \phi\rangle \\
&+ \tfrac{1}{2}(\operatorname{tr}_g p \nu^i - p_{ij}\nu^j)\langle e^i \cdot e^0 \cdot \phi, \phi\rangle \\
=&\langle \nu^\flat \cdot D\phi, \phi\rangle + \langle \phi, \nabla_\nu \phi\rangle - \tfrac{1}{2}\pi_{ij}\nu^j \langle e^i \cdot e^0 \cdot \phi, \phi\rangle,
\end{aligned}$$

where ν is the unit normal of $\partial M_r \cap \operatorname{int} M$ pointing to the infinity. Because that ϕ converges to a constant spinor ϕ_0, we know that as $r \to \infty$

$$\int_{\partial M_r \cap \operatorname{int} M} \langle \nu^\flat \cdot D\phi, \phi\rangle + \langle \phi, \nabla_\nu \phi\rangle \to \tfrac{1}{4} E |\phi_0|^2_\delta$$

from [ABdL16, Section 5.2] and

$$-\tfrac{1}{2}\int_{\partial M_r \cap \operatorname{int} M} \pi_{ij}\nu^j \langle e^i \cdot e^0 \cdot \phi, \phi\rangle \to -\tfrac{1}{4} P_i \langle \mathrm{d}x^i \cdot \mathrm{d}x^0 \cdot \phi_0, \phi_0\rangle_\delta$$

since $\langle e^i \cdot e^0 \cdot \phi, \phi\rangle$ converges to the constant $\langle \mathrm{d}x^i \cdot \mathrm{d}x^0 \cdot \phi_0, \phi_0\rangle_\delta$. Here δ is the standard Euclidean metric. From (3.5) that

$$P_n \langle \mathrm{d}x^n \cdot \mathrm{d}x^0 \cdot \phi_0, \phi_0\rangle_\delta = \mp \cos\theta P_n |\phi_0|^2_\delta,$$

and from (3.6) that

$$P_\gamma \langle \mathrm{d}x^\gamma \cdot \mathrm{d}x^0 \cdot \phi_0, \phi_0\rangle_\delta = \mp \sin\theta P_\gamma \langle \sqrt{-1} \mathrm{d}x^\gamma \cdot \mathrm{d}x^n \cdot \mathrm{d}x^0 \cdot \phi_0, \phi_0\rangle_\delta. \tag{4.3}$$

By the lemma below, we can make a choice of the constant spinor ϕ_0 and the sign of θ such that

$$P_\gamma \langle \mathrm{d}x^\gamma \cdot \mathrm{d}x^0 \cdot \phi_0, \phi_0\rangle_\delta = \sin|\theta| |\hat{P}| |\phi_0|_\delta.$$

So taking limits as $r \to \infty$, we have

$$
\begin{aligned}
&\tfrac{1}{4}(E \pm \cos\theta P_n - \sin|\theta||\hat{P}|)|\phi_0|_\delta^2 \\
=&\tfrac{1}{2}\int_M \langle(\mu - J\cdot e^0\cdot)\phi, \phi\rangle \\
&+\tfrac{1}{2}\int_{\partial M}(H \pm \cos\theta \operatorname{tr}_{\partial M} p)|\phi|^2 \pm \sin\theta\langle\sqrt{-1}p_{n\gamma}e^\gamma\cdot e^n\cdot e^0\cdot\phi, \phi\rangle.
\end{aligned}
\tag{4.4}
$$

By the dominant energy conditions (1.1) and (1.2), we obtain that $E \pm \cos\theta P_n \geqslant \sin|\theta||\hat{P}|$. □

Lemma 4.2. *There exists a choice of ϕ_0 and the sign of θ such that*

$$\mp\sin\theta P_\gamma\langle\sqrt{-1}\mathrm{d}x^\gamma\cdot\mathrm{d}x^n\cdot\mathrm{d}x^0\cdot\phi_0, \phi_0\rangle_\delta = \sin|\theta||\hat{P}||\phi_0|_\delta.$$

Proof. We work with the flat metric in this proof. Let

$$A = P_\alpha\sqrt{-1}\mathrm{d}x^\alpha\cdot\mathrm{d}x^n\cdot\mathrm{d}x^0,$$

we know from the last item of Lemma 3.4 that $\bar{Q}$ commutes with A. So they have the same eigen spinors. Then for some $\phi_0 \neq 0$ such that $\bar{Q}\phi_0 = \pm\phi_0$, so $A\phi_0 = \lambda\phi_0$ for some $\lambda \in \mathbb{C}$. It is easy to check that A is Hermitian and so $\lambda \in \mathbb{R}$. We calculate λ by the following:

$$
\begin{aligned}
\lambda^2|\phi_0|^2 &= P_\alpha P_\beta\langle\sqrt{-1}\mathrm{d}x^\alpha\cdot\mathrm{d}x^n\cdot\mathrm{d}x^0\cdot\phi_0, \sqrt{-1}\mathrm{d}x^\beta\cdot\mathrm{d}x^n\cdot\mathrm{d}x^0\cdot\phi_0\rangle_\delta \\
&= P_\alpha P_\beta\langle\mathrm{d}x^\alpha\cdot\mathrm{d}x^n\cdot\mathrm{d}x^0\cdot\phi_0, \mathrm{d}x^\beta\cdot\mathrm{d}x^n\cdot\mathrm{d}x^0\cdot\phi_0\rangle_\delta \\
&= P_\alpha P_\beta\langle\mathrm{d}x^n\cdot\mathrm{d}x^\alpha\cdot\mathrm{d}x^0\cdot\phi_0, \mathrm{d}x^n\cdot\mathrm{d}x^\beta\cdot\mathrm{d}x^0\cdot\phi_0\rangle_\delta \\
&= P_\alpha P_\beta\langle\mathrm{d}x^\alpha\cdot\mathrm{d}x^0\cdot\phi_0, \mathrm{d}x^\beta\cdot\mathrm{d}x^0\cdot\phi_0\rangle_\delta \\
&= P_\alpha P_\beta\langle\mathrm{d}x^0\cdot\mathrm{d}x^\alpha\cdot\phi_0, \mathrm{d}x^0\cdot\mathrm{d}x^\beta\cdot\phi_0\rangle_\delta \\
&= P_\alpha P_\beta\langle\mathrm{d}x^\alpha\cdot\phi_0, \mathrm{d}x^\beta\cdot\phi_0\rangle_\delta \\
&= -P_\alpha P_\beta\langle\mathrm{d}x^\alpha\cdot\mathrm{d}x^\beta\cdot\phi_0, \phi_0\rangle_\delta \\
&= -P_\alpha P_\beta\langle\tfrac{1}{2}(\mathrm{d}x^\beta\cdot\mathrm{d}x^\alpha\cdot + \mathrm{d}x^\alpha\cdot\mathrm{d}x^\beta\cdot)\phi_0, \phi_0\rangle_\delta \\
&= |\hat{P}|^2|\phi_0|_\delta^2.
\end{aligned}
$$

Hence $\lambda = \pm|\hat{P}|$. Using this choice of ϕ_0, from (4.3), we get

$$
\begin{aligned}
&\langle P_\gamma\mathrm{d}x^\gamma\cdot\mathrm{d}x^n\cdot\mathrm{d}x^0\cdot\phi_0, \phi_0\rangle_\delta \\
=&\mp\sin\theta\lambda|\phi_0|_\delta^2 \\
=&(\mp1)(\pm1)|\hat{P}|\sin\theta|\phi_0|_\delta^2.
\end{aligned}
$$

If $|\hat{P}|$ does not vanish, we can always make $\mp\sin\theta\lambda$ positive regardless of the sign in $\bar{Q}\phi = \pm\phi$ by making a free choice of the sign of θ, fixing such θ, we have

$$\langle P_\gamma\mathrm{d}x^\gamma\cdot\mathrm{d}x^n\cdot\mathrm{d}x^0\cdot\phi_0, \phi_0\rangle_\delta = \sin|\theta||\hat{P}||\phi_0|_\delta^2.$$

And the proof is done. □

4.3. **Some consequences of vanishing mass.** We assume that $\theta \neq 0$ in this subsection, however, the results and the proof work through for the case $\theta = 0$ with only minor changes.

By (1.1), (1.2), (3.3) and (4.4), there exists a nonzero spinor ϕ which satisfies

$$\tilde{\nabla}\phi = 0, \tag{4.5}$$

$$\mathrm{Re}\langle(\mu - J\cdot e^0\cdot)\phi, \phi\rangle = 0 \text{ in } M, \tag{4.6}$$

$$Q\phi = \pm\phi,$$

$$\mathrm{Re}\langle\tilde{H}\phi, \phi\rangle = 0 \text{ on } \partial M, \tag{4.7}$$

where $\tilde{H}$ is a shorthand given by

$$\tilde{H} = H \pm \cos\theta \,\mathrm{tr}_{\partial M}\, p \mp \sin\theta\sqrt{-1}p_{\alpha n}e^\alpha \cdot e^n \cdot e^0\cdot,$$

We define

$$N = \langle\phi, \phi\rangle, X = \sum_j (e^j \cdot \phi, \phi)e_j = X^j e_j.$$

As the naming of N and X suggests, they are related to the lapse function and shift vector in Section 2. In the interior of M, we have the following consequence of (4.5).

Proposition 4.3. *If* (4.5) *holds, then*

$$L_X g + 2Np = 0,\ \mathrm{d}(N^2 - |X|^2) = 0 \tag{4.8}$$

in M*. If* (1.1) *and* (4.6) *hold, then*

$$\mu N + \langle J, X\rangle = 0,\ \mu X^k + NJ^k = 0$$

in M*.*

Proof. We show by direct calculation. We choose a geodesic normal frame e_i at an interior point of M, combining with (4.5), we have

$$\begin{aligned} e_i(N) &= \tilde{\nabla}_i(e^0\cdot\phi, \phi) = (\tilde{\nabla}_i e^0 \cdot \phi, \phi) \\ &= -p_{ij}(e^j\cdot\phi, \phi) = -p_{ij}X^j, \end{aligned}$$

and

$$\begin{aligned} \nabla_i X &= e_i[(e^j\cdot\phi, \phi)e_j] = (\tilde{\nabla}_i e^j \cdot \phi, \phi)e_j \\ &= -p_{ij}(e^0\cdot\phi, \phi)e_j = -p_{ij}Ne_j. \end{aligned}$$

This proves (4.8). From (4.6), we obtain

$$\begin{aligned} 0 &= \langle(\mu - J\cdot e^0\cdot)\phi, \phi\rangle \\ &= \mu\langle\phi, \phi\rangle - J_i\langle e^i\cdot e^0\cdot\phi, \phi\rangle \\ &= \mu\langle\phi, \phi\rangle + J_i(e^i\cdot\phi, \phi) \\ &= \mu N + J_i X^i. \end{aligned}$$

For any C^2 spinor ψ and $s \in \mathbb{R}$, by (1.1), we have

$$\mathrm{Re}\langle(\mu - J\cdot e^0\cdot)(\phi + s\psi), \phi + s\psi\rangle \geqslant 0.$$

Subtracting (4.6) from the above, we have

$$\begin{aligned} &s\,\mathrm{Re}\langle(\mu - J\cdot e^0\cdot)\phi,\psi\rangle + s\,\mathrm{Re}\langle(\mu - J\cdot e^0\cdot\psi),\phi\rangle \\ &+s^2\,\mathrm{Re}\langle(\mu - J\cdot e^0\cdot)\psi,\psi\rangle \geqslant 0, \end{aligned}$$

which forces

$$\mathrm{Re}\langle(\mu - J\cdot e^0\cdot)\phi,\psi\rangle + \mathrm{Re}\langle(\mu - J\cdot e^0\cdot\psi),\phi\rangle = 0.$$

Because $\mu - J\cdot e^0\cdot$ is Hermitian with respect to $\langle\cdot,\cdot\rangle$, so

$$\mathrm{Re}\langle(\mu - J\cdot e^0\cdot)\phi,\psi\rangle = 0. \tag{4.9}$$

Setting ψ to be $e^0\cdot e^k\cdot\phi$ gives

$$\begin{aligned} 0 &= \mathrm{Re}\langle(\mu - J\cdot e^0\cdot)\phi, e^0\cdot e^k\cdot\phi\rangle \\ &= \mu\langle\phi, e^0\cdot e^k\cdot\phi\rangle - J_i\,\mathrm{Re}\langle e^i\cdot e^0\cdot\phi, e^0\cdot e^k\cdot\phi\rangle \\ &= \mu\langle e^0\cdot e^k\cdot\phi,\phi\rangle + J_i\,\mathrm{Re}\langle e^i\cdot\phi, e^k\cdot\phi\rangle \\ &= \mu(e^k\cdot\phi,\phi) + J_i\delta^{ik}|\phi|^2 \\ &= \mu X^k + NJ^k. \end{aligned}$$

And the lemma is proved. □

Now we study the consequences of (4.1) at the boundary ∂M. First, we show some basic calculations along ∂M.

Lemma 4.4. *We have*

$$\begin{aligned} \tilde\nabla_\alpha(Q\phi) = &\cos\theta[-p_{\alpha\beta}e^\beta\cdot e^n\cdot\phi + h_{\alpha\beta}e^0\cdot e^\beta\cdot\phi] \\ &+\sqrt{-1}\sin\theta[-p_{\alpha n}e^0\cdot\phi + h_{\alpha\beta}e^\beta\cdot\phi]. \end{aligned}$$

Proof. First,

$$\tilde\nabla_i e^0 = -p_{ij}e^j, \tilde\nabla_\alpha e^n = -p_{\alpha n}e^0 + h_{\alpha\beta}e^\beta.$$

By product rule and that $\tilde\nabla\phi = 0$, we have

$$\begin{aligned} &\tilde\nabla_\alpha(\cos\theta e^0\cdot e^n\cdot\phi + \sqrt{-1}\sin\theta e^n\cdot\phi) \\ =&\cos\theta[\tilde\nabla_\alpha e^0\cdot e^n\cdot\phi + e^0\cdot\tilde\nabla_\alpha e^n\cdot\phi] + \sqrt{-1}\sin\theta\tilde\nabla_\alpha e^n\cdot\phi \\ =&\cos\theta[-p_{\alpha i}e^i\cdot e^n\cdot\phi - e^0\cdot p_{\alpha n}e^0\cdot\phi + e^0\cdot h_{\alpha\beta}e^\beta\cdot\phi] \\ &+\sqrt{-1}\sin\theta[-p_{\alpha n}e^0\cdot\phi + h_{\alpha\beta}e^\beta\cdot\phi] \\ =&\cos\theta[-p_{\alpha\beta}e^\beta\cdot e^n\cdot\phi + e^0\cdot h_{\alpha\beta}e^\beta\cdot\phi] + \sqrt{-1}\sin\theta[-p_{\alpha n}e^0\cdot\phi + h_{\alpha\beta}e^\beta\cdot\phi]. \end{aligned}$$

□

Lemma 4.5. *We have*

$$\langle e^\alpha\cdot\phi,\phi\rangle = X^\alpha\sqrt{-1}\cot\theta.$$

Proof. From the relation of Q with $e^\alpha\cdot e^0\cdot$ (see Lemma 3.4), we see that

$$X^\alpha = (e^\alpha\cdot\phi,\phi) = \langle e^0\cdot e^\alpha\cdot\phi,\phi\rangle = \pm\sqrt{-1}\sin\theta\langle e^0\cdot e^\alpha\cdot e^n\cdot\phi,\phi\rangle.$$

Similarly,

$$\begin{aligned}&\langle e^\alpha\cdot\phi,\phi\rangle\\ =&\pm\cos\theta\langle e^\alpha\cdot e^0\cdot e^n\cdot\phi,\phi\rangle\\ =&-\cos\theta(\pm\langle e^0\cdot e^\alpha\cdot e^n\cdot\phi,\phi\rangle)\\ =&-\cos\theta\tfrac{X^\alpha}{\sqrt{-1}\sin\theta}=X^\alpha\sqrt{-1}\cot\theta.\end{aligned}$$

□

Proposition 4.6. *We have*

$$p_{\alpha n}N\sin^2\theta=\pm p_{\alpha\beta}X^\beta\cos\theta+h_{\alpha\beta}X^\beta=(H\pm\cos\theta\operatorname{tr}_{\partial M}p)X_\alpha \tag{4.10}$$

and

$$p_{\alpha n}X^\gamma\mp\cos\theta p_{\alpha\gamma}N-h_{\alpha\gamma}N=0 \tag{4.11}$$

along ∂M.

Proof. Taking the product $\langle\tilde\nabla_\alpha(Q\phi),e^0\cdot\phi\rangle$, we get

$$\begin{aligned}&\langle\tilde\nabla_\alpha(Q\phi),e^0\cdot\phi\rangle\\ =&-\cos\theta p_{\alpha\beta}\langle e^\beta\cdot e^n\cdot\phi,e^0\cdot\phi\rangle\\ &+\cos\theta h_{\alpha\beta}\langle e^0\cdot e^\beta\cdot\phi,e^0\cdot\phi\rangle\\ &-\sqrt{-1}\sin\theta p_{\alpha n}\langle e^0\cdot\phi,e^0\cdot\phi\rangle\\ &+\sqrt{-1}\sin\theta h_{\alpha\beta}\langle e^\beta\cdot\phi,e^0\cdot\phi\rangle\\ =&\mp\cos\theta p_{\alpha\beta}\tfrac{X^\beta}{\sqrt{-1}\sin\theta}\\ &+\cos\theta h_{\alpha\beta}\sqrt{-1}X^\beta\cot\theta\\ &-\sqrt{-1}p_{\alpha n}N\sin\theta\\ &+\sqrt{-1}\sin\theta h_{\alpha\beta}X^\beta.\end{aligned}$$

Since $\tilde\nabla\phi=0$ and $Q\phi=\pm\phi$ along ∂M, so $\tilde\nabla_\alpha(Q\phi)=0$ along ∂M. So we get

$$p_{\alpha n}N=\pm p_{\alpha\beta}X^\beta\tfrac{\cos\theta}{\sin^2\theta}+h_{\alpha\beta}X^\beta\tfrac{\cos^2\theta}{\sin^2\theta}+h_{\alpha\beta}X^\beta$$

which leads to the first identity of (4.10). The product $\langle\tilde\nabla_\alpha(Q\phi),e^\gamma\cdot\phi\rangle$ leads to

$$\begin{aligned}&\langle\tilde\nabla_\alpha(Q\phi),e^\gamma\cdot\phi\rangle\\ =&-\cos\theta p_{\alpha\beta}\langle e^\beta\cdot e^n\cdot\phi,e^\gamma\cdot\phi\rangle+\cos\theta h_{\alpha\beta}\langle e^0\cdot e^\beta\cdot\phi,e^\gamma\cdot\phi\rangle\\ &-\sqrt{-1}\sin\theta p_{\alpha n}\langle e^0\cdot\phi,e^\gamma\cdot\phi\rangle+\sqrt{-1}\sin\theta h_{\alpha\beta}\langle e^\beta\cdot\phi,e^\gamma\cdot\phi\rangle\\ =&\mp\sin\theta\cos\theta\sqrt{-1}p_{\alpha\beta}\langle e^\gamma\cdot e^\beta\cdot\phi,\phi\rangle\\ &-\sqrt{-1}\sin\theta p_{\alpha n}X^\gamma\\ &-\sqrt{-1}\sin\theta h_{\alpha\beta}\langle e^\gamma\cdot e^\beta\cdot\phi,\phi\rangle.\end{aligned}$$

Considering $\tilde\nabla_\alpha(Q\phi)=0$, we have

$$\sin\theta p_{\alpha n}X^\gamma\pm\cos\theta\sin\theta p_{\alpha\beta}\langle e^\gamma\cdot e^\beta\cdot\phi,\phi\rangle+h_{\alpha\beta}\sin\theta\langle e^\gamma\cdot e^\beta\cdot\phi,\phi\rangle=0.$$

Taking the real part and dividing by $\sin\theta$,

$$p_{\alpha n}X^\gamma\mp\cos\theta p_{\alpha\gamma}N-h_{\alpha\gamma}N=0.$$

This is (4.11).

Similar to the derivation of (4.9), we obtain from (1.2) and (4.7) that for any C^2 spinor ψ that $\mathrm{Re}\langle \tilde{H}\phi, \psi\rangle = 0$. Taking ψ to be $e^\gamma \cdot \phi$, we obtain

$$\begin{aligned}
&\langle \tilde{H}\phi, e^\gamma \cdot \phi\rangle \\
=&\langle H\phi \pm \cos\theta \,\mathrm{tr}_{\partial M}\, p\phi \pm \sin\theta p_{n\alpha}\sqrt{-1}e^\alpha \cdot e^0 \cdot e^n \cdot \phi, e^\gamma \cdot \phi\rangle \\
=&-(H \pm \cos\theta \,\mathrm{tr}_{\partial M}\, p)\langle e^\gamma \cdot \phi, \phi\rangle \mp \sqrt{-1}\sin\theta p_{\alpha n}\langle e^\gamma \cdot e^\alpha \cdot e^0 \cdot e^n \cdot \phi, \phi\rangle \\
=&-(H \pm \cos\theta \,\mathrm{tr}_{\partial M}\, p)X^\gamma \sqrt{-1}\cot\theta \mp \sqrt{-1}\sin\theta p_{\alpha n}(\pm 1)\cos\theta \langle e^\gamma \cdot e^\alpha \cdot \phi, \phi\rangle \\
=&-(H \pm \cos\theta \,\mathrm{tr}_{\partial M}\, p)X^\gamma \sqrt{-1}\cot\theta - \sqrt{-1}\sin\theta p_{\alpha n}\cos\theta \langle e^\gamma \cdot e^\alpha \cdot \phi, \phi\rangle.
\end{aligned}$$

So taking the imaginary part of the above, we arrive

$$(H \pm \cos\theta \,\mathrm{tr}_{\partial M}\, p)X^\gamma + \sin^2\theta p_{\alpha n}\langle e^\gamma \cdot e^\alpha \cdot \phi, \phi\rangle = 0.$$

Taking the real part of the above leads to the second identity of (4.11). □

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