BAND WIDTH ESTIMATE WITH LOWER SPECTRAL CURVATURE BOUNDS

XIAOXIANG CHAI AND YUKAI SUN

Abstract. In this work, we use the warped μ-bubble method to study the consequences of a spectral curvature bound. In particular, with a lower spectral Ricci curvature bound and lower spectral scalar curvature bound, we show that the band width of a torical band is bounded above. We also obtain some rigidity results.

1. Introduction

A Riemannian *band* (M^n, g) is a compact connected orientable smooth manifold with a metric g and nonempty boundary ∂M such that

$$\partial M = \partial_- M \cup \partial_+ M, \quad \partial_- M \neq \emptyset, \quad \partial_+ M \neq \emptyset, \quad \partial_- M \cap \partial_+ M = \emptyset,$$

where $\partial_- M$ and $\partial_+ M$ are unions of boundary components. The *width* of (M, g) is defined as

$$\operatorname{width}_g(M) = \operatorname{dist}_g(\partial_+ M, \partial_- M),$$

where dist_g is the distance on M with respect to the metric g.

Let $\gamma > 0, \Lambda \in \mathbb{R}$ be two constants. On the Riemannian band (M^n, g), if there is a positive smooth function u on $M \backslash \partial M$ with

$$-\gamma \Delta_g u + \tfrac{1}{2} \operatorname{Sc}_g u \geq \Lambda u, \quad u|_{\partial M} = 0, \tag{1.1}$$

where Sc_g is the scalar curvature of g, then we say that (M, g) satisfies a lower *spectral scalar curvature bound*. Similarly, if

$$-\gamma \Delta_g u + 2 \operatorname{Ric}(x) u \geq \Lambda u, \quad u|_{\partial M} = 0, \tag{1.2}$$

where

$$\operatorname{Ric}(x) := \inf_{e \in T_x M, e \neq 0} \operatorname{Ric}(e, e)|e|^{-2}, \tag{1.3}$$

then we say that (M, g) satisfies a lower *spectral Ricci curvature bound*. We use the convention that if Ric is written without argument or with one argument, then it means (1.3), otherwise, it means the Ricci curvature tensor.

The spectral curvature bounds (1.1) and (1.2) are easily seen to be weaker than their pointwise counterparts. The effects of a positive scalar curvature bound on a band of the form $M^n = [-1, 1] \times T^{n-1}$ with a metric g. Here, we denote by T^n the n-dimensional torus. The Riemannian band $([-1, 1] \times T^{n-1}, g)$ is also called a *torical band* which was studied by Gromov [Gro18]. For the torical band, he proved that if the scalar curvature is bounded by $n(n-1)$, then the band width is bounded above by $\frac{2\pi}{n}$ for $2 \leq n \leq 7$. Zhu [Zhu21] proved that the width of a 3-dimensional torical band is less than or equal $\frac{\pi}{2}$ when the Ricci curvature is bounded below by 2, in particular, when the sectional curvature is bounded below by 1. There were other interesting developments such as [HKKZ25], [CZ24], [Rad23].

2020 *Mathematics Subject Classification.* 53C24, 53C21.

Key words and phrases. Band width, torical band, warped μ-bubble, scalar curvature rigidity, spectral curvature bound, Ricci curvature.

The spectral curvature bound, while weaker, has found its important applications in the stable Bernstein theorem [CL21, CLMS24, Maz24]; see also the related works of Gilles-Christian [CR21], Antonelli-Xu [AX24, APX24], Hong-Wang [HW25] for diameter estimates, volume comparison theorems, splitting theorems under lower spectral curvature.

A natural question is that what effects (1.1) and (1.2) have on the band width of a torical band. This question was raised by Gromov [Gro23, Section 6.1.2] (vaguely, in Item 1), and via several methods, Hirsch-Kazaras-Khuri-Zhang [HKKZ24] first proved a band width estimate of a torical band under a lower spectral scalar curvature bound.

In this work, we establish some band width estimates for torical bands using the warped μ-bubble methods under the spectral Ricci curvature bound and the spectral scalar curvature bound. We also have a generalization of [Zhu21] to the cases under the negative and zero Ricci curvature bound. It is not the main theme of our article, so we put this result in Appendix B.

Theorem 1.1. *For a Riemannian band $M^3 = [-1,1] \times T^2$ with the metric g, let u be a positive smooth function on $M \setminus \partial M$ with $u = 0$ on ∂M such that*

$$-\gamma \Delta_g u + 2 \operatorname{Ric} u \geq \Lambda u, \tag{1.4}$$

where $\Lambda > 0$, $0 < \gamma < 4$. Then

$$\operatorname{width}_g(M) \leq \frac{2\pi}{\sqrt{\Lambda(4-\gamma)}}.$$

By taking $\gamma = 0$ and $\Lambda = 4$, we see that this band width estimate generalizes the band width estimate by Zhu [Zhu21]. Our proof strategy is by warped μ-bubble methods. However, the equality case of this band width estimate seems difficult to characterize by warped μ-bubble methods. Nonetheless, we have the following Ricci curvature rigidity.

Theorem 1.2. *Let M, Λ, γ and u be given as in Theorem 1.1, and ζ be the coordinate representing the interval $[-1,1]$. If further $|\nabla \zeta| \leq 1/\ell_0$ with $\ell_0 = \pi/\sqrt{\Lambda(4-\gamma)}$, then the interior of (M^3, g) is isometric to the band $((-\ell_0, \ell_0) \times T^2, g_0)$ with*

$$g_0 = dt^2 + \phi_1(t)^2 ds_1^2 + \phi_2(t)^2 ds_2^2, \ t \in (-\ell_0, \ell_0)$$

where s_1, s_2 are arc-length parameters of the circles, and ϕ_1, ϕ_2 are positive functions given by

$$\phi_1(t) = \phi_1(0) \left(\cos(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}t)\right)^{\frac{1-\gamma/2}{2-\gamma/8}} \exp\left(\frac{\phi_1'(0)}{\phi_1(0)} \int_0^t \left(\cos(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}s)\right)^{-\frac{1-\gamma/2}{1-\gamma/4}} ds\right),$$

$$\phi_2(t) = \phi_2(0) \left(\cos(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}t)\right)^{\frac{1-\gamma/2}{2-\gamma/8}} \exp\left(\frac{\phi_2'(0)}{\phi_2(0)} \int_0^t \left(\cos(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}s)\right)^{-\frac{1-\gamma/2}{1-\gamma/4}} ds\right),$$

where $\phi_1'(0)\phi_1(0)^{-1} = -\phi_2'(0)\phi_2(0)^{-1}$ and $\frac{1}{2}(1-\frac{\gamma}{2})\Lambda \geq 2(\frac{\phi_1'}{\phi_1}(0))^2$.

Now we state an estimate of band width when the Ricci curvature is replaced by the scalar curvature.

Theorem 1.3. *For a Riemannian band $M^n = [-1,1] \times T^{n-1}$ ($3 \leq n \leq 7$) with the metric g, let u be a positive smooth function on $M \setminus \partial M$ with $u = 0$ on ∂M such that*

$$-\gamma \Delta_g u + \tfrac{1}{2} \operatorname{Sc}_g u \geq \Lambda u, \tag{1.5}$$

where $\Lambda > 0, 0 < \gamma < \frac{2n}{n-1}$. *Then*

$$\mathrm{width}_g(M) \le \frac{\pi}{\sqrt{\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}\Lambda}}.$$

As mentioned earlier, the above band width estimate was proven by Hirsch-Kazaras-Khuri-Zhang [HKKZ24] via three different methods. In particular, in three dimensions, they showed full rigidity (see [HKKZ24, Theorem 1.1]) via spacetime harmonic functions. And, in dimensions $3 \le n \le 7$ they showed for the case $\gamma = 1$ (see [HKKZ24, Theorem 1.3]) using the warped μ-bubble method. Here, we use the μ-bubble method as well, but with a wider range of γ. Moreover, we have the following scalar curvature rigidity.

Theorem 1.4. *For a Riemannian band $M^n = [-1,1] \times T^{n-1}$ ($3 \le n \le 7$) with the metric g, let u be a positive smooth function on $M \setminus \partial M$ with $u = 0$ on ∂M such that*

$$-\gamma\Delta_g u + \tfrac{1}{2}\,\mathrm{Sc}_g\, u \ge \Lambda u,$$

where $\Lambda > 0, 0 < \gamma < \frac{2n}{n-1}$. Let ζ represents the coordinate of the interval $[-1,1]$ and define

$$\ell_1 = \frac{\pi}{2\sqrt{\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}\Lambda}}$$

to be half of the band width earlier in Theorem 1.3. If further $|\nabla\zeta| \le 1/\ell_1$, then the interior of the band (M,g) is isometric to $(-\ell_1,\ell_1) \times T^{n-1}$ with the metric $g = dt^2 + \xi^2(t) g_{T^{n-1}}$ where $t \in (-\ell_1,\ell_1)$,

$$\xi(t) = \left(\cos\left(\sqrt{\frac{\Lambda(-n\gamma+\gamma+2n)}{4(n-1)+2\gamma(2-n)}}\ t\right)\right)^{2(2-\gamma)(-n\gamma+\gamma+2n)^{-1}},$$

and $g_{T^{n-1}}$ is a flat metric on the T^{n-1}.

The article is organized as follows:

In Section 2, we introduce our main technical tool warped μ-bubble and recall its basics. In Section 3, we show Theorems 1.1 and 1.2. The determination of the metric in Theorem 1.2 is deferred to Appendix A. In Section 4, we show Theorems 1.3 and 1.4. In Appendix B, we prove band width estimate under the pointwise Ricci curvature bound for a 3-dimensional torical band.

Acknowledgment. X.C. is partially supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. RS-2024-00337418) and an NRF grant No. 2022R1C1C1013511. Y.S. is partially funded by the National Key R&D Program of China Grant 2020YFA0712800.

2. Basics of warped μ-bubble

In this section, we collect the basics of the warped μ-bubble including the first and second variations, and the existence theorem.

2.1. **First and second variations of warped μ-bubble.** On the Riemannian band (M^n, g), let h be a smooth function on the interior of M, denoted by $\mathring{M}$ or on M. Choose a Caccioppoli set $\bar{\Omega} \subset M$ with smooth boundary $\partial\bar{\Omega} \subset \mathring{M}$ and

$\partial_- M \subset \bar{\Omega}$. Let $\partial^*\Omega$ denote the reduced boundary of the Caccioppoli set Ω and u be a positive smooth function on M. We consider the following functional

$$E(\Omega) = \int_{\partial^*\Omega} u^\gamma \mathcal{H}^{n-1} - \int (\chi_\Omega - \chi_{\bar{\Omega}}) h u^\gamma \mathcal{H}^n \tag{2.1}$$

for $\Omega \in \mathcal{C}$, where $\mathcal{C}$ is defined as

$$\mathcal{C} = \{\Omega : \text{ all Caccioppoli sets } \Omega \subset M \text{ and } \Omega \triangle \bar{\Omega} \Subset \mathring{M}\},$$

where $\mathcal{H}^n$ denotes n-dimensional Hausdorff measure. We usually omit the measure $\mathcal{H}^n$ in the integral. Then the warped μ-bubbles with the warped function u^γ are critical points of the functional $E(\Omega)$. If $u \equiv 1$ or $\gamma = 0$, then this becomes the usual μ-bubble.

Lemma 2.1 (First variation of warped μ-bubbles)**.** *Let Ω_t be a smooth 1-parameter family of region in $\mathcal{C}$ with $\Omega_0 = \Omega$ and normal speed ϕ at $t = 0$, then*

$$\tfrac{\mathrm{d}}{\mathrm{d}t} E(\Omega_t)|_{t=0} = \int_{\partial\Omega} (H + \gamma u^{-1} u_\nu - h) u^\gamma \phi,$$

where H is the mean curvature of $\partial\Omega$ and ν is the outwards pointing unit normal vector on $\partial\Omega$, $u_\nu = \nabla^g_\nu u$. In particular, a warped μ-bubble Ω satisfies

$$H = -\gamma u^{-1} u_\nu + h.$$

Lemma 2.2 (Second variation of warped μ-bubbles)**.** *Consider a warped μ-bubble Ω with $\partial\Omega = \Sigma$. Assume that Ω_t is a smooth 1-parameter family of regions in $\mathcal{C}$ with $\Omega_0 = \Omega$ and normal speed ϕ at $t = 0$. Then*

$$\begin{aligned}\tfrac{\mathrm{d}^2}{\mathrm{d}t^2} E(\Omega_t)|_{t=0} = \int_{\partial\Omega} & [-\Delta_{\partial\Omega}\phi - |A|^2\phi - \mathrm{Ric}(\nu,\nu)\phi - \gamma u^{-2} u_\nu^2 \phi \\ & + \gamma u^{-1}\phi(\Delta u - \Delta_{\partial\Omega} u - H u_\nu) - \gamma u^{-1}\langle \nabla_{\partial\Omega} u, \nabla_{\partial\Omega}\phi\rangle - h_\nu \phi] u^\gamma \phi.\end{aligned} \tag{2.2}$$

If $\frac{\mathrm{d}^2}{\mathrm{d}t^2} E(\Omega_t)|_{t=0} \geq 0$, we call that the warped μ-bubble Ω is stable. And we also call $\partial\Omega$ a stable hypersurface of prescribed mean curvature $-\gamma u^{-1} u_\nu + h$, or just a stable hypersurface when the underlying u and h are clear from the context.

Lemma 2.3. *The second variation for the functional $E(\Omega)$ in Lemma 2.2, that is,* (2.2) *can be rewritten as*

$$\begin{aligned}\tfrac{\mathrm{d}^2}{\mathrm{d}t^2} E(\Omega_t)|_{t=0} = & \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2 + \int_{\partial\Omega} [\gamma\psi\langle\nabla_{\partial\Omega} w, \nabla_{\partial\Omega}\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2 |\nabla_{\partial\Omega} w|^2] \\ & + \int_{\partial\Omega} [\gamma u^{-1}\Delta u - (|A|^2 + \mathrm{Ric}(\nu,\nu))]\psi^2 \\ & - \int_{\partial\Omega} [\gamma H w_\nu + h_\nu + \gamma w_\nu^2]\psi^2.\end{aligned}$$

where $w = \log u$ and ψ is any smooth function on $\partial\Omega$.

Proof. We do this by setting $\psi = \phi u^{\gamma/2}$ in Lemma 2.2. We collect the following three terms

$$\int_{\partial\Omega} [\Delta_{\partial\Omega}\phi + \gamma u^{-1}\phi\Delta_{\partial\Omega} u + \gamma u^{-1}\langle\nabla_{\partial\Omega} u, \nabla_{\partial\Omega}\phi\rangle] u^\gamma \phi,$$

and now let $\phi = u^{-\gamma/2}\psi$, we have

$$\begin{aligned}
&\int_{\partial\Omega}[\Delta_{\partial\Omega}\phi + \gamma u^{-1}\phi\Delta_{\partial\Omega}u + \gamma u^{-1}\langle\nabla_{\partial\Omega}u, \nabla_{\partial\Omega}\phi\rangle]u^{\gamma}\phi \\
=&\int_{\partial\Omega}[\psi\Delta_{\partial\Omega}u^{-\gamma/2} + 2\langle\nabla_{\partial\Omega}u^{-\gamma/2}, \nabla_{\partial\Omega}\psi\rangle + u^{-\gamma/2}\Delta_{\partial\Omega}\psi + \gamma u^{-1-\gamma/2}\psi\Delta_{\partial\Omega}u \\
&\quad + \gamma u^{-1}\langle\nabla_{\partial\Omega}u, \nabla_{\partial\Omega}(u^{-\gamma/2}\psi)\rangle]u^{\gamma/2}\psi \\
=&\int_{\partial\Omega}\psi\Delta_{\partial\Omega}\psi \\
&\quad + \int_{\partial\Omega}[u^{\gamma/2}\psi^2\Delta_{\partial\Omega}u^{-\gamma/2} + 2u^{\gamma/2}\psi\langle\nabla_{\partial\Omega}u^{-\gamma/2}, \nabla_{\partial\Omega}\psi\rangle \\
&\qquad + \gamma u^{-1}\psi^2\Delta_{\partial\Omega}u + \gamma u^{\gamma/2-1}\psi\langle\nabla_{\partial\Omega}u, \nabla_{\partial\Omega}(u^{-\gamma/2}\psi)\rangle] \\
=&-\int_{\partial\Omega}|\nabla_{\partial\Omega}\psi|^2 \\
&\quad + \int_{\partial\Omega}[-\langle\nabla_{\partial\Omega}(u^{\gamma/2}\psi^2), \nabla_{\partial\Omega}u^{-\gamma/2}\rangle + 2u^{\gamma/2}\psi\langle\nabla_{\partial\Omega}u^{-\gamma/2}, \nabla_{\partial\Omega}\psi\rangle \\
&\qquad - \gamma\langle\nabla_{\partial\Omega}(u^{-1}\psi^2), \nabla_{\partial\Omega}u\rangle + \gamma u^{\gamma/2-1}\psi\langle\nabla_{\partial\Omega}u, \nabla_{\partial\Omega}(u^{-\gamma/2}\psi)\rangle].
\end{aligned}$$

In the last line, we have used integration by parts on the two terms containing $\Delta_{\partial\Omega}u$ and the term containing $\Delta_{\partial\Omega}\psi$. By a direct calculation, we conclude that

$$\begin{aligned}
&\int_{\partial\Omega}[\Delta_{\partial\Omega}\phi + \gamma u^{-1}\phi\Delta_{\partial\Omega}u + \gamma u^{-1}\langle\nabla_{\partial\Omega}u, \nabla_{\partial\Omega}\phi\rangle]u^{\gamma}\phi \\
=&-\int_{\partial\Omega}|\nabla_{\partial\Omega}\psi|^2 + \int_{\partial\Omega}[-\gamma\psi\langle\nabla_{\partial\Omega}w, \nabla_{\partial\Omega}\psi\rangle + (\gamma - \tfrac{\gamma^2}{4})\psi^2|\nabla_{\partial\Omega}w|^2]
\end{aligned}$$

in (2.2). The remaining calculation is direct. □

2.2. **Existence theorem.** For $u \equiv 1$, from [Zhu21, Proposition 2.1] and [Gro23, Section 5.1], we have the following existence result of the μ-bubble.

Lemma 2.4 (Existence of μ-bubble)**.** *For a Riemannian band (M^n, g) with $3 \leq n \leq 7$, if either $h \in C^\infty(\overset{\circ}{M})$ with $h \to \pm\infty$ on $\partial_\mp M$, or $h \in C^\infty(M)$ with*

$$h|_{\partial_- M} > H_{\partial_- M}, \quad h|_{\partial_+ M} < H_{\partial_+ M}$$

where $H_{\partial_- M}$ is the mean curvature of $\partial_- M$ with respect to the inward normal and $H_{\partial_+ M}$ is the mean curvature of $\partial_+ M$ with respect to the outward normal. Then there exists an $\Omega \in \mathcal{C}$ with smooth boundary such that

$$E(\Omega) = \inf_{\Omega' \in \mathcal{C}} E(\Omega').$$

For the warped μ-bubble, i.e. $u \geq \delta > 0$ in equation (2.1), from [CL24, Proposition 12], we have

Lemma 2.5 (Existence of warped μ-bubble)**.** *For a Riemannian band (M^n, g) with $3 \leq n \leq 7$, if $h \in C^\infty(\overset{\circ}{M})$ with $h \to \pm\infty$ on $\partial_\mp M$ in the functional $E(\Omega)$. Then there exists an $\Omega \in \mathcal{C}$ with smooth boundary such that*

$$E(\Omega) = \inf_{\Omega' \in \mathcal{C}} E(\Omega').$$

3. Band width estimate with the spectral Ricci curvature condition

In this section, we prove the band width estimate with a spectral Ricci curvature bound (Theorem 1.1) and the scalar curvature rigidity Theorem 1.2. The proof of Theorem 1.2 is quite involved, and we divide the proof into several subsections.

3.1. **Proof of the band width estimate.** We first prove Theorem 1.1.

Proof of Theorem 1.1. We prove by contradiction. We assume that

$$\mathrm{width}_g(M) > 2\ell_0 := \frac{2\pi}{\sqrt{\Lambda(4-\gamma)}},$$

then there exists a small $\delta > 0$ such that the band

$$M \setminus \{x \in M : \ \mathrm{dist}_g(x, \partial M) < \delta\}.$$

We can perturb this band into a smooth band $\tilde{M}$ so that the band width of $\tilde{M}$ is greater than $2\ell_0$. By [CZ24, Lemma 7.2], there exists a smooth function $\chi : M \to [-\ell_0, \ell_0]$ on $\tilde{M}$ such that the Lipschitz constant $\mathrm{Lip}\,\chi < 1$, $\chi^{-1}(-\ell_0) = \partial_- \tilde{M}$ and $\chi^{-1}(\ell_0) = \partial_+ \tilde{M}$. Then the function

$$h := -\frac{2\sqrt{\Lambda}}{\sqrt{4-\gamma}} \tan\left(\frac{1}{2}\sqrt{(4-\gamma)\Lambda}\,\chi\right)$$

on $\tilde{M} \setminus \partial\tilde{M}$ tends to $-\infty$ as $x \to \partial_+\tilde{M}$ and to ∞ as $x \to \partial_-\tilde{M}$. We also have u bounded below by a positive number on $\tilde{M}$.

Using Lemma 2.5, we have a stable warped μ-bubble Ω such that $H = -\gamma u^{-1} u_\nu + h$ along $\partial\Omega$, then it follows from Lemma 2.3 that

$$\begin{aligned} 0 \leq & \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2 + \int_{\partial\Omega} [\gamma\psi\langle\nabla_{\partial\Omega} w, \nabla_{\partial\Omega}\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2 |\nabla_{\partial\Omega} w|^2] \\ & + \int_{\partial\Omega} [\gamma u^{-1}\Delta u - (|A|^2 + \mathrm{Ric}(\nu,\nu))]\psi^2 \\ & - \int_{\partial\Omega} [\gamma H w_\nu + h_\nu + \gamma w_\nu^2]\psi^2, \end{aligned}$$

where $w = \log u$ and ψ is any smooth function on $\partial\Omega$. We use [Zhu21, (5.2)]: let $\{e_1, e_2\}$ be a local orthonormal frame of $\partial\Omega$, then

$$\mathrm{Ric}(\nu,\nu) + |A|^2 = \mathrm{Ric}(e_1,e_1) + \mathrm{Ric}(e_2,e_2) + h^2 - \mathrm{Sc}_{\partial\Omega}. \tag{3.1}$$

Since

$$\gamma u^{-1}\Delta u - \mathrm{Ric}(e_1,e_1) - \mathrm{Ric}(e_2,e_2) \leq \gamma u^{-1}\Delta u - 2\,\mathrm{Ric} \leq -\Lambda$$

by (1.3) and (1.4), we have

$$\begin{aligned} & \int_{\partial\Omega} [\gamma u^{-1}\Delta u - (|A|^2 + \mathrm{Ric}(\nu,\nu))]\psi^2 \\ = & \int_{\partial\Omega} [\gamma u^{-1}\Delta u - (\mathrm{Ric}(e_1,e_1) + \mathrm{Ric}(e_2,e_2) + H^2 - \mathrm{Sc}_{\partial\Omega})]\psi^2 \\ = & \int_{\partial\Omega} (\mathrm{Sc}_{\partial\Omega} - H^2)\psi^2 + \int_{\partial\Omega} [\gamma u^{-1}\Delta u - \mathrm{Ric}(e_1,e_1) - \mathrm{Ric}(e_2,e_2)]\psi^2 \\ \leq & \int_{\partial\Omega} (\mathrm{Sc}_{\partial\Omega} - H^2 - \Lambda)\psi^2. \end{aligned}$$

Using the above and that $H = -\gamma w_\nu + h$ in (4.1), we arrive

$$
\begin{aligned}
0 \leq & \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2 + \int_{\partial\Omega} [\gamma\psi\langle\nabla_{\partial\Omega} w, \nabla_{\partial\Omega}\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_{\partial\Omega} w|^2] \\
& + \int_{\partial\Omega} [\mathrm{Sc}_{\partial\Omega} - (-\gamma w_\nu + h)^2 - \Lambda]\psi^2 \\
& - \int_{\partial\Omega} [\gamma(-\gamma w_\nu + h) w_\nu + h_\nu + \gamma w_\nu^2]\psi^2 \\
= & \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2 + \mathrm{Sc}_{\partial\Omega}\,\psi^2 + \int_{\partial\Omega} [\gamma\psi\langle\nabla_{\partial\Omega} w, \nabla_{\partial\Omega}\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_{\partial\Omega} w|^2] \\
& - \int_{\partial\Omega} [\gamma w_\nu^2 - \gamma h w_\nu + h^2 - |\nabla h| + \Lambda]\psi^2.
\end{aligned}
$$

By Cauchy-Schwarz inequality,

$$
\int_{\partial\Omega} [\gamma\psi\langle\nabla_{\partial\Omega} w, \nabla_{\partial\Omega}\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_{\partial\Omega} w|^2] \leq \tfrac{1}{4}\gamma(1 - \tfrac{\gamma}{4})^{-1} \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2, \tag{3.2}
$$

and

$$
\gamma w_\nu^2 - \gamma h w_\nu + h^2 = \gamma(w_\nu - \tfrac{1}{2}h)^2 + (1 - \tfrac{1}{4}\gamma)h^2 \geq (1 - \tfrac{1}{4}\gamma)h^2,
$$

which is positive by the assumption $0 < \gamma < 4$. Therefore,

$$
\begin{aligned}
0 \leq & (1 + \tfrac{1}{4}\gamma(1 - \gamma/4)^{-1}) \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2 + \int_{\partial\Omega} \mathrm{Sc}_{\partial\Omega}\,\psi^2 \\
& - \int_{\partial\Omega} \left[(1 - \tfrac{1}{4}\gamma)h^2 - |\nabla h| + \Lambda\right]\psi^2.
\end{aligned} \tag{3.3}
$$

Then

$$
\begin{aligned}
& \tfrac{4}{4-\gamma} \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2 + \int_{\partial\Omega} \mathrm{Sc}_{\partial\Omega}\,\psi^2 \\
> & \int_{\partial\Omega} \left[(1 - \tfrac{1}{4}\gamma)h^2 + h' + \Lambda\right]\psi^2 = 0.
\end{aligned}
$$

The operator $-\frac{4}{4-\gamma}\Delta_{\partial\Omega} + \mathrm{Sc}_{\partial\Omega}$ is positive on $\partial\Omega$ which is contraction. Since $\partial\Omega$ is homologous to T^2. □

3.2. **Existence of a non-trivial minimiser.**

From here in this section, we are devoted to the proof of Theorem 1.2. First, we construct a non-trivial minimiser to the weighted functional (2.1) by using an argument of J. Zhu [Zhu21].

Lemma 3.1. *Let M be as in Theorem 1.2, then there exists a stable critical point Ω of the functional* (2.1).

Proof. For convenience, we multiply ζ by ℓ_0 and still denote the new resulting function by ζ. Hence,

$$
\zeta(\partial_\pm M) = \pm\ell_0, \text{ and } |\nabla\zeta| \leq 1.
$$

We choose an odd, smooth function $\alpha(t) : [-\ell_0, \ell_0] \to \mathbb{R}$ such that $\alpha(t) > 0$ on $(0, \ell_0]$, $\alpha'(t) > 0$ on $[0, \frac{\ell_0}{2})$, $\alpha'(t) < 0$ on $(\frac{\ell_0}{2}, \ell_0]$.

Let η be the function that satisfies

$$
(1 - \tfrac{1}{4}\gamma)\eta^2 + \eta' + \Lambda = 0,\ \eta' < 0. \tag{3.4}
$$

We define $\eta_\varepsilon(t) = \eta(t + \varepsilon\alpha(t))$ on a sub-interval $(-T_\varepsilon, T_\varepsilon)$ of $[-\ell_0, \ell_0]$ such that $\eta_\varepsilon(t) \to \pm\infty$ as $t \to \pm T_\varepsilon$, and we easily find that

$$
(1 - \tfrac{1}{4}\gamma)\eta_\varepsilon^2 + \eta_\varepsilon' + \Lambda = \varepsilon\alpha'(t)\eta'(t + \varepsilon\alpha(t)),
$$

and

$$(1-\tfrac{1}{4}\gamma)\eta_\varepsilon^2+\eta_\varepsilon'+\Lambda>0 \text{ if } \tfrac{\ell_0}{2}<|t|\leq T_\varepsilon,$$
$$(1-\tfrac{1}{4}\gamma)\eta_\varepsilon^2+\eta_\varepsilon'+\Lambda<0 \text{ if } |t|<\tfrac{\ell_0}{2}. \tag{3.5}$$

Define $h_\varepsilon(x)=\eta_\varepsilon(\zeta(x))$, where $x\in M$. Fix a sufficiently small number $\varepsilon_0>0$. By Sard's lemma, $\zeta^{-1}(\pm T_\varepsilon)$ are both regular surfaces of M for almost all $\varepsilon\in(0,\varepsilon_0)$. We use such ε. Due to the condition $\eta_\varepsilon(t)\to\pm\infty$ as $t\to\pm T_\varepsilon$, and the existence result of Lemma 2.5, we can construct a stable warped μ-bubble $\Omega_\varepsilon\subset\zeta^{-1}((-T_\varepsilon,T_\varepsilon))$. Let $\Sigma_\varepsilon=\partial\Omega_\varepsilon$. Note Σ_ε and and h_ε satisfy the inequality

$$\begin{aligned}0\leq&(1+\tfrac{1}{4}\gamma(1-\gamma/4)^{-1})\int_{\partial\Omega_\varepsilon}|\nabla_{\partial\Omega_\varepsilon}\psi|^2+\int_{\partial\Omega}\mathrm{Sc}_{\partial\Omega_\varepsilon}\psi^2\\&-\int_{\partial\Omega_\varepsilon}\left[(1-\tfrac{1}{4}\gamma)h_\varepsilon^2+\nabla_{\nu_\varepsilon}h_\varepsilon+\Lambda\right]\psi^2,\end{aligned}$$

by the proof of Theorem 1.1. We claim that Σ_ε cannot lie entirely in the region

$$\{x\in M:\ \tfrac{\ell_0}{2}<|\zeta(x)|<T_\varepsilon\}.$$

That is, Σ_ε has a non-empty intersection with the compact set

$$K:=\{x\in M:|\zeta(x)|\leq\tfrac{\ell_0}{2}.\}$$

Indeed, if Σ_ε lies in this region, then by (3.5), taking $\psi=1$ on Σ_ε, applying the Gauss-Bonnet theorem, we see $4\pi\chi(\Sigma_\varepsilon)>0$ which would contradict the construction of Σ_ε.

By curvature estimates (see [ZZ20, Theorem 3.6]) and compactness, we can pick a sequence $\varepsilon_k\to0$ such that $\Sigma_{\varepsilon_k}:=\partial\Omega_{\varepsilon_k}$ converges locally and smoothly to a surface $\hat\Sigma$ such that $\hat\Sigma$ has non-empty intersection with K. We now use the subscript k on every geometric quantity associated with Σ_{ε_k}. We see that each Σ_{ε_k} satisfies the inequality

$$\begin{aligned}\int_{\Sigma_{\varepsilon_k}}((\nu_{\varepsilon_k},\nabla\zeta)-1)\eta_{\varepsilon_k}'\circ\zeta\leq&-\int_{\Sigma_{\varepsilon_k}}\left[(1-\tfrac{\gamma}{4})(\eta_{\varepsilon_k}\circ\zeta)^2+\Lambda+\eta_{\varepsilon_k}'\circ\zeta\right]\\=&-\varepsilon_k\int_{\Sigma_{\varepsilon_k}}\alpha'(\zeta)\eta'(\zeta+\varepsilon\alpha(\zeta))\\=&-\varepsilon_k\left(\int_{\Sigma_{\varepsilon_k}\cap K}+\int_{\Sigma_{\varepsilon_k}\backslash K}\right)\alpha'(\zeta)\eta'(\zeta+\varepsilon\alpha(\zeta))\\\leq&C\varepsilon_k\,\mathrm{Area}(\Sigma_{\varepsilon_k}\cap K)\to0,\end{aligned}$$

as $k\to\infty$ where we have dropped the negative part, that is, the integration on $\Sigma_{\varepsilon_k}\backslash K$. Note that $\eta_{\varepsilon_k}'<0$ for sufficiently large k and $\langle\nu_{\varepsilon_k},\nabla\zeta\rangle\leq1$. We can conclude that the limit of ν_{ε_k} exists and we denote it by ν. Hence $\mathrm{d}\zeta(\nu)=1$, and so $\nabla_\Sigma\zeta=0$ by $\mathrm{Lip}\,\zeta\leq1$. Therefore the surface Σ has a connected component contained in $\zeta^{-1}(t_0)$ for some $t_0\in[-\frac{\ell_0}{2},\frac{\ell_0}{2}]$. □

3.3. **Construction of foliation.** First, we show that Σ are infinitesimally rigid (see Lemma 3.3), then we apply the inverse function theorem to obtain a foliation near Σ. For convenience, we define a notion of infinitesimal rigidity.

Definition 3.2. *We call a surface Σ infinitesimally rigid if Σ satisfies all the following relations:*

$$\begin{aligned} \mathrm{Ric}(e_1,e_1) &= \mathrm{Ric}(e_2,e_2) = \mathrm{Ric}, \\ -\gamma u^{-1}\Delta u + 2\,\mathrm{Ric} &= \Lambda, \\ \mathrm{Sc}_\Sigma &= 0, \\ H &= -\gamma w_\nu + h, \\ (1-\tfrac{1}{4}\gamma)h^2 + h_\nu + \Lambda &= 0,\ h' = -|\nabla h|,\ \nabla t = \nu, \\ w = \log u, w_\nu &= \tfrac{1}{2}h,\ h \text{ are constants along } \Sigma, \\ \Sigma &\text{ is a level set of } h. \end{aligned}$$

Lemma 3.3. *Let Ω be constructed in Lemma 3.1, then $\Sigma = \partial\Omega$ is infinitesimally rigid.*

Proof. First, according to (3.3), Σ satisfies the following inequality

$$\begin{aligned} 0 \leq &(1+\tfrac{1}{4}\gamma(1-\gamma/4)^{-1})\int_\Sigma |\nabla_\Sigma\psi|^2 + \int_\Sigma \mathrm{Sc}_\Sigma\,\psi^2 \\ &- \int_\Sigma \left[(1-\tfrac{1}{4}\gamma)h^2 + h_\nu + \Lambda\right]\psi^2 =: \int_\Sigma \psi L\psi =: B(\psi,\psi) \end{aligned} \tag{3.6}$$

for all $\psi \in C^\infty(\Sigma)$. Here,

$$L = -(1+\tfrac{1}{4}\gamma(1-\gamma/4)^{-1})\Delta_\Sigma + \mathrm{Sc}_\Sigma - \left[(1-\tfrac{1}{4}\gamma)h^2 + h_\nu + \Lambda\right].$$

We only show $\mathrm{Sc}_\Sigma = 0$, $\nabla t = \nu$ and $(1-\frac{1}{4}\gamma)h^2 + h_\nu + \Lambda = 0$ using a now standard argument due to [FCS80]. The rest follows easily by tracing back the derivation of (3.6).

By taking $\psi = 1$ on Σ and applying the Gauss-Bonnet theorem in (3.6),

$$4\pi\chi(\Sigma) \geq \int_\Sigma \mathrm{Sc}_\Sigma \geq \int_\Sigma \left[(1-\tfrac{1}{4}\gamma)h^2 + h_\nu + \Lambda\right] \geq 0.$$

By construction, the Euler characteristic of Σ satisfies the bound $\chi(\Sigma) \leq 0$. And the last inequality follows from $h_\nu \geq \eta'|\nabla t| \geq \eta'$ and the ODE (3.4). Hence all inequalities in the above have to be equalities, and we obtain that $\chi(\Sigma) = 0$ and $\nabla t = \nu$. By considering $\psi = 1$ in $B(\psi,\psi) \geq 0$, we obtain that $B(1,1) = 0$. Hence, constants are the first eigenfunction of the operator L with zero as the eigenvalue, that is, $L1 = 0$. Hence $\mathrm{Sc}_\Sigma = 0$. □

Now we use the inverse function to construct a foliation near Σ.

Lemma 3.4. *Let Ω be a stable critical point of $E(\Omega)$ and $\Sigma = \partial\Omega$. We can construct a local foliation $\{\Sigma_t\}_{-\epsilon<t<\epsilon}$ such that Σ_t is of constant $H + \gamma w_\nu - h$, $\Sigma_0 = \Sigma$,*

(1) *each Σ_t is a graph over Σ with graph function ρ_t along outward unit normal vector field ν such that*
$$\left.\frac{\partial\rho_t}{\partial t}\right|_{t=0} = 1 \quad \text{and} \quad \frac{1}{\mathrm{vol}(\Sigma)}\int_\Sigma \rho_t dv = t;$$
(2) *$H_t + \gamma\omega_\nu - h$ is a constant function on Σ_t, where H_t is the mean curvature of Σ_t.*

Here, dv is the volume element of Σ.

Proof. Let

$$\hat{C}^\alpha(\Sigma) = \left\{\phi \in C^\alpha(\Sigma) : \int_\Sigma \phi dv = 0\right\}$$

for some $\alpha > 0$. Consider the map

$$\begin{aligned} \tilde{\Phi} : C^{2,\alpha}(\Sigma) &\to \hat{C}^{\alpha}(\Sigma) \times \mathbb{R}, \\ \rho &\mapsto \left(H_\rho + \gamma\omega_\nu - h - \frac{\int_\Sigma (H_\rho + \gamma\omega_\nu - h)\, dv}{\mathrm{vol}(\Sigma)}, \frac{\int_\Sigma \rho dv}{\mathrm{vol}(\Sigma)} \right), \end{aligned}$$

where H_ρ is the mean curvature of the graph over Σ with graph function ρ. We use the infinitesimal rigidity of Σ (see Definition 3.2) to calculate the first variation of $H - \gamma u^{-1} u_\nu - h$:

$$\begin{aligned} &\delta_{\phi\nu}(H - \gamma u^{-1} u_\nu - h) \\ = &- \Delta_\Sigma \phi - (\mathrm{Ric}(\nu,\nu) + |A|^2)\phi - \gamma u^{-2} u_\nu^2 \phi - \gamma u^{-1} \langle \nabla_\Sigma u, \nabla_\Sigma \phi \rangle \\ &+ \gamma u^{-1}(\Delta u - \Delta_\Sigma u - H u_\nu)\phi - h_\nu \phi \\ = &- \Delta_\Sigma \phi - \mathrm{Ric}(e_1, e_1)\phi - \mathrm{Ric}(e_2, e_2)\phi + operatornameSc_\Sigma \phi - H^2 \phi - \gamma w_\nu^2 \phi \\ &+ \gamma u^{-1} \phi \Delta u - \gamma u^{-1} \phi \Delta_\Sigma u - \gamma w_\nu (h - \gamma w_\nu)\phi - h_\nu \phi \\ = &- \Delta_\Sigma \phi - \Lambda \phi - (h - \gamma \tfrac{1}{2} h)^2 \phi - \gamma (\tfrac{1}{2} h)^2 \phi \\ &- \gamma (\tfrac{1}{2} h)(h - \tfrac{1}{2}\gamma h)\phi - h_\nu \phi \\ = &- \Delta_\Sigma \phi - ((1 - \tfrac{1}{4}\gamma) h^2 + h' + \Lambda)\phi \\ = &- \Delta_\Sigma \phi. \end{aligned}$$

Then, the linearization of $\tilde{\Phi}$ given by

$$D\tilde{\Phi}|_{\rho=0} : C^{2,\alpha}(\Sigma) \to \hat{C}^{\alpha}(\Sigma) \times \mathbb{R}, \quad \psi \mapsto \left(\Delta_\Sigma \psi, \frac{1}{\mathrm{vol}(\Sigma)} \int_\Sigma \psi dv \right),$$

at $\rho = 0$ is invertible. By the inverse function theorem, we can find a family of functions $\rho_t : \Sigma \to \mathbb{R}$ with $t \in (-\epsilon, \epsilon)$ with the following properties:

(1) The function ρ_t satisfies $\rho_0 \equiv 0$,

$$\left. \frac{\partial \rho_t}{\partial t} \right|_{t=0} \equiv 1, \quad \text{and} \quad \frac{1}{\mathrm{vol}(\Sigma)} \int_\Sigma \rho_t dv = t$$

(2) The graphs Σ_t over Σ with graph function ρ_t has $H + \gamma u^{-1} u_\nu - h$.

From (3.11), with the value of ϵ decreased a little bit, the speed $\partial_t \rho_t$ will be positive for $t \in (-\epsilon, \epsilon)$, from which it follows that the graphs $\{\Sigma_t\}_{t \in (-\epsilon,\epsilon)}$ form a foliation around Σ. □

Lemma 3.5. *Let $\tilde{\Omega}_t$ be the region enclosed by Σ_t and Σ, and*

$$\Omega_t = \begin{cases} \Omega \cup \tilde{\Omega}_t, & \text{if } 0 < t < \epsilon, \\ \Omega \setminus \tilde{\Omega}_t, & \text{if } -\epsilon < t < 0. \end{cases}$$

Then Ω_t is also a stable critical point of $E(\Omega)$.

Proof. The first step is to prove that $H_t + \gamma\omega_\nu - h = 0$ on Σ_t for $t \in (-\epsilon, \epsilon)$. Then the stable minimal property of Ω_t follows. We do that in Proposition 3.6 and 3.7. □

Proposition 3.6. *There exists a continuous function $\Psi(t)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\exp\left(\int_0^t \Psi(\tau) \mathrm{d}\tau\right) \tilde{H} \right) \leq 0 \tag{3.13}$$

where

$$\tilde{H} = H + \gamma w_\nu - h. \tag{3.14}$$

Proof. Let $\Phi : \Sigma \times (-\varepsilon, \varepsilon) \to M$ parametrize the foliation, $Y = \frac{\partial \Phi}{\partial t}$, $\phi_t = \langle Y, \nu_t \rangle$. Since we have shown that ϕ_0 is a constant. We can fix ε sufficiently small that $\phi_t > 0$ for all $t \in (-\varepsilon, \varepsilon)$. Recall that the first variation gives

$$\begin{aligned}
& - \tilde{H}'(t) \\
= & - \tfrac{\mathrm{d}}{\mathrm{d}t}(H_t + \gamma w_\nu - h) \\
= & \Delta_{\Sigma_t}\phi_t + (\mathrm{Ric}(\nu_t, \nu_t) + |A_t|^2)\phi_t + \gamma u^{-2} u_\nu^2 \phi_t \\
& - \gamma u^{-1}(\Delta u - \Delta_{\Sigma_t} u - H u_\nu)\phi_t + \gamma u^{-1}\langle \nabla_{\Sigma_t} u, \nabla_{\Sigma_t}\phi_t \rangle \\
& + h_{\nu_t}\phi_t.
\end{aligned}$$

Using the rewrite (3.1), the definition of Ric, (3.14), and the spectral bound (1.4), and with suitable grouping of terms, we see

$$\begin{aligned}
& - \tilde{H}'(t)\phi_t^{-1} \\
\geq & (\phi_t^{-1}\Delta_{\Sigma_t}\phi_t + \gamma u^{-1}\Delta_{\Sigma_t} u + \gamma u^{-1}\langle \nabla_{\Sigma_t} u, \nabla_{\Sigma_t}\phi_t \rangle \phi_t^{-1}) \\
& + \Lambda - \mathrm{Sc}_{\Sigma_t} + h_{\nu_t} + H^2 + \gamma w_\nu^2 + \gamma H w_\nu.
\end{aligned}$$

Inserting (3.14) in the above and using

$$\begin{aligned}
& (\phi_t^{-1}\Delta_{\Sigma_t}\phi_t + \gamma u^{-1}\Delta_{\Sigma_t} u + \gamma u^{-1}\langle \nabla_{\Sigma_t} u, \nabla_{\Sigma_t}\phi_t \rangle \phi_t^{-1}) \\
= & \operatorname{div}_{\Sigma_t}\left(\frac{\nabla_{\Sigma_t}\phi_t}{\phi_t} + \gamma \frac{\nabla_{\Sigma_t} u}{u} \right) + (1 - \tfrac{\gamma}{4}) \left| \tfrac{\nabla_{\Sigma_t}\phi_t}{\phi_t} \right|^2 + \gamma \left| \frac{\nabla_{\Sigma_t} u}{u} - \frac{\nabla_{\Sigma_t}\phi_t}{2\phi_t} \right|^2 \\
\geq & \operatorname{div}_{\Sigma_t}\left(\frac{\nabla_{\Sigma_t}\phi_t}{\phi_t} + \gamma \frac{\nabla_{\Sigma_t} u}{u} \right)
\end{aligned}$$

yields

$$\begin{aligned}
& - \tilde{H}'(t)\phi_t^{-1} \\
\geq & \operatorname{div}_{\Sigma_t}\left(\frac{\nabla_{\Sigma_t}\phi_t}{\phi_t} + \gamma \frac{\nabla_{\Sigma_t} u}{u} \right) \\
& + \Lambda - \mathrm{Sc}_{\Sigma_t} + h_{\nu_t} + \tilde{H}^2 + \tilde{H}(-\gamma w_\nu + 2h) + (\gamma w_\nu^2 + h^2 - \gamma h w_\nu).
\end{aligned}$$

Applying the Cauchy-Schwarz inequality

$$\gamma w_\nu^2 + h^2 - \gamma h w_\nu = \gamma(w_\nu - \tfrac{1}{2}h)^2 + (1 - \tfrac{1}{4}\gamma)h^2 \geq (1 - \tfrac{1}{4}\gamma)h^2$$

on the last term on the bracket and the trivial bound $\tilde{H}^2 \geq 0$, we obtain that

$$\begin{aligned}
& - \tilde{H}'(t)\phi_t^{-1} \\
\geq & \tilde{H}(-\gamma w_\nu + 2h) + \operatorname{div}_{\Sigma_t}\left(\frac{\nabla_{\Sigma_t}\phi_t}{\phi_t} + \gamma \frac{\nabla_{\Sigma_t} u}{u} \right) - \mathrm{Sc}_{\Sigma_t} \\
& + \left(\Lambda + h_{\nu_t} + (1 - \tfrac{1}{4}\gamma)h^2 \right).
\end{aligned}$$

Note that $\Lambda + h_{\nu_t} + (1 - \frac{1}{4}\gamma)h^2 \geq 0$, and finally,

$$-\tilde{H}'(t)\phi_t^{-1} \geq \tilde{H}(-\gamma w_\nu + 2h) + \operatorname{div}_{\Sigma_t}\left(\frac{\nabla_{\Sigma_t}\phi_t}{\phi_t} + \gamma \frac{\nabla_{\Sigma_t} u}{u} \right) - \mathrm{Sc}_{Sigma_t}.$$

We integrate the above on Σ_t and we find by the divergence theorem and Gauss-Bonnet theorem (Σ_t is a torus) that,

$$-\tilde{H}'(t) \int_{\Sigma_t} \tfrac{1}{\phi_t} \geq \tilde{H}(t) \int_{\Sigma_t} (-\gamma w_\nu + 2h).$$

We set $\Psi(t) = (\int_{\Sigma_t} \frac{1}{\phi_t})^{-1} \int_{\Sigma_t} (-\gamma w_{\nu_t} + 2h)$, then

$$\tilde{H}' + \Psi(t)\tilde{H} \leq 0.$$

By solving this inequality, we finish the proof of the lemma. □

Proposition 3.7. *Every Ω_t is a minimiser to $E(\Omega)$.*

Proof. Let ∂_t be the variation vector field of the foliation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$, and $\phi_t = \langle \partial_t, \nu_t\rangle$. Recall that the first variation E given in Lemma 2.1, and

$$\tfrac{\mathrm{d}}{\mathrm{d}t}E(\Omega_t) = \int_{\Sigma_t} \tilde{H}(t)u^\gamma\phi_t.$$

It follows from $\tilde{H}(0)=0$ and (3.13) that $\tilde{H}(t)\le 0$ for $t\ge 0$ and $\tilde{H}(t)\ge 0$ for $t\le 0$. So $E(\Omega_t)\le E(\Omega_0)$ for all $t\in(-\varepsilon,\varepsilon)$ and hence

$$E(\Omega_t) = E(\Omega_0)$$

for all $t\in(-\varepsilon,\varepsilon)$. Hence, all the foliation analysis on Ω_0 can be applied to Ω_t. Since M is connected, we can conclude that M is foliated by the boundaries $\partial\Omega_t$ of the minimisers of E. $\square$

3.4. **Rigidity.** The foliation process tells us that the band has a local isometric to $(-\epsilon,\epsilon)\times\Sigma$ with the metric $g=\phi_t(x)dt^2+g_t$ where g_t is a flat metric on Σ, $x\in\Sigma$ and ϕ_t is a function defined in the proof of Proposition 3.6. Since Ω_t is stable, by calculating in the proof of Lemma 3.4, there is

$$\Delta_{g_t}\phi_t(x) = 0.$$

I.e., ϕ_t is a constant function on Σ. Since

$$\begin{aligned} 2\operatorname{Ric} =& \Lambda + \gamma u^{-1}\Delta u,\\ =&\Lambda + \gamma u^{-1}\left(\Delta_\Sigma + H\partial_t + \partial_t^2 - \nabla_{\partial_t}\partial_t\right)u,\end{aligned}$$

we see H and u depend on t. Therefore, Ric only depends on t.

Recall that the local orthonormal basis $\{\nu=\partial_t, e_1, e_2\}$ on M is chosen such that

$$\operatorname{Ric}(\nu, e_\alpha) = 0,\quad \operatorname{Ric}(e_1,e_2)=0$$

for $\alpha=1,2$. By Gauss equation or Schoen-Yau's rewrite (4.2) and $\operatorname{Ric}(e_1,e_1)=\operatorname{Ric}(e_2,e_2)=\operatorname{Ric}$, definition of the scalar curvature $\operatorname{Sc}=\operatorname{Ric}(\nu,\nu)+2\operatorname{Ric}$, we have

$$\begin{aligned} 0 =& \operatorname{Sc} - 2\operatorname{Ric}(\nu,\nu) + H^2 - |A|^2\\ =& -\operatorname{Sc} + 4\operatorname{Ric} + H^2 - |A|^2.\end{aligned}$$

By the contracted Bianchi identify,

$$\begin{aligned} \frac{1}{2}e_1(\operatorname{Ric}(\nu,\nu)) =& \frac{1}{2}e_1(\operatorname{Sc})\\ =& \nabla_\nu \operatorname{Ric}(\nu,e_1) + \sum_{\alpha=1}^2 \nabla_{e_\alpha}\operatorname{Ric}(e_1,e_\alpha)\\ =& -\operatorname{Ric}(\nabla_\nu\nu, e_1) - \operatorname{Ric}(\nabla_\nu e_1,\nu) - \sum_{\alpha=1}^2 \operatorname{Ric}(\nabla_{e_\alpha}e_1, e_\alpha)\\ & - \sum_{\alpha=1}^2 \operatorname{Ric}(e_1, \nabla_{e_\alpha}e_\alpha)\\ =& 0.\end{aligned}$$

Similarly, $e_2(\operatorname{Ric}(\nu,\nu))=0$. Thus, $\operatorname{Ric}(\nu,\nu)$ only depends on t, and so does $|A|^2$. By

$$\begin{aligned} R(\nu,e_1,e_1,\nu) + R(e_2,e_1,e_1,e_2) =& \operatorname{Ric}(e_1,e_1) = \operatorname{Ric}\\ R(\nu,e_2,e_2,\nu) + R(e_2,e_1,e_1,e_2) =& \operatorname{Ric}(e_2,e_2) = \operatorname{Ric},\end{aligned}$$

we see $R(\nu,e_1,e_1,\nu) = R(\nu,e_2,e_2,\nu)$.

However $\mathrm{Ric}(\nu,\nu)=R(\nu,e_1,e_1,\nu)+R(\nu,e_2,e_2,\nu)$, then $R(\nu,e_1,e_1,\nu)$ and $R(\nu,e_2,e_2,\nu)$ also only depend on t. Again because the frame $\{e_1,e_2\}$ is arbitrary, we can conclude that $R(e_1,\nu,\nu,e_2)=0$. We also have the following equation

$$\begin{cases}\partial_t A - A^2 = -R(\cdot,\nu,\nu,\cdot);\\ \partial_t g_t = 2A.\end{cases}$$

Because of the above ODE, and that $R(e_1,\nu,\nu,e_1)=R(e_2,\nu,\nu,e_2)$ depends only on t and that $R(e_1,\nu,\nu,e_2)=0$, we can write the metric $g=dt^2+\phi_1^2ds_1^2+\phi_2^2ds_2^2$ for some positive functions ϕ_1 and ϕ_2 which only depend on t. The calculation of explicit forms of ϕ_1 and ϕ_2 are given in Appendix A.

4. Band width estimate with the spectral scalar curvature condition

In this section, we prove Theorems 1.3 and 1.4. The proof of Theorem 1.4 is similar to that of Theorem 1.2, in particular for three dimensions. However, we need to modify some of the arguments to adapt to higher dimensions.

4.1. **Proof of the band width estimate.**

Proof of Theorem 1.3. We prove by contradiction. We assume that

$$\mathrm{width}_g(M) > 2\ell_1 := \frac{\pi}{\sqrt{\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}\Lambda}},$$

then there exists a small $\delta>0$ such that the band

$$M\setminus\{x\in M:\ \mathrm{dist}_g(x,\partial M)<\delta\}.$$

We can perturb this band into a smooth band $\tilde M$ so that the band width of $\tilde M$ is greater than l. By [CZ24, Lemma 7.2], there exists a smooth function $\chi: M\to[\ell_1,\ell_1]$ on $\tilde M$ such that the Lipschitz constant $\mathrm{Lip}\,\chi<1$, $\chi^{-1}(-\ell_1)=\partial_-\tilde M$ and $\chi^{-1}(\ell_1)=\partial_+\tilde M$. Then the function

$$h := -\frac{\sqrt{\Lambda}}{\sqrt{\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}}}\tan\left(\sqrt{\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}\Lambda}\,\chi\right)$$

on $\tilde M\setminus\partial\tilde M$ tends to $-\infty$ as $x\to\partial_+\tilde M$ and to ∞ as $x\to\partial_-\tilde M$. We also have u bounded below by a positive number on $\tilde M$. By Lemma 2.5, there exists a stable warped μ-bubble Ω such that $H_{\partial\Omega}=-\gamma u^{-1}u_\nu+h$ and it follows from Lemma 2.3 that

$$\begin{aligned}0\le&\int_{\partial\Omega}|\nabla_{\partial\Omega}\psi|^2+\int_{\partial\Omega}[\gamma\psi\langle\nabla_{\partial\Omega}w,\nabla_{\partial\Omega}\psi\rangle+(\tfrac{\gamma^2}{4}-\gamma)\psi^2|\nabla_{\partial\Omega}w|^2]\\&+\int_{\partial\Omega}[\gamma u^{-1}\Delta u-(|A|^2+\mathrm{Ric}(\nu,\nu))]\psi^2\\&-\int_{\partial\Omega}[\gamma H w_\nu+h_\nu+\gamma w_\nu^2]\psi^2,\end{aligned}\tag{4.1}$$

for $w=\log u$ and ψ is any smooth function on $\partial\Omega$. Using the Schoen-Yau's trick of the Gauss equation, we have

$$|A|^2+\mathrm{Ric}(\nu,\nu)=\tfrac12(\mathrm{Sc}_g-\mathrm{Sc}_{\partial\Omega}+|A|^2+H^2)\tag{4.2}$$

$$\ge\tfrac12(\mathrm{Sc}_g-\mathrm{Sc}_{\partial\Omega}+\frac{n}{n-1}H^2).\tag{4.3}$$

Using the above, (1.5) and that $H_{\partial\Omega} = -\gamma w_\nu + h$ in (4.1), we arrive

$$\begin{aligned} 0 \leq & \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2 + \int_{\partial\Omega} [\gamma\psi\langle\nabla_{\partial\Omega} w, \nabla_{\partial\Omega}\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_{\partial\Omega} w|^2] \\ & + \int_{\partial\Omega} [\tfrac{1}{2}\operatorname{Sc}_{\partial\Omega} - \tfrac{n}{2(n-1)}(-\gamma w_\nu + h)^2 - \Lambda]\psi^2 \\ & - \int_{\partial\Omega} [\gamma(-\gamma w_\nu + h)w_\nu + h_\nu + \gamma w_\nu^2]\psi^2 \\ = & \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2 + \tfrac{1}{2}\operatorname{Sc}_{\partial\Omega}\psi^2 + \int_{\partial\Omega} [\gamma\psi\langle\nabla_{\partial\Omega} w, \nabla_{\partial\Omega}\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_{\partial\Omega} w|^2] \\ & - \int_{\partial\Omega} \left[(\tfrac{n}{2(n-1)}\gamma^2 - \gamma^2 + \gamma)w_\nu^2 - \tfrac{1}{n-1}\gamma h w_\nu + \tfrac{n}{2(n-1)}h^2 + h_\nu + \Lambda\right]\psi^2. \end{aligned} \tag{4.4}$$

Since $\gamma^2/4 - \gamma < 0$, so by Cauchy-Schwarz inequality,

$$\int_{\partial\Omega} [\gamma\psi\langle\nabla_{\partial\Omega} w, \nabla_{\partial\Omega}\psi\rangle + (\tfrac{\gamma^2}{4} - \gamma)\psi^2|\nabla_{\partial\Omega} w|^2] \leq \tfrac{1}{4}\gamma(1 - \tfrac{\gamma}{4})^{-1} \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2$$

as done in (3.2) and

$$\begin{aligned} & (\tfrac{n}{2(n-1)}\gamma^2 - \gamma^2 + \gamma)w_\nu^2 - \tfrac{1}{n-1}\gamma h w_\nu + \tfrac{n}{2(n-1)}h^2 \\ \geq & \left[-\frac{\gamma^2}{4(\tfrac{n}{2(n-1)}\gamma^2 - \gamma^2 + \gamma)(n-1)^2} + \tfrac{n}{2(n-1)}\right] h^2 \\ = & \frac{-n\gamma + \gamma + 2n}{4(\tfrac{n}{2(n-1)}\gamma - \gamma + 1)(n-1)} h^2 > 0. \end{aligned}$$

The positive sign in the last line is due to the assumption $0 < \gamma < \frac{2n}{n-1}$. Therefore,

$$\begin{aligned} 0 \leq & \tfrac{4}{4-\gamma} \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2 + \tfrac{1}{2} \int_{\partial\Omega} \operatorname{Sc}_{\partial\Omega}\psi^2 \\ & - \int_{\partial\Omega} \left[\frac{-n\gamma + \gamma + 2n}{4(n-1) + 2\gamma(2-n)} h^2 + h_\nu + \Lambda\right]\psi^2 \end{aligned} \tag{4.5}$$

by the estimate $h_\nu > -|\nabla h| = h'$. Then

$$\begin{aligned} \tfrac{4}{4-\gamma} \int_{\partial\Omega} |\nabla_{\partial\Omega}\psi|^2 + \tfrac{1}{2} \int_{\partial\Omega} \operatorname{Sc}_{\partial\Omega}\psi^2 > & \int_{\partial\Omega} \left[\frac{-n\gamma + \gamma + 2n}{4(n-1) + 2\gamma(2-n)} h^2 + h' + \Lambda\right]\psi^2 \\ = & 0. \end{aligned}$$

Because $\frac{4(n-2)}{n-3} > \frac{8}{4-\gamma}$ for $3 < n \leq 7$, then the operator $-\Delta_{\partial\Omega} + \frac{4(n-2)}{n-3}\operatorname{Sc}_{\partial\Omega}$ is positive. Hence, there is a positive scalar curvature metric on $\partial\Omega$ by the conformal change which is a contradiction. Since $\partial\Omega$ is homologous to T^{n-1} and there is no positive scalar curvature metric on T^{n-1}. If $n = 3$, since $\partial\Omega$ homological to T^2, the operator $-\frac{8}{4-\gamma}\Delta_{\partial\Omega} + \operatorname{Sc}_{\partial\Omega}$ is always non-positive which is also contradiction. □

4.2. **Existence of a non-trivial minimiser.** From here in this section, we prove our scalar curvature rigidity result (Theorem 1.4). We construct a non-trivial minimiser to the functional (2.1) by modifying an argument of J. Zhu [Zhu20].

Lemma 4.1. *Let M be as in Theorem 1.4, then there exists a stable critical point Ω of the weighted functional* (2.1) *which admits a map to T^{n-1} of non-zero degree.*

Proof. For convenience, we multiply ζ by ℓ_1 and still denote the resulting function by ζ, then

$$\zeta(\partial_\pm M) = \pm \frac{\pi}{2\sqrt{\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}\Lambda}} = \pm\ell_1, \ |\nabla\zeta| \leq 1.$$

We can choose an odd, smooth function $\alpha(t) : [-\ell_1, \ell_1] \to \mathbb{R}$ such that $\alpha(t) > 0$ on $(0, \ell_1]$, $\alpha'(t) > 0$ on $[0, \frac{\ell_1}{2})$, $\alpha'(t) < 0$ on $(\frac{\ell_1}{2}, \ell_1]$.

Let

$$\eta(t) := -\frac{\sqrt{\Lambda}}{\sqrt{\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}}} \tan\left(t\sqrt{\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}\Lambda}\right),$$

we see from the proof of Theorem 1.4 that η satisfies the ODE

$$\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}\eta^2 + \eta' + \Lambda = 0 \text{ and } \eta' < 0.$$

We define $\eta_\varepsilon(t) = \eta(t + \varepsilon\alpha(t))$ on a sub-interval $(-T_\varepsilon, T_\varepsilon)$ of $[-\ell_1, \ell_1]$ such that $\eta_\varepsilon(t) \to \pm\infty$ as $t \to \pm T_\varepsilon$, and we easily find that

$$\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}\eta_\varepsilon^2 + \eta_\varepsilon' + \Lambda = \varepsilon\alpha'(t)\eta'(t+\varepsilon\alpha(t)), \tag{4.6}$$

and

$$\begin{aligned} &\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}\eta_\varepsilon^2 + \eta_\varepsilon' + \Lambda > 0 \text{ if } \tfrac{\ell_1}{2} < |t| < T_\varepsilon, \\ &\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}\eta_\varepsilon^2 + \eta_\varepsilon' + \Lambda < 0 \text{ if } |t| < \tfrac{\ell_1}{2}. \end{aligned} \tag{4.7}$$

Define $h_\varepsilon(x) = \eta_\varepsilon(\zeta(x))$, where $x \in M$. Fix a sufficiently small number $\varepsilon_0 > 0$. By Sard's lemma, $\zeta^{-1}(\pm T_\varepsilon)$ are both regular surfaces of M for almost all $\varepsilon \in (0, \varepsilon_0)$. We use such ε. Due to the condition $\eta_\varepsilon(t) \to \pm\infty$ as $t \to \pm T_\varepsilon$, and the existence result of Lemma 2.5, we can construct a stable warped μ-bubble $\Omega_\varepsilon \subset \zeta^{-1}((-T_\varepsilon, T_\varepsilon))$. Let $\Sigma_\varepsilon = \partial\Omega_\varepsilon$, then Σ_ε is a stable surface Σ_ε of prescribed mean curvature $-\gamma u^{-1}u_\nu + h_\varepsilon$. Similar to the proof of Theorem 1.3, we see

$$\begin{aligned} 0 \leq &\tfrac{4}{4-\gamma}\int_{\partial\Omega_\varepsilon} |\nabla_{\partial\Omega_\varepsilon}\psi|^2 + \tfrac{1}{2}\int_{\partial\Omega} \mathrm{Sc}_{\partial\Omega_\varepsilon}\,\psi^2 \\ &- \int_{\partial\Omega_\varepsilon}\left[\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}h_\varepsilon^2 + \nabla_{\nu_\varepsilon}h_\varepsilon + \Lambda\right]\psi^2. \end{aligned}$$

For $n = 3$, we can argue similarly as Lemma 3.1 and finish the proof. Hence, hereafter, we focus only on the dimensions $3 < n \leq 7$.

We claim that Σ_ε cannot lie entirely in the region $\{x \in M : \frac{\ell_1}{2} < |\zeta(x)| < T_\varepsilon\}$. That is, Σ_ε has a non-empty intersection with the compact set

$$K := \{x \in M : |\zeta(x)| \leq \tfrac{\ell_1}{2}\}.$$

Let α be the positive constant given by

$$\frac{4(n-2)}{n-3}\alpha = \frac{8}{4-\gamma}. \tag{4.8}$$

By the range $0 < \gamma < \frac{2n}{n-1}$, we have $\alpha < 1$. Let

$$Q_\varepsilon = \frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}h_\varepsilon^2 + h_{\nu_\varepsilon} + \Lambda,$$

and

$$L_\varepsilon = -\tfrac{4}{4-\gamma}\Delta_{\partial\Omega_\varepsilon} + \tfrac{1}{2}\mathrm{Sc}_{\partial\Omega_\varepsilon} - Q_\varepsilon.$$

By (4.5), we see that the first eigenvalue $\lambda_{1,\varepsilon}$ of the operator L_ε is non-negative. Assume that $v = v_\varepsilon$ is the first eigenfunction, that is,

$$L_\varepsilon v = \lambda_{1,\varepsilon} v.$$

We consider the metric $\hat{g}_\varepsilon = (v^\alpha)^{\frac{4}{n-3}} g|_{\partial\Omega_\varepsilon}$, by the well known conformal change of the scalar curvature, it follows that the scalar curvature of the metric $\hat{g}$ is given by

$$\begin{aligned}(v^\alpha)^{\frac{n+1}{n-3}} \mathrm{Sc}_{\partial\Omega}(\hat{g}_\varepsilon) &= \mathrm{Sc}_{\partial\Omega} v^\alpha - \tfrac{4(n-2)}{n-3}\Delta v^\alpha \\ &= v^{\alpha-1}\left(\mathrm{Sc}_{\partial\Omega} v - \tfrac{4(n-2)}{n-3}\alpha\Delta v - \tfrac{4(n-2)}{n-3}\alpha(\alpha-1)v^{-1}|\nabla v|^2\right). \qquad (4.9)\end{aligned}$$

By the definition of α in (4.8) and that $\alpha < 1$, we see

$$\begin{aligned}& v^{\frac{4\alpha}{n-3}+1}\,\mathrm{Sc}_{\partial\Omega}(\hat{g}_\varepsilon) \\ \geq & \mathrm{Sc}_{\partial\Omega} v - \tfrac{4(n-2)}{n-3}\alpha\Delta v \\ = & \mathrm{Sc}_{\partial\Omega}\, v - \tfrac{8}{4-\gamma}\Delta_{\partial\Omega_\varepsilon} v \\ = & 2(\lambda_{1,\varepsilon} v + Q_\varepsilon v) \\ \geq & 2Q_\varepsilon v. \qquad (4.10)\end{aligned}$$

If Σ_ε lies entirely in $\{x \in M : \frac{\ell_1}{2} < \zeta(x) < T_\varepsilon\}$, then $Q_\varepsilon > 0$ by (4.7) and hence $\mathrm{Sc}_{\partial\Omega}(\hat{g}_\varepsilon) > 0$. This contradicts that Σ_ε admits a map of non-zero degree to T^{n-1} and that such $\partial\Omega$ does not admit a metric of positive scalar curvature. Hence, the claim is proved.

We need to ensure that $\hat{g}_\varepsilon$ has a limit as $\varepsilon \to 0$, so we assume that $\sup_{K\cap\Sigma_\varepsilon} v_\varepsilon = 1$. Then there exists some point $p_\varepsilon \in K \cap \Sigma_\varepsilon$ such that $v_\varepsilon(p_\varepsilon) = 1$ by compactness of K.

By curvature estimates [ZZ20, Theorem 3.6], there exists a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}}$ such that $\varepsilon_k \to 0$, Ω_{ε_k} converges to some smooth Ω local graphically and with multiplicity one as $k \to \infty$. Up to a subsequence, the locally graphical convergence with multiplicity one implies that the pointed $(\Sigma_{\varepsilon_k}, p_{\varepsilon_k})$ converges to $(\Sigma := \partial\Omega, p)$ in the pointed smooth topology. From the Harnack inequality for v_{ε_k}, we conclude that v_{ε_k} converges smoothly to some positive u with $u(p) = 1$. Assume that the limit metric of $\hat{g}_{\varepsilon_k}$ is $\hat{g}$, then $\hat{g} = u^{\frac{4\alpha}{n-3}} g|_\Sigma$.

From (4.6) and (4.10), we see that $\mathrm{Sc}_{\partial\Omega_{\varepsilon_k}}(\hat{g}_{\varepsilon_k}) \geq -C\varepsilon_k$ on the compact set K for some positive constant C. And outside K, we have already shown that $\mathrm{Sc}_{\partial\Omega_{\varepsilon_k}}(\hat{g}_{\varepsilon_k})$ is non-negative. Now we can apply [Zhu20, Proposition 3.2] to obtain that the limit $(\Sigma, \hat{g})$ is Ricci flat.

Considering the Ricci flatness of the limit $(\Sigma, \hat{g})$, (4.9) and (4.10), we see that u is a constant. Hence $u = 1$ on Σ by the fact that $u(p) = 1$.

Denote by ν_0 the limit of ν_{ε_k}. Considering again the limit of (4.10), we see that $Q_0 = 0$ (setting $\varepsilon = 0$). Therefore, we must have $\langle\nabla\zeta, \nu_0\rangle = 1$. Recall that $|\nabla\zeta| \leq 1$, we must have $\nabla\zeta = \nu_0$, which implies that Σ is a level set of ζ. And we conclude the proof. □

4.3. **Rigidity.** We can construct a foliation near Σ as well. The proof is similar to Lemma 3.4. Then we use the foliation to extend the rigidity of Σ to every leaf. It remains to calculate the metric.

Lemma 4.2. *Let Σ be constructed as in Lemma 4.1, then there exists a foliation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ near Σ such that $H + \gamma u^{-1}u_\nu - h$ is constant along Σ_t.*

Proof. Let

$$Q = \frac{-n\gamma + \gamma + 2n}{4(n-1) + 2\gamma(2-n)}h^2 + h_\nu + \Lambda,$$

and

$$L = -\tfrac{4}{4-\gamma}\Delta_\Sigma + \tfrac{1}{2}\mathrm{Sc}_\Sigma - Q.$$

We have $Q \geq 0$ along Σ.

Since Ω is a stable critical point of $E(\Omega)$, (4.5) holds for $\partial\Omega = \Sigma$. It follows then that the first eigenvalue λ_1 of L is non-negative. We set u to be the first eigenfunction, that is $Lu = \lambda_1 u$. We consider the metric $\hat{g} = (u^\alpha)^{\frac{4}{n-3}} g_{|\Sigma}$ on Σ, where α is defined in (4.8). By the conformal change formula for the scalar curvature,

$$(u^\alpha)^{\frac{n+1}{n-3}} \mathrm{Sc}_\Sigma(\hat{g}) = u^{\alpha-1}\left(\mathrm{Sc}_\Sigma u - \tfrac{4(n-2)}{n-3}\alpha\Delta u - \tfrac{4(n-2)}{n-3}\alpha(\alpha-1)u^{-1}|\nabla u|^2\right).$$

We see by $0<\alpha<1$ that

$$(u^\alpha)^{\frac{n+1}{n-3}} \mathrm{Sc}_\Sigma(\hat{g}) \ge u^{\alpha-1}(Lu+Qu) \ge 0.$$

However, Σ admits a map of non-zero degree to T^{n-1}, hence $\mathrm{Sc}_\Sigma(\hat{g}) = 0$. Then $\lambda_1 = 0$, and v is a constant. Hence

$$\mathrm{Sc}_\Sigma(g_{|\Sigma}) = 0. \tag{4.11}$$

Denote by A^0 the traceless part of the second fundamental form. Now tracing back the equalities leads to that

$$h_\nu = -|\nabla h|, \tag{4.12}$$

$$\nabla\phi = \nu, \tag{4.13}$$

$$A^0 = 0, \tag{4.14}$$

$$Q = \frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}h^2 + h_\nu + \Lambda = 0, \tag{4.15}$$

$$w_\nu = (\log u)_\nu = \frac{1}{2(n-1)\left(\frac{n}{2(n-1)}\gamma-\gamma+1\right)}h, \tag{4.16}$$

$$u \text{ is a constant along } \Sigma. \tag{4.17}$$

The above leads the linearization of $H+\gamma u^{-1}u_\nu - h$ to just $-\Delta_\Sigma$, and this is sufficient to show the existence of a foliation $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ near Σ. The rest of the proof is similar to the proof of Lemma 3.4 and we omit it. □

For convenience, we give the following definition.

Definition 4.3. *We call a hypersurface Σ in M infinitesimally rigid if it satisfies*

$$H = -\gamma u^{-1}u_\nu + h,$$

(4.11), (4.12), (4.13), (4.14), (4.15), (4.16) *and* (4.17).

We can determine the sign of $H+\gamma u^{-1}u_\nu - h$ for every Σ_t.

Lemma 4.4. *Let $\{\Sigma_t\}_{t\in(-\varepsilon,\varepsilon)}$ be constructed as in Lemma 4.2, then*

$$\tilde{H}_t := H + \gamma u^{-1}u_\nu - h \le 0 \tag{4.18}$$

for $t\in[0,\varepsilon)$ and $\tilde{H}_t \ge 0$ for $t\in(-\varepsilon,0)$.

Proof. It suffices to prove for $t\in(0,\varepsilon)$, and it is similar for $t\in(-\varepsilon,0)$. We do this by establishing an ordinary differential inequality for $\tilde{H}$.

Let ν_t be the unit normal of Σ_t pointing to the direction of the foliation and $\phi_t = \langle\partial_t,\nu_t\rangle$ is then a positive function on Σ_t. We set

$$\phi_t = u^{-\gamma/2}e^{\xi_t}, \tag{4.19}$$

$w = \log u$ and

$$P_t = -|A_{\Sigma_t}|^2 - \mathrm{Ric}(\nu_t,\nu_t) - \gamma w_\nu^2 + \gamma u^{-1}\Delta u - \gamma H w_\nu - h_\nu. \tag{4.20}$$

The first variation formula for $\tilde{H}_t$ (see Lemma 2.2) gives

$$\phi_t^{-1}\tilde{H}' = -\phi_t^{-1}\Delta_{\Sigma_t}\phi_t - \gamma u^{-1}\Delta_{\Sigma_t}u - \gamma\phi_t^{-1}\langle\nabla_{\Sigma_t}w,\nabla_{\Sigma_t}\phi_t\rangle + P_t.$$

Using (4.19) in the above and after a tedious calculation, we see

$$\phi_t^{-1}\tilde{H}' = -|\nabla_{\Sigma_t}\xi_t|^2 - \Delta_{\Sigma_t}\xi_t + (\tfrac{\gamma^2}{4} - \gamma)|\nabla_{\Sigma_t} w|^2 - \tfrac{\gamma}{2}\Delta_{\Sigma_t} w + P_t.$$

The estimate below follows

$$2(|A_{\Sigma_t}|^2 + \mathrm{Ric}(\nu_t,\nu_t)) \geq \mathrm{Sc}_g - \mathrm{Sc}_{\Sigma_t} + \tfrac{n}{n-1}(\tilde{H}_t - \gamma w_{\nu_t} + h)^2$$

from (4.3) and (4.18). Using (4.20), the above, (1.5) in (4.18), and with suitable regrouping of terms, we obtain

$$\begin{aligned}\phi_t^{-1}\tilde{H}' \leq & -|\nabla_{\Sigma_t}\xi_t|^2 - \Delta_{\Sigma_t}\xi_t + (\tfrac{\gamma^2}{4} - \gamma)|\nabla_{\Sigma_t} w|^2 + \tfrac{\gamma}{2}\Delta_{\Sigma_t} w \\ & - \Lambda + \tfrac{1}{2}\mathrm{Sc}_{\Sigma_t} - \tfrac{n}{2(n-1)}(-\gamma w_\nu + h)^2 - [\gamma(-\gamma w_\nu + h) w_{\nu_t} + h_{\nu_t} + \gamma w_{\nu_t}^2] \\ & - \tfrac{n}{2(n-1)}\tilde{H}_t^2 - \tilde{H}_t(\tfrac{n}{n-1}(-\gamma w_{\nu_t} + h) + \gamma w_{\nu_t}) \\ =: & L(t) - \tfrac{n}{2(n-1)}\tilde{H}_t^2 - \tilde{H}_t(\tfrac{n}{n-1}(-\gamma w_{\nu_t} + h) + \gamma w_{\nu_t}).\end{aligned}$$

We set $q(t) = \phi_t(\frac{n}{n-1}(-\gamma w_{\nu_t} + h) + \gamma w_{\nu_t})$. Then

$$\tilde{H}' + q(t)\tilde{H}_t \leq \phi_t L(t).$$

For any positive smooth function φ on Σ_t, we have

$$\tilde{H}' + \frac{\int_{\Sigma_t} q(t)\varphi}{\int_{\Sigma_t}\varphi}\tilde{H}_t \leq \frac{\int_{\Sigma_t}\varphi\phi_t L(t)}{\int_{\Sigma_t}\varphi},$$

since $\tilde{H}_t$ depends only on t. Let $\Phi(t) = \frac{\int_{\Sigma_t} q(t)\varphi}{\int_{\Sigma_t}\varphi}$. Then

$$\left(\tilde{H}_t e^{\int_0^t \Phi(s)\mathrm{d}s}\right)' = (\tilde{H}' + \Phi(t)\tilde{H}_t)e^{\int_0^t \Phi(s)\mathrm{d}s} \leq e^{\int_0^t \Phi(s)\mathrm{d}s}\frac{\int_{\Sigma_t}\varphi\phi_t L(t)}{\int_{\Sigma_t}\varphi}.$$

The proof is concluded if we can show that there exists a positive function φ such that $\int_{\Sigma_t}\varphi\phi_t L(t) \leq 0$ for each $t \in (0,\varepsilon)$. Indeed, $\tilde{H}_t e^{\int_0^t \Phi(s)\mathrm{d}s}$ is a non-increasing function on $[0,\varepsilon)$, so

$$\tilde{H}_t e^{\int_0^t \Phi(s)\mathrm{d}s} \leq \tilde{H}_0 = 0 \text{ for } t \in [0,\varepsilon).$$

Now show that there exists a positive function φ such that $\int_{\Sigma_t}\varphi\phi_t L(t) \leq 0$ for each $t \in [0,\varepsilon)$. We assume otherwise: $\int_{\Sigma_t}\varphi\phi_t L(t) > 0$ for some $t \in [0,\varepsilon)$ and any $\varphi > 0$. Let $\psi^2 = \varphi\phi_t$. Then

$$\int_{\Sigma_t} L(t)\psi^2 > 0 \tag{4.21}$$

for any non-zero positive smooth function $\psi \in C^\infty(\Sigma_t)$.

We analyze only terms involving $\frac{\gamma}{2}\Delta_{\Sigma_t} w$ and $|\nabla_{\Sigma_t}\xi|^2 + \Delta_{\Sigma_t}\xi_t$. First, we see that

$$\begin{aligned}&(|\nabla_{\Sigma_t}\xi|^2 + \Delta_{\Sigma_t}\xi_t)\psi^2 \\ =&|\nabla_{\Sigma_t}\xi|^2\psi^2 - 2\langle\nabla_{\Sigma_t}\psi, \psi\nabla_{\Sigma_t}\xi\rangle + \mathrm{div}_{\Sigma_t}(\psi^2\nabla_{\Sigma_t}\xi_t) \\ \geq& -|\nabla_{\Sigma_t}\psi|^2 + \mathrm{div}_{\Sigma_t}(\psi^2\nabla_{\Sigma_t}\xi_t).\end{aligned}$$

So it follows from integration by parts that

$$-\int_{\Sigma_t}(|\nabla_{\Sigma_t}\xi|^2 + \Delta_{\Sigma_t}\xi_t)\psi^2 \leq \int_{\Sigma_t}|\nabla_{\Sigma_t}\psi|^2.$$

And also

$$\tfrac{\gamma}{2}\int_{\Sigma_t}\psi^2\Delta_{\Sigma_t} w = -\int_{\Sigma_t}\gamma\psi\langle\nabla_{\Sigma_t} w, \nabla_{\Sigma_t}\psi\rangle.$$

Putting the above two formulas into (4.21), we see

$$\begin{aligned}0<&\int_{\Sigma_t}|\nabla_{\Sigma_t}\psi|^2+\int_{\Sigma_t}[\gamma\psi\langle\nabla_{\Sigma_t}w,\nabla_{\Sigma_t}\psi\rangle+(\tfrac{\gamma^2}{4}-\gamma)\psi^2|\nabla_{\Sigma_t}w|^2]\\&+\int_{\Sigma_t}[\tfrac12\mathrm{Sc}_{\Sigma_t}-\tfrac{n}{2(n-1)}(-\gamma w_\nu+h)^2-\Lambda]\psi^2\\&-\int_{\Sigma_t}[\gamma(-\gamma w_\nu+h)w_\nu+h_\nu+\gamma w_\nu^2]\psi^2.\end{aligned}$$

This is the same form as the inequality that appeared in the Theorem 1.3 (see the lines near (4.4)). Note also that the inequality is strict.

Now we argue similarly as in Theorem 1.3. The inequality would imply that Σ_t admits a metric of positive scalar curvature. However, this is not possible because Σ_t admits a map of non-zero degree to T^{n-1}. Here, the proof is complete. □

Now we give the proof of Theorem 1.4.

Proof of Theorem 1.4. First, we define Ω_t by setting

$$\Omega_t=\begin{cases}\Omega\cup\tilde\Omega_t, & \text{if } 0<t<\epsilon,\\ \Omega\setminus\tilde\Omega_t, & \text{if } -\epsilon<t<0.\end{cases}$$

from the foliation $\{\Sigma_t\}_{t\in(\epsilon,\epsilon)}$ constructed in Lemma 4.2. Here $\tilde\Omega_t$ is the region bounded by Σ_t and Σ. We show that every Ω_t is a stable critical point of the functional E. It is enough to show for $t>0$, and it is similar for $t<0$.

Let ∂_t be the variation vector field of the foliation and $\phi_t=\langle\nu_t,\partial_t\rangle$, by the first variation of E (Lemma 2.1),

$$\tfrac{\mathrm{d}}{\mathrm{d}t}E(\Omega_t)=\int_{\Sigma_t}\tilde H_t u^\gamma\phi_t\le 0.$$

where we used $\tilde H_t\le 0$ from Lemma 4.4. Therefore, $E(\Omega_t)\le E(\Omega_0)$ and every Ω_t is a minimiser to the functional E.

Hence, every Σ_t is infinitesimally rigid. We use the definition of infinitesimal rigidity to finish the proof. By umbilicity, we find that the metric g is a warped product $\mathrm{d}t^2+\xi(t)^2 g_{T^{n-1}}$ for some flat metric on T^{n-1}, so the mean curvature of the t-level set is given by

$$H=(n-1)\tfrac{\xi'}{\xi}.$$

By the fact that $\nabla\phi=\nu$, we see that ϕ differs t by some constant. We can solve the ordinary differential equation (4.15) (note that $h_\nu=h'$) and obtain

$$h(t)=-\frac{\sqrt{\Lambda}}{\sqrt{\frac{-n\gamma+\gamma+2n}{4(n-1)+2\gamma(2-n)}}}\tan\left(\sqrt{\tfrac{\Lambda(-n\gamma+\gamma+2n)}{4(n-1)+2\gamma(2-n)}}(t+c)\right)$$

in terms of t for some constant c. Using (4.16), we find an expression for $(\log u)_\nu$ and hence an expression for the mean curvature

$$H=-\gamma u^{-1}u_\nu+h=\tfrac{(2-\gamma)(n-1)}{2(n-1)+\gamma(2-n)}h=(n-1)\xi'/\xi,$$

so

$$\xi'\xi^{-1}=\tfrac{2-\gamma}{\gamma(2-n)+2(n-1)}h.$$

We easily find that

$$\xi(t)=\left(\cos(\sqrt{\tfrac{\Lambda(-n\gamma+\gamma+2n)}{4(n-1)+2\gamma(2-n)}}(t+c))\right)^{2(2-\gamma)(-n\gamma+\gamma+2n)^{-1}}.$$

By connectedness, we can extend the rigidity to the whole of M and considering the boundary values of ζ (i.e. t), we see that $c=0$. Hence the theorem is proved. □

Appendix A. Determine the metric in the spectral Ricci case

Here, we give details of how to determine the metric in Theorem 1.2. As shown earlier in Section 3, the metric g is a doubly warped product

$$\mathrm{d}t^2 + \phi(t)^2 \mathrm{d}s_1^2 + \varphi(t)^2 \mathrm{d}s_2^2. \tag{A.1}$$

Here, to avoid using subscripts, we have set $\phi = \phi_1$ and $\varphi = \phi_2$. The function $h = h(t)$ satisfies the ODE

$$(1 - \gamma/4)h^2 + h' + \Lambda = 0. \tag{A.2}$$

The function u is constant on each t-level set,

$$w' = (\log u)' = \frac{1}{2}h, \tag{A.3}$$

and the mean curvature $H(t)$ of the level set is given by

$$H = -\gamma u^{-1}u' + h = (-\frac{1}{2}\gamma + 1)h. \tag{A.4}$$

Let $\{e_1, e_2\}$ be an arbitrary orthonormal frame of the t-level set, then

$$\mathrm{Ric}(e_1, e_1) = \mathrm{Ric}(e_2, e_2) \leq \mathrm{Ric}(\partial_t, \partial_t). \tag{A.5}$$

We put our calculation in the form of a lemma.

Lemma A.1. *Let g be a metric of the form* (A.1) *which satisfies* (A.2), (A.3), (A.4) *and* (A.5), *then*

(A.6)
$$\phi(t) = \phi(0) \left(\cos(\sqrt{\Lambda(1 - \tfrac{\gamma}{4})}t)\right)^{\frac{1-\gamma/2}{2-\gamma/8}} \exp\left(\frac{\phi'(0)}{\phi(0)} \int_0^t \left(\cos(\sqrt{\Lambda(1 - \tfrac{\gamma}{4})}s)\right)^{-\frac{1-\gamma/2}{1-\gamma/4}} ds\right)$$

(A.7)
$$\varphi(t) = \varphi(0) \left(\cos(\sqrt{\Lambda(1 - \tfrac{\gamma}{4})}t)\right)^{\frac{1-\gamma/2}{2-\gamma/8}} \exp\left(\frac{\varphi'(0)}{\varphi(0)} \int_0^t \left(\cos(\sqrt{\Lambda(1 - \tfrac{\gamma}{4})}s)\right)^{-\frac{1-\gamma/2}{1-\gamma/4}} ds\right),$$

where $\frac{\phi'(0)}{\phi(0)} = -\frac{\varphi'(0)}{\varphi(0)}$ *and*

$$\tfrac{1}{2}(1 - \tfrac{\gamma}{2})\Lambda \geq 2\left(\frac{\phi'(0)}{\phi(0)}\right)^2. \tag{A.8}$$

We find it convenient to record the non-zero components of the Ricci curvatures of the metric g given in (A.1) here. They are given by

$$\mathrm{Ric}(\partial_t, \partial_t) = -(\phi\varphi)^{-1}(\phi\varphi'' + \varphi''\phi), \tag{A.9}$$

$$\mathrm{Ric}(\phi(t)^{-1}\frac{\partial}{\partial s_1}, \phi(t)^{-1}\frac{\partial}{\partial s_1}) = -(\phi\varphi)^{-1}(\varphi\phi'' + \varphi'\phi'), \tag{A.10}$$

$$\mathrm{Ric}(\varphi(t)^{-1}\frac{\partial}{\partial s_2}, \varphi(t)^{-1}\frac{\partial}{\partial s_2}) = -(\phi\varphi)^{-1}(\varphi''\phi + \varphi'\phi'). \tag{A.11}$$

Note that

$$\{\phi(t)^{-1}\frac{\partial}{\partial s_1}, \varphi(t)^{-1}\frac{\partial}{\partial s_2}, \partial_t\}$$

is an orthonormal frame with respect to the metric g.

Proof of Lemma A.1. The solution $h = h(t)$ to the ODE (A.2) is given by

$$h(t) = -\sqrt{\frac{\Lambda}{1 - \frac{1}{4}\gamma}} \tan(\sqrt{\Lambda(1 - \tfrac{\gamma}{4})}t).$$

It follows from (A.3) that

$$u = \cos\left(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}t\right)^{\frac{1}{2(1-\gamma/4)}}.$$

Choosing $e_1 = \phi(t)^{-1}\frac{\partial}{\partial s_1}$ and $e_2 = \varphi(t)^{-1}\frac{\partial}{\partial s_2}$ in (A.5), and using (A.10) and (A.11) leads to

$$\phi''/\phi = \varphi''/\varphi. \tag{A.12}$$

The mean curvature of the t-slice by (A.1) and (A.4) is

$$H = \phi'/\phi + \varphi'/\varphi = (-\tfrac{1}{2}\gamma + 1)h. \tag{A.13}$$

By (A.12) and (A.13),

$$\begin{aligned}\left(\frac{\phi'}{\phi} - \frac{\varphi'}{\varphi}\right)' &= \frac{\phi''}{\phi} - \left(\frac{\phi'}{\phi}\right)^2 - \frac{\varphi''}{\varphi} + \left(\frac{\varphi'}{\varphi}\right)^2 \\ &= \left(\frac{\varphi'}{\varphi} - \frac{\phi'}{\phi}\right)\left(\frac{\varphi'}{\varphi} + \frac{\phi'}{\phi}\right) \\ &= -\left(\frac{\phi'}{\phi} - \frac{\varphi'}{\varphi}\right)(-\tfrac{1}{2}\gamma + 1)h\end{aligned}$$

Then,

$$\frac{\phi'}{\phi} - \frac{\varphi'}{\varphi} = 2\beta\left(\cos(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}t)\right)^{-\frac{1-\gamma/2}{1-\gamma/4}}$$

for some constant β. We have by the above and (A.13) that

$$\begin{aligned}\frac{\phi'}{\phi} &= \frac{\phi'(0)}{\phi(0)}\left(\cos(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}t)\right)^{-\frac{1-\gamma/2}{1-\gamma/4}} + \tfrac{1}{2}(-\tfrac{1}{2}\gamma + 1)h, \\ \frac{\varphi'}{\varphi} &= \frac{\varphi'(0)}{\varphi(0)}\left(\cos(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}t)\right)^{-\frac{1-\gamma/2}{1-\gamma/4}} + \tfrac{1}{2}(-\tfrac{1}{2}\gamma + 1)h, \\ \beta &= \frac{\phi'(0)}{\phi(0)} = -\frac{\varphi'(0)}{\varphi(0)} \\ \left(\log(\tfrac{\phi}{\varphi})\right)' &= 2\frac{\phi'(0)}{\phi(0)}\left(\cos(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}t)\right)^{-\frac{1-\gamma/2}{1-\gamma/4}}.\end{aligned}$$

Then

$$\frac{\phi}{\varphi} = \frac{\phi(0)}{\varphi(0)} e^{2\frac{\phi'(0)}{\phi(0)}\int_0^t \left(\cos(\sqrt{\Lambda(1-\frac{\gamma}{4})}s)\right)^{-\frac{1-\gamma/2}{1-\gamma/4}} ds}.$$

We also have,

$$\phi\varphi = \phi(0)\varphi(0)\left(\cos(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}t)\right)^{\frac{1-\gamma/2}{1-\gamma/4}}.$$

From here, we easily obtain (A.6) and (A.7). It remains to derive a consequence of the inequality in (A.5). This would put a restriction on $\beta = \phi'(0)/\phi(0)$.

By choosing $e_1 = \phi(t)^{-1}\frac{\partial}{\partial s_1}$ and $e_2 = \varphi(t)^{-1}\frac{\partial}{\partial s_2}$ in (A.5) again, and using (A.9), (A.10) and (A.11), we have

$$\frac{\phi''}{\phi} = \frac{\varphi''}{\varphi} \geq \frac{\varphi'\phi'}{\varphi\phi}.$$

We express ϕ''/ϕ in terms of ϕ'/ϕ and φ'/φ as follows:

$$\phi''/\phi = \varphi''/\varphi = \tfrac{1}{2}[(\tfrac{\phi'}{\phi} + \tfrac{\varphi'}{\varphi})' + (\tfrac{\phi'}{\phi})^2 + (\tfrac{\varphi'}{\varphi})^2].$$

Since $\phi'/\phi + \varphi'/\varphi = 0$ at $t = 0$, we can set

$$\begin{aligned}a =& \beta\cos\left(\sqrt{\Lambda(1-\gamma/4)}t\right)^{-\frac{1-\gamma/2}{1-\gamma/4}} \\ d =& \tfrac{1}{2}(-\tfrac{\gamma}{2} + 1)h\end{aligned}$$

$$\phi'/\phi = a + d, \varphi'/\varphi = -a + d,$$

Then

$$\begin{aligned}
0 \geq & \frac{\phi''}{\phi} - \frac{\varphi'\phi'}{\varphi\phi} \\
= & \tfrac{1}{2}[(\tfrac{\phi'}{\phi} + \tfrac{\varphi'}{\varphi})' + (\tfrac{\phi'}{\phi})^2 + (\tfrac{\varphi'}{\varphi})^2] - \frac{\varphi'\phi'}{\varphi\phi} \\
= & \tfrac{1}{2}(\tfrac{\phi'}{\phi} + \tfrac{\varphi'}{\varphi})' + \tfrac{1}{2}(a+d)^2 + \tfrac{1}{2}(-a+d)^2 - (a+d)(-a+d) \\
= & \tfrac{1}{2}(-\tfrac{\gamma}{2} + 1)h' + 2a^2 \\
= & -\tfrac{1}{2}(-\tfrac{\gamma}{2} + 1)\Lambda \cos(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}t)^{-2} \\
& + 2\beta^2 \cos(\sqrt{\Lambda(1-\tfrac{\gamma}{4})}t)^{-2\frac{1-\gamma/2}{1-\gamma/4}}
\end{aligned}$$

which gives the restriction (A.8). □

Appendix B. Band width estimate with Ricci curvature bound

We are able to generalize the band width estimate [Zhu21, Theorem 5.1] to zero and negative Ricci curvature lower bound. See also [HKKZ25, Main Theorem C] for a method using spacetime harmonic functions.

Theorem B.1. *For each $\kappa \in \{-1, 0, 1\}$, we define $\eta = \eta_\kappa$ to be the function which satisfies*

$$\eta' + \eta^2 + 4\kappa = 0, \ \eta' < 0.$$

If $(M = [-1, 1] \times T^2, g)$ is a torical band with Ricci curvature bound $\mathrm{Ric} \geq 2\kappa$, and there exist constants $0 < t_- < t_+$ for $\kappa \in \{0, -1\}$ and $0 < t_- < t_+ < \frac{\pi}{2}$ for $\kappa = 1$ such that the $H_{\partial_+ M} \geq \eta(t_+)$ and $H_{\partial_- M} \leq \eta(t_-)$, then

$$\mathrm{width}(M, g) \leq t_+ - t_-.$$

Here the mean curvature of $H_{\partial_- M}$ is computed about the unit inner normal vector and $H_{\partial_+ M}$ is computed about the unit outer normal vector. Equality is achieved if and only if (M, g) is isometric to the models $([t_-, t_+] \times T^2, g)$ where g is given by

$$g = \begin{cases} \mathrm{d}t^2 + \sin^{1+c} t \cos^{1-c} t \mathrm{d}s_1^2 + \sin^{1-c} t \cos^{1+c} t \mathrm{d}s_2^2 & \kappa = 1; \\ \mathrm{d}t^2 + t^{1+c}\mathrm{d}s_1^2 + t^{1-c}\mathrm{d}s_2^2 & \kappa = 0; \\ \mathrm{d}t^2 + \sinh^{1+c} t \cosh^{1-c} t \mathrm{d}s_1^2 + \sinh^{1-c} t \cosh^{1+c} t \mathrm{d}s_2^2 & \kappa = -1, \end{cases} \tag{B.1}$$

where $0 \leq c \leq 1$.

Remark B.2. The metric (B.1) for $\kappa = 1$ is easily checked to be consistent with [HKKZ25, (2.8)], while Zhu [Zhu21] only gave the metric (B.1) with $c = 0, 1$ for [Zhu21, Theorem 5.1]. We obtain a band width estimate when we replace the Ricci curvature with the sectional curvature.

Proposition B.3. *Let (M, g) be a band given in Theorem B.1 and d be a Lipschitz function such that*

$$|\nabla d| \leq 1 \ \text{in } M, \ \text{and } d(\partial_\pm M) = t_\pm \ \text{along } \partial_\pm M.$$

Assume that Σ is a stable $C^{2,\alpha}$-surface ($0 < \alpha < 1$) of mean curvature $\eta \circ d$ and homologous to $\partial_- M$. Then (M, g) is isometric to the models given in Theorem B.1. In particular, the band with is equal to $t_+ - t_-$.

Proof. Some parts of the proof already appeared in [Zhu21]. Here, we make use of the arguments already written down for Theorem 1.2. We set $h = \eta \circ d$, since Σ is a stable surface of mean curvature $\eta \circ d$, then Σ satisfies the inequality (3.3) with $\gamma = 0$ and $\Lambda = 2\kappa$. By following the proof of Theorem 1.2, (note that we do not need Lemma 3.1 because we assumed apriori the existence of a stable surface) we see that (M, g) is isometric to a doubly warped product in the form of (A.1) and $H_{\partial_\pm M} = \eta(t_\pm)$. The rest is a calculation similar to Lemma A.1. The mean curvature of the t-level set is

$$H = h(t) = \phi'/\phi + \varphi'/\varphi,$$

which satisfies the ODE

$$h^2 + h' + \Lambda = 0. \tag{B.2}$$

And let $\{e_1, e_2\}$ be an arbitrary orthonormal frame of the t-level set, then

$$0 = \mathrm{Ric}(e_1, e_1) = \mathrm{Ric}(e_2, e_2) \leq \mathrm{Ric}(\partial_t, \partial_t).$$

We calculate for $\kappa = 0$ as an example and leave the calculations for the case $\kappa = \pm 1$. When $\kappa = 0$, $\Lambda = 0$ as well and so

$$h^2 + h' = 0, \ h' < 0$$

by (B.2). This leads to

$$h = \tfrac{1}{t+t_0} = \phi'/\phi + \varphi'/\varphi$$

for some constant t_0. We set $t_0 = 0$ and adjust the range of t later. Setting $e_1 = \phi(t)^{-1}\frac{\partial}{\partial s_1}$ and $e_2 = \varphi(t)^{-1}\frac{\partial}{\partial s_2}$ in $\mathrm{Ric}(e_1, e_1) = \mathrm{Ric}(e_2, e_2) = 0$ and using (A.10), (A.11) gives $\varphi''/\varphi = \phi''/\phi$. So

$$\begin{aligned}\left(\tfrac{\phi'}{\phi} - \tfrac{\varphi'}{\varphi}\right)' &= \tfrac{\phi''}{\phi} - \left(\tfrac{\phi'}{\phi}\right)^2 - \tfrac{\varphi''}{\varphi} + \left(\tfrac{\varphi'}{\varphi}\right)^2 \\ &= \left(\tfrac{\varphi'}{\varphi} - \tfrac{\phi'}{\phi}\right)\left(\tfrac{\varphi'}{\varphi} + \tfrac{\phi'}{\phi}\right) \\ &= -\left(\tfrac{\phi'}{\phi} - \tfrac{\varphi'}{\varphi}\right) h,\end{aligned}$$

then

$$\tfrac{\phi'}{\phi} - \tfrac{\varphi'}{\varphi} = c\tfrac{1}{t}.$$

By symmetry of ϕ and φ, we set $c \geq 0$. So $\phi'/\phi = \frac{1+c}{2t}, \varphi'/\varphi = \frac{1-c}{2t}$, which gives

$$\phi = c_1 t^{\frac{1+c}{2}}, \varphi = c_2 t^{\frac{1-c}{2}}$$

for some positive constants c_1 and c_2. We can set $c_1 = c_2 = 1$ by adjusting the length of the circles represented by s_1 and s_2. By (A.9), $\mathrm{Ric}(\partial_t, \partial_t) = \frac{1-c^2}{2t^2}$. Another condition that $\mathrm{Ric}(\partial_t, \partial_t) \geq 0$ gives $c \leq 1$. So the metric takes

$$\mathrm{d}t^2 + t^{1+c}\mathrm{d}s_1^2 + t^{1-c}\mathrm{d}s_2^2.$$

We see from the mean curvatures of $H_{\partial_\pm M}$ that $t \in [t_-, t_+]$. In particular, the band width is $t_+ - t_-$. □

Now we use an argument inspired by [EGM21, AM09] to show that there exists indeed a stable surface which would conclude the proof of Theorem B.1.

Proof of Theorem B.1. We define $d = \min\{\mathrm{dist}_g(\partial_- M, x) + t_-, t_+\}$ and $h = \eta \circ d$. We prove the band width estimate by contradiction: we assume that

$$\mathrm{width}(M, g) > t_+ - t_-. \tag{B.3}$$

Then

$$\begin{aligned} H_{\partial_+ M} \geq& \eta(t_+) = h(\partial_+ M), \\ H_{\partial_- M} \leq& \eta(t_-) = h(\partial_- M).\end{aligned}$$

First, we note that none of $\partial_\pm M$ is stable (see Lemma 2.2, take $u \equiv 1$). Indeed, if $\partial_- M$ or $\partial_+ M$ is stable, then the band width is $t_+ - t_-$ by Proposition B.3 contradicting our assumption (B.3).

Claim: $\partial_- M$ can be perturbed graphically to a surface Σ_- which satisfies

$$H_{\Sigma_-} < h \text{ along } \Sigma_-.$$

Since $\partial_- M$ is not stable, then either (1) $H_{\partial_- M} \lneqq h$ or (2) $\partial_- M$ satisfies $H_{\partial_- M} - h \equiv 0$ but is not stable. In case (1), we can run the following mean curvature type flow $F(t, \cdot) : S \to M$ with

$$\partial_t F = -(H - h)\nu$$

starting from $\partial_- M$ for a very short time. Here, S is a manifold diffeomorphic to $\partial_- M$ serving as a coordinate space. By the strong maximum principle, for all sufficiently small enough t, $H - h < 0$ (see [AM09, Lemma 5.2] and h is Lipschitz, which is enough for their argument). It suffices to put $\Sigma_- = F(t, \cdot)$ with a small t.

It remains to prove the claim in case (2) when $H_{\partial_- M} - h = 0$ along $\partial_- M$ but $\partial_- M$ is not stable. Let u be the first eigenfunction of

$$L = -\Delta_{\partial_- M} - (\mathrm{Ric}(\nu, \nu) + |A|^2) - \nabla_\nu h.$$

We can choose $u > 0$. Then $Lu < 0$. Let X be a vector field such that $X = u\nu$ along $\partial_- M$. The first variation of $H - h$ is $\delta_X (H - h) = Lu < 0$. Hence, there exists a Σ_- such that $H_{\Sigma_-} - h < 0$.

We can argue similarly that there exists a perturbation Σ_+ of $\partial_+ M$ such that $H_{\Sigma_+} - h > 0$. (Observe that it is convenient to reverse the choice of normal, the sign of the mean curvature. And the roles of $\partial_\pm M$ are switched.)

Because of $\Sigma_\pm$, we can apply Theorem 2.4, and there exists a stable surface Σ. (We remark that Σ is $C^{2,\alpha}$ by [Duz93, Theorem 4.4] and sufficient for our purpose.) By Proposition B.3, the band width is again $t_+ - t_-$ contradicting (B.3).

Now we turn to the rigidity case. If $\mathrm{width}(M, g) = t_+ - t_-$, then by the above proof, either one of $\partial_\pm M$ is stable or there exists a stable surface disjoint from $\partial_\pm M$. In any case, we can apply Proposition B.3, and the proof is done.

□

References

[AM09] Lars Andersson and Jan Metzger. The area of horizons and the trapped region. *Comm. Math. Phys.*, 290(3):941–972, 2009.

[APX24] Gioacchino Antonelli, Marco Pozzetta, and Kai Xu. A sharp spectral splitting theorem. *arXiv:2412.12707*, 2024.

[AX24] Gioacchino Antonelli and Kai Xu. New spectral bishop-gromov and bonnet-myers theorems and applications to isoperimetry. *arXiv: 2405.08918*, 2024.

[CL21] Otis Chodosh and Chao Li. Stable minimal hypersurfaces in four-dimensions. *arXiv: 2108.11462*, 2021.

[CL24] Otis Chodosh and Chao Li. Generalized soap bubbles and the topology of manifolds with positive scalar curvature. *Ann. of Math. (2)*, 199(2):707–740, 2024.

[CLMS24] Otis Chodosh, Chao Li, Paul Minter, and Douglas Stryker. Stable minimal hypersurfaces in $\mathbf{R}^5$. *arXiv: 2401.01492v1*, 2024.

[CR21] Gilles Carron and Christian Rose. Geometric and spectral estimates based on spectral Ricci curvature assumptions. *J. Reine Angew. Math.*, 772:121–145, 2021.

[CZ24] Simone Cecchini and Rudolf Zeidler. Scalar and mean curvature comparison via the Dirac operator. *Geom. Topol.*, 28(3):1167–1212, 2024.

[Duz93] Frank Duzaar. On the existence of surfaces with prescribed mean curvature and boundary in higher dimensions. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 10(2):191–214, 1993.

[EGM21] Michael Eichmair, Gregory J. Galloway, and Abraão Mendes. Initial data rigidity results. *Comm. Math. Phys.*, 386(1):253–268, 2021.

[FCS80] Doris Fischer-Colbrie and Richard Schoen. The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. *Communications on Pure and Applied Mathematics*, 33(2):199–211, 1980.

[Gro18] Misha Gromov. Metric inequalities with scalar curvature. *Geom. Funct. Anal.*, 28(3):645–726, 2018.

[Gro23] Misha Gromov. Four lectures on scalar curvature. In *Perspectives in scalar curvature. Vol. 1*, pages 1–514. World Sci. Publ., Hackensack, NJ, [2023] ©2023.

[HKKZ24] Sven Hirsch, Demetre Kazaras, Marcus Khuri, and Yiyue Zhang. Spectral torical band inequalities and generalizations of the Schoen–Yau black hole existence theorem. *International Mathematics Research Notices*, 2024(4):3139–3175, 2024.

[HKKZ25] Sven Hirsch, Demetre Kazaras, Marcus Khuri, and Yiyue Zhang. Rigid comparison geometry for Riemannian bands and open incomplete manifolds. *Mathematische Annalen*, 391(2):2587–2652, 2025.

[HW25] Han Hong and Gaoming Wang. A splitting theorem for manifolds with spectral nonnegative Ricci curvature and mean-convex boundary. *arXiv:2503.07009*, 2025.

[Maz24] Laurent Mazet. Stable minimal hypersurfaces in $\mathbb{R}^6$. *arXiv:2405.14676*, 2024.

[Rad23] Daniel Rade. Scalar and mean curvature comparison via μ-bubbles. *Calculus of Variations and Partial Differential Equations*, 62(7):187, 2023.

[Zhu20] Jintian Zhu. Rigidity results for complete manifolds with nonnegative scalar curvature. *arXiv: 2008.07028*, August 2020.

[Zhu21] Jintian Zhu. Width estimate and doubly warped product. *Trans. Amer. Math. Soc.*, 374(2):1497–1511, 2021.

[ZZ20] Xin Zhou and Jonathan J. Zhu. Existence of hypersurfaces with prescribed mean curvature I – generic min-max. *Cambridge Journal of Mathematics*, 8(2):311–362, 2020.

Department of Mathematics, POSTECH, Pohang, Gyeongbuk, South Korea
Email address: xxchai@kias.re.kr, xxchai@postech.ac.kr

Key Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China
Email address: sunyukai@math.pku.edu.cn