SCALAR CURVATURE RIGIDITY OF DOMAINS IN A WARPED PRODUCT

XIAOXIANG CHAI AND GAOMING WANG

Abstract. A warped product with a spherical factor and a logarithmically concave warping function satisfies a scalar curvature rigidity of Llarull type. We develop a scalar curvature rigidity of Llarull type for a general class of domains in a three dimensional spherical warped product. In the presense of rotationally symmetry, we identify the condition analogous to the logarithmic concavity of the warping function on the boundary.

1. Introduction

Llarull proved a scalar curvature rigidity theorem for the standard n-spheres.

Theorem 1.1 ([Lla98]). *Let g be a smooth metric on the n-sphere with the metric comparison $g \geqslant \bar{g}$ and the scalar curvature comparison $R_g \geqslant n(n-1)$. Then $g = \bar{g}$.*

A distinct feature of this scalar curvature rigidity for spheres comparing to that of torus [SY79a], the Euclidean space [SY79b] and the hyperbolic space [MO89] is the requirement of a metric comparison $g \geqslant \bar{g}$. A counterexample without the requirement $g \geqslant \bar{g}$ was given in [BM11] (for half n-sphere fixing the geometry of the boundary), but $g \geqslant \bar{g}$ can be weakened, see [Lis10]. Llarull also showed that the condition $g \geqslant \bar{g}$ can be reformulated more generally as the existence of a distance non-increasing map $F : (M, g) \to (\mathbb{S}^n, \bar{g})$ of non-zero degree.

Recently, there were efforts in extending Llarull's theorem to a spherical warped product

$$(\bar{M}^n, \bar{g}) := ([t_-, t_+] \times S^{n-1}, \mathrm{d}t^2 + \psi(t)^2 g_{\mathbb{S}^{n-1}}) \text{ with } t_- < t_+,\ (\log \psi)'' < 0, \tag{1.1}$$

by spinors [CZ24], [BBHW24], [WX23b], by μ-bubbles [Gro21], [HLS23] and by spacetime harmonic functions [HKKZ].

We are interested in Llarull type theorems of domains in the spherical warped product (1.2). Although the form (1.2) can also be considered as a domain in a larger spherical warped product, our focus will be on such domains with boundaries that are not necessarily given by t-level sets. Previously, this direction has been explored by Lott [Lot21], Wang-Xie [WX23a] and Chai-Wan [CW24], which all involved spinors. Gromov first suggested the use of stable capillary minimal surface in studying the scalar curvature rigidity of Euclidean balls (see Section 5.8.1 of [Gro21]; *Spin-Extremality of Doubly Punctured Balls*) and Li [Li20] in three-dimensional Euclidean dihedral rigidity. In this article, we make use of the stable capillary surfaces with prescribed (varying) contact angle and prescribed mean curvature, or in the terminology of Gromov [Gro21], (part of) the boundary of a stable capillary μ-bubble. This is also a further development of the previous work [CW23] and the recent work of Ko-Yao [KY24] in the Euclidean case.

We consider three dimensions, fix a metric g_{S^2} of positive Gauss curvature on the 2-sphere S^2, and the following

$$(\bar{M}^3, \bar{g}) := ([t_-, t_+] \times S^2, \mathrm{d}t^2 + \psi(t)^2 g_{S^2}) \text{ with } t_- < t_+,\ (\log \psi)'' < 0. \tag{1.2}$$

We reserve $g_{\mathbb{S}^2}$ for the standard round metric on the 2-sphere S^2. We use t_x to indicate the t coordinate and p_x to denote the S^2 coordinate of a point $x \in M \subset [t_-, t_+] \times S^2$. Let $K(p)$ be the Gauss curvature at $p \in (S^2, g_{S^2})$.

Let $P_\pm = \{t_\pm\} \times S^2$, we always assume that M lies between $P_\pm$ such that $P_\pm \cap \partial M$ is non-empty. Let $\partial_s M = \partial M \backslash (P_+ \cup P_-)$ and $\bar{X}$ be the unit outward normal of $\partial_s M$ in M with respect to $\bar{g}$. We fix $\bar{h}(t) = 2\psi'(t)/\psi(t)$ and $\bar{\gamma}$ to be the dihedral angles formed by $\partial_s M$ and $\Sigma_t = (\{t\} \times S^2) \cap M$ that is given by

$$\cos \bar{\gamma} = \bar{g}(\bar{X}, \partial_t). \tag{1.3}$$

We need to fix some more conventions for the direction of the unit normal, the sign of the mean curvatures and the dihedral angles. Let Σ be a surface with boundary on $\partial_s M$ and separates $P_+ \cap \partial M$ and $P_- \cap \partial M$, we always fix the direction of the unit normal N of Σ to be the direction which points inside of the region bounded by Σ, $P_+ \cap \partial M$ and $\partial_s M$. The mean curvature is then the trace of the second fundamental form ∇N. We fix γ_Σ to be the contact angle formed by Σ and $\partial_s M$, that is, $\cos \gamma_\Sigma = \langle X, N \rangle$. For the mean curvature of $\partial_s M$, it is always computed with respect to the outward unit normal. The geometric quantity on $(M, \bar{g})$ comes with a bar unless otherwise specified (see Figure 2.1).

As is well known, the warped product metric (1.2) is conformal to a direct product metric. Indeed, let $s = \int^t \frac{1}{\psi(\tau)} \mathrm{d}\tau$, then $\mathrm{d}s = \frac{1}{\psi(t)} \mathrm{d}t$ and

$$\mathrm{d}t^2 + \psi(t)^2 g_{S^2} = \psi(t)^2 \mathrm{d}s^2 + \psi(t)^2 g_{S^2} = \psi(t)^2 (\mathrm{d}s^2 + g_{S^2}) \tag{1.4}$$

where $t = t(s)$ is implicitly given by $s = \int^t \frac{1}{\psi(\tau)} \mathrm{d}\tau$.

Now we state our first scalar curvature rigidity result.

Theorem 1.2. *Let M be a smooth domain in $[t_-, t_+] \times S^2$ with the metric $\mathrm{d}t^2 + \psi(t)^2 g_{S^2}$ where $(\log \psi)'' < 0$, $\psi(t) > 0$ on $[t_-, t_+]$. Assume that M is convex with respect to the conformally related metric $\mathrm{d}s^2 + g_{S^2}$ where s is given in* (1.4) *and g is another metric on M which satisfies the comparisons of:*

(1) the scalar curvature $R_g \geqslant R_{\bar{g}}$,
(2) the mean curvatures $H_g \geqslant H_{\bar{g}}$ of the boundary ∂M in M,
(3) and the metrics $g \geqslant \bar{g}$,

then $g = \bar{g}$.

We use a special case of $(M, \bar{g})$ to illustrate the convexity of $\partial_s M$ with respect to the metric $\mathrm{d}s^2 + g_{S^2}$ given in (1.4). Assume that g_{S^2} is just the standard round metric, that is,

$$g_{S^2} = g_{\mathbb{S}^2} = \mathrm{d}r^2 + \sin^2 r \mathrm{d}\theta^2,\ r \in [0, \pi],\ \theta \in \mathbb{S}^1, \tag{1.5}$$

using the polar coordiates and M is given by

$$M = \{(t, r, \theta):\ t \in [t_-, t_+],\ r \leqslant \rho(t) < \tfrac{\pi}{2},\ \theta \in \mathbb{S}^1\}$$

for some positive function $\rho(t)$ on $[t_-, t_+]$. In this case, the prescribed contact angle $\bar{\gamma}$ only depends on t. It is easy to check that the convexity of $\partial_s M$ with respect to $\mathrm{d}s^2 + g_{\mathbb{S}^2}$ is equivalent to $\frac{\mathrm{d}\bar{\gamma}}{\mathrm{d}s} < 0$. Also, we find that

$$\frac{\mathrm{d}\bar{\gamma}}{\mathrm{d}t} < 0 \tag{1.6}$$

using (1.4) or that the angles are conformally invariant. The mean curvature of a t-level set is given by

$$\bar{h}(t) := 2\psi^{-1}\tfrac{\mathrm{d}\psi}{\mathrm{d}t} = 2\tfrac{\mathrm{d}(\log\psi)}{\mathrm{d}t}.$$

The logarithmic concavity of ψ is equivalent to the more geometric statement that the mean curvature of the t-level set is monotonically decreasing as t increases. The condition $\frac{\mathrm{d}\bar{\gamma}}{\mathrm{d}t} < 0$ can be viewed as a boundary analog of the logarithmic concavity. Geometrically, (1.6) says that the dihedral angles (1.3) formed by Σ_t and $\partial_s M$ monotonically decreases along the ∂_t direction with respect to the metric $\bar{g}$. This condition answers a question raised by Gromov at the end of [Gro21, Section 5.8.1]. See also Chai-Wan [CW24, Theorem 1.1].

Theorem 1.4 does not yet generalize Theorem 1.1 genuinely, since in the case of round metric, $\psi(t) = \sin t$, $t \in [0, \pi]$ is allowed to take zero values at $t = 0$ and $t = \pi$. We have the following.

Theorem 1.3. *Assume that $\bar{g}$ is a metric in* (1.2) *with* $(\log\psi)''(t) < 0$, $\psi(t_+) > 0$ *and*

$$\psi(t) = a(t - t_-) + o(|t - t_-|),$$

where $a > 0$ is a constant such that the Ricci curvature of the tangent cone at $t = t_-$ is non-negative. Let M be a region in $\bar{M}$ such that $\partial_s M$ is convex with respect to the conformally related metric (1.4)*, and g be another metric on M satisfying the following comparisons of:*

1. *metrics $g \geqslant \bar{g}$ in M; scalar curvatures $R_g \geqslant R_{\bar{g}}$ in M;*
2. *the mean curvatures $H_{\partial_s M} \geqslant \bar{H}_{\partial_s M}$ of $\partial_s M$, mean curvatures $H_{P_+\cap\partial M} \geqslant \bar{h}|_{P_+\cap\partial M} = \bar{h}(t_+)$ of $P_+ \cap \partial M$;*
3. *the dihedral angles $\gamma_{P_+\cap\partial M} \geqslant \bar{\gamma}|_{P_+\cap\partial M}$ forming by $P_+ \cap \partial M$ and $\partial_s M$ along $\partial(P_+ \cap \partial M)$.*

Then $g = \bar{g}$.

The mean curvature comparisons can be reformulated as $H_{\partial M} \geqslant \bar{H}_{\partial M}$ on ∂M if all mean curvatures are computed with respect to the outward unit normal. Our approach toward Theorems 1.2 and 1.3 (including Theorem 1.6) is by construction of surfaces of prescribed mean curvature and prescribed contact angles $\bar{\gamma}$ near $t = t_-$ which serves as barriers (see Definition 2.7). This is a purely local construction near $t = t_-$. In this way, our proof of Theorems 1.2, 1.3, 1.6 can be reduced to the following case with barriers.

Theorem 1.4. *Let $\bar{g}$ be the metric in* (1.2) *with* $\psi(t_\pm) > 0$, $(\log\psi)''(t) < 0$, *and $\partial_s M$ is convex with respect to the metric $\mathrm{d}s^2 + g_{S^2}$ given in* (1.4)*. Let g be another smooth metric on M which satisfies the following comparisons of:*

a) *the metrics $g \geqslant \bar{g}$ in M; the scalar curvatures $R_g \geqslant R_{\bar{g}}$ in M;*
b) *the mean curvatures $H_{\partial_s M} \geqslant \bar{H}_{\partial_s M}$ of $\partial_s M$, mean curvatures $H_{P_+\cap\partial M} \geqslant \bar{h}|_{P_+\cap\partial M} = \bar{h}(t_+)$ of $P_+ \cap \partial M$, mean curvatures $H_{P_-\cap\partial M} \leqslant \bar{h}|_{P_-\cap\partial M} = \bar{h}(t_-)$ of $P_- \cap \partial M$;*
c) *The dihedral angles $\gamma_{P_+\cap\partial M} \geqslant \bar{\gamma}|_{P_+\cap\partial M}$ of $\partial_s M$ and $P_+ \cap \partial M$ and the dihedral angles $\gamma_{P_-\cap\partial M} \leqslant \bar{\gamma}|_{P_-\cap\partial M}$ of $\partial_s M$ and $P_- \cap \partial M$.*

Then $g = \bar{g}$.

Hu-Liu-Shi [HLS23] (see also Gromov [Gro21]) used a μ-bubble approach for Theorem 1.1. As indicated earlier, we use the capillary version of the μ-bubble approach. However, our method differs from theirs in a technical manner when handling $\psi(t) = t + o(|t|)$ near $t = 0$. They constructed a family of perturbations on the function $2\psi'/\psi$ while our strategy is to perform a careful tangent cone analysis near $t = t_\pm$. As a result of this new strategy, we are able to generalize the Llarull Theorem 1.1 to the case where the background metric $\bar{g}$ are equipped with antipodal conical points.

Theorem 1.5. *Let $n = 3$ and $(\bar{M}, \bar{g})$ be a three dimensional warped product given in* (1.1) *such that*

$$\psi(t_\pm) = a_\pm |t - t_\pm| + o(|t - t_\pm|),\ 0 < a_\pm \leqslant 1,$$

If g is another smooth metric on $\bar{M}$ with possible cone singularity at only $t = t_\pm$ which satisfies the comparisons of metrics $g \geqslant \bar{g}$ and scalar curvatures $R_g \geqslant R_{\bar{g}}$, then $g = \bar{g}$.

Theorem 1.5 directly follows from the proof of Theorem 1.3 with only slight changes and we omit its proof. See Remark 3.13. The condition $0 < a_\pm \leqslant 1$ ensures that the Ricci curvature of the tangent cone with respect to $\bar{g}$ at $t = t_\pm$ is non-negative. The scalar curvature rigidity of Llarull type for $a_\pm > 1$ is an interesting question. One could also compare Theorem 1.5 with [CLZ24] where conical singularities with respect to the metric g are allowed at multiple points on S^n.

An interesting case when $\psi(t_-) \neq 0$ (we assume that $\rho(t_+) \neq 0$ and $\psi(t_+) \neq 0$) and the set $P_- \cap \partial M$ only contains a single point remains. We have already have Theorem 1.2 when ∂M is smooth at $P_+ \cap \partial M$, but, more conditions are needed in our Llarull type rigidity Theorem 1.6 when ∂M is conical at $P_+ \cap \partial M$.

Theorem 1.6. *Let g_{S^2} be the standard round metric written in polar coordinates* (1.5) *and M be the domain in $\bar{M}$ given by*

$$M = \cup_{t \in (t_-, t_+)} \{(t, r, \theta) : r \in [0, \rho(t)), \theta \in \mathbb{S}^1\},$$

where $0 < \rho(t) < \frac{\pi}{2}$ on $(t_-, t_+]$ and near $t = t_-$, ρ satisfies the asymptotics

$$\rho(t) = a_1 |t - t_-| + o(|t - t_-|),\ a_1 > 0.$$

Assume that $\bar{g}$ is a metric in (1.2) *with $\psi(t) > 0$, $(\log \psi)''(t) < 0$, $\bar{\gamma}'(t) < 0$ on $[t_-, t_+]$. If g is another metric on M satisfying the same comparisons as in Theorem 1.3, then $g = \bar{g}$.*

Remark 1.7. Here, the condition $\rho(t) < \frac{\pi}{2}$ ensures that each t-level set Σ_t has a convex boundary $\partial\Sigma_t$ in the metric $\bar{g}$. This fact is used in Lemma 2.1 if one were to go through the proof in detail.

Although Theorem 1.6 is stated only for a domain with rotational symmetry, it can actually be generalized slightly assuming a condition on the tangent cone at $t = t_-$ (see Remark 4.3).

Some of the essential difficulties of Theorems 1.2 and 1.6 were already present in [CW23, Theorem 1.2 *(2)* and *(3)*]. In light of this, we only give a proof sketch for Theorems 1.2 and 1.6 in Section 4 , and refer relevant details to [CW23].

It is possible that the inequalities in $(\log \psi)'' < 0$ and the convexity of $\partial_s M$ can be weakened in some cases. For instance, we can consider the direct product metric $\mathrm{d}t^2 + g_D$ where $t \in [0, 1]$ and (D, g_D) is a convex disk in the 2-sphere (S^2, g_{S^2}). In this case, $\log \psi$ vanishes. The Llarull type rigidity Theorem 1.4 is still valid for this metric.

Now one could naturally ask what are other shapes of point singularities, in particular, asymptotics of ψ, such that Theorem 1.3 and 1.6 remain valid. However, it is a quite intricate matter to which we do not have an answer at the moment. It is also desirable to find a proof for higher dimensional analogs of our results using the stable capillary μ-bubbles. This seems to be a promising direction to investigate being aware of the recent works [CWXZ24, WWZ24].

The article is organized as follows:

In Section 2, we introduce basics of stable capillary μ-bubble and we use it to show Theorem 1.4.

In Section 3, we use the tangent cone analysis at $t = t_-$ to construct barriers and reduce Theorem 1.3 to Theorem 1.4.

In Section 4, we revisit our constructions in [CW23] and use the techniques developed there to show Theorem 1.6.

Acknowledgments X. Chai was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. RS-2024-00337418) and an NRF grant No. 2022R1C1C1013511. G. Wang was supported by the China Postdoctoral Science Foundation (No. 2024M751604).

2. Stable capillary μ-bubble

In this section, we introduce the functional (2.1) whose minimiser is a stable capillary μ-bubble. We introduce the *barrier* condition which combining with a maximum principle ensures the existence of a regular minimiser to (2.1). By a rigidity analysis on the second variation of (2.1), we conclude the proof of Theorem 1.4.

2.1. **Notations.** We set up some notations. Let $E \subset M$ be be a set such that $\partial E \cap M$ is a regular surface with boundary which we name it Σ. We set

- N, unit normal vector of Σ pointing inside E;
- ν, unit normal vector of $\partial\Sigma$ in Σ pointing outside of Σ;
- η, unit normal vector of $\partial\Sigma$ in ∂M pointing outside of $\partial E \cap \partial M$;
- X: unit normal vectors of ∂M in M pointing outside of M;
- γ: the contact angle formed by Σ and ∂M and the magnitude of the angle is given by $\cos\gamma = \langle X, N\rangle$,
- $\langle Y, Z\rangle = g(Y, Z)$, the inner product of vectors Y and Z with respect to the metric g;
- $\langle Y, Z\rangle_{\bar{g}} = \bar{g}(Y, Z)$, the inner product of vectors Y and Z with respect to the metric $\bar{g}$.

See Figure 2.1. We use a bar on every quantity to denote that the quantity is computed with respect to the metric $\bar{g}$ given in (1.2).

2.2. **Functional and first variation.** We fix $\bar{h} = 2\psi'/\psi$ and $\bar{\gamma}$ to be given by (1.3). We define the functional

$$I(E) = \mathcal{H}^2(\partial^* E \cap \operatorname{int} M) - \int_E \bar{h} - \int_{\partial^* E\cap\partial M} \cos\bar{\gamma}, \tag{2.1}$$

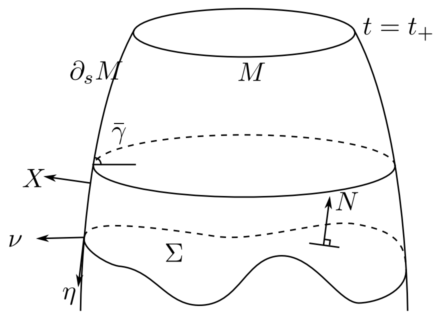

FIGURE 2.1. Notations.

where $\partial^* E$ denotes the reduced boundary of E and the variational problem

$$\mathcal{I} = \inf\{I(E) : \ E \in \mathcal{E}\}, \tag{2.2}$$

where $\mathcal{E}$ is the collection of contractible open subsets E' such that $P_+ \subset E'$. Let Σ be a surface with boundary $\partial\Sigma$ such that $\partial\Sigma$ separates $P_\pm$. Then Σ separates M into two components and the component closer to P_+ is just E. We reformulate the functional (2.1) in terms of Σ. We define

$$F(\Sigma) = I(E) = |\Sigma| - \int_E \bar{h} - \int_{\partial E \cap \partial M} \cos\bar{\gamma}.$$

Let ϕ_t be a family of immersions $\phi_t : \Sigma \to M$ such that $\phi_t(\partial\Sigma) \subset \partial M$ and $\phi_0(\Sigma) = \Sigma$. Let $\Sigma_t = \phi_t(\Sigma)$ and E_t be the corresponding component separated by Σ_t. Let Y be the vector field $\frac{\partial \phi_t}{\partial t}$. Define $\mathcal{A}(t) = F(\Sigma_t)$ and $f = \langle Y, N\rangle$, then by the first variation

$$\mathcal{A}'(0) = \int_\Sigma f(H - \bar{h}) + \int_{\partial\Sigma} \langle Y, \nu - \eta\cos\bar{\gamma}\rangle. \tag{2.3}$$

We know that if Σ is regular, then it is of mean curvature $\bar{h}$ and meets ∂M at a prescribed angle $\bar{\gamma}$. And E is called a *capillary μ-bubble*. The second variation at such Σ is

$$\mathcal{A}''(0) = Q(f,f) := -\int_\Sigma (f\Delta f + (|A|^2 + \mathrm{Ric}(N) + \partial_N \bar{h}) f^2) + \int_{\partial\Sigma} f(\tfrac{\partial f}{\partial \nu} - qf). \tag{2.4}$$

where $f \in C^\infty(\Sigma)$ and

$$q := \tfrac{1}{\sin\bar{\gamma}} A_{\partial M}(\eta,\eta) - \cot\bar{\gamma} A(\nu,\nu) + \tfrac{1}{\sin^2\bar{\gamma}} \partial_\eta \cos\bar{\gamma}. \tag{2.5}$$

We define two operators

$$L = -\Delta - (|A|^2 + \mathrm{Ric}(N) + \partial_N \bar{h}) \text{ in } \Sigma,$$

and

$$B = \tfrac{\partial}{\partial\nu} - q \text{ on } \partial\Sigma.$$

The surface Σ is called *stable* if

$$Q(f,f) \geqslant 0 \tag{2.6}$$

for all $f \in C^\infty(M)$. The second variation (2.4) is closely related to the variation of $H - \bar{h}$ and $\cos\gamma - \cos\bar{\gamma}$. Indeed, let $f = \langle Y, N\rangle$, we have that the first variation of

$H - \bar{h}$ is

$$\begin{aligned} \nabla_Y (H - \bar{h}) &= Lf + \nabla_{Y^\top}(H - \bar{h}) \\ &= -\Delta f - (|A|^2 + \mathrm{Ric}(N) + \partial_N \bar{h}) f + \nabla_{Y^\top}(H - \bar{h}). \end{aligned} \tag{2.7}$$

And the first variation of the angle difference $\langle X, N\rangle - \cos\bar{\gamma}$ is

$$\begin{aligned} &\nabla_Y (\cos\gamma - \cos\bar{\gamma}) = -\sin\bar{\gamma}\tfrac{\partial f}{\partial \nu} \\ &+ (A_{\partial M}(\eta,\eta) - \cos\bar{\gamma} A(\nu,\nu) + \tfrac{1}{\sin\bar{\gamma}} \partial_\eta \cos\bar{\gamma}) f + \nabla_{Y^\top}(\langle X, N\rangle - \cos\bar{\gamma}). \end{aligned} \tag{2.8}$$

For Σ, Schoen-Yau [SY79b] rewrote the term $|A|^2 + \mathrm{Ric}(N)$ as

$$|A|^2 + \mathrm{Ric}(N) = \tfrac{1}{2}(R_g - 2K + |A|^2 + H^2) \tag{2.9}$$

where K is the Gauss curvature of Σ. Along the boundary $\partial\Sigma$, we have the rewrite (see [RS97, Lemma 3.1] or [Li20, (4.13)])

$$\tfrac{1}{\sin\bar{\gamma}} A_{\partial M}(\eta,\eta) - \cos\bar{\gamma} A(\nu,\nu) = -H\cot\bar{\gamma} + \tfrac{H_{\partial M}}{\sin\bar{\gamma}} - \kappa \tag{2.10}$$

where κ is the geodesic curvature of $\partial\Sigma$ in Σ.

2.3. **Analysis of stability.** Starting from now on, we assume that Σ is a regular stable capillary μ-bubble in (M, g) which satisfies the assumptions of Theorem 1.4.

Lemma 2.1. *Let Σ be a regular stable capillary μ-bubble, then Σ is a t-level set.*

Proof. First, we note that the second variation $\mathcal{A}''(0) \geqslant 0$ as in (2.4). First, using Schoen-Yau's rewrite (2.9) we see that

$$\begin{aligned} & |A|^2 + \mathrm{Ric}(N) + \partial_N \bar{h} \\ =& \tfrac{1}{2}(R - 2K + |A|^2 + H^2) + \partial_N \bar{h} \\ =& \tfrac{1}{2}(R - 2K + |A^0|^2 + \tfrac{H^2}{2} + H^2) + \partial_N \bar{h} \\ =& \tfrac{1}{2}(R + \tfrac{3}{2}\bar{h}^2 + 2\partial_N \bar{h}) - K + \tfrac{1}{2}|A^0|^2, \end{aligned} \tag{2.11}$$

where A^0 is the traceless part of the second fundamental form. Similarly using (2.10), we see

$$q = -H\cot\bar{\gamma} + \tfrac{H_{\partial M}}{\sin\bar{\gamma}} - \kappa + \tfrac{1}{\sin^2\bar{\gamma}} \partial_\eta \cos\bar{\gamma}.$$

We obtain by letting $f \equiv 1$ in the (2.6) (also using (2.4) and (2.5)),

$$\begin{aligned} 2\pi\chi(\Sigma) &= \int_\Sigma K + \int_{\partial\Sigma} \kappa \\ &\geqslant \int_\Sigma \left[\tfrac{1}{2}(R + \tfrac{3}{2}\bar{h}^2 + 2\partial_N \bar{h}) + \tfrac{1}{2}|A^0|^2\right] + \int_{\partial\Sigma} \left(\tfrac{H_{\partial M}}{\sin\bar{\gamma}} - \bar{h}\cot\bar{\gamma} + \tfrac{1}{\sin^2\bar{\gamma}} \partial_\eta \cos\bar{\gamma}\right) \\ &\geqslant \int_\Sigma \tfrac{1}{2}\left(R + \tfrac{3}{2}\bar{h}^2 + 2\partial_N \bar{h}\right) + \int_{\partial\Sigma} \left(\tfrac{H_{\partial M}}{\sin\bar{\gamma}} - \bar{h}\cot\bar{\gamma} + \tfrac{1}{\sin^2\bar{\gamma}} \partial_\eta \cos\bar{\gamma}\right) \\ &\geqslant \int_\Sigma \tfrac{1}{2}\left(R_{\bar{g}} + \tfrac{3}{2}\bar{h}^2 + 2\partial_N \bar{h}\right) + \int_{\partial\Sigma} \left(\tfrac{\bar{H}_{\partial M}}{\sin\bar{\gamma}} - \bar{h}\cot\bar{\gamma} + \tfrac{1}{\sin^2\bar{\gamma}} \partial_\eta \cos\bar{\gamma}\right), \end{aligned} \tag{2.12}$$

where in the last line we have incorporated the comparisons $R_g \geqslant R_{\bar{g}}$ in M and $H_{\partial M} \geqslant \bar{H}_{\partial M}$ on ∂M.

Now we estimate $R_{\bar{g}} + \frac{3}{2}\bar{h}^2 + 2\partial_N \bar{h}$. We have that

$$\partial_N \bar{h} = \bar{g}(N, \nabla^{\bar{g}} \bar{h}) \geqslant -|N|_{\bar{g}} |\nabla^{\bar{g}} \bar{h}|_{\bar{g}} = |N|_{\bar{g}} \bar{h}',$$

since $g \geqslant \bar{g}$, so

$$1 = |N|_g \geqslant |N|_{\bar{g}},$$

and we get

$$\partial_N \bar{h} \geqslant \bar{h}'.$$

So

$$R_{\bar{g}} + \tfrac{3}{2}\bar{h}^2 + 2\partial_N \bar{h} \geqslant R_{\bar{g}} + \tfrac{3}{2}\bar{h}^2 + 2\bar{h}'.$$

For any point $x \in \Sigma$, the right hide side is just $\frac{2K(p_x)}{\psi^2(t_x)}$. Recall that $x = (t_x, p_x)$ is the coordinate of $x \in \bar{\Sigma}_t$. This is by a direct calculation of the scalar curvature of the warped product metric (1.2). So

$$R_{\bar{g}} + \tfrac{3}{2}\bar{h}^2 + 2\partial_N \bar{h} \geqslant \tfrac{2K(p_x)}{\psi^2(t_x)}. \tag{2.13}$$

Let $\hat{g} = \mathrm{d}s^2 + g_{S^2}$ and it is conformally related to $\bar{g}$ via (1.4). Let $\hat{X}$ be the unit outward normal of $\partial_s M$ in M and $\hat{H}_{\partial_s M}$ be the mean curvature of $\partial_s M$ in M with respect to $\hat{g}$. Since $\bar{g}$ is conformal to $\hat{g}$, by a well known formula of conformal change of mean curvature,

$$\bar{H}_{\partial_s M} = \tfrac{1}{\varphi(s)}(\hat{H}_{\partial_s M} + 2\partial_{\hat{X}} \log \varphi). \tag{2.14}$$

Similarly, the mean curvature $\bar{h}$ of Σ_t in M is

$$\bar{h}(t) = \tfrac{2}{\varphi(s)^2}\varphi'(s). \tag{2.15}$$

Hence, by (2.14), (2.15), (1.4) and that $\hat{g}(\partial_s, \hat{X}) = \cos\bar{\gamma}$,

$$\tfrac{\bar{H}_{\partial M}}{\sin\bar{\gamma}} - \bar{h}\cot\bar{\gamma} + \tfrac{1}{\sin^2\bar{\gamma}}\partial_\eta \cos\bar{\gamma} = \tfrac{1}{\psi(t_x)\sin\bar{\gamma}}(\hat{H}_{\partial_s M} - \partial_{\psi\eta}\bar{\gamma}).$$

Inserting (2.13) and the above in (2.12) yields

$$2\pi\chi(\Sigma) \geqslant \int_\Sigma \tfrac{K(p_x)}{\psi(t_x)^2}\mathrm{d}\sigma + \int_{\partial\Sigma} \tfrac{1}{\psi(t_x)\sin\bar{\gamma}}(\hat{H}_{\partial_s M} - \partial_{\psi\eta}\bar{\gamma})\mathrm{d}\lambda,$$

where we have written the area element $\mathrm{d}\sigma$ and line length element $\mathrm{d}\lambda$ explicitly in the metric g. The rest of the proof is deferred to the next Lemma 2.2. □

Lemma 2.2. *Assume that $g \geqslant \bar{g}$. If Σ is a surface in M whose boundary $\partial\Sigma$ is a simple smooth curve that separates $\partial(P_+ \cap \partial M)$ and $\partial(P_- \cap \partial M)$, then*

$$\int_\Sigma \tfrac{K(p_x)}{\psi(t_x)^2}\mathrm{d}\sigma + \int_{\partial\Sigma} \tfrac{1}{\psi(t_x)\sin\bar{\gamma}}(\hat{H}_{\partial_s M} - \partial_{\psi\eta}\bar{\gamma})\mathrm{d}\lambda \geqslant 2\pi, \tag{2.16}$$

where equality occurs if and only if Σ is a t-level set.

Proof. It suffices to prove (2.16) for $\psi \equiv 1$ since $g \geqslant \bar{g} = \psi^2\hat{g}$. In this case, $s = t$, we just use t. We also suppress the subscript $\partial_s M$ for clarity in this proof. In addition, we assume that every t-coordinate of Σ is strictly less than t_+, since we can increase t_+ a little. Let e_1 be the unit tangent vector of $\partial\Sigma_t$ with respect to $\hat{g}$ and e_2 be the unit outward normal of $\partial\Sigma_t$ in $\partial_s M$ with respect to $\hat{g}$. Let T (resp. $\hat{T}$) be the unit tangent vector of $\partial\Sigma$ with respect to g (resp. $\hat{g}$). We recall an ingenious inequality of [KY24, Lemma 3.2],

$$\hat{H} - \nabla_\eta\bar{\gamma} \geqslant \langle T, e_2\nabla_1\bar{\gamma} + (\hat{H} - \nabla_2\bar{\gamma})e_1\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product with respect to $\hat{g}$.

Let λ be an arc-length parameter of $\partial\Sigma$ with respect to g, then the length element of $\partial\Sigma$ with respect to $\hat{g}$ is given by $|T|_{\hat{g}}\mathrm{d}\lambda =: \mathrm{d}\hat{\lambda}$ and $T = |T|_{\hat{g}}\hat{T}$. Therefore,

$$\begin{aligned} & \int_{\partial\Sigma} \tfrac{1}{\sin\bar{\gamma}}(\hat{H} - \partial_\eta\bar{\gamma})\mathrm{d}\lambda \\ \geqslant & \int_{\partial\Sigma} \tfrac{1}{\sin\bar{\gamma}}\langle T, \partial_1\bar{\gamma}e_2 + (\hat{H} - \partial_2\bar{\gamma})e_1\rangle\mathrm{d}\lambda \\ = & \int_{\partial\Sigma} \tfrac{1}{\sin\bar{\gamma}}\langle \hat{T}, \nabla_1\bar{\gamma}e_2 + (\hat{H} - \nabla_2\bar{\gamma})e_1\rangle\mathrm{d}\hat{\lambda}. \end{aligned}$$

Let $\hat{\eta}$ be the unit outward normal of $\partial\Sigma$ in $\partial_s M$ with respect to $\hat{g}$. Recall the notation $\hat{A} = \hat{A}_{\partial_s M}$, and we know that the components of $\hat{A}$ satisfy

$$\partial_1\bar{\gamma} = \hat{A}_{12},\ \partial_2\bar{\gamma} = \hat{A}_{22},\ \hat{H} - \partial_2\bar{\gamma} = \hat{A}_{11}, \tag{2.17}$$

in the frame $\{e_1, e_2\}$. So

$$\langle \hat{T}, \partial_1\bar{\gamma}e_2 + (\hat{H} - \partial_2\bar{\gamma})e_1\rangle = \langle \hat{T}, \hat{A}_{12}e_2 + \hat{A}_{11}e_1\rangle = \hat{A}_{1\hat{T}}.$$

Since the orthonormal frames $\{\hat{T}, \hat{\eta}\}$ and $\{e_1, e_2\}$ have the same orientation, we can set $\hat{T} = a_1 e_1 + a_2 e_2$ and so $\hat{\eta} = -a_2 e_1 + a_1 e_2$ for some a_1 and a_2 which satisfy $a_1^2 + a_2^2 = 1$. It then follows that

$$\hat{A}_{1\hat{T}} = -(\hat{A} - \hat{H}\hat{g})(e_2, \hat{\eta}) =: -\hat{W}(e_2, \hat{\eta}), \tag{2.18}$$

by considering also that $\hat{H} = \hat{A}_{11} + \hat{A}_{22}$. Hence

$$\int_{\partial\Sigma} \tfrac{1}{\sin\bar{\gamma}}(\hat{H} - \partial_\eta\bar{\gamma})\mathrm{d}\lambda \geqslant -\int_{\partial\Sigma} \tfrac{1}{\sin\bar{\gamma}}\hat{W}(e_2, \hat{\eta})\mathrm{d}\hat{\lambda}.$$

Suppose that $\partial\Sigma(t_+)$ (we set $\Sigma(t_{+\pm}) = \Sigma_{t_\pm}$ to avoid double subscripts) and $\partial\Sigma$ enclose a region S in $\partial_s M$. By the divergence theorem,

$$\begin{aligned} & -\int_{\partial\Sigma} \tfrac{1}{\sin\bar{\gamma}}\hat{W}(e_2, \hat{\eta})\mathrm{d}\hat{\lambda} \\ = & -\int_S \hat{\nabla}_i^S\left(\hat{W}_{ij}\langle \tfrac{1}{\sin\bar{\gamma}}e_2, e_j\rangle\right)\mathrm{d}\hat{\sigma} - \int_{\partial\Sigma(t_+)} \tfrac{1}{\sin\bar{\gamma}}(\hat{A} - \hat{H}\hat{g})(e_2, e_2)\mathrm{d}\hat{\lambda} \\ =: & -\int_S \hat{\nabla}_i^S(\hat{A}_{ij} - \hat{H}\hat{g}_{ij})\langle \tfrac{1}{\sin\bar{\gamma}}e_2, e_j\rangle\mathrm{d}\hat{\sigma} - \int_S \hat{W}_{ij}\langle\hat{\nabla}_i^S(\tfrac{1}{\sin\bar{\gamma}}e_2), e_j\rangle\mathrm{d}\hat{\sigma} - I_3 \\ =: & - I_1 - I_2 - I_3, \end{aligned}$$

where $\hat{\nabla}^S$ is the induced connection on S with respect to the metric $\hat{g}$. For I_1, we use Gauss-Codazzi equation,

$$I_1 = -\int_S \mathrm{Ric}_{\hat{g}}(\hat{X}, \tfrac{1}{\sin\bar{\gamma}}e_2)\mathrm{d}\hat{\sigma} = -\int_S K(p_x)\cos\bar{\gamma}\mathrm{d}\hat{\sigma} = -\int_S K(p_x)\langle\partial_t, \hat{X}\rangle\mathrm{d}\hat{\sigma},$$

where the fact that $\hat{g} = \mathrm{d}t^2 + g_{S^2}$ was also used in determining the Ricci curvature. For I_3, we use (2.18), we see

$$I_3 = \int_{\partial\Sigma(t_+)} \tfrac{1}{\sin\bar{\gamma}}\hat{A}_{11}\mathrm{d}\hat{\lambda} = \int_{\partial\Sigma(t_+)} \hat{\kappa}(p_x, t)\mathrm{d}\hat{\lambda},$$

where $\hat{\kappa}(p, t)$ is the geodesic curvature of $\partial\Sigma_t$ in Σ_t at $(p, t) \in \partial\Sigma_t$. It seems tricky to calculate $\hat{\nabla}_i\left(\tfrac{1}{\sin\bar{\gamma}}e_2\right)$ in I_2 directly at a first sight, but we can convert to terms that are easier. By the identity,

$$\tfrac{1}{\sin\bar{\gamma}}e_2 = \partial_t - \tfrac{\cos\bar{\gamma}}{\sin\bar{\gamma}}\hat{\nu} = \partial_t - \tfrac{\cos\bar{\gamma}}{\sin^2\bar{\gamma}}\hat{X} + \tfrac{\cos^2\bar{\gamma}}{\sin^2\bar{\gamma}}\partial_t$$

at $x \in \partial_s M$, we see

$$\begin{aligned}
&\hat{W}_{ij}\langle\hat{\nabla}_i(\tfrac{1}{\sin\bar{\gamma}}e_2), e_j\rangle \\
=&\hat{W}_{ij}\left\langle\hat{\nabla}_i\left(\partial_t - \tfrac{\cos\bar{\gamma}}{\sin^2\bar{\gamma}}\hat{X} + \tfrac{\cos^2\bar{\gamma}}{\sin^2\bar{\gamma}}\partial_t\right), e_j\right\rangle \\
=& -\tfrac{\cos\bar{\gamma}}{\sin^2\bar{\gamma}}\hat{W}_{ij}\hat{A}_{ij} + \hat{W}_{ij}\partial_i\left(\tfrac{\cos^2\bar{\gamma}}{\sin^2\bar{\gamma}}\right)\langle\partial_t, e_j\rangle \\
=& -\tfrac{\cos\bar{\gamma}}{\sin^2\bar{\gamma}}\hat{W}_{ij}\hat{A}_{ij} - 2\hat{W}_{i2}\tfrac{\cos\bar{\gamma}}{\sin^3\bar{\gamma}}\partial_i\bar{\gamma}\langle\partial_t, e_2\rangle
\end{aligned}$$

at x. It is now a tedious task to check from the definition of $\hat{W}$, $\langle\partial_t, e_2\rangle = -\sin\bar{\gamma}$ and (2.17) that the above vanishes. Therefore, $I_2 = 0$ and to sum up, we have shown that

$$\int_{\partial\Sigma}\tfrac{1}{\sin\bar{\gamma}}(\hat{H} - \partial_\eta\bar{\gamma})\mathrm{d}\lambda \geqslant -\int_S K(p_x)\langle\partial_t, \hat{X}\rangle\mathrm{d}\hat{\sigma} + \int_{\partial\Sigma(t_+)}\hat{\kappa}(p_x, t)\mathrm{d}\hat{\lambda}. \tag{2.19}$$

Now we set the region enclosed by $\Sigma(t_+)$, Σ and $\partial_s M$ to be Ω. Let $\hat{G} = \mathrm{Ric}_{\hat{g}} - \frac{1}{2}R_{\hat{g}}\hat{g}$. Using the divergence free property of $\hat{G}$ (twice-contracted Gauss-Codazzi equation), ∂_t, and the divergence theorem,

$$0 = \int_\Omega \hat{\nabla}_i(\hat{G}_{ij}(\partial_t)_j) = \int_{\partial\Omega}\hat{G}(\partial_t, \hat{X}) = -\frac{1}{2}\int_{\partial\Omega}R_{\hat{g}}\langle\partial_t, \hat{X}\rangle, = -\int_{\partial\Omega}K(p_x)\langle\partial_t, \hat{X}\rangle,$$

where $\hat{X}$ now also denotes the unit outward normal of $\partial\Omega$ with respect to $\hat{g}$ and $(\partial_t)_j$ denotes the j-th component of the vector field ∂_t. Note that $\partial\Omega = S \cup \Sigma(t_+) \cup \Sigma$ and so

$$0 = \int_{\Sigma(t_+)}K(p_x)\langle\partial_t, \partial_t\rangle - \int_\Sigma K(p_x)\langle\partial_t, \hat{N}\rangle + \int_S K(p_x)\langle\partial_t, \hat{X}\rangle,$$

and it follows from that $K(p_x) > 0$ that

$$\int_\Sigma K(p_x) \geqslant \int_\Sigma K(p_x)\langle\partial_t, \hat{N}\rangle = \int_{\Sigma(t_+)}K(p_x) + \int_S K(p_x)\langle\partial_t, \hat{X}\rangle. \tag{2.20}$$

Finally, it follows from (2.19) and (2.20) that

$$\int_\Sigma K(p_x) + \int_{\partial\Sigma}\tfrac{1}{\sin\bar{\gamma}}(\hat{H} - \partial_\eta\bar{\gamma})\mathrm{d}\lambda \geqslant \int_{\Sigma(t_+)}K(p_x) + \int_{\partial\Sigma(t_+)}\hat{\kappa}(p_x, t)\mathrm{d}\hat{\lambda}.$$

An application of the Gauss-Bonnet theorem on the right hand finishes the proof of (2.16). The equality case is easy to trace. □

2.4. **Infinitesimally rigid surface.** The surface Σ be a stable capillary μ-bubble has more consequences than the mere Lemma 2.1. We can conclude that Σ is a so-called infinitesimally rigid surface. See Definition 2.3.

All inequalities are in fact equalities by Lemma 2.2 and tracing the equalities in (2.12), we arrive that

$$R_g = R_{\bar{g}}, N = \bar{N}, |A^0| = 0 \text{ in } \Sigma \tag{2.21}$$

and

$$H_{\partial M} = \bar{H}_{\partial M} \text{ along } \partial\Sigma. \tag{2.22}$$

It then follows from the equality case of Lemma 2.2 that

$$t_x = t_0 \text{ at all } x \in \bar{\Sigma} \tag{2.23}$$

for some constant $t_0 \in [t_-, t_+]$. Because Σ is stable (equivalently $Q(f,f) \geqslant 0$), so the eigenvalue problem

$$\begin{cases} Lf &= \mu f \text{ in } \Sigma \\ Bf &= 0 \text{ on } \partial\Sigma \end{cases} \tag{2.24}$$

has a nonnegative first eigenvalue $\mu_1 \geqslant 0$.

The analysis now is similar to [FCS80]. Letting $f \equiv 1$ in (2.6), using (2.21), (2.22) and (2.23), we get

$$\begin{aligned} Q(1,1) = &\int_\Sigma \left[K - \tfrac{1}{2}(R + \tfrac{3}{2}\bar{h}^2 + 2\partial_N \bar{h})\right] \\ &+ \int_{\partial\Sigma} \left[\kappa - (\tfrac{H_{\partial M}}{\sin\bar{\gamma}} - \bar{h}\cot\bar{\gamma} - \tfrac{1}{\sin\bar{\gamma}}\tfrac{\partial\bar{\gamma}}{\partial\eta})\right] = 0. \end{aligned} \tag{2.25}$$

And so the first eigenvalue μ_1 is zero, and the constant 1 is its corresponding eigenfunction.

By (2.21) and (2.11), the stability operator L reduces to

$$L = -\Delta - \left(\tfrac{K(p_x)}{\psi(t_0)^2} - K\right);$$

by considering (2.22) and that $t_x = t_0$, the boundary stability operator B reduces to

$$B = \partial_\nu - \left(\frac{\hat{\kappa}(p_x, t_0)}{\psi(t_0)} - \kappa\right).$$

Putting $f = 1$ and $\mu_1 = 0$ in the eigenvalue problem (2.24), we get

$$K = \tfrac{K(p_x)}{\psi^2(t_0)} \text{ in } \Sigma,\ \kappa = \tfrac{\hat{\kappa}(p_x,t_0)}{\psi(t_0)} \text{ on } \partial\Sigma. \tag{2.26}$$

Now we summarize the properties of Σ in the definition of an *infinitesimally rigid surface.*

Definition 2.3. *We say that Σ is infinitesimally rigid if it satisfies* (2.21)*,* (2.22)*,* (2.23) *and* (2.26)*.*

2.5. **Capillary foliation of constant $H - \bar{h}$.** See for instance the works [Ye91], [BBN10] and [Amb15] on constructing CMC foliations. First, we construct a foliation with prescribed angles $\bar{\gamma}$ whose leaf is of constant $H - \bar{h}$. Let $\phi(x,t)$ be a local flow of a vector field Y which is tangent to ∂M and transverse to Σ and that $\langle Y, N\rangle = 1$.

In the following theorem, we only require that the scalar curvature of (M, g) and the mean curvature of ∂M are bounded below.

Theorem 2.4. *Suppose (M, g) is a three manifold with boundary, if Σ is an infinitesimally rigid surface, then there exists $\varepsilon > 0$ and a function $w(x,t)$ on $\Sigma \times (-\varepsilon, \varepsilon)$ such that for each $t \in (-\varepsilon, \varepsilon)$, the surface*

$$\Sigma_t = \{\phi(x, w(x,t)) : x \in \Sigma\}$$

is a surface of constant $H - \bar{h}$ intersecting ∂M with prescribed angle $\bar{\gamma}$. Moreover, for every $x \in \Sigma$ and every $t \in (-\varepsilon, \varepsilon)$,

$$w(x,0) = 0,\ \int_\Sigma (w(x,t) - t) = 0 \text{ and } \tfrac{\partial}{\partial t} w(x,t)|_{t=0} = 1.$$

Proof. Given a function in the Hölder space $C^{2,\alpha}(\Sigma)\cap C^{1,\alpha}(\bar{\Sigma})$ $(0<\alpha<1)$, we consider

$$\Sigma_u=\{\phi(x,u(x)):x\in\Sigma\},$$

which is a properly embedded surface if the norm of u is small enough. We use the subscript u to denote the quantities associated with Σ_u.

Consider the space

$$\mathcal{Y}=\left\{u\in C^{2,\alpha}(\Sigma)\cap C^{1,\alpha}(\bar{\Sigma}):\int_\Sigma u=0\right\}$$

and

$$\mathcal{Z}=\left\{u\in C^{0,\alpha}(\Sigma):\int_\Sigma u=0\right\}.$$

Given small $\delta>0$ and $\varepsilon>0$, we define the map

$$\Phi:(-\varepsilon,\varepsilon)\times B(0,\delta)\to\mathcal{Z}\times C^{0,\alpha}(\partial\Sigma)$$

given by

$$\begin{aligned}&\Phi(t,u)\\ =&\left((H_{t+u}-\bar{h}_{t+u})-\tfrac{1}{|\Sigma|}\int_\Sigma(H_{t+u}-\bar{h}_{t+u}),\langle X_{t+u},N_{t+u}\rangle-\cos\bar{\gamma}_{t+u}\right).\end{aligned}$$

Here, $B(0,\delta)$ is a ball of radius $\delta>0$ centered at the zero function in $\mathcal{Y}$. For each $v\in\Sigma$, the map

$$f:(x,s)\in\Sigma\times(-\varepsilon,\varepsilon)\to\phi(x,sv(x))\in M$$

gives a variation with

$$\tfrac{\partial f}{\partial s}|_{s=0}=\tfrac{\partial}{\partial s}\phi(x,sv(x))|_{s=0}=vN.$$

Since Σ is infinitesimally rigid and using also (2.7) and (2.8), we obtain that

$$D\Phi_{(0,0)}(0,v)=\tfrac{\mathrm{d}}{\mathrm{d}s}\Phi(0,sv)|_{s=0}=\left(-\Delta v+\tfrac{1}{|\Sigma|}\int_{\partial\Sigma}\Delta v,-\sin\bar{\gamma}\tfrac{\partial v}{\partial\nu}\right).$$

It follows from the elliptic theory for the Laplace operator with Neumann type boundary conditions that $D\Phi(0,0)$ is an isomorphism when restricted to $0\times\mathcal{Y}$.

Now we apply the implicit function theorem: For some smaller ε, there exists a function $u(t)\in B(0,\delta)\subset\mathcal{X}$, $t\in(-\varepsilon,\varepsilon)$ such that $u(0)=0$ and $\Phi(t,u(t))=\Phi(0,0)=(0,0)$ for every t. In other words, the surfaces

$$\Sigma_{t+u(t)}=\{\phi(x,t+u(t)):x\in\Sigma\}$$

are of constant $H-\bar{h}$ with prescribed angles $\bar{\gamma}$.

Let $w(x,t)=t+u(t)(x)$ where $(x,t)\in\Sigma\times(-\varepsilon,\varepsilon)$. By definition, $w(x,0)=0$ for every $x\in\Sigma$ and $w(\cdot,t)-t=u(t)\in B(0,\delta)\subset\mathcal{X}$ for every $t\in(-\varepsilon,\varepsilon)$. Observe that the map $s\mapsto\phi(x,w(x,s))$ gives a variation of Σ with variational vector field given by

$$\tfrac{\partial\phi}{\partial t}\tfrac{\partial w}{\partial s}|_{s=0}=\tfrac{\partial w}{\partial s}|_{s=0}Y.$$

Since for every t we have that

$$\begin{aligned}0=&\Phi(t,u(t))\\ =&\left((H_{w(\cdot,t)}-\bar{h}_{w(\cdot,t)})-\tfrac{1}{|\Sigma|}\int_\Sigma(H_{w(\cdot,t)}-\bar{h}_{w(\cdot,t)}),\langle X_{t+u},N_{t+u}\rangle-\cos\bar{\gamma}_{t+u}\right),\end{aligned}$$

by taking the derivative at $t = 0$ we conclude that

$$\langle \tfrac{\partial w}{\partial t}|_{t=0} Y, N\rangle = \tfrac{\partial w}{\partial t}|_{t=0}$$

satisfies the homogeneous Neumann problem. Therefore, it is constant on Σ. Since

$$\int_\Sigma (w(x,t) - t) = \int_\Sigma u(x,t) = 0$$

for every t, by taking derivatives at $t = 0$ again, we conclude that

$$\int_\Sigma \tfrac{\partial w}{\partial t}|_{t=0} = |\Sigma|.$$

Hence, $\frac{\partial w}{\partial t}|_{t=0} = 1$. Taking ε small, we see that $\phi(x, w(x,t))$ parameterize a foliation near Σ. □

Theorem 2.5. *There exists a continuous function $\Psi(t)$ such that*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\exp(-\int_0^t \Psi(\tau)\mathrm{d}\tau)(H - \bar{h})\right) \leqslant 0.$$

Proof. Let $\psi : \Sigma \times I \to M$ parameterize the foliation, $Y = \frac{\partial \psi}{\partial t}$, $v_t = \langle Y, N_t\rangle$. Then

$$-\tfrac{\mathrm{d}}{\mathrm{d}t}(H - \bar{h}) = \Delta_t v_t + (\mathrm{Ric}(N_t) + |A_t|^2)v_t + v_t \nabla_{N_t}\bar{h} \text{ in } \Sigma_t, \tag{2.27}$$

and

$$\tfrac{\partial v_t}{\partial \nu_t} = [-\cot\bar{\gamma} A_t(\nu_t,\nu_t) + \tfrac{1}{\sin\bar{\gamma}} A_{\partial M}(\eta_t,\eta_t) + \tfrac{1}{\sin^2\bar{\gamma}} \nabla_{\eta_t}\cos\bar{\gamma}]v_t. \tag{2.28}$$

By shrinking the interval if if needed, we assume that $v_t > 0$ for $t \in I$. By multiplying of (2.27) and integrate on Σ_t, we deduce by integration by parts and applying the Schoen-Yau rewrite (2.9) that

$$\begin{aligned}
& -(H - \bar{h})' \int_{\Sigma_t} \tfrac{1}{v_t} \\
= & \int_{\Sigma_t} \tfrac{\Delta_t v_t}{v_t} + (\mathrm{Ric}(N_t) + |A_t|^2 + \nabla_{N_t}\bar{h}) \\
= & \int_{\partial\Sigma_t} \tfrac{1}{v_t}\tfrac{\partial v_t}{\partial \nu_t} + \tfrac{1}{2}\int_{\Sigma_t}(R_g + |A_t|^2 + H_t^2 + 2\nabla_{N_t}\bar{h}) - \int_{\Sigma_t} K_{\Sigma_t} + \int_{\Sigma_t} \tfrac{|\nabla v_t|^2}{v_t^2}.
\end{aligned}$$

Let $\chi = A - \frac{1}{2}\bar{h}\sigma$, we have that

$$\begin{aligned}
& |A_t|^2 \\
= & |\chi + \tfrac{1}{2}\bar{h}\sigma|^2 \\
= & |\chi|^2 + \langle \chi, \bar{h}\sigma\rangle + \tfrac{1}{2}\bar{h}^2, \\
= & |\chi^0|^2 + \tfrac{1}{2}(\mathrm{tr}_\sigma \chi)^2 + \bar{h}\,\mathrm{tr}_\sigma \chi + \tfrac{1}{2}\bar{h}^2,
\end{aligned}$$

where χ^0 is the traceless part of χ. Also,

$$H^2 = (\mathrm{tr}_\sigma \chi + \bar{h})^2 = (\mathrm{tr}_\sigma \chi)^2 + 2\,\mathrm{tr}_\sigma \chi \bar{h} + \bar{h}^2.$$

So

$$
\begin{aligned}
&-(H-\bar{h})'\int_{\Sigma_t}\tfrac{1}{v_t}\\
=&\int_{\partial\Sigma_t}\tfrac{1}{v_t}\tfrac{\partial v_t}{\partial\nu_t}+\tfrac{1}{2}\int_{\Sigma_t}(R_g+|A_t|^2+H_t^2+2\nabla_{N_t}\bar{h})-\int_{\Sigma_t}K_{\Sigma_t}+\int_{\Sigma_t}\tfrac{|\nabla v_t|^2}{v_t^2}\\
=&\int_{\partial\Sigma_t}\tfrac{1}{v_t}\tfrac{\partial v_t}{\partial\nu_t}+\tfrac{1}{2}\int_{\Sigma_t}(R_g+\tfrac{3}{2}\bar{h}^2+2\nabla_{N_t}\bar{h})\\
&+\tfrac{1}{2}\int_{\Sigma_t}|\chi^0|^2+\tfrac{3}{2}(\operatorname{tr}_\sigma\chi)^2+3\bar{h}\operatorname{tr}_\sigma\chi-\int_{\Sigma_t}K_{\Sigma_t}+\int_{\Sigma_t}\tfrac{|\nabla v_t|^2}{v_t^2}\\
\geqslant&\int_{\partial\Sigma_t}\tfrac{1}{v_t}\tfrac{\partial v_t}{\partial\nu_t}+\int_{\Sigma_t}\tfrac{K(p_x)}{\psi^2(t_x)}+\tfrac{3}{2}(H-\bar{h})\int_{\Sigma_t}\bar{h}-\int_{\Sigma_t}K_{\Sigma_t},
\end{aligned}
$$

where in the last line we have also used the bound (2.13). Now we use (2.28) and also the rewrite (2.10), we see that

$$
\begin{aligned}
&-(H-\bar{h})'\int_{\Sigma_t}\tfrac{1}{v_t}\\
\geqslant&\int_{\partial\Sigma_t}[-\cot\bar{\gamma}A_t(\nu_t,\nu_t)+\tfrac{1}{\sin\bar{\gamma}}A_{\partial M}(\eta_t,\eta_t)+\tfrac{1}{\sin^2\bar{\gamma}}\nabla_{\eta_t}\cos\bar{\gamma}]\\
&+\int_{\Sigma_t}\tfrac{K(p_x)}{\psi^2(t_x)}+\tfrac{3}{2}(H-\bar{h})\int_{\Sigma_t}\bar{h}-\int_{\Sigma_t}K_{\Sigma_t}\\
\geqslant&\int_{\partial\Sigma_t}[-\kappa_{\partial\Sigma_t}-H(t)\cot\bar{\gamma}+\tfrac{1}{\sin\bar{\gamma}}H_{\partial M}+\tfrac{1}{\sin^2\bar{\gamma}}\nabla_{\eta_t}\cos\bar{\gamma}]\\
&+\int_{\Sigma_t}\tfrac{K(p_x)}{\psi^2(t_x)}+\tfrac{3}{2}(H-\bar{h})\int_{\Sigma_t}\bar{h}-\int_{\Sigma_t}K_{\Sigma_t}\\
=&-\left(\int_{\Sigma_t}K_{\Sigma_t}+\int_{\partial\Sigma_t}\kappa_{\partial\Sigma_t}\right)+\left[\int_{\Sigma_t}\tfrac{K(p_x)}{\psi^2(t_x)}+\int_{\partial\Sigma_t}\left(\tfrac{1}{\sin\bar{\gamma}}H_{\partial M}-\bar{h}\cot\bar{\gamma}+\tfrac{1}{\sin^2\bar{\gamma}}\nabla_{\eta_t}\cos\bar{\gamma}\right)\right]\\
&+\tfrac{3}{2}(H-\bar{h})\int_{\Sigma_t}\bar{h}-(H-\bar{h})\int_{\partial\Sigma_t}\cot\bar{\gamma}.
\end{aligned}
$$

It follows from Lemma 2.2 and the proof of Lemma 2.1 that the second term in the big bracket is bounded below by 2π. Using also the Gauss-Bonnet theorem on the first term in the bracket, we see that

$$-(H-\bar{h})'\int_{\Sigma_t}\tfrac{1}{v_t}\geqslant(H-\bar{h})(\tfrac{3}{2}\int_{\Sigma_t}\bar{h}-\int_{\partial\Sigma_t}\cot\bar{\gamma}).$$

Let

$$\Psi(t)=\left(\int_{\Sigma_t}\tfrac{1}{v_t}\right)^{-1}(\int_{\partial\Sigma_t}\cot\bar{\gamma}-\tfrac{3}{2}\int_{\Sigma_t}\bar{h}), \tag{2.29}$$

then note that we have assume that $v_t > 0$ near $t = 0$, so $H - \bar{h}$ satisfies the ordinary differential inequality

$$(H - \bar{h})' - \Psi(t)(H - \bar{h}) \leqslant 0. \tag{2.30}$$

We see then

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(-\int_0^t \Psi(\tau)\mathrm{d}\tau\right)(H - \bar{h})\right) \leqslant 0.$$

So the function $\exp(-\int_0^t \Psi(\tau)\mathrm{d}\tau)(H - \bar{h})$ is nonincreasing. □

2.6. **From local foliation to rigidity.** Let Σ_t be the constant mean curvature surfaces with prescribed contact angles $\bar{\gamma}$ with ∂M.

Proposition 2.6. *Every Σ_t constructed in Theorem 2.4 is infinitesimally rigid.*

Proof. Let Ω_t be the component of $M \backslash \Sigma_t$ whose clossure contains $P_+ \cap \partial M$. We abuse the notation and define

$$F(t) = |\Sigma_t| - \int_{\Omega_t} \bar{h} - \int_{\partial\Omega_t} \cos\bar{\gamma}.$$

By the first variation formula (2.3),

$$F(t_2) - F(t_1) = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} (H - \bar{h}) v_t.$$

By Theorem 2.5,

$$H - \bar{h} \leqslant 0 \text{ if } t \geqslant 0;\ H - \bar{h} \geqslant 0 \text{ if } t \leqslant 0,$$

which in turn implies that

$$F(t) \leqslant 0 \text{ if } t \geqslant 0;\ F(t) \leqslant 0 \text{ if } t \leqslant 0.$$

However, Ω_t is a minimiser to the functional (2.1), hence

$$F(t) \equiv F(0).$$

It then follows every Σ_t is a minimiser, hence infinitesimally rigid. □

Now we introduce the *barrier* condition which enables to find a stable capillary μ-bubble.

Definition 2.7. *We say that a surface Σ_+ (Σ_-) whose boundary separates $\partial(P_+ \cap \partial M)$ and $\partial(P_- \cap \partial M)$ is an upper (lower) barrier if $H_{\Sigma_+} \geqslant \bar{h}|_{\Sigma_+}$ ($H_{\Sigma_-} \leqslant \bar{h}|_{\Sigma_-}$) and $\gamma_{\Sigma_+} \geqslant \bar{\gamma}|_{\partial\Sigma_+ \cap \partial M}$ ($\gamma_{\Sigma_-} \leqslant \bar{\gamma}|_{\partial\Sigma_- \cap \partial M}$) along $\partial\Sigma_+$ ($\partial\Sigma_-$). We call Σ_+ and Σ_- are a set of barriers if Σ_+ and Σ_- are respectively an upper barrier and a lower barrier and Σ_+ is closer to P_+ than Σ_-.*

We can conclude the proof of Theorem 1.4.

Proof of Theorem 1.4. We note easily by the assumptions of Theorem 1.4 that $\Sigma_\pm = \partial_\pm M$ are a set of barriers (Definition 2.7), by the maximum principle, there exists a minimiser E to (2.2) such that E is either empty or $\partial E \backslash \partial_s M$ or lies entirely away from $P_\pm$. Without loss of generality, we assume that $\Sigma = \partial E \cap \operatorname{int} M$ non-empty. By [DPM15], Σ is a regular stable surface of prescribed mean curvature $\bar{h}$ and prescribed contact angle $\bar{\gamma}$. Moreover, the second variation $\mathcal{A}''(0) \geqslant 0$ in (2.4) for any smooth family Σ_s such that $\Sigma_0 = \Sigma$.

Let $Y = \frac{\mathrm{d}}{\mathrm{d}t}\phi(x, w(x,t))$ where ϕ and w are as Theorem 2.4, we show first that N_t is conformal. It suffices to show that $Y^\perp$ is conformal.

Since every Σ_t is infinitesimally rigid by Proposition 2.6, from (2.24) and (2.25), we know that $\langle Y, N_t\rangle$ is a constant. Let ∂_i, $i = 1, 2$ be vector fields induced by local coordinates on Σ, ∂_i also extends to a neighborhood of Σ via the diffeomorphism ϕ. We have $\nabla_{\partial_i}\langle Y, N\rangle = 0$. Note that Σ_t are umbilical with constant mean curvature $\bar{h}$, so

$$\nabla_{\partial_i} N \equiv \tfrac{1}{2}\bar{h}\partial_i$$

and

$$\begin{aligned} 0 &= \nabla_{\partial_i}\langle Y, N\rangle \\ &= \langle \nabla_{\partial_i} Y, N\rangle + \langle Y, \nabla_{\partial_i} N\rangle \\ &= \langle \nabla_{\partial_i} Y, N\rangle + \tfrac{1}{2}\bar{h}\langle Y, \partial_i\rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 &= \langle \nabla_{\partial_i} Y, N\rangle = \langle \nabla_Y \partial_i, N\rangle \\ &= Y\langle \partial_i, N\rangle - \langle \partial_i, \nabla_Y N\rangle \\ &= -\langle \partial_i, \nabla_Y N\rangle \\ &= -\langle \partial_i, \nabla_{Y^\top} N\rangle - \langle \partial_i, \nabla_{Y^\perp} N\rangle \\ &= -\tfrac{1}{2}\bar{h}\langle Y^\top, \partial_i\rangle - \langle \partial_i, \nabla_{Y^\perp} N\rangle. \end{aligned}$$

Combining the two equations above, we conclude that $\nabla_{Y^\perp} N = 0$ which implies that Σ foliates a warped product under the diffeomorphism ϕ (parameterized by t). Considering that the induced metric on Σ agrees with the induced metric from $\bar{g}$, we conclude that $g = \bar{g}$. □

3. Construction of barriers (I)

In this section, we prove Theorem 1.3. Our strategy is to construct a surface Σ_- which together with $\Sigma_+ := P_+ \cap \partial M$ serve as barriers, and to use Theorem 1.4 to finish the proof. This section is occupied by such a construction of Σ_-.

3.1. **Setting up coordinates and notations.** For convenience, we set $t_- = 0$. As before, for any $t > 0$, we set Σ_t to be the t-level set of t and Ω_t to be the t-sublevel set, that is, all points of M which lie below Σ_t. Since both (M, g) and $(M, \bar{g})$ has cone structures near where $t_- = 0$ where each cross-section of the cone is a topological disk and it collapses to a point which we denote by p_0. For convergence of sequences of Riemannian manifolds and notions of tangent cones of at a point of a Riemannian manifold, we refer to the textbook [BBI01, Chapter 8].

In the following subsections, we construct graphical perturbations Σ_{t,t^2u} of Σ_t. Let Σ_{t,t^2u} be the surface which consists of points $x + t^2u(x,t)N_t(x)$ where N_t is the unit normal of Σ_t with respect to the metric g at $x \in \Sigma_t$. The boundary $\partial\Sigma_{t,t^2u}$ might not lie in $\partial_s M$, we can compensate this by expanding or shrinking Σ_{t,t^2u} a little, and we still denote the resulting surface Σ_{t,t^2u}.

We use a t subscript on every geometric quantity on Σ_t and a t, t^2u subscript on every geometric quantity on Σ_{t,t^2u}. We will explicitly indicate when there was a confusion or change.

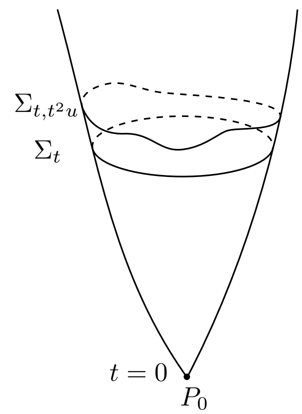

FIGURE 3.1. Construction of Σ_{t,t^2u}.

3.2. **Capillary foliation with constant $H-\bar{h}$.** We assume that (M,g) and $(M,\bar{g})$ have isometric tangent cones at p_0 and we construct a foliation of constant $H-\bar{h}$ with prescribed angles $\bar{\gamma}$ near p_0. In fact, later in Subsection 3.3, it is shown that this is the only case.

By the first variation formula of the mean curvatures

$$H_{t,t^2u} - H_t = -\Delta_t u - t^2(\mathrm{Ric}(N_t) + |A_t|^2)u + O(t), \tag{3.1}$$

where Δ_t is the Laplacian with respect to the induced rescaled metric $t^{-2}g|_{\Sigma_t}$. Note that $\mathrm{Ric}(N_t) = O(t^{-1})$ by the fact the tangent cone is $\mathrm{d}t^2 + a^2t^2 g_{S^2}$. By the Taylor expansion of the function $\bar{h}$, we see that

$$\bar{h}_{t,t^2u} - \bar{h}_t = \bar{h}'(t)t^2u = t^2u\nabla_{N_t}\bar{h} + O(t).$$

So

$$(H_{t,t^2u} - \bar{h}_{t,t^2u}) - (H_t - \bar{h}_t) = -\Delta_t u - t^2(\mathrm{Ric}(N_t) + |A_t|^2 + \nabla_{N_t}\bar{h})u + O(t). \tag{3.2}$$

Note that both $H_t - \bar{h}_t$ and $H_{t,t^2u} - \bar{h}_{t,t^2u}$ are finite and $|A_t|^2 + \nabla_{N_t}\bar{h} = O(t^{-1})$ considering that (M,g) and $(M,\bar{g})$ have isometric tangent cones at p_0.

Remark 3.1. We elaborate a bit more on (3.1) and its $O(t)$ remainder term. Since the metric g is close to $\mathrm{d}t^2 + \psi(t)^2 g_{S^2}$ when $t \to 0^+$, we calculate the expansions with respect to the rescaled metric $t^{-2}g$ when computing for small $t > 0$. This is similar to [Ye91]. Then we rescale back and we obtain (3.1). The term $O(t)$ involves products of $|A_t|$ which is of order t^{-1} with terms of order at most $O(1)$. That is why the remainder is only of order $O(t)$ instead of $O(t^2)$.

Also, the variation of angles give

$$\begin{aligned} & t^{-1}[\langle X_{t,t^2u}, N_{t,t^2u}\rangle - \langle X_t, N_t\rangle] \\ = & -\sin\gamma \tfrac{\partial u}{\partial \nu_t} + t(-\cos\gamma A(t^{-1}\nu_t, t^{-1}\nu_t) + A_{\partial M}(\eta_t,\eta_t))u + O(t^2), \end{aligned} \tag{3.3}$$

where ν_t is the outward unit normal of $\partial\Sigma_t$ in Σ_t with respect to the rescaled induced metric $t^{-2}g|_{\Sigma_t}$ (note that $t^{-1}\nu_t$ is of unit length with respect to g). Other geometric quantities are not rescaled. By the variation of the prescribed angle $\bar{\gamma}$,

$$t^{-1}(\cos\bar{\gamma}_{t,t^2u} - \cos\bar{\gamma}_t) = -\tfrac{tu}{\sin\bar{\gamma}}\partial_{\bar{\eta}_t}\cos\bar{\gamma} + O(t^2).$$

So

$$
\begin{aligned}
&t^{-1}[(\langle X_{t,t^2u}, N_{t,t^2u}\rangle - \cos\bar{\gamma}_{t,t^2u}) - (\langle X_t, N_t\rangle - \cos\bar{\gamma}_t)]\\
=& -\sin\gamma \tfrac{\partial u}{\partial \nu_t}\\
&+ t(-\cos\gamma A(t^{-1}\nu_t, t^{-1}\nu_t) + A_{\partial M}(\eta_t, \eta_t) + \tfrac{1}{\sin\bar{\gamma}}\partial_{\bar{\eta}_t}\cos\bar{\gamma})u + O(t^2).
\end{aligned} \tag{3.4}
$$

Remark 3.2. The term $A(t^{-1}\nu_t, t^{-1}\nu_t) = O(t^{-1})$, however, we observe that $\lim_{t\to 0}\bar{\gamma}_t = \pi/2$, and $A_{\partial M}(\eta_t, \eta_t) = O(1)$ since $A_{\partial M}(\bar{\eta}_t, \bar{\eta}_t) = O(1)$. Or we can calculate with respect to the rescaling metric as in Remark 3.1.

Since g and $\bar{g}$ has isometric tangent cone at p_0, we see that the limit of the surface $(\Sigma_t, t^{-2}g|_{\Sigma_t})$ as $t \to 0$ is $(\Sigma, a^2 g_{S^2})$ where Σ is a scaling copy of a geodesic disk of radius $\rho(0) = \lim_{t\to 0}\rho(t) > 0$ in the standard 2-sphere. Consider the spaces

$$\mathcal{Y} = \left\{u \in C^{2,\alpha}(\Sigma) \cap C^{1,\alpha}(\bar{\Sigma}) : \int_\Sigma u = 0\right\}$$

and

$$\mathcal{Z} = \left\{u \in C^{0,\alpha}(\Sigma) : \int_D u = 0\right\}.$$

Given small $\delta > 0$ and $\varepsilon > 0$, we define the map

$$\Phi : (-\varepsilon, \varepsilon) \times B(0, \delta) \to \mathcal{Z} \times C^{1,\alpha}(\partial\Sigma)$$

given by $\Phi(t, u) = (\Phi_1(t, u), \Phi_2(t, u))$ where Φ_i, $i = 1, 2$ are given by

$$
\begin{aligned}
\Phi_1(t, u) =& (H_{t,t^2u} - \bar{h}_{t,t^2u}) - \frac{1}{|\Sigma|}\int_\Sigma (H_{t,t^2u} - \bar{h}_{t,t^2u}),\\
\Phi_2(t, u) =& t^{-1}(\langle X_{t,t^2u}, N_{t,t^2u}\rangle - \cos\bar{\gamma}_{t,t^2u})
\end{aligned}
$$

for $t \neq 0$. Here $B(0, \delta) \subset \mathcal{Y}$ is an open ball with radius δ in the $C^{2,\alpha}$ norm and the integration on Σ is with respect to the metric $g_{\mathbb{S}^2}$. We extend $\Phi(t, u)$ to $t = 0$ by taking limits, that is,

$$\Phi(0, u) = \lim_{t\to 0}\Psi(t, u).$$

We have the following proposition.

Proposition 3.3. *For each $t \in [0, \varepsilon)$ with ε small enough, we can find $u_t = u(\cdot, t) \in C^{2,\alpha}(\Sigma) \cap C^{1,\alpha}(\bar{\Sigma})$ such that $\int_\Sigma u(\cdot, t) = 0$ and*

$$\Phi(t, u_t) = (0, 0).$$

In particular, each of the surfaces Σ_{t,t^2u} have constant $\lambda_t := H_{t,t^2u} - \bar{h}_{t,t^2u}$ and prescribed angles $\gamma_{t,t^2u} = \bar{\gamma}_{t,t^2u}$. Moreover, $\lambda_t \leqslant 0$ for all small $t \in [0, \varepsilon)$.

Before proving this proposition, we give a variational lemma.

Lemma 3.4. *Suppose that $(\Omega, \hat{g})$ is a compact manifold with piecewise smooth boundary $\partial\Omega$ and Σ is a relatively open, smooth subset of $\partial\Omega$. Let g_s be a smooth family of metrics indexed by $s \in [0, \varepsilon)$ such that $g_s \to \hat{g}$ as $s \to 0$, let $h_s = g_s - \hat{g}$. We now omit the subscript on h_s. Let ν be the unit outward normal of $\partial\Omega$ in (Ω, g), H_g and A_g be the mean curvatures and the second fundamental form of $\partial\Omega$ in (Ω, g) computed with respect to the unit normal pointing outward of Ω, and γ be the dihedral angles formed by Σ and $\partial\Omega\backslash\Sigma$ with respect to the metric g. We put a hat at appropriate places for the geometric quantities with respect to $\hat{g}$.*

Then

$$2\left[-\int_\Sigma (H_g - H_{\hat g}) + \int_{\partial\Sigma} \tfrac{1}{\sin\gamma_{\hat g}}(\cos\gamma_{\hat g} - \cos\gamma_g)\right]$$
$$= \int_\Omega ((R_g - R_{\hat g}) + \langle \mathrm{Ric}_{\hat g}, h\rangle_{\hat g}) + 2\int_{\partial\Omega\setminus\Sigma}(H_g - H_{\hat g}) + \int_{\partial\Omega}\langle h, A_{\hat g}\rangle + O(s^2).$$

Here, we have used $O(s^2)$ to denote a remainder term comparable to $|h|^2_{\hat g} + |h|_{\hat g}|\hat\nabla h|_{\hat g} + |\hat\nabla h|^2_{\hat g}$.

Proof. From the variational formulas of the scalar curvature and the mean curvature, we have

$$R_g - R_{\hat g} = -\langle \mathrm{Ric}_{\hat g}, h\rangle_{\hat g} - \mathrm{div}_{\hat g}(\mathrm{d}(\mathrm{tr}_{\hat g} h) - \mathrm{div}_{\hat g} h) + O(s^2),$$

and

$$2(H_g - H_{\hat g}) = (\mathrm{d}(\mathrm{tr}_{\hat g} h) - \mathrm{div}_{\hat g} h)(\hat\nu) - \mathrm{div}_\sigma Y - \langle h, A_{\hat g}\rangle_\sigma + O(s^2) \tag{3.5}$$

where Y is the tangential component dual to the 1-form $h(\cdot, \hat\nu)$. For the explicit form of the remainder terms, refer to [BM11, Proposition 4] and [MP21].

We integrate the variation of the mean curvature (3.5) on the boundary $\partial\Omega$ with respect to the metric $\hat g$, we see

$$\int_{\partial\Omega}[(\mathrm{d}(\mathrm{tr}_{\hat g} h) - \mathrm{div}_{\hat g} h)(\hat\nu) - \mathrm{div}_{\hat\sigma} Y - \langle h, A_{\hat g}\rangle] = 2\int_{\partial\Omega}(H_g - H_{\hat g}) + O(s^2).$$

By the divergence theorem and the variation of the scalar curvature,

$$\int_{\partial\Omega}(\mathrm{d}(\mathrm{tr}_{\hat g} h) - \mathrm{div}_{\hat g} h)(\hat g) = \int_\Omega[-(R_g - R_{\hat g}) - \langle \mathrm{Ric}_{\hat g}, h\rangle_{\hat g}] + O(s^2).$$

For the term $\int_{\partial\Omega}\mathrm{div}_{\hat\sigma} Y$, we follow [MP21, (3.18)] and obtain

$$\int_{\partial\Omega}\mathrm{div}_{\hat\sigma} Y = \int_\Sigma \mathrm{div}_{\hat g} Y + \int_{\partial\Omega\setminus\Sigma}\mathrm{div}_{\hat\sigma} Y = 2\int_{\partial\Sigma}\tfrac{1}{\sin\hat\gamma}(\cos\hat\gamma - \cos\gamma) + O(s^2).$$

Collecting all the formulas in the proof proves the lemma. □

Lemma 3.5 implies the following by taking the difference of two families of metrics.

Corollary 3.5. *Assume $(\Omega, \hat g)$ is the manifold from Lemma 3.5, for two family of metrics $\{g_i\}_{i=1,2}$ close to $\hat g$ indexed both by a small parameter s, we have*

$$2\left[-\int_\Sigma (H_{g_2} - H_{g_1}) + \int_{\partial\Sigma}\tfrac{1}{\sin\hat\gamma}(\cos\gamma_{g_1} - \cos\gamma_{g_2})\right]$$

$$= \int_\Omega((R_{g_2} - R_{g_1}) + \langle \mathrm{Ric}_{\hat g}, g_2 - g_1\rangle_{\hat g}) + 2\int_{\partial\Omega\setminus\Sigma}(H_{g_2} - H_{g_1}) + \int_{\partial\Omega}\langle g_2 - g_1, A_{\hat g}\rangle + O(s^2).$$

Now we are ready to prove Proposition 3.3.

Proof of Proposition 3.3. The proof is similar to [CW23]. We bring up only the main differences.

Because that the right hand of both (3.2) and (3.4) converge to Δu and $\frac{\partial u}{\partial \nu}$ (up to a constant) respectively, so we can first follow [CW23, Proposition 4.2] to construct a foliation $\{\Sigma_{t,t^2u}\}_{t\in[0,\varepsilon)}$ near p_0 with constant $H-\bar{h}$ and $\gamma_{t,t^2u}=\bar{\gamma}_{t,t^2u}$ along $\partial\Sigma_{t,t^2u}$, and then [CW23, Lemma 4.3] to obtain that

$$-\lambda_t|\Sigma_t| = \int_{\Sigma_t}(H_t-\bar{h}_t)+\int_{\partial\Sigma_t}\tfrac{1}{\sin\gamma_t}(\cos\bar{\gamma}_t-\cos\gamma_t)+O(t^3). \tag{3.6}$$

Now we show that $\lim_{t\to 0}\lambda_t \leqslant 0$.

We consider the rescaled set $t^{-1}\Omega_t$ with two rescaled metrics $t^{-2}g$ and $t^{-2}\bar{g}$. Since $\bar{g}=\mathrm{d}t^2+\psi(t)^2 g_{S^2}$ and $\psi(t)=at+o(t)$, it is easy to see that $(t^{-1}\Omega_t, t^{-2}\bar{g})$ converges to a truncated metric cone $\Lambda=(0,1]\times D$ with the metric $\varrho:=\mathrm{d}s^2+a^2s^2g_{S^2}$ where $s\in(0,1]$ and $(D,a^2g_{S^2})$ is some convex disk in a 2-sphere $(S^2,a^2g_{S^2})$. We set $D_s=\{s\}\times D$. Since g and $\bar{g}$ has isometric tangent cone at p_0, $(t^{-1}\Omega_t,t^{-2}g)$ converges to (Λ,ϱ) as well. Therefore, we can view $g_1=t^{-2}g$ and $g_2=t^{-2}\bar{g}$ (indexed by t) as two metrics on Λ getting closer to ϱ as $t\to 0$. We rescale (3.6) by a factor of t^{-2}, we obtain

$$-\lambda_t|\Sigma_t|t^{-2}=\int_{\Sigma_t}(H_t-\bar{h}_t)t^{-2}+\int_{\partial\Sigma_t}\tfrac{1}{\sin\gamma_t}(\cos\bar{\gamma}_t-\cos\gamma_t)t^{-2}+O(t)$$

which is equivalent to

$$-\lambda_t|D|_{g_1}=\int_D(H_{g_2}-H_{g_1})+\int_{\partial D}\tfrac{1}{\sin\gamma_t}(\cos\bar{\gamma}_t-\cos\gamma_t)+O(t).$$

In the above the integration done is with respect to the metric g_1 and H_{g_i} are the mean curvature of $\{1\}\times D$ in (Λ,g_i) computed with respect to the normal pointing inside of Λ.

All the comparisons in Theorem 1.3 carry over to the rescaled metrics g_1 and g_2 on Λ, and that (Λ,ϱ) has nonnegative Ricci curvature by the assumptions of Theorem 1.3. We use Corollary 3.5 and arrive that $\lambda_t\leqslant O(t)$, that is,

$$\lim_{t\to 0}\lambda(t)\leqslant 0.$$

Since λ_t satisfies the differential inequality (2.30) and considering the asymptotics $u(\cdot,t)=1+O(t)$, $\cot\bar{\gamma}=O(t)$ and $\bar{h}=2/t+O(1)$ in (2.29), we see that $\lambda_t\leqslant 0$ for all $t\in(0,\varepsilon)$. □

Remark 3.6. The Ricci curvature in Corollary 3.5 blows up near $\{0\}\times D$, however, because we are integrating with respect to the metric ϱ, the volume near $\{0\}\times D$ is small. Also, the difference g_2-g_1 is small. So the blowing up of the Ricci curvature will not cause an issue.

3.3. **Barrier construction with non-isometric tangent cones.** Since $\bar{g}=\mathrm{d}t^2+\psi(t)^2g_{S^2}$, the manifold $(M,\bar{g})$ is topologically a cone near $t=0$ and it is a point at $t=0$. According to the assumptions of Theorem 1.3, (M,g) at p_0 also locally resembles a cone, that is,

$$g=\mathrm{d}s^2+s^2g_0+g_1, \tag{3.7}$$

where s is a parameter, g_0 is a metric on a two dimensional disk D and g_1 is small compare to $\mathrm{d}s^2+s^2g_0$. In other words, the tangent cone at p_0 is a cone with the metric $\mathrm{d}s^2+s^2g_0$.

Now we can also identify M near p_0 as $(0,\varepsilon)\times D$ and t as a function on $(0,\varepsilon)\times D$. Let $(s,x)\in(0,\varepsilon)\times D$, we see that $\tau:=s/t$ as a function on M only depends on $x\in D$. So we view τ as a function on D. Since $g\geqslant\bar{g}$ on M, we have that $\tau(x)\geqslant 1$. Now we discuss the case that $\tau(x)\equiv 1$ on D.

Lemma 3.7. *If $\tau \equiv 1$ on D, then $g_0 = a^2 g_{S^2}$. That is, (M, g) and $(M, \bar{g})$ have isometric tangent cones at p_0.*

Proof. Since $\tau \equiv 1$, so we can rescale $(M, \bar{g})$ and (M, g) by the same scale to obtain a cone $\mathcal{C} = (0, \infty) \times D$ but with two different metrics $\chi_1 = \mathrm{d}t^2 + a^2 t^2 g_{S^2}$ and $\chi_2 = \mathrm{d}t^2 + t^2 g_0$. For $s > 0$, set $D_s = \{s\} \times D \subset \mathcal{C}$. Since the metric comparison, the mean curvature and the scalar curvature comparison are preserved by rescaling, so $g_0 \geqslant a^2 g_{S^2}$, the scalar curvature $R_{\chi_2} \geqslant R_{\chi_1}$ and the mean curvature of $\partial\mathcal{C}$ at ∂D_1 satisfies $H_{\chi_2} \geqslant H_{\chi_1}$.

Since both χ_i, $i = 1, 2$ are warped product metrics, the comparison $R_{\chi_2} \geqslant R_{\chi_1}$ reduces to Gaussian curvature comparison $K_2 \geqslant K_1 = a^{-2}$ of (D_1, g_0) and $(D_1, a^2 g_{S^2})$ by a direct computation of scalar curvature (or Gauss equation). Let κ_i be the geodesic curvatures of ∂D_1 with respect to $\chi_i|_{D_1}$. By direct calculation, the second fundamental form $A^{(i)}_{\partial\mathcal{C}}$ of $\partial\mathcal{C}$ in the direction ∂_t vanishes with respect to both metrics χ_i and the second fundamental form $A^{(i)}_{D_1}$ of D_1 in $\mathcal{C}$ with respect to χ_i agree. It then follows from $H_{\chi_2} \geqslant H_{\chi_1}$ and (2.10) that $\kappa_2 \geqslant \kappa_1$.

To summarize, we have comparisons on D_1 that $g_0 \geqslant a^2 g_{S^2}$, $K_2 \geqslant K_1$ and $\kappa_2 \geqslant \kappa_1$ along ∂D_1. By Gauss-Bonnet theorem, $g_0 \equiv a^2 g_{S^2}$ on D_1 and it follows that $\chi_1 \equiv \chi_2$. Therefore, (M, g) and $(M, \bar{g})$ have isometric tangent cones at p_0. $\square$

By the above lemma, the case $\tau \equiv 1$ is the case which implies isometric tangent cones of (M, g) and $(M, \bar{g})$ at p_0. This is the case we have already addressed in Subsection 3.2. Without loss of generality, we assume that $\tau \not\equiv 1$.

We first consider the difference of $H - \bar{h}$ of the perturbation for D_s. We now represent $\bar{h}$ at D_s and its value at the graphical perturbations of D_s by ζ to avoid notational confusion. By the first variation of the mean curvatures,

$$
\begin{aligned}
&(H_{s,s^2 u} - \zeta_{s,s^2 u}) - (H_s - \zeta_s) \\
= &- \Delta_s u - s^2(\mathrm{Ric}(N_s) + |A_s|^2 + s^{-2}(\zeta_{s,s^2 u} - \zeta_s))u + O(s),
\end{aligned}
$$

where Δ_s is the Laplacian with respect to the metric $s^{-2} g|_{D_s}$.

Remark 3.8. We have $\{(s^{-1} D_s, s^{-2} g|_{D_s})\}_{s>0}$ converges to (D, g_0) as $s \to 0$ by the metric (3.7) near p_0, and to indicate that the limit carries the metric g_0, we use D_0 instead of D only.

Lemma 3.9. *We have that*

$$s^2(|A_s|^2 - s^{-2}(\zeta_{s,s^2 u} - \zeta_s)) = (2 - 2\tau) + O(s).$$

Proof. Since $\{(s^{-1}\Lambda_s, s^{-2} g)\}_{s>0}$ converges to a truncated radial cone and $\{(s^{-1} D_s, s^{-2} g|_{D_s})\}_{s>0}$ converges to the section of the radial cone with unit distance to p_0, so the section has second fundamental form -2 and by rescaling,

$$|A_s|^2 = 2s^{-2} + O(s^{-1})$$

as $s \to 0$.

At a point $p = (s, x) \in D_s$, the value of t is given by $t = s\tau(x)$ where x is the projection of p to the second coordinate. Since τ as a function on M only depends on x, we see that the value of the function t at the graphical perturbation $s + s^2 u$ of D_s is given by $(s + s^2 u)\tau$. Since $\bar{h}(t) = 2t^{-1} + O(1)$, so

$$\zeta_{s,s^2 u} - \zeta_s = \tfrac{2}{(s+s^2 u)\tau} - \tfrac{2}{s\tau} + O(1) = -\tfrac{2\tau}{s^2}(s^2 u) + O(1).$$

Hence

$$s^2(|A_s|^2 + s^{-2}(\zeta_{s,s^2u} - \zeta_s)) = (2 - 2\tau) + O(s),$$

which proves the lemma. □

Let $f = \lim_{s\to 0} s^2(\mathrm{Ric}(N_s) + |A_s|^2 + s^{-2}(\zeta_{s,s^2u} - \zeta_s))$ which is a function on the limit D_0, so

$$\lim_{s\to 0}[(H_{s,s^2u} - \zeta_{s,s^2u}) - (H_s - \zeta_s)] = -\Delta_0 u - fu,$$

where Δ_0 is the Laplacian of D_0. Recall that $\mathrm{Ric}(N_s) = O(s^{-1})$, so

$$f = 2 - 2\tau \text{ on } D_0.$$

Let α_s be the dihedral angles formed by ∂M and D_s, and α_{s,s^2u} be the angles formed by ∂M and the graphical perturbation of D_s.

Lemma 3.10. *The dihedral angles α_s formed by ∂M and D_s approach $\pi/2$ as $s \to 0$.*

Proof. Since $\{(s^{-1}\Lambda_s, s^{-2}g)\}_{s>0}$ converges to a truncated radial cone, $\{(s^{-1}D_s, s^{-2}g|_{D_s})\}_{s>0}$ converges to the section of the radial cone with unit distance to p_0, and this section is orthogonal to the radial direction in the limit, so the intersection angles of ∂M and D_s approaches $\pi/2$ as $s \to 0$. □

Lemma 3.11. *We have that $A_{\partial M}(\eta, \eta) = O(1)$.*

Proof. The lemma can be deduced from that η is approximately the radial direction ∂_s as $s \to 0$, the scaling property of $A_{\partial M}$ and the following lemma. □

Lemma 3.12. *Let (S, σ) be a 2-surface with boundary and $(C = [0, \infty) \times S, \mathrm{d}s^2 + s^2\sigma)$ be the cone over (S, σ). Then the second fundamental form of ∂C in C in the direction ∂_t vanishes.*

Proof. Let Z be a tangent vector field over Σ, then by direct calculation $\nabla_{\partial_t} Z = \nabla_X \partial_t = s^{-1}Z$. So $\langle \nabla_{\partial_t} Z, \partial_t\rangle = 0$ since on C the metric is $\mathrm{d}t^2 + t^2\sigma$. Due to the same reason, the unit normal vector Z of ∂C in M is tangent to Σ, so the claim is proved. □

We are interested in the difference between α_{s,s^2u} and the value of $\bar{\gamma}$ which to avoid confusion we denote by β_s (β_{s,s^2u}) at (the graphical perturbation s^2u of) D_s. Using the relation of s and t, $\beta = \bar{\gamma}_{s/\tau, s^2u/\tau}$. By the expansion of angles (see (3.3)), we see

$$\cos\alpha_{s,s^2u} - \cos\alpha_s = -\sin\alpha_s \tfrac{\partial u}{\partial \nu_s} + s(-\cos\alpha_s A(s^{-1}\nu_s, s^{-1}\nu_s) + A_{\partial M}(\eta_s,\eta_s))u + O(s^2).$$

And

$$s^{-1}(\cos\beta_{s,s^2u} - \cos\beta_s) = su\tau^{-1}\nabla_{\eta_{s/\tau}} \cos\bar{\gamma}_{s/\tau, s^2u/\tau} + O(s^2)$$

Since each Σ_t is stable capillary minimal surface under the metric $\bar{g}$, so we know that

$$\tfrac{1}{\sin\bar{\gamma}}\nabla_{\eta_t}\cos\bar{\gamma} = -\cos\bar{\gamma} A(\nu_t, \nu_t) + A_{\partial M}(\eta_t,\eta_t).$$

Based on the above asymptotic analysis and Lemmas 3.10 and 3.11, we see

$$\lim_{s\to 0}[s^{-1}(\cos\alpha_{s,s^2u} - \cos\alpha_s) - s^{-1}(\cos\beta_{s,s^2u} - \cos\beta_s)] = -\tfrac{\partial u}{\partial \nu_0}$$

on ∂D_0 where ν_0 is the outward normal of ∂D_0 in D_0. By elliptic strong maximum principle, the operator

$$(-\Delta_0 - f, -\tfrac{\partial}{\partial \nu_0}) : C^{2,\alpha}(D_0) \cap C^{1,\alpha}(\bar{D}_0) \to C^{0,\alpha}(D_0) \times C^{0,\alpha}(\partial D_0)$$

is an isomorphism since $f \leqslant 0$ in D_0 due to Lemma 3.9 and $\tau \gneqq 1$. In other words, we can specify the limits

$$\begin{aligned} &\lim_{s\to 0}[(H_{s,s^2u} - \zeta_{s,s^2u}) - (H_s - \zeta_s)] \\ &\quad \text{and } \lim_{s\to 0}[s^{-1}(\cos\alpha_{s,s^2u} - \cos\alpha_s) - s^{-1}(\cos\beta_{s,s^2u} - \cos\beta_s)] \end{aligned}$$

by choosing a suitable $u \in C^{2,\alpha}(D_0) \cap C^{1,\alpha}(\bar{D}_0)$.

We have these facts: by Lemma 3.10, both α_s and β_s tend to $\pi/2$ as $s \to 0$, so $\lim_{s\to 0} s^{-1}(\alpha_s - \beta_s)$ is a function on ∂D_0; $H_s - \zeta_s = (2-2\tau)s^{-1} + O(1)$;

$$H_{s,s^2u} - \zeta_{s,s^2u} = (2-2\tau)s^{-1} + O(1) \tag{3.8}$$

for small $s > 0$ with a remainder term depending on u. Hence, we can specify a function u to counter-effect the $O(1)$ remainder term in $H_s - \zeta_s$ and make the remainder term in (3.8) strictly negative. That is, we can specify a function u such that

$$\begin{aligned} &\lim_{s\to 0}(H_{s,s^2u} - \zeta_{s,s^2u} - (2-2\tau)s^{-1}) = u_0 \text{ in } D_0, \\ &\quad \lim_{s\to 0} s^{-1}(\cos\alpha_{s,s^2u} - \cos\beta_{s,s^2u}) < 0 \text{ along } \partial D_0, \end{aligned}$$

for some negative function $u_0 \in C^{0,\alpha}(\bar{D}_0)$. Recall the definitions of ζ, τ, β, and by continuity, there exists a surface $\Sigma_- \subset M$ satisfying

$$H - \bar{h} < 0 \text{ in } \Sigma_- \text{ and } \alpha > \bar{\gamma} \text{ along } \partial\Sigma_-.$$

This surface Σ_- is a a lower barrier in the sense of Definition 2.7.

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. Assume that g and $\bar{g}$ do not have isometric tangent cone at p_0, then we can construct a barrier Σ_- such that $H - \bar{h} < 0$ in Σ_- and the angle $\alpha > \bar{\gamma}$ along $\partial\Sigma_-$. But due to Theorem 1.4 (see also Remark **??**), this is not possible. So g and $\bar{g}$ have isometric tangent cones at p_0, then by the construction of the foliation in Theorem 3.3, again we have a barrier near $t = 0$, but the barrier condition is now not strict. We can extend the rigidity $g = \bar{g}$ in Theorem 1.4 beyond the barrier and to all of M. □

Remark 3.13. By considering only the mean curvature, this provide an alternative proof of Theorem 1.1 in dimension 3. Moreover, we allow conical metrics of $(\mathbb{S}^3, g)$ at two antipodal points.

Remark 3.14. During the construction of barriers in the case of non-isometric cones, the Gauss-Bonnet theorem is only used in Lemma 3.7.

4. Construction of barriers (II)

In this section, we prove Theorems Theorem 1.2 and 1.6. Our method is similar to the previous work [CW23].

4.1. **Proof of Theorem 1.6.** For convenience, we set $t_- = 0$. As before, for any $t > 0$, we set Σ_t to be the t-level set of t and Ω_t to be the t-sublevel set, that is, all points of M which lie below Σ_t. The sequence $\{(t^{-1}M, t^{-2}\bar{g})\}_{t>0}$ converges to some right circular cone $\bar{C}$ in $\mathbb{R}^3$ equipped with a flat metric $g_{\mathbb{R}^3}$ as $t \to 0$. Then $\{(t^{-1}M, t^{-2}g)\}_{t>0}$ converges to the same cone $\bar{C}$ but with a different constant metric g_0. The cone $(\bar{C}, g_0)$ is also a circular cone, which might be oblique if represented in $(\mathbb{R}^3, g_{\mathbb{R}^3})$. To see what $g_{\mathbb{R}^3}$ is, we make use of another coordinate. We write the metric $g_{\mathbb{S}^2}$ of 2-spheres of (1.1) in a conformal form. It is well known that there exists a diffeomorphism $\Phi : \mathbb{R}^2 \cup \{\infty\} \to \mathbb{S}^2$ such that the pull back metric of the round metric $g_{\mathbb{S}^2}$ on $\mathbb{S}^2$ is

$$\Phi^*(g_{\mathbb{S}^2}) = 4|\mathrm{d}y|^2(1 + |y|^2)^{-2},\ y \in \mathbb{R}^2.$$

It is easy to see that in this coordinate system that

$$g = \mathrm{d}t^2 + 4\psi(t)^2|\mathrm{d}y|^2(1 + |y|^2)^{-2} \tag{4.1}$$

and $g_{\mathbb{R}^3}$ is just $\mathrm{d}t^2 + 4\psi(t_-)^2|\mathrm{d}y|^2$.

We have the existence of a barrier if (M, g) and $(M, \bar{g})$ have non-isometric tangent cones at p_0.

Lemma 4.1. *Let M be given as in Theorem 1.6. If the tangent cones of (M, g) and $(M, \bar{g})$ at p_0 are not isometric, assume that the mean curvature comparison and the metric comparison hold near p_0, then there exists a surface Σ_- satisfying*

$$H - \bar{h} < 0 \text{ in } \Sigma_- \text{ and } \gamma_{\Sigma_-} > \bar{\gamma} \text{ along } \partial\Sigma_-$$

as the above. This surface Σ_- is a barrier in the sense of Definition 2.7.

Proof. First, we note that the mean curvature comparison and the metric comparison (we only need boundary metric comparison) are preserved in the limits. By non-isometry of tangent cones and by the angle comparison of [CW23, Proposition 4.9], there exists a plane P in $\bar{C}$ such that the the dihedral angles formed by $\partial\bar{C}$ and P in the metric g_0 are everywhere larger than $\bar{\gamma}(t_-)$.

We gain a lot of freedom to construct the barrier from the *strict* comparison of angles. The rest of the argument is complete analogous to [CW23, Proposition 4.10]. All is needed is a coordinate system to carry out the construction of Σ_t, and the coordinate system (4.1) suffices for our purpose. □

Remark 4.2. Note that the scalar curvature comparison is not needed here.

Proof of Theorem 1.6. First, the tangent cones of (M, g) and $(M, \bar{g})$ at p_0 must be isometric. Indeed, by Lemma 4.1 and Theorem 1.4, the barrier constructed in Lemma 4.1 cannot have $H - \bar{h} < 0$ in Σ_- and $\gamma_{\Sigma_-} < \bar{\gamma}$ hold strictly along $\partial\Sigma_-$.

By following Subsection 3.2, we can construct graphical perturbations Σ_{t,t^2u} of Σ_t which satisfy Proposition 3.3. For every sufficiently small $t > 0$, Σ_{t,t^2u} is a barrier in the sense of Definition 2.7, we conclude that $g = \bar{g}$ for the region bounded by Σ_{t,t^2u} and $P_+ \cap \partial M$ for every $t > 0$ from Theorem 1.4. Hence, Theorem 1.6 is proved. □

Remark 4.3. More generally, let M be a domain as in Theorem 1.2 with the only difference that ∂M is conical at $P_+ \cap \partial M$. The proof of Theorem 1.6 works as well if the limit $(t^{-1}\Omega_t, t^{-1}\bar{g})$ as $t \to 0$ is a right circular solid cross-section of the tangent cone at $t = 0$.

4.2. **Proof of Theorem 1.2.** This part is a slightly extension of the argument in Section 5 in our previous paper [CW23] and Ko-Yao's paper [KY24]. So we only sketch the key steps here and refer to the above papers for more details.

Suppose M is given by

$$M = \{(t,p) \in [0,\varepsilon) \times S^2 : t \le f(p)\},$$

near $p^- = O$, where f is a smooth function such that $f(p^-) = 0$ and $\mathrm{Hess} f$ is negative definite at p^-. Note that we need to assume $\psi(0) \neq 0$, otherwise the manifold M will have a cusp at point $O = (0,0,0)$.

To better illustrate the situation, we choose the coordinate (x_1, x_2) on S^2 such that the expansion of metric $\bar{g}$ at O is given by

$$\bar{g} = dt^2 + dx_1^2 + dx_2^2 + O(t) + O(|x|^2),$$

where $|x| = \sqrt{x_1^2 + x_2^2}$.

For simplicity, we denote $\bar{g}_0 = dt^2 + dx_1^2 + dx_2^2$ as the linearised part of $\bar{g}$ at O. After a suitable rotation, we can write $g = g_0 + th + O(t^2)$ for some constant metric g_0 defined as

$$g_0 = a_{33} dt^2 + (a_{11} dx_1^2 + a_{22} dx_2^2) + 2a_{13} dx_1 dt + 2a_{23} dx_2 dt, \tag{4.2}$$

where the matrix

$$\begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

is positive definite and satisfies $a_{11}, a_{22}, a_{33} \ge 1$.

We assume the manifold M can be written as

$$M = \{(t, x_1, x_2) : t \in [0,\varepsilon), t \le \zeta(x_1, x_2)\},$$

where $\zeta(x_1, x_2) = c_{11} x_1^2 + 2c_{12} x_1 x_2 + c_{22} x_2^2 + O(|x|^3)$ is a smooth function with $c_{11}, c_{22}, c_{11}c_{22} - c_{12}^2 > 0$.

We write a^{ij} as the inverse matrix of a_{ij}, and define several constants as

$$\begin{aligned} B &= \sqrt{a^{33}((\sqrt{a_{11}} c_{22} + \sqrt{a_{22}} c_{11})^2 + (\sqrt{a_{11}} - \sqrt{a_{22}})^2 c_{12}^2)} \\ b_{11} &= a^{33} B^{-1} c_{11}^{-1} (a_{11}(c_{11}c_{22} - c_{12}^2) + \sqrt{a_{11}a_{22}}(c_{11}^2 + c_{12}^2)) \\ b_{12} &= b_{21} = a^{33} B^{-1} \sqrt{a_{11}a_{22}} (c_{11} + c_{22}) \\ b_{22} &= a^{33} B^{-1} c_{22}^{-1} (a_{22}(c_{11}c_{22} - c_{12}^2) + \sqrt{a_{11}a_{22}}(c_{12}^2 + c_{22}^2)). \end{aligned}$$

and consider the function $G_{s,t}$ defined by

$$\begin{aligned} G_{s,t}(x_1, x_2) = {} & c_{11}(b_{11}(1+s) - 1)x_1^2 + c_{22}(b_{22}(1+s) - 1)x_2^2 \\ & + 2c_{12}(b_{12}(1+s) - 1)x_1 x_2 - t^2. \end{aligned}$$

and the surface $\Sigma_{s,t}$ is defined by

$$\Sigma_{s,t} = \left\{(G_{s,t}(x), x) : x \in \mathbb{R}^2 \text{ and } G_{s,t}(x) \ge \zeta(|x|^2)\right\}.$$

We use an ellipse E_s to parameterize $\Sigma_{s,t}$ where $E_s \subset \mathbb{R}^2$ is given by

$$E_s := \{\hat{x} \in \mathbb{R}^2 : c_{11} b_{11} \hat{x}_1^2 + c_{22} b_{22} \hat{x}_2^2 + 2c_{12} b_{12} \hat{x}_1 \hat{x}_2 < \frac{1}{1+s}\}.$$

Then, the surface $\Sigma_{s,t}$ can be written as a map $E_s \to \Sigma_{s,t}$ such that

$$\Sigma_{s,t}(\hat{x}) := (G_{s,t}(\Phi_{s,t}(\hat{x})), \Phi_{s,t}(\hat{x}))$$

where $\Phi_{s,t} : E_s \to \mathbb{R}^2$ satisfies

$$\Phi_{s,t}(\hat{x}) = t\hat{x} + O(t^3).$$

We also use $\Sigma_t = \Sigma_{0,t}$ for short. We have the following result by the argument in [CW23].

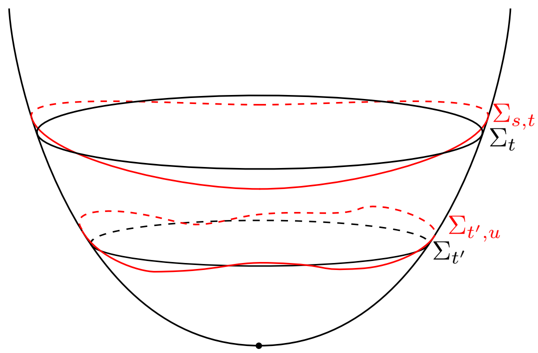

FIGURE 4.1. Construction of $\Sigma_{s,t}$ and $\Sigma_{t,u}$.

Proposition 4.4. *Suppose the metric g can be written as $g = g_0 + th + O(t^2)$ where g_0 is the constant metric defined in* (4.2) *and h is a bounded symmetric two-tensor. Then, we have*

$$\begin{aligned}\cos\gamma_{s,t}(\hat{x}) = &\cos\bar{\gamma}_{s,t}(\hat{x}) - 4st^2\left[\frac{(\hat{x}_1c_{11}b_{11} + \hat{x}_2c_{12}b_{12})^2}{a_{11}a^{33}} + \frac{(\hat{x}_1c_{12}b_{12} + \hat{x}_2c_{22}b_{22})^2}{a_{22}a^{33}}\right] \\ &+ t^2O(s^2) + A(\hat{x})t^3 + L(h)t^3 + O(t^4).\end{aligned}$$

for any $\hat{x} \in E_s$. Here, $A(\hat{x})$ is a bounded term (not related to t and h) which is also odd symmetric with respect to $\hat{x}$, $L(h)$ is a bounded term (not related to t) relying on h linearly.

Sketch of the proof. We use the same argument for Proposition 5.1 in [KY24] and also track the term t^3 to to get the following expansions

$$\begin{aligned}\cos\measuredangle_{g_0}(\Sigma_{s,t}, \partial M) &= 1 - \frac{2(\hat{x}_\beta c_{\alpha\beta}b_{\alpha\beta}(1+s))^2}{a_{\alpha\alpha}a^{33}}t^2 + A(\hat{x})t^3 + O(t^4) \\ \cos\bar{\gamma}_{s,t}(\hat{x}) &= 1 - 2t^2\left[(\hat{x}_1c_{11} + \hat{x}_2c_{12})^2 + (\hat{x}_1c_{12} + \hat{x}_2c_{22})^2\right] + A(\hat{x})t^3 + O(t^4).\end{aligned}$$

where we assume $c_{21} = c_{12}$. Together with the remaining computation for Proposition 5.1 in [KY24] and the Corollary 5.5 in [CW23] (see the proof for Corollary 5.17 in [CW23]), we can establish the result. □

As a corollary of Proposition 4.4, we can easily establish the following results for $\sin\gamma_t$,

$$\begin{aligned}\sin\gamma_t(\hat{x}) = &\sin\bar{\gamma}_t(\hat{x}) + O(t^2) \\ = &4t\sqrt{(\hat{x}_1c_{11} + \hat{x}_2c_{12})^2 + (\hat{x}_1c_{12} + \hat{x}_2c_{22})^2} + O(t^2), \end{aligned} \tag{4.3}$$

and the following proposition.

Proposition 4.5. *Suppose the conditions in Proposition 4.4 holds. Then, for any $s > 0$, we can find $t_0 > 0$ (might rely on s) such that for any $t < t_0$, we have*

$$\gamma_{s,t}(\hat{x}) > \bar{\gamma}_{s,t}(\hat{x})$$

for any $\hat{x} \in \partial E_s$.

We need to analyze the asymptotic behavior of mean curvature. We define the following mean curvatures:

$$H^g_{s,t}(\hat{x}) := \text{Mean curvature of } \Sigma_{s,t} \text{ at } \Sigma_{s,t}(\hat{x}) \text{ under metric } g,$$
$$H^g_{s,t,\partial M}(\hat{x}) := \text{Mean curvature of } \partial M \text{ at } (\varphi(|\Phi_{s,t}(\hat{x})|^2), \Phi_{s,t}(\hat{x})) \text{ under metric } g.$$

Using the same computation for Corollary 5.2 in [KY24], we have

Proposition 4.6. *Suppose the metric g can be written as $g = g_0 + th + O(t^2)$ where g_0 is a constant metric defined in (4.2), and h is a bounded symmetric two-tensor. Then, we have the following formula for the behavior of mean curvature*

$$H^g_t(\hat{x}) = H^g_{t,\partial M}(\hat{x}) - H^{\bar{g}}_{t,\partial M}(\hat{x}) - 2(c_{11} + c_{22}) + \frac{2B}{\sqrt{a_{11}a_{22}a^{33}}} + tL(\hat{x}) + O(t^2), \tag{4.4}$$

for any $\hat{x} \in E$. Here, we write $H^g_t = H^g_{0,t}$ and $H^g_{t,\partial M} = H^g_{0,t,\partial M}$ for short.

Now, we consider

$$H_0 := \lim_{t \to 0} H^g_t(\hat{x}),$$

which is well-defined by (4.4) (the limit does not depend on the choice of $\hat{x}$.)

We have two subcases to consider.

If $H_0 < \bar{h}(0)$, then we can use the continuation of $H^g_{s,t}$ with respect to s and t, together with Proposition 4.5, we can show the following results (cf. Proposition 5.10 in [CW23]).

Proposition 4.7. *Suppose the metric g can be written as $g = g_0 + th + O(t^2)$ where g_0 is a constant metric defined in (4.2), and h is a bounded symmetric two-tensor. If $H_0 < \bar{h}(0)$, we can choose some $s > 0, t > 0$ small such that $H^g_{s,t}(\hat{x}) > \bar{h}(\Sigma_{s,t}(\hat{x}))$ for any $\hat{x} \in \partial E_s$ and $\gamma_{s,t}(\hat{x}) < \bar{\gamma}_{s,t}(\hat{x})$ for each $\hat{x} \in \partial E_s$.*

Now, we focus on the case $H_0 = \bar{h}(0)$. In particular, it implies $a_{11} = a_{22} = 1$ and $H^g_{\partial M}(O) = H^{\bar{g}}_{\partial M}(O)$.

Then, we need to construct a foliation near O. We define the vector field $Y_t(\hat{x}) := \frac{\partial}{\partial t}\Sigma_t(\hat{x})$. Given $u \in C^{1,\alpha}(\bar{E}) \cap C^{2,\alpha}(E)$ where $E = E_0$, we can define the perturbation surface $\Sigma_{t,u}$ by

$$\Sigma_{t,u} := \left\{ \Sigma_{t + \frac{u}{\langle Y_t(\hat{x}), N_t(\hat{x}) \rangle}}(\hat{x}) : \hat{x} \in E \right\}$$

where $N_t(\hat{x})$ is the unit normal vector field of Σ_t.

We write $E = E_0$. Replacing u by $t^3 u$ and assuming that $u = O(1)$, we have

$$\begin{aligned}
\frac{H_{t,t^3u} - \bar{h}_{t,t^3u}}{t} &= -\Delta^E_t u + \frac{H_t - \bar{h}_t}{t} + O(t), \\
\frac{\cos\gamma_{t,t^3u} - \cos\bar{\gamma}_{t,t^3u}}{t^3} &= -4\sqrt{\zeta'(0)}\psi(0)|\hat{x}|\frac{\partial u}{\partial \nu^E_t} + (A_{\partial M}(\eta_t, \eta_t) - \cos\gamma_t A(\nu_t, \nu_t) \\
&\quad - \bar{A}_{\partial M}(\bar{\eta}_t, \bar{\eta}_t))u + \frac{\cos\gamma_t - \cos\bar{\gamma}_t}{t^3} + O(t),
\end{aligned}$$

where Δ_t^E denotes the Laplacian-Beltrami operator on E under the metric $\frac{1}{t^2}\Sigma_t^*(g)$, and ν_t^E is the unit normal vector field of ∂E under the metric $\frac{1}{t^2}\Sigma_t^*(g)$. Here, we have used (4.3).

By using the same argument for Proposition 5.27 in [CW23], together with the asymptotic behavior of mean curvature, for each $t \in (0, \varepsilon)$ sufficiently small, we can find $u_t(\cdot) = u(\cdot, t)$ such that the mean curvature $H_{t,t^3 u_t}$ is $\bar{h}_{t,t^3 u} + t\lambda(t)$ where $\lambda(t)$ is a function only depends on t, the contact angle $\gamma_{t,t^3 u_t} = \bar{\gamma}_{t,t^3 u_t}$, and u satisfies the following

$$\lim_{t\to 0}(u(\hat{x}, t) + u(-\hat{x}, t)) = 0$$

for any $\hat{x} \in E$. A finer analysis of λ_t will give $\lambda_t < 0$ for t sufficiently small (cf. Proposition 5.28 in [CW23]), and it leads to the following.

Proposition 4.8. *We can construct a surface Σ_- near O such that the mean curvature of Σ_- is not greater than $\bar{h}$ and it has prescribed contact angle $\bar{\gamma}$ with ∂M.*

Proof of Theorem 1.2. If $\rho(t)$ satisfies b) in Theorem 1.6, then we can use Proposition 4.7 or Proposition 4.8 depending on the value of H_0 to construct a barrier surface Σ_- with mean curvature not greater than $\bar{h}$ and prescribed contact angle $\bar{\gamma}$ with ∂M. Then, we can use Theorem 1.4 to extend the rigidity to all of M. $\square$

References

[Amb15] Lucas C. Ambrozio. Rigidity of area-minimizing free boundary surfaces in mean convex three-manifolds. *J. Geom. Anal.*, 25(2):1001–1017, 2015.

[BBHW24] Christian Bär, Simon Brendle, Bernhard Hanke, and Yipeng Wang. Scalar curvature rigidity of warped product metrics. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 20:Paper No. 035, 26, 2024.

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in Metric Geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2001.

[BBN10] Hubert Bray, Simon Brendle, and Andre Neves. Rigidity of area-minimizing two-spheres in three-manifolds. *Comm. Anal. Geom.*, 18(4):821–830, 2010.

[BM11] Simon Brendle and Fernando C. Marcques. Scalar Curvature Rigidity of Geodesic Balls in s^n. *Journal of Differential Geometry*, 88(3):379–394, 2011.

[CLZ24] Jianchun Chu, Man-Chun Lee, and Jintian Zhu. Llarull's theorem on punctured sphere with $L\infty$ metric. *arXiv: 2405.19724*, 2024.

[CW23] Xiaoxiang Chai and Gaoming Wang. Scalar curvature comparison of rotationally symmetric sets. *arXiv/2304.13152*, 2023.

[CW24] Xiaoxiang Chai and Xueyuan Wan. Scalar curvature rigidity of domains in a warped product. *arXiv:2407.10212 [math]*, 2024.

[CWXZ24] Simone Cecchini, Jinmin Wang, Zhizhang Xie, and Bo Zhu. Scalar curvature rigidity of the four-dimensional sphere. *arXiv: 2402.12633v2*, 2024.

[CZ24] Simone Cecchini and Rudolf Zeidler. Scalar and mean curvature comparison via the Dirac operator. *Geom. Topol.*, 28(3):1167–1212, 2024.

[DPM15] G. De Philippis and F. Maggi. Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. *Arch. Ration. Mech. Anal.*, 216(2):473–568, 2015.

[FCS80] Doris Fischer-Colbrie and Richard Schoen. The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. *Communications on Pure and Applied Mathematics*, 33(2):199–211, 1980.

[Gro21] Misha Gromov. Four Lectures on Scalar Curvature. *arXiv:1908.10612 [math]*, 2021.

[HKKZ] Sven Hirsch, Demetre Kazaras, Marcus Khuri, and Yiyue Zhang. Rigid comparison geometry for riemannian bands and open incomplete manifolds. (arXiv:2209.12857).

[HLS23] Yuhao Hu, Peng Liu, and Yuguang Shi. Rigidity of 3D spherical caps via μ-bubbles. *Pacific J. Math.*, 323(1):89–114, 2023.

[KY24] Dongyeong Ko and Xuan Yao. Scalar curvature comparison and rigidity of 3-dimensional weakly convex domains. *arXiv:2410.20548*, 2024.

[Li20] Chao Li. A polyhedron comparison theorem for 3-manifolds with positive scalar curvature. *Invent. Math.*, 219(1):1–37, 2020.

[Lis10] Mario Listing. Scalar curvature on compact symmetric spaces. *arXiv:1007.1832 [math]*, 2010.

[Lla98] Marcelo Llarull. Sharp estimates and the Dirac operator. *Math. Ann.*, 310(1):55–71, 1998.

[Lot21] John Lott. Index theory for scalar curvature on manifolds with boundary. *Proc. Amer. Math. Soc.*, 149(10):4451–4459, 2021.

[MO89] Maung Min-Oo. Scalar curvature rigidity of asymptotically hyperbolic spin manifolds. *Math. Ann.*, 285(4):527–539, 1989.

[MP21] Pengzi Miao and Annachiara Piubello. Mass and Riemannian Polyhedra. *arXiv:2101.02693 [gr-qc]*, 2021.

[RS97] Antonio Ros and Rabah Souam. On stability of capillary surfaces in a ball. *Pacific J. Math.*, 178(2):345–361, 1997.

[SY79a] R. Schoen and Shing Tung Yau. Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. *Ann. of Math. (2)*, 110(1):127–142, 1979.

[SY79b] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65(1):45–76, 1979.

[WWZ24] Jinmin Wang, Zhichao Wang, and Bo Zhu. Scalar-mean rigidity theorem for compact manifolds with boundary. *arXiv: 2409.14503*, 2024.

[WX23a] Jinmin Wang and Zhizhang Xie. Dihedral rigidity for submanifolds of warped product manifolds. *arXiv: 2303.13492*, 2023.

[WX23b] Jinmin Wang and Zhizhang Xie. Scalar curvature rigidity of degenerate warped product spaces. *arXiv: 2306.05413*, 2023.

[Ye91] Rugang Ye. Foliation by constant mean curvature spheres. *Pacific Journal of Mathematics*, 147(2):381–396, 1991.

Department of Mathematics, POSTECH, Pohang, Gyeongbuk, South Korea
Email address: `xxchai@kias.re.kr, xxchai@postech.ac.kr`

Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, China
Email address: `gmwang@tsinghua.edu.cn`