SCALAR CURVATURE RIGIDITY OF ROTATIONALLY SYMMETRIC DOMAINS IN A WARPED PRODUCT

XIAOXIANG CHAI AND GAOMING WANG

Abstract. A warped product with a spherical factor and a logarithmically concave warping function satisfies a scalar curvature rigidity of Llarull type. We develop a scalar curvature rigidity of Llarull type for doamins rotationally symmetric with respect to the warping direction in a three dimensional spherical warped product. We identify the condition analogous to the logarithmic concavity of the warping function on the boundary.

1. Introduction

Llarull [Lla98] proved a scalar curvature rigidity theorem for the standard n-spheres. A distinct feature of this scalar curvature rigidity for spheres comparing to that of torus [SY79a], the Euclidean space [SY79b] and the hyperbolic space [MO89] is the requirement of a metric comparison $g \geqslant \bar{g}$. The assumptions were weakened by Listing [Lis10]. Llarull's theorem is as follows.

Theorem 1.1. *Let g be a smooth metric on the n-sphere with the metric comparison $g \geqslant \bar{g}$ and the scalar curvature comparison $R_g \geqslant n(n-1)$. Then $g = \bar{g}$.*

Recently, there were efforts in extending Llarull's theorem to a spherical warped product

$$(1.1)\quad (\bar{M}^n, \bar{g}) := ([t_-, t_+] \times S^{n-1}, \mathrm{d}t^2 + \psi(t)^2 g_{\mathbb{S}^{n-1}}) \text{ with } t_- < t_+,\ (\log\psi)'' < 0,$$

by spinors [CZ24], [BBHW24], [WX23b], by μ-bubbles [Gro21], [HLS23] and by spacetime harmonic functions [HKKZ].

We are interested in Llarull type theorems of domains in the spherical warped product (1.1). Although the form (1.1) can also be considered as a domain in a larger spherical warped product, our focus will be on such domains with boundaries that are not necessarily given by t-level sets. Previously, this direction has been explored by Lott [Lot21], Wang-Xie [WX23a] and Chai-Wan [CW24], which all involved spinors.

In this article, we utilize the stable capillary surfaces with prescribed (varying) contact angle and prescribed mean curvature, or in short, stable capillary μ-bubble in the terminology of Gromov [Gro21]. Note that Gromov first suggested the use of stable capillary μ-bubble in studying the scalar curvature rigidity of Euclidean balls (see Section 5.8.1 of [Gro21]; *Spin-Extremality of Doubly Punctured Balls*) and Li [Li20] in three-dimensional Euclidean dihedral rigidity. This is also a further development of our previous work [CW23] in the rotationally symmetric settings. For a scalar curvature rigidity for weakly convex domains, see the recent work [KY24].

We consider the three dimensional spherical warped product (1.1) which takes the form

$$(\bar{M},\bar{g}) = ([t_-,t_+]\times[0,\pi]\times\mathbb{S}^1, \mathrm{d}t^2+\psi(t)^2(\mathrm{d}r^2+\sin^2 r\mathrm{d}\theta^2)), \tag{1.2}$$

where the standard metric $g_{\mathbb{S}^2}$ on the 2-sphere is written in polar coordinates. Our domain of interest M in $\bar{M}$ is given by

$$M=\cup_{t\in[t_-,t_+]}\Sigma_t := \cup_{t\in[t_-,t_+]}\{(t,r,\theta)\in[t_-,t_+]\times[0,\pi]\times\mathbb{S}^1:\ r\leqslant\rho(t)\} \tag{1.3}$$

for some positive function $\rho(t)$, where $\mathbb{S}^1$ represents the circle. We call such M *rotationally symmetric* with respect to the warping direction. Let

$$\partial_s M=\cup_{t\in[t_-,t_+]}\{(t,r,\theta)\in[t_-,t_+]\times[0,\pi]\times\mathbb{S}^1:\ r=\rho(t)\}.$$

It is easy to check that the outward unit normal of $\partial_s M$ is $\bar{X}=\partial_r-\rho'\psi^2\partial_t$. The dihedral angles $\bar{\gamma}$ formed by $\partial_s M$ and Σ_t is then

$$\cos\bar{\gamma}=\langle\bar{X},\partial_t\rangle=-\frac{\rho'\psi}{\sqrt{(\rho'\psi)^2+1}}. \tag{1.4}$$

We set

$$P_\pm=\{(t_\pm,r,\theta):0<r<\pi,\theta\in\mathbb{S}^1\}\subset\bar{M},$$

and $\bar{\gamma}_\pm=\bar{\gamma}(t_\pm)$ if well defined. It is easy to see that the boundary ∂M is given by the union of $P_\pm\cap\partial M=\{(t_\pm,r,\theta)\in\bar{M}:0\leqslant r\leqslant\rho(t_\pm)\}$ and $\partial_s M$.

Before stating our first scalar curvature rigidity, we fix some more conventions for the direction of the unit normal, the sign of the mean curvatures and the dihedral angles. Let Σ be a surface with boundary on $\partial_s M$ and separates $P_+\cap\partial M$ and $P_-\cap\partial M$, we always fix the direction of the unit normal N of Σ to be the direction which points inside of the region bounded by Σ, $P_+\cap\partial M$ and $\partial_s M$. The mean curvature is then the trace of the second fundamental form ∇N. We fix γ_Σ to be the angle formed by N and X, that is, $\cos\gamma_\Sigma=\langle X,N\rangle$. For the mean curvature of $\partial_s M$, it is always computed with respect to the outward unit normal. The geometric quantity on $(M,\bar{g})$ comes with a bar unless otherwise specified (see Figure 1).

Theorem 1.2. *Let M be be given in* (1.3) *where $0<\rho(t)<\frac{\pi}{2}$, $t\in[t_-,t_+]$, and $\bar{g}$ be the metric in* (1.2) *with $\psi(t_\pm)>0$, $(\log\psi)''(t)<0$, $\bar{\gamma}'(t)<0$ on $[t_-,t_+]$. Let g be another metric on M which satisfies the following comparisons of:*

a) the metrics $g\geqslant\bar{g}$ in M; the scalar curvatures $R_g\geqslant R_{\bar{g}}$ in M;
b) the mean curvatures $H_{\partial_s M}\geqslant\bar{H}_{\partial_s M}$ of $\partial_s M$, mean curvatures $H_{P_+\cap\partial M}\geqslant\bar{h}|_{P_+\cap\partial M}=\bar{h}(t_+)$ of $P_+\cap\partial M$, mean curvatures $H_{P_-\cap\partial M}\leqslant\bar{h}|_{P_-\cap\partial M}=\bar{h}(t_-)$ of $P_-\cap\partial M$;
c) The dihedral angles $\gamma_{P_+\cap\partial M}\geqslant\bar{\gamma}|_{P_+\cap\partial M}$ of $\partial_s M$ and $P_+\cap\partial M$ and the dihedral angles $\gamma_{P_-\cap\partial M}\leqslant\bar{\gamma}|_{P_-\cap\partial M}$ of $\partial_s M$ and $P_-\cap\partial M$.

Then $g=\bar{g}$.

The mean curvature comparisons can be reformulated as $H_{\partial M}\geqslant\bar{H}_{\partial M}$ on ∂M if all mean curvatures are computed with respect to the outward unit normal. We emphasize here that the condition

$$\bar{\gamma}'(t)<0 \tag{1.5}$$

geometrically says that the dihedral angles (1.4) formed by Σ_t and $\partial_s M$ monotonically decreases along the ∂_t direction with respect to the metric $\bar{g}$. It is the desired boundary analog of the logarithmic concavity $(\log\psi)'' < 0$. This settles a question raised by Gromov at the end of [Gro21, Section 5.8.1].

To fully understand its geometric meaning, we use another parametrization of (1.1). Let $s = \int^t \frac{1}{\psi(\tau)} \mathrm{d}\tau$, then $\mathrm{d}s = \frac{1}{\psi(t)}\mathrm{d}t$ and

$$\mathrm{d}t^2 + \psi(t)^2 g_{\mathbb{S}^2} = \psi(t)^2 \mathrm{d}s^2 + \psi(t)^2 g_{\mathbb{S}^2} = \psi(t)^2(\mathrm{d}s^2 + g_{\mathbb{S}^2})$$

where $t = t(s)$ is implicitly given by $s = \int^t \frac{1}{\psi(\tau)}\mathrm{d}\tau$. Note that angle is conformally invariant, the condition (1.5) together with $0 < \rho(t) < \frac{\pi}{2}$ in fact is equivalent to the *convexity* of $\partial_s M$ with respect to the conformally related metric $\mathrm{d}s^2 + g_{\mathbb{S}^2}$. This has been already observed by Chai-Wan [CW24, Theorem 1.1]. Alternatively, the logarithmic concavity $\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\log\psi) < 0$ can be formulated in terms of s: let $\bar{\psi}(s) = \psi(t(s))$, $(\log\psi)''(t) < 0$ is equivalent to $(\bar{\psi}'\bar{\psi}^{-2})'(s) < 0$. In geometric terms, as it is easy to check, the mean curvature of a t-level set or an s-level set decreases as t or s increases.

The condition $g \geqslant \bar{g}$ in Theorem 1.2 can be reformulated more generally as the existence of a distance non-increasing map $F : (M_1, g) \to (M, \bar{g})$ preserving the boundaries, and other conditions can be reformulated accordingly. Such generalizations are straightforward. See [Gro21] for a wealth of results which are generalized in this fashion.

It is possible that the inequalities in $(\log\psi)'' < 0$ and $\bar{\gamma}'(t) < 0$ can be weakened in some cases. For instance, we can consider $\mathrm{d}t^2 + t^2 g_D$ where $t \in (0, 1]$ and g_D is a geodesic disk smaller than half of the standard 2-sphere. In this case $\log\psi$ vanishes. The Llarull type rigidity Theorem 1.2 is still valid for this metric with $\bar{\gamma}' < 0$.

Theorem 1.2 does not yet generalize Theorem 1.1 genuinely, since in the case of round metric, $\psi(t) = \sin t$, $t \in [0, \pi]$ is allowed to take zero value at $t = 0$ and $t = \pi$. We have the following.

Theorem 1.3. *Let M be the region in $\bar{M}$ given by*

$$M = \cup_{t\in(t_-,t_+)}\{(t, r, \theta) : r \in [0, \rho(t)), \theta \in \mathbb{S}^1\}.$$

where $0 < \rho(t) < \frac{\pi}{2}$, $t \in [t_-, t_+]$. Assume that $\bar{g}$ is a metric in (1.2) *with $(\log\psi)''(t) < 0$, $\psi(t_+) > 0$, $\bar{\gamma}'(t) < 0$ on $[t_-, t_+]$ and*

$$\psi(t) = a(t - t_-) + o(|t - t_-|), \tag{1.6}$$

$a \in (0, 1]$. Let g be another metric on M satisfying the following comparisons of:

(1) metrics $g \geqslant \bar{g}$ in M; scalar curvatures $R_g \geqslant R_{\bar{g}}$ in M;
(2) the mean curvatures $H_{\partial_s M} \geqslant \bar{H}_{\partial_s M}$ of $\partial_s M$, mean curvatures $H_{P_+\cap\partial M} \geqslant \bar{h}|_{P_+\cap\partial M} = \bar{h}(t_+)$ of $P_+ \cap \partial M$;
(3) the dihedral angles $\gamma_{P_+\cap\partial M} \geqslant \bar{\gamma}|_{P_+\cap\partial M}$ forming by $P_+ \cap \partial M$ and $\partial_s M$ along $\partial(P_+ \cap \partial M)$.

Then $g = \bar{g}$.

Hu-Liu-Shi [HLS23] (see also Gromov [Gro21]) used a μ-bubble approach to show Theorem 1.1. However, our method differs from theirs in a technical manner when handling $\psi(t) = t + o(|t|)$ near $t = 0$. They constructed a family of small perturbations on the function $2\psi'/\psi$ while we develop a careful tangent cone analysis near $t = t_\pm$. As a result, we are able to generalize the Llarull Theorem 1.1.

Theorem 1.4. *Let $(\bar{M}, \bar{g})$ be the metric given in* (1.1) *such that*

$$\psi(t_\pm) = a_\pm |t - t_\pm| + o(|t - t_\pm|), \ 0 < a_\pm \leqslant 1,$$

If g is another smooth metric on $\bar{M}$ with possible cone singularity at only $t = t_\pm$ which satisfies the comparisons of metrics $g \geqslant \bar{g}$ and scalar curvatures $R_g \geqslant R_{\bar{g}}$, then $g = \bar{g}$.

Theorem 1.4 directly follows from the proof of Theorem 1.3 with only slight changes and we omit its proof. See Remark 3.13. The condition $0 < a_\pm \leqslant 1$ seems reasonable since it ensures that tangent cone with respect to $\bar{g}$ at $t = t_\pm$ is convex. The scalar curvature rigidity of Llarull type for $a_\pm > 1$ is an interesting question. One could compare Theorem 1.4 with [CLZ24] where conical singularities with respect to the metric g are allowed at multiple points on S^n.

There are two more cases when $\psi(t_-) \neq 0$ (we assume that $\rho(t_+) \neq 0$ and $\psi(t_+) \neq 0$): $\rho(t) = a|t - t_-| + o(|t - t_-|)$ and $\rho(t) = a(t - t_-)^2 + o(|t - t_-|^2)$ for some number $a > 0$, which we can handle using the techniques developped in [CW23]. We summarize the results in the following theorem.

Theorem 1.5. *Let M be the region in $\bar{M}$ given by*

$$M = \cup_{t \in (t_-, t_+)} \{(t, r, \theta) : r \in [0, \rho(t)), \theta \in \mathbb{S}^1\},$$

where $\rho(t_+) > 0$ and near $t = t_-$, ρ satisfies either of the asymptotics

a) $\rho(t) = a_1 |t - t_-| + o(|t - t_-|)$*;*
b) $\rho(t) = a_2 (t - t_-)^2 + o(|t - t_-|^2)$*, $a_1 > 0$, $a_2 > 0$.*

Assume that $\bar{g}$ is a metric in (1.2) *with $\psi(t) > 0$, $(\log \psi)''(t) < 0$, $\bar{\gamma}'(t) < 0$ on $[t_-, t_+]$, $a \in (0, 1]$. If g is another metric on M satisfying the same comparisons as in Theorem 1.3, then $g = \bar{g}$.*

Our approach toward Theorems 1.3 and 1.5 is by construction of surfaces of prescribed mean curvature and prescribed contact angles $\bar{\gamma}$ near $t = t_-$ which serves as barriers (see Definition 2.7). This is a local construction near $t = t_-$, so near $t = t_+$, the functions ψ and ρ can also take similar asymptotics as in (1.6) of Theorem 1.3 and in Items a) and b) of Theorem 1.5. Our proof does apply to these simple variations.

The essential difficulties of Theorem 1.5 were already present in [CW23, Theorem 1.2 *(2)* and *(3)*]. In some sense, our novel contributions are Theorems 1.2 and 1.3. In light of this, we only give a proof sketch for Theorem 1.5 in Section 4 and refer relevant details to [CW23].

Now one could naturally ask what are other possible asymptotics of ψ and ρ such that Theorem 1.3 and 1.5 remain valid. However, it is a quite intricate matter to which we do not have an answer at the moment. As a final remark, it is desirable to seek higher dimensional analogs of our results using the stable capillary μ-bubbles. This seems to be a promising direction to investigate being aware of the recent works [CWXZ24, WWZ24].

The article is organized as follows:

In Section 2, we introduce basics of stable capillary μ-bubble and we use it to show Theorem 1.2.

In Section 3, we use the tangent cone analysis at $t = t_-$ to construct barriers and reduce Theorem 1.3 to Theorem 1.2.

In Section 4, we revisit our constructions in [CW23] and use the techniques developed there to show Theorem 1.5.

Acknowledgments X. Chai was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. RS-2024-00337418) and an NRF grant No. 2022R1C1C1013511. G. Wang was supported by the China Postdoctoral Science Fundation (No. 2024M751604).

2. Stable capillary μ-bubble

In this section, we introduce the functional (2.1) whose minimiser is a stable capillary μ-bubble. We introduce a *barrier* condition which combining with a maximum principle ensures the existence of a regular minimiser to (2.1). By a rigidity analysis on the second variation of (2.1), we give the proof of Theorem 1.2.

2.1. **Notations.** We set up some notations. Let $E \subset M$ be be a set such that $\partial E \cap M$ is a regular surface with boundary which we name it Σ. We set

- N, unit normal vector of Σ pointing inside E;
- ν, unit normal vector of $\partial\Sigma$ in Σ pointing outside of Σ;
- η, unit normal vector of $\partial\Sigma$ in ∂M pointing outside of $\partial E \cap \partial M$;
- X: unit normal vectors of ∂M in M pointing outside of M;
- γ: the contact angle formed by Σ and ∂M and the magnitude of the angle is given by $\cos\gamma = \langle X, N\rangle$,
- $\langle Y, Z\rangle = g(Y, Z)$, the inner product of vectors Y and Z with respect to the metric g;
- $\langle Y, Z\rangle_{\bar g} = \bar g(Y, Z)$, the inner product of vectors Y and Z with respect to the metric $\bar g$.

See Figure 1. We use a bar on every quantity to denote that the quantity is computed with respect to the metric $\bar g$ given in (1.2).

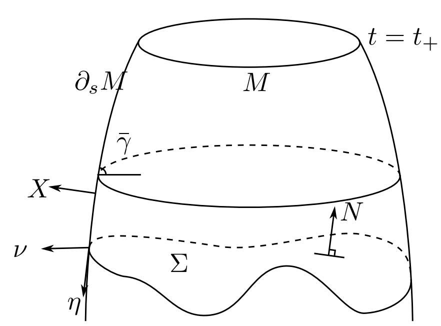

Figure 1. Notations.

2.2. **Functional and first variation.** We fix $\bar h = 2\psi'/\psi$ and $\bar\gamma$ to be given by (1.4). We define the functional

$$I(E) = \mathcal{H}^2(\partial^* E \cap \operatorname{int} M) - \int_E \bar h - \int_{\partial^* E \cap \partial M} \cos\bar\gamma, \tag{2.1}$$

where $\partial^* E$ denotes the reduced boundary of E and the variational problem

$$\mathcal{I} = \inf\{I(E) : \ E \in \mathcal{E}\}, \tag{2.2}$$

where $\mathcal{E}$ is the collection of contractible open subsets E' such that $P_+ \subset E'$. Let Σ be a surface with boundary $\partial\Sigma$ such that $\partial\Sigma$ separates $P_\pm$. Then Σ separates M into two components and the component closer to P_+ is just E. We reformulate the functional (2.1) in terms of Σ. We define

$$F(\Sigma) = I(E) = |\Sigma| - \int_E \bar{h} - \int_{\partial E\cap\partial M} \cos\bar{\gamma}.$$

Let ϕ_t be a family of immersions $\phi_t : \Sigma \to M$ such that $\phi_t(\partial\Sigma) \subset \partial M$ and $\phi_0(\Sigma) = \Sigma$. Let $\Sigma_t = \phi_t(\Sigma)$ and E_t be the corresponding component separated by Σ_t. Let Y be the vector field $\frac{\partial\phi_t}{\partial t}$. Define $\mathcal{A}(t) = F(\Sigma_t)$ and $f = \langle Y, N\rangle$, then by the first variation

$$\mathcal{A}'(0) = \int_\Sigma f(H - \bar{h}) + \int_{\partial\Sigma} \langle Y, \nu - \eta\cos\bar{\gamma}\rangle. \tag{2.3}$$

We know that if Σ is regular, then it is of mean curvature $\bar{h}$ and meets ∂M at a prescribed angle $\bar{\gamma}$, that is, a *capillary μ-bubble.* The second variation at such Σ is

$$\mathcal{A}''(0) = Q(f,f) := -\int_\Sigma (f\Delta f + (|A|^2 + \mathrm{Ric}(N) + \partial_N \bar{h})f^2) + \int_{\partial\Sigma} f(\tfrac{\partial f}{\partial\nu} - qf). \tag{2.4}$$

where $f \in C^\infty(\Sigma)$ and

$$q := \tfrac{1}{\sin\bar{\gamma}} A_{\partial M}(\eta,\eta) - \cot\bar{\gamma} A(\nu,\nu) + \tfrac{1}{\sin^2\bar{\gamma}} \partial_\eta \cos\bar{\gamma}. \tag{2.5}$$

We define two operators

$$L = -\Delta - (|A|^2 + \mathrm{Ric}(N) + \partial_N \bar{h}) \text{ in } \Sigma,$$

and

$$B = \tfrac{\partial}{\partial\nu} - q \text{ on } \partial\Sigma.$$

The surface Σ is called *stable* if

$$Q(f,f) \geqslant 0 \tag{2.6}$$

for all $f \in C^\infty(M)$. The second variation (2.4) is closely related to the variation of $H - \bar{h}$ and $\cos\gamma - \cos\bar{\gamma}$. Indeed, let $f = \langle Y, N\rangle$, we have that the first variation of $H - \bar{h}$ is

$$\begin{aligned}\nabla_Y(H - \bar{h}) &= Lf + \nabla_{Y^\top}(H - \bar{h}) \\ &= -\Delta f - (|A|^2 + \mathrm{Ric}(N) + \partial_N \bar{h})f + \nabla_{Y^\top}(H - \bar{h}).\end{aligned} \tag{2.7}$$

And the first variation of the angle difference $\langle X, N\rangle - \cos\bar{\gamma}$ is

$$\begin{aligned}&\nabla_Y(\cos\gamma - \cos\bar{\gamma}) = -\sin\bar{\gamma}\tfrac{\partial f}{\partial\nu} \\ &+(A_{\partial M}(\eta,\eta) - \cos\bar{\gamma}A(\nu,\nu) + \tfrac{1}{\sin\bar{\gamma}}\partial_\eta\cos\bar{\gamma})f + \nabla_{Y^\top}(\langle X, N\rangle - \cos\bar{\gamma}).\end{aligned} \tag{2.8}$$

For Σ, Schoen-Yau [SY79b] rewrote the term $|A|^2 + \mathrm{Ric}(N)$ as

$$|A|^2 + \mathrm{Ric}(N) = \tfrac{1}{2}(R_g - 2K + |A|^2 + H^2) \tag{2.9}$$

where K is the Gauss curvature of Σ. Along the boundary $\partial\Sigma$, we have the rewrite (see [RS97, Lemma 3.1] or [Li20, (4.13)])

$$\tfrac{1}{\sin\bar{\gamma}} A_{\partial M}(\eta,\eta) - \cos\bar{\gamma}A(\nu,\nu) = -H\cot\bar{\gamma} + \tfrac{H_{\partial M}}{\sin\bar{\gamma}} - \kappa \tag{2.10}$$

where κ is the geodesic curvature of $\partial\Sigma$ in Σ.

2.3. **Analysis of stability.** Starting from now on, we assume that Σ is a regular stable capillary μ-bubble in (M,g) which satisfies the assumptions of Theorem 1.2.

Lemma 2.1. *Let Σ be a regular stable capillary μ-bubble, then Σ is a t-level set.*

Proof. First, we note that the second variation $\mathcal{A}''(0)\geqslant 0$ as in (2.4). First, using Schoen-Yau's rewrite (2.9) we see that

$$\begin{aligned}
&|A|^2+\mathrm{Ric}(N)+\partial_N\bar{h}\\
=&\tfrac{1}{2}(R-2K+|A|^2+H^2)+\partial_N\bar{h}\\
=&\tfrac{1}{2}(R-2K+|A^0|^2+\tfrac{H^2}{2}+H^2)+\partial_N\bar{h}\\
=&\tfrac{1}{2}(R+\tfrac{3}{2}\bar{h}^2+2\partial_N\bar{h})-K+\tfrac{1}{2}|A^0|^2,
\end{aligned}\tag{2.11}$$

where A^0 is the traceless part of the second fundamental form. Similarly using (2.10), we see

$$q=-H\cot\bar{\gamma}+\tfrac{H_{\partial M}}{\sin\bar{\gamma}}-\kappa+\tfrac{1}{\sin^2\bar{\gamma}}\partial_\eta\cos\bar{\gamma}.$$

We obtain by letting $f\equiv 1$ in the (2.6) (also using (2.4) and (2.5)),

$$\begin{aligned}
2\pi\chi(\Sigma)=&\int_\Sigma K+\int_{\partial\Sigma}\kappa\\
\geqslant&\int_\Sigma\left[\tfrac{1}{2}(R+\tfrac{3}{2}\bar{h}^2+2\partial_N\bar{h})+\tfrac{1}{2}|A^0|^2\right]+\int_{\partial\Sigma}\left(\tfrac{H_{\partial M}}{\sin\bar{\gamma}}-\bar{h}\cot\bar{\gamma}+\tfrac{1}{\sin^2\bar{\gamma}}\partial_\eta\cos\bar{\gamma}\right)\\
\geqslant&\int_\Sigma\tfrac{1}{2}\left(R+\tfrac{3}{2}\bar{h}^2+2\partial_N\bar{h}\right)+\int_{\partial\Sigma}\left(\tfrac{H_{\partial M}}{\sin\bar{\gamma}}-\bar{h}\cot\bar{\gamma}+\tfrac{1}{\sin^2\bar{\gamma}}\partial_\eta\cos\bar{\gamma}\right)\\
\geqslant&\int_\Sigma\tfrac{1}{2}\left(R_{\bar{g}}+\tfrac{3}{2}\bar{h}^2+2\partial_N\bar{h}\right)+\int_{\partial\Sigma}\left(\tfrac{\bar{H}_{\partial M}}{\sin\bar{\gamma}}-\bar{h}\cot\bar{\gamma}+\tfrac{1}{\sin^2\bar{\gamma}}\partial_\eta\cos\bar{\gamma}\right),
\end{aligned}\tag{2.12}$$

where in the last line we have incorporated the comparisons $R_g\geqslant R_{\bar{g}}$ in M and $H_{\partial M}\geqslant\bar{H}_{\partial M}$ on ∂M.

Now we estimate $R_{\bar{g}}+\frac{3}{2}\bar{h}^2+2\partial_N\bar{h}$. We have that

$$\partial_N\bar{h}=\bar{g}(N,\nabla^{\bar{g}}\bar{h})\geqslant-|N|_{\bar{g}}|\nabla^{\bar{g}}\bar{h}|_{\bar{g}}=|N|_{\bar{g}}\bar{h}',$$

since $g\geqslant\bar{g}$, so

$$1=|N|_g\geqslant|N|_{\bar{g}},$$

and we get

$$\partial_N\bar{h}\geqslant\bar{h}'.$$

So

$$R_{\bar{g}}+\tfrac{3}{2}\bar{h}^2+2\partial_N\bar{h}\geqslant R_{\bar{g}}+\tfrac{3}{2}\bar{h}^2+2\bar{h}'.$$

For any point $x\in\Sigma$, the right hide side is just $\frac{2}{\psi^2(t_x)}$, where t_x is the number such that $x\in\bar{\Sigma}_t$. This is by a direct calculation of the scalar curvature of the warped product metric (1.1). So

$$R_{\bar{g}}+\tfrac{3}{2}\bar{h}^2+2\partial_N\bar{h}\geqslant\tfrac{2}{\psi^2(t_x)}.\tag{2.13}$$

Since $\bar{\gamma}'<0$ and that $\partial_{\bar{\eta}}\bar{\gamma}=|\bar{\nabla}^{\partial M}\bar{\gamma}|_{\bar{\sigma}}$, so

$$-\partial_\eta\bar{\gamma}=-\bar{\sigma}(\eta,\bar{\nabla}\bar{\gamma})\geqslant-|\eta|_{\bar{\sigma}}|\bar{\nabla}^{\partial M}\bar{\gamma}|_{\bar{\sigma}}=-|\eta|_{\bar{\sigma}}\partial_{\bar{\eta}}\bar{\gamma}.$$

It follows from $g\geqslant\bar{g}$ that $\sigma\geqslant\bar{\sigma}$ on ∂M, hence $|\eta|_{\bar{\sigma}}\leqslant|\eta|_\sigma=1$. So

$$-\partial_\eta\bar{\gamma}\geqslant-|\eta|_{\bar{\sigma}}\partial_{\bar{\eta}}\bar{\gamma}\geqslant-|\eta|_\sigma\partial_{\bar{\eta}}\bar{\gamma}=-\partial_{\bar{\eta}}\bar{\gamma},$$

Using that $\bar{\gamma} \in (0, \pi)$, we see

$$\tfrac{\bar{H}_{\partial M}}{\sin \bar{\gamma}} - \bar{h} \cot \bar{\gamma} + \tfrac{1}{\sin^2 \bar{\gamma}} \partial_\eta \cos \bar{\gamma} \geqslant \tfrac{\bar{H}_{\partial M}}{\sin \bar{\gamma}} - \bar{h} \cot \bar{\gamma} + \tfrac{1}{\sin^2 \bar{\gamma}} \partial_{\bar{\eta}} \cos \bar{\gamma}.$$

Using the rewrite of q for the background metric $\bar{g}$, we see the right hand side is the geodesic curvature $\bar{\kappa}$ of the curve $\partial \Sigma_t$ in Σ_t at $x \in \Sigma_t$ in the model metric $\bar{g}$. It is easy to see that $\bar{\kappa} = \frac{\cos \rho(t_x)}{\psi(t_x) \sin \rho(t_x)}$. So

$$\tfrac{\bar{H}_{\partial M}}{\sin \bar{\gamma}} - \bar{h} \cot \bar{\gamma} + \tfrac{1}{\sin^2 \bar{\gamma}} \partial_\eta \cos \bar{\gamma} \geqslant \tfrac{\cos \rho(t_x)}{\sin \rho(t_x) \psi(t_x)}. \tag{2.14}$$

Using both (2.13) and (2.14) in the inequality (2.12), we arrive

$$2\pi \chi(\Sigma) \geqslant \int_\Sigma \tfrac{1}{\psi^2(t_x)} \mathrm{d}\sigma + \int_{\partial \Sigma} \tfrac{\cos \rho(t_x)}{\sin \rho(t_x) \psi(t_x)} \mathrm{d}\lambda.$$

We have written the area element and line length element explicitly. Because of $g \geqslant \bar{g}$, we know

$$\mathrm{d}\sigma \geqslant \mathrm{d}\bar{\sigma} \text{ on } \Sigma,\ \mathrm{d}\lambda \geqslant \mathrm{d}\bar{\lambda} \text{ along } \partial \Sigma.$$

Let $\hat{g} = \mathrm{d}t^2 + g_{\mathbb{S}^2}$, then

$$\tfrac{1}{\psi^2(t_x)} \mathrm{d}\bar{\sigma} \geqslant \mathrm{d}\hat{\sigma} \text{ on } \Sigma,\ \tfrac{1}{\psi(t_x)} \mathrm{d}\bar{\lambda} \geqslant \mathrm{d}\hat{\lambda} \text{ along } \partial \Sigma.$$

So

$$2\pi \chi(\Sigma) \geqslant \int_\Sigma \mathrm{d}\hat{\sigma} + \int_{\partial \Sigma} \tfrac{\cos \rho(t_x)}{\sin \rho(t_x)} \mathrm{d}\hat{\lambda}. \tag{2.15}$$

The lower bound of $\int_\Sigma \mathrm{d}\hat{\sigma} + \int_{\partial \Sigma} \frac{\cos \rho(t_x)}{\sin \rho(t_x)} \mathrm{d}\hat{\lambda}$ is 2π and its proof is deferred to Lemma 2.2.

Since Σ has at least one boundary component, so $\chi(\Sigma) \leqslant 1$. Combining with (2.15) and (2.16), we see that

$$2\pi \geqslant 2\pi \chi(\Sigma) \geqslant \int_\Sigma \tfrac{1}{\psi^2(t_x)} \mathrm{d}\sigma + \int_{\partial \Sigma} \tfrac{\cos \rho(t_x)}{\sin \rho(t_x) \psi(t_x)} \mathrm{d}\lambda \geqslant 2\pi.$$

Hence, equality must hold and by Lemma 2.2, Σ is a t-level set. $\square$

Lemma 2.2. *Let Σ be as in Lemma 2.1, then*

$$\int_\Sigma \mathrm{d}\hat{\sigma} + \int_{\partial \Sigma} \tfrac{\cos \rho(t_x)}{\sin \rho(t_x)} \mathrm{d}\hat{\lambda} \geqslant 2\pi, \tag{2.16}$$

with equality occurring if and only if Σ is a t-level set.

Proof. Now we work under the direct product metric $\hat{g}$ whose full form is

$$\hat{g} = \mathrm{d}t^2 + (\mathrm{d}r^2 + \sin^2 r \mathrm{d}\theta^2). \tag{2.17}$$

Without loss of generality, we can modify $\rho(t)$ such that $\rho(t)$ remains unchanged on $(\inf_{x \in \Sigma}(t_x - \varepsilon), \sup_{x \in \Sigma}(t_x + \varepsilon))$ for sufficiently small $\varepsilon > 0$, and then we arbitrarily and smoothly extend $\rho(t)$ to all $t \in \mathbb{R}$ such that $0 < \rho(t) < \frac{\pi}{2}$. We fix $t_1 = \sup_{x \in \Sigma}(t_x + \varepsilon)$ and denote the t_1-level by Σ_1. Let Ω be the region bounded below Σ_1 and above Σ and satisfies $0 \leqslant r \leqslant \rho(t)$ for all $x = (t, r, \theta)$ in the closure $\bar{\Omega}$.

By the divergence theorem,

$$0 = \int_\Omega \mathrm{div}_{\hat{g}}(\partial_t) \mathrm{d}\hat{v} = \int_{\partial \Omega} \langle \partial_t, \hat{\nu} \rangle_{\hat{g}} \mathrm{d}\hat{\sigma},$$

where $\hat{\nu}$ is the unit outward normal of $\partial \Omega$ in Ω under the metric $\hat{g}$.

The boundary $\partial\Omega$ consists of three portions: Σ_1, Σ and $S := \partial\Omega\backslash(\Sigma_1 \cup \Sigma)$. Since Σ_1 is just some level set lying above all Ω, we see

$$\int_{\Sigma_1} \langle \partial_t, \hat{\nu} \rangle_{\hat{g}} \mathrm{d}\hat{\sigma} = |\Sigma_1|_{\hat{\sigma}}.$$

Along S, the unit outward normal is easily seen to be $\hat{\nu} = \frac{1}{\sqrt{1+(\rho')^2}}(-\rho', 1, 0)$ in the coordinate (t, r, θ). Letting (t, θ) parameterize S, we see that the area element $\mathrm{d}\hat{\sigma}$ on S (induced by $\hat{g}$) is

$$\mathrm{d}\hat{\sigma} = \sqrt{1 + (\rho')^2} \sin \rho \mathrm{d}t \mathrm{d}\theta$$

with (t, θ) defined on some appropriate domain S_0 in $\mathbb{R} \times \mathbb{S}^1$. Evidently,

$$\int_S \langle \partial_t, \hat{\nu} \rangle \mathrm{d}\hat{\sigma} = \int_{S_0} (-\rho' \sin \rho) \mathrm{d}t \mathrm{d}\theta.$$

We deal with a simple case such that $\partial\Sigma$ can be parameterized by $\theta \in \mathbb{S}^1$ such that

$$\partial\Sigma = \{(t = \tau(\theta), r = \rho(\tau(\theta)), \theta) \in \bar{M} : \ \theta \in \mathbb{S}^1\} \tag{2.18}$$

for some smooth function $\tau(\theta)$ to illustrate the idea.

We see then $S_0 = \{(t, \theta) \in \mathbb{R} \times \mathbb{S}^1 : \tau(\theta) \leqslant t \leqslant t'\}$. Now we integrate with respect to t first, and

$$\begin{aligned}\int_{S_0} (-\rho' \sin \rho) \mathrm{d}t \mathrm{d}\theta &= \int_{\theta \in \mathbb{S}^1} \int_{\tau(\theta)}^{t_1} (-\rho'(t) \sin \rho(t)) \mathrm{d}t \mathrm{d}\theta \\ &= \int_{\theta \in \mathbb{S}^1} \cos \rho(t_1) \mathrm{d}\theta - \int_{\theta \in \mathbb{S}^1} \cos \rho(\tau(\theta)) \mathrm{d}\theta.\end{aligned}$$

To collect the integration on $\partial\Omega$ on all of its three portions, we see that

$$0 = \int_\Sigma \langle \partial_t, \hat{\nu} \rangle_{\hat{g}} \mathrm{d}\hat{\sigma} + |\Sigma_1|_{\hat{\sigma}} + \int_{\theta \in \mathbb{S}^1} \cos \rho(t_1) \mathrm{d}\theta - \int_{\theta \in \mathbb{S}^1} \cos \rho(\tau(\theta)) \mathrm{d}\theta.$$

It is not difficult to see from the metric (2.17) that

$$\int_{\theta \in \mathbb{S}^1} \cos \rho(t_1) \mathrm{d}\theta = \int_{\partial\Sigma_1} \tfrac{\cos \rho(t_1)}{\sin \rho(t_1)} \mathrm{d}\hat{\lambda}.$$

Since Σ_1 is a geodesic disk in $\mathbb{S}^2$, it then follows from the Gauss-Bonnet theorem that

$$|\Sigma_1|_{\hat{\sigma}} + \int_{\theta \in \mathbb{S}^1} \cos \rho(t_1) \mathrm{d}\theta = 2\pi.$$

Hence

$$-\int_\Sigma \langle \partial_t, \hat{\nu} \rangle_{\hat{g}} \mathrm{d}\hat{\sigma} + \int_{\theta \in \mathbb{S}^1} \cos \rho(\tau(\theta)) \mathrm{d}\theta \geqslant 2\pi.$$

From $\langle \partial_t, \hat{\nu} \rangle_{\hat{g}} < 0$ and $|\partial_t|_{\hat{g}} = |\hat{\nu}|_{\hat{g}} = 1$, we see

$$|\Sigma|_{\hat{\sigma}} \geqslant \int_\Sigma -\langle \partial_t, \hat{\nu} \rangle_{\hat{g}} \mathrm{d}\hat{\sigma}.$$

Now it suffices to show

$$\int_{\partial\Sigma} \tfrac{\cos \rho(t_x)}{\sin \rho(t_x)} \mathrm{d}\hat{\lambda} \geqslant \int_{\theta \in \mathbb{S}^1} \cos \rho(\tau(\theta)) \mathrm{d}\theta$$

to finish our proof of (2.16). Indeed, in terms of $\mathrm{d}\theta$, the length element $\mathrm{d}\hat{\lambda}$ of $\partial\Sigma$ is

$$\sqrt{\mathrm{d}\tau(\theta)^2 + \mathrm{d}(\rho(\tau(\theta)))^2 + \sin^2(\rho(\tau(\theta)))}\mathrm{d}\theta \geqslant \sin\rho(\tau(\theta))\mathrm{d}\theta.$$

Also with $\rho < \pi/2$, $\cos(\rho(\tau(\theta))) > 0$, we can see that

$$\int_{\partial\Sigma} \tfrac{\cos\rho(t_x)}{\sin\rho(t_x)}\mathrm{d}\hat{\lambda} \geqslant \int_{\theta\in\mathbb{S}^1} \frac{\cos\rho(\tau(\theta))}{\sin\rho(\tau(\theta))}\sin\rho(\tau(\theta))\mathrm{d}\theta = \int_{\theta\in\mathbb{S}^1}\cos\rho(\tau(\theta))\mathrm{d}\theta.$$

The equality case of (2.16) is easy to trace from the above proof.

Now we handle the case such that (2.18) might not hold. Using the definition of S_0, we find that the boundary $\partial S_0 = L_1 \cup L_2$ is given by

$$L_0 = \{(t,\theta) \in \mathbb{R}\times\mathbb{S}^1 : \ (t,r,\theta)\in\Sigma\},$$
$$L_1 = \{(t,\theta) \in \mathbb{R}\times\mathbb{S}^1 : \ (t,r,\theta)\in\Sigma_1\} = \{t_1\}\times\mathbb{S}^1.$$

Let $g_0 = \mathrm{d}t^2 + \mathrm{d}\theta^2$ and L_0 be parametrized by g_0-arc-length as

$$\ell : [0,\ell_0] \to (t(\ell),\theta(\ell)) \in L_0$$

such that $\ell(0) = \ell(\ell_0)$. The unit normal of L_0 in S_0 pointing outward of S_0 is $(-\frac{\partial\theta}{\partial\ell}, \frac{\partial t}{\partial\ell})$ and the unit normal of L_1 pointing outward of S_0 is ∂_t. Let $Z = \cos\rho(t)\partial_t$ be the vector field on S_0, then $-\rho'(t)\sin\rho(t) = \mathrm{div}_{g_0} Z$, where div_{g_0} is the divergence with respect to g_0. Using the divergence theorem,

$$\int_{S_0}(-\rho'\sin\rho)\mathrm{d}t\mathrm{d}\theta = \int_{L_1}\cos\rho(t_1)\mathrm{d}\theta - \int_0^{\ell_0}\cos\rho(t(\ell))\frac{\partial\theta}{\partial\ell}\mathrm{d}\ell.$$

Using the parameter ℓ, we set

$$\partial\Sigma = \{(t(\ell),\rho(t(\ell)),\theta(\ell)) : \ell\in[0,\ell_0]\}.$$

The length element $\mathrm{d}\hat{\lambda}$ is then

$$\mathrm{d}\hat{\lambda} \geqslant \sin^2\rho(t(\ell))\mathrm{d}\theta(\ell).$$

The rest of the proof proceeds in the same way as before. □

2.4. **Infinitesimally rigid surface.** The surface Σ be a stable capillary μ-bubble has more consequences than the mere Lemma 2.1. We can conclude that Σ is a so-called infinitesimally rigid surface. See Definition 2.3.

All inequalities are in fact equalities by Lemma 2.2 and tracing the equalities in (2.4), we arrive that

$$R_g = R_{\bar{g}}, N = \bar{N}, |A^0| = 0 \text{ in } \Sigma \tag{2.19}$$

and

$$H_{\partial M} = \bar{H}_{\partial M}, \eta = \bar{\eta} \text{ along } \partial\Sigma. \tag{2.20}$$

It then follows from the equality case of Lemma 2.2 and $N = \bar{N}$ that

$$g = \bar{g}, t_x = t_0 \text{ at all } x \in \bar{\Sigma} \tag{2.21}$$

for some constant $t_0 \in [t_-, t_+]$. Because Σ is stable (equivalently $Q(f,f) \geqslant 0$), so the eigenvalue problem

$$\begin{cases} Lf & = \mu f \text{ in } \Sigma \\ Bf & = 0 \text{ on } \partial\Sigma \end{cases} \tag{2.22}$$

has a nonnegative first eigenvalue $\mu_1 \geqslant 0$.

The analysis now is similar to [FCS80]. Letting $f \equiv 1$ in (2.6), using (2.19), (2.20) and (2.21), we get

$$\begin{aligned} Q(1,1) = &\int_\Sigma \left[K - \tfrac{1}{2}(R + \tfrac{3}{2}\bar{h}^2 + 2\partial_N \bar{h})\right] \\ &+ \int_{\partial\Sigma} \left[\kappa - (\tfrac{H_{\partial M}}{\sin\bar{\gamma}} - \bar{h}\cot\bar{\gamma} - \tfrac{1}{\sin\bar{\gamma}}\tfrac{\partial\bar{\gamma}}{\partial\eta})\right] = 0. \end{aligned} \tag{2.23}$$

And so the first eigenvalue μ_1 is zero, hence the constant 1 is its corresponding eigenfunction.

By (2.19) and (2.11), the stability operator L reduces to

$$L = -\Delta - \left[\tfrac{1}{\psi(t_x)^2} - K\right];$$

by considering (2.20) and equality in (2.14). The boundary stability operator B reduces to

$$B = \partial_\nu - \left[\tfrac{\cos\rho(t_x)}{\sin\rho(t_x)\psi(t_x)} - \kappa\right].$$

Putting $f = 1$ and $\mu_1 = 0$ in the eigenvalue problem (2.22), we get

$$K = \tfrac{1}{\psi^2(t_0)} \text{ in } \Sigma, \ \kappa = \tfrac{\cos\rho(t_0)}{\sin\rho(t_0)\psi(t_0)} \text{ on } \partial\Sigma. \tag{2.24}$$

The above says that is a scaling of a geodesic disk in the standard unit 2-sphere with radius $\rho(t_0)$ by a factor of $\psi(t_0)$. Now we summarize the properties of Σ in the definition of an *infinitesimally rigid surface.*

Definition 2.3. *We say that Σ is infinitesimally rigid if it satisfies* (2.19)*,* (2.20)*,* (2.21) *and* (2.24)*.*

2.5. **Capillary foliation of constant $H - \bar{h}$.** See for instance the works [Ye91], [BBN10] and [Amb15] on constructing CMC foliations. First, we construct a foliation with prescribed angles $\bar{\gamma}$ whose leaf is of constant $H - \bar{h}$. Let $\phi(x,t)$ be a local flow of a vector field Y which is tangent to ∂M and transverse to Σ and that $\langle Y, N\rangle = 1$.

In the following theorem, we only require that the scalar curvature of (M, g) and the mean curvature of ∂M are bounded below.

Theorem 2.4. *Suppose (M, g) is a three manifold with boundary, if Σ is an infinitesimally rigid surface, then there exists $\varepsilon > 0$ and a function $w(x,t)$ on $\Sigma \times (-\varepsilon, \varepsilon)$ such that for each $t \in (-\varepsilon, \varepsilon)$, the surface*

$$\Sigma_t = \{\phi(x, w(x,t)) : x \in \Sigma\}$$

is a surface of constant $H - \bar{h}$ intersecting ∂M with prescribed angle $\bar{\gamma}$. Moreover, for every $x \in \Sigma$ and every $t \in (-\varepsilon, \varepsilon)$,

$$w(x,0) = 0, \quad \int_\Sigma (w(x,t) - t) = 0 \ \textit{and} \ \tfrac{\partial}{\partial t} w(x,t)|_{t=0} = 1.$$

Proof. Given a function in the Hölder space $C^{2,\alpha}(\Sigma) \cap C^{1,\alpha}(\bar{\Sigma})$ $(0 < \alpha < 1)$, we consider

$$\Sigma_u = \{\phi(x, u(x)) : x \in \Sigma\},$$

which is a properly embedded surface if the norm of u is small enough. We use the subscript u to denote the quantities associated with Σ_u.

Consider the space

$$\mathcal{Y}=\left\{u\in C^{2,\alpha}(\Sigma)\cap C^{1,\alpha}(\bar{\Sigma}):\int_\Sigma u=0\right\}$$

and

$$\mathcal{Z}=\left\{u\in C^{0,\alpha}(\Sigma):\int_\Sigma u=0\right\}.$$

Given small $\delta>0$ and $\varepsilon>0$, we define the map

$$\Phi:(-\varepsilon,\varepsilon)\times B(0,\delta)\to\mathcal{Z}\times C^{0,\alpha}(\partial\Sigma)$$

given by

$$\begin{aligned}&\Phi(t,u)\\ =&\left((H_{t+u}-\bar{h}_{t+u})-\tfrac{1}{|\Sigma|}\int_\Sigma(H_{t+u}-\bar{h}_{t+u}),\langle X_{t+u},N_{t+u}\rangle-\cos\bar{\gamma}_{t+u}\right).\end{aligned}$$

Here, $B(0,\delta)$ is a ball of radius $\delta>0$ centered at the zero function in $\mathcal{Y}$. For each $v\in\Sigma$, the map

$$f:(x,s)\in\Sigma\times(-\varepsilon,\varepsilon)\to\phi(x,sv(x))\in M$$

gives a variation with

$$\tfrac{\partial f}{\partial s}|_{s=0}=\tfrac{\partial}{\partial s}\phi(x,sv(x))|_{s=0}=vN.$$

Since Σ is infinitesimally rigid and using also (2.7) and (2.8), we obtain that

$$D\Phi_{(0,0)}(0,v)=\tfrac{\mathrm{d}}{\mathrm{d}s}\Phi(0,sv)|_{s=0}=\left(-\Delta v+\tfrac{1}{|\Sigma|}\int_{\partial\Sigma}\Delta v,-\sin\bar{\gamma}\tfrac{\partial v}{\partial\nu}\right).$$

It follows from the elliptic theory for the Laplace operator with Neumann type boundary conditions that $D\Phi(0,0)$ is an isomorphism when restricted to $0\times\mathcal{Y}$.

Now we apply the implicit function theorem: For some smaller ε, there exists a function $u(t)\in B(0,\delta)\subset\mathcal{X}$, $t\in(-\varepsilon,\varepsilon)$ such that $u(0)=0$ and $\Phi(t,u(t))=\Phi(0,0)=(0,0)$ for every t. In other words, the surfaces

$$\Sigma_{t+u(t)}=\{\phi(x,t+u(t)):x\in\Sigma\}$$

are of constant $H-\bar{h}$ with prescribed angles $\bar{\gamma}$.

Let $w(x,t)=t+u(t)(x)$ where $(x,t)\in\Sigma\times(-\varepsilon,\varepsilon)$. By definition, $w(x,0)=0$ for every $x\in\Sigma$ and $w(\cdot,t)-t=u(t)\in B(0,\delta)\subset\mathcal{X}$ for every $t\in(-\varepsilon,\varepsilon)$. Observe that the map $s\mapsto\phi(x,w(x,s))$ gives a variation of Σ with variational vector field given by

$$\tfrac{\partial\phi}{\partial t}\tfrac{\partial w}{\partial s}|_{s=0}=\tfrac{\partial w}{\partial s}|_{s=0}Y.$$

Since for every t we have that

$$\begin{aligned}0=&\Phi(t,u(t))\\ =&\left((H_{w(\cdot,t)}-\bar{h}_{w(\cdot,t)})-\tfrac{1}{|\Sigma|}\int_\Sigma(H_{w(\cdot,t)}-\bar{h}_{w(\cdot,t)}),\langle X_{t+u},N_{t+u}\rangle-\cos\bar{\gamma}_{t+u}\right),\end{aligned}$$

by taking the derivative at $t=0$ we conclude that

$$\langle\tfrac{\partial w}{\partial t}|_{t=0}Y,N\rangle=\tfrac{\partial w}{\partial t}|_{t=0}$$

satisfies the homogeneous Neumann problem. Therefore, it is constant on Σ. Since

$$\int_\Sigma(w(x,t)-t)=\int_\Sigma u(x,t)=0$$

for every t, by taking derivatives at $t=0$ again, we conclude that

$$\int_{\Sigma} \tfrac{\partial w}{\partial t}|_{t=0} = |\Sigma|.$$

Hence, $\frac{\partial w}{\partial t}|_{t=0} = 1$. Taking ε small, we see that $\phi(x, w(x,t))$ parameterize a foliation near Σ. $\square$

Theorem 2.5. *There exists a continuous function $\Psi(t)$ such that*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(-\int_0^t \Psi(\tau)\mathrm{d}\tau\right)(H-\bar{h})\right) \leqslant 0.$$

Proof. Let $\psi : \Sigma \times I \to M$ parameterize the foliation, $Y = \frac{\partial \psi}{\partial t}$, $v_t = \langle Y, N_t \rangle$. Then

$$-\tfrac{\mathrm{d}}{\mathrm{d}t}(H-\bar{h}) = \Delta_t v_t + (\mathrm{Ric}(N_t) + |A_t|^2) v_t + v_t \nabla_{N_t} \bar{h} \text{ in } \Sigma_t, \tag{2.25}$$

and

$$\tfrac{\partial v_t}{\partial \nu_t} = [-\cot\bar{\gamma} A_t(\nu_t, \nu_t) + \tfrac{1}{\sin\bar{\gamma}} A_{\partial M}(\eta_t, \eta_t) + \tfrac{1}{\sin^2\bar{\gamma}} \nabla_{\eta_t} \cos\bar{\gamma}] v_t. \tag{2.26}$$

By shrinking the interval if if needed, we assume that $v_t > 0$ for $t \in I$. By multiplying of (2.25) and integrate on Σ_t, we deduce by integration by parts and applying the Schoen-Yau rewrite (2.9) that

$$\begin{aligned}
&-(H-\bar{h})' \int_{\Sigma_t} \tfrac{1}{v_t} \\
=& \int_{\Sigma_t} \tfrac{\Delta_t v_t}{v_t} + (\mathrm{Ric}(N_t) + |A_t|^2 + \nabla_{N_t}\bar{h}) \\
=& \int_{\partial\Sigma_t} \tfrac{1}{v_t}\tfrac{\partial v_t}{\partial \nu_t} + \tfrac{1}{2}\int_{\Sigma_t} (R_g + |A_t|^2 + H_t^2 + 2\nabla_{N_t}\bar{h}) - \int_{\Sigma_t} K_{\Sigma_t} + \int_{\Sigma_t} \tfrac{|\nabla v_t|^2}{v_t^2}.
\end{aligned}$$

Let $\chi = A - \frac{1}{2}\bar{h}\sigma$, we have that

$$\begin{aligned}
&|A_t|^2 \\
=&|\chi + \tfrac{1}{2}\bar{h}\sigma|^2 \\
=&|\chi|^2 + \langle \chi, \bar{h}\sigma \rangle + \tfrac{1}{2}\bar{h}^2, \\
=&|\chi^0|^2 + \tfrac{1}{2}(\mathrm{tr}_\sigma \chi)^2 + \bar{h}\,\mathrm{tr}_\sigma \chi + \tfrac{1}{2}\bar{h}^2,
\end{aligned}$$

where χ^0 is the traceless part of χ. Also,

$$H^2 = (\mathrm{tr}_\sigma \chi + \bar{h})^2 = (\mathrm{tr}_\sigma \chi)^2 + 2\,\mathrm{tr}_\sigma \chi \bar{h} + \bar{h}^2.$$

So

$$\begin{aligned}
&-(H-\bar{h})' \int_{\Sigma_t} \tfrac{1}{v_t} \\
=& \int_{\partial\Sigma_t} \tfrac{1}{v_t}\tfrac{\partial v_t}{\partial \nu_t} + \tfrac{1}{2}\int_{\Sigma_t} (R_g + |A_t|^2 + H_t^2 + 2\nabla_{N_t}\bar{h}) - \int_{\Sigma_t} K_{\Sigma_t} + \int_{\Sigma_t} \tfrac{|\nabla v_t|^2}{v_t^2} \\
=& \int_{\partial\Sigma_t} \tfrac{1}{v_t}\tfrac{\partial v_t}{\partial \nu_t} + \tfrac{1}{2}\int_{\Sigma_t} (R_g + \tfrac{3}{2}\bar{h}^2 + 2\nabla_{N_t}\bar{h}) \\
&+ \tfrac{1}{2}\int_{\Sigma_t} |\chi^0|^2 + \tfrac{3}{2}(\mathrm{tr}_\sigma \chi)^2 + 3\bar{h}\,\mathrm{tr}_\sigma \chi - \int_{\Sigma_t} K_{\Sigma_t} + \int_{\Sigma_t} \tfrac{|\nabla v_t|^2}{v_t^2} \\
\geqslant& \int_{\partial\Sigma_t} \tfrac{1}{v_t}\tfrac{\partial v_t}{\partial \nu_t} + \int_{\Sigma_t} \tfrac{1}{\psi^2(t_x)} + \tfrac{3}{2}(H-\bar{h}) \int_{\Sigma_t} \bar{h} - \int_{\Sigma_t} K_{\Sigma_t},
\end{aligned}$$

where in the last line we have also used the bound (2.13). Now we use (2.26) and also the rewrite (2.10), we see that

$$
\begin{aligned}
& -(H-\bar{h})' \int_{\Sigma_t} \tfrac{1}{v_t} \\
\geqslant & \int_{\partial\Sigma_t} [-\cot\bar{\gamma} A_t(\nu_t,\nu_t) + \tfrac{1}{\sin\bar{\gamma}} A_{\partial M}(\eta_t,\eta_t) + \tfrac{1}{\sin^2\bar{\gamma}} \nabla_{\eta_t}\cos\bar{\gamma}] \\
& + \int_{\Sigma_t} \tfrac{1}{\psi^2(t_x)} + \tfrac{3}{2}(H-\bar{h}) \int_{\Sigma_t} \bar{h} - \int_{\Sigma_t} K_{\Sigma_t} \\
\geqslant & \int_{\partial\Sigma_t} [-\kappa_{\partial\Sigma_t} - H(t)\cot\bar{\gamma} + \tfrac{1}{\sin\bar{\gamma}} H_{\partial M} + \tfrac{1}{\sin^2\bar{\gamma}} \nabla_{\eta_t}\cos\bar{\gamma}] \\
& + \int_{\Sigma_t} \tfrac{1}{\psi^2(t_x)} + \tfrac{3}{2}(H-\bar{h}) \int_{\Sigma_t} \bar{h} \\
= & -\left(\int_{\Sigma_t} K_{\Sigma_t} + \int_{\partial\Sigma_t} \kappa_{\partial\Sigma_t}\right) + \left[\int_{\Sigma_t} \tfrac{1}{\psi^2(t_x)} + \int_{\partial\Sigma_t} \left(\tfrac{1}{\sin\bar{\gamma}} H_{\partial M} - \bar{h}\cot\bar{\gamma} + \tfrac{1}{\sin^2\bar{\gamma}} \nabla_{\eta_t}\cos\bar{\gamma}\right)\right] \\
& + \tfrac{3}{2}(H-\bar{h}) \int_{\Sigma_t} \bar{h} - (H-\bar{h}) \int_{\partial\Sigma_t} \cot\bar{\gamma}.
\end{aligned}
$$

By Gauss-Bonnet theorem and that the second term in the big bracket is bounded below by 2π, we see that

$$
-(H-\bar{h})' \int_{\Sigma_t} \tfrac{1}{v_t} \geqslant (H-\bar{h})(\tfrac{3}{2}\int_{\Sigma_t} \bar{h} - \int_{\partial\Sigma_t} \cot\bar{\gamma}).
$$

Let

$$
\Psi(t) = \left(\int_{\Sigma_t} \tfrac{1}{v_t}\right)^{-1} (\int_{\partial\Sigma_t} \cot\bar{\gamma} - \tfrac{3}{2}\int_{\Sigma_t} \bar{h}), \tag{2.27}
$$

then note that we have assume that $v_t > 0$ near $t = 0$, so $H - \bar{h}$ satisfies the ordinary differential inequality

$$
(H-\bar{h})' - \Psi(t)(H-\bar{h}) \leqslant 0. \tag{2.28}
$$

We see then

$$
\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\exp\left(-\int_0^t \Psi(\tau)\mathrm{d}\tau\right)(H-\bar{h})\right) \leqslant 0.
$$

So the function $\exp(-\int_0^t \Psi(\tau)\mathrm{d}\tau)(H-\bar{h})$ is nonincreasing. □

2.6. **From local foliation to rigidity.** Let Σ_t be the constant mean curvature surfaces with prescribed contact angles $\bar{\gamma}$ with ∂M.

Proposition 2.6. *Every Σ_t constructed in Theorem 2.4 is infinitesimally rigid.*

Proof. Let Ω_t be the component of $M \backslash \Sigma_t$ whose clossure contains $P_+ \cap \partial M$. We abuse the notation and define

$$F(t) = |\Sigma_t| - \int_{\Omega_t} \bar{h} - \int_{\partial \Omega_t} \cos \bar{\gamma}.$$

By the first variation formula (2.3),

$$F(t_2) - F(t_1) = \int_{t_1}^{t_2} \mathrm{d}t \int_{\Sigma_t} (H - \bar{h}) v_t.$$

By Theorem 2.5,

$$H - \bar{h} \leqslant 0 \text{ if } t \geqslant 0; \; H - \bar{h} \geqslant 0 \text{ if } t \leqslant 0,$$

which in turn implies that

$$F(t) \leqslant 0 \text{ if } t \geqslant 0; \; F(t) \leqslant 0 \text{ if } t \leqslant 0.$$

However, Ω_t is a minimiser to the functional (2.1), hence

$$F(t) \equiv F(0).$$

It then follows every Σ_t is a minimiser, hence infinitesimally rigid. □

Now we introduce the *barrier* condition which enables to find a stable capillary μ-bubble.

Definition 2.7. *We say that a surface Σ_+ (Σ_-) whose boundary separates $\partial(P_+ \cap \partial M)$ and $\partial(P_- \cap \partial M)$ is an upper (lower) barrier if $H_{\Sigma_+} \geqslant \bar{h}|_{\Sigma_+}$ ($H_{\Sigma_-} \leqslant \bar{h}|_{\Sigma_-}$) and $\gamma_{\Sigma_+} \geqslant \bar{\gamma}|_{\partial\Sigma_+ \cap \partial M}$ ($\gamma_{\Sigma_-} \leqslant \bar{\gamma}|_{\partial\Sigma_- \cap \partial M}$) along $\partial\Sigma_+$ ($\partial\Sigma_-$). We call Σ_+ and Σ_- are a set of barriers if Σ_+ and Σ_- are respectively an upper barrier and a lower barrier and Σ_+ is closer to P_+ than Σ_-.*

We can conclude the proof of Theorem 1.2.

Proof of Theorem 1.2. We note easily by the assumptions of Theorem 1.2 that $\Sigma_\pm = \partial_\pm M$ are a set of barriers (Definition 2.7), by the maximum principle, there exists a minimiser E to (2.2) such that E is either empty or $\partial E \backslash \partial_s M$ or lies entirely away from $P_\pm$. Without loss of generality, we assume that $\Sigma = \partial E \cap \operatorname{int} M$ non-empty. By [DPM15], Σ is a regular stable μ-bubble. Moreover, the second variation $\mathcal{A}''(0) \geqslant 0$ in (2.4) for any smooth family Σ_s such that $\Sigma_0 = \Sigma$.

Let $Y = \frac{\mathrm{d}}{\mathrm{d}t}\phi(x, w(x,t))$ where ϕ and w are as Theorem 2.4, we show first that N_t is conformal. It suffices to show that $Y^\perp$ is conformal.

Since every Σ_t is infinitesimally rigid by Proposition 2.6, from (2.22) and (2.23), we know that $\langle Y, N_t \rangle$ is a constant. Let ∂_i, $i = 1, 2$ be vector fields induced by local coordinates on Σ, ∂_i also extends to a neighborhood of Σ via the diffeomorphism ϕ. We have $\nabla_{\partial_i} \langle Y, N \rangle = 0$. Note that Σ_t are umbilical with constant mean curvature $\bar{h}$, so

$$\nabla_{\partial_i} N \equiv \tfrac{1}{2} \bar{h} \partial_i$$

and

$$\begin{aligned} 0 &= \nabla_{\partial_i} \langle Y, N \rangle \\ &= \langle \nabla_{\partial_i} Y, N \rangle + \langle Y, \nabla_{\partial_i} N \rangle \\ &= \langle \nabla_{\partial_i} Y, N \rangle + \tfrac{1}{2} \bar{h} \langle Y, \partial_i \rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} 0 &= \langle \nabla_{\partial_i} Y, N\rangle = \langle \nabla_Y \partial_i, N\rangle \\ &= Y\langle \partial_i, N\rangle - \langle \partial_i, \nabla_Y N\rangle \\ &= -\langle \partial_i, \nabla_Y N\rangle \\ &= -\langle \partial_i, \nabla_{Y^\top} N\rangle - \langle \partial_i, \nabla_{Y^\perp} N\rangle \\ &= -\tfrac{1}{2}\bar{h}\langle Y^\top, \partial_i\rangle - \langle \partial_i, \nabla_{Y^\perp} N\rangle. \end{aligned}$$

Combining the two equations above, we conclude that $\nabla_{Y^\perp} N = 0$ which implies that Σ foliates a warped product under the diffeomorphism ϕ (parameterized by t). Considering that the induced metric on Σ agrees with the induced metric from $\bar{g}$, we conclude that $g = \bar{g}$. □

Remark 2.8. Observe that the proof of Theorem 1.2 works if M is replaced by a domain bounded by $\partial_s M$ and $\Sigma_\pm$ which satisfy the barrier condition (see Definition 2.7).

3. Construction of barriers (I)

In this section, we prove Theorem 1.3. Our strategy is to construct a surface Σ_- which together with $\Sigma_+ := P_+ \cap \partial M$ serve as barriers, and to use Theorem 1.2 to finish the proof. This section is occupied by such a construction of Σ_-.

3.1. **Setting up coordinates and notations.** For convenience, we set $t_- = 0$. Since both (M, g) and $(M, \bar{g})$ has a cone structure near where $t_- = 0$ where the cross-section of the cone is an $\mathbb{S}^2$, we can denote the point $\{0\} \times \{(r, \theta) \in \mathbb{S}^2 : r \leqslant \rho(0)\}$ as p_0. For any $t > 0$, we set

$$\Sigma_t = \{(t, r, \theta) \in M : \ r \leqslant \rho(t)\}, \tag{3.1}$$

$$\Omega_t = \{(s, r, \theta) \in M : \ 0 \leqslant s \leqslant t, \ r \leqslant \rho(s)\}, \tag{3.2}$$

using the polar coordinates of $\mathbb{S}^2$ as in (1.2).

In the following subsections, we will construct graphical perturbations $\Sigma_{t,t^2 u}$ of Σ_t. Let $\Sigma_{t+t^2 u}$ be the surface which consists of points $x + t^2 u(x,t) N_t(x)$ where N_t is the unit normal of Σ_t with respect to the metric g at $x \in \Sigma_t$. The boundary $\partial\Sigma_{t+t^2 u}$ might not lie in $\partial_s M$, we can compensate this by expanding or shrinking $\Sigma_{t+t^2 u}$ a little, we denote the resulting surface $\Sigma_{t,t^2 u}$.

We use a t subscript on every geometric quantity on Σ_t and a $t, t^2 u$ subscript on every geometric quantity on $\Sigma_{t,t^2 u}$. We will explicitly indicate when there was a confusion or change.

3.2. **Capillary foliation with constant $H - \bar{h}$.** We assume that (M, g) and $(M, \bar{g})$ have isometric tangent cones at p_0 and we construct a foliation of constant $H - \bar{h}$ with prescribed angles $\bar{\gamma}$ near p_0. In fact, later in Subsection 3.3, we will show that this is the only case.

By the first variation formula of the mean curvatures

$$H_{t,t^2 u} - H_t = -\Delta_t u - t^2(\mathrm{Ric}(N_t) + |A_t|^2) u + O(t), \tag{3.3}$$

where Δ_t is the Laplacian with respect to the induced rescaled metric $t^{-2} g|_{\Sigma_t}$. Note that $\mathrm{Ric}(N_t) = O(t^{-1})$ by the fact the tangent cone is $\mathrm{d}t^2 + a^2 t^2 g_{\mathbb{S}^2}$. By the Taylor

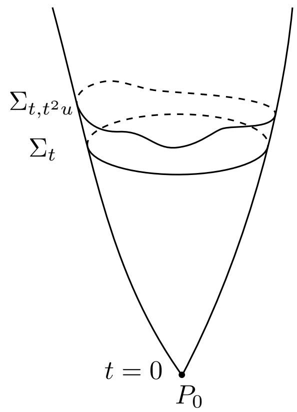

FIGURE 2. Construction of Σ_{t,t^2u}.

expansion of the function $\bar{h}$, we see that

$$\bar{h}_{t,t^2u} - \bar{h}_t = \bar{h}'(t)t^2u = t^2u\nabla_{N_t}\bar{h} + O(t).$$

So

$$(H_{t,t^2u} - \bar{h}_{t,t^2u}) - (H_t - \bar{h}_t) = -\Delta_t u - t^2(\mathrm{Ric}(N_t) + |A_t|^2 + \nabla_{N_t}\bar{h})u + O(t). \tag{3.4}$$

Note that both $H_t - \bar{h}_t$ and $H_{t,t^2u} - \bar{h}_{t,t^2u}$ are finite and $|A_t|^2 + \nabla_{N_t}\bar{h} = O(t^{-1})$ considering that (M, g) and $(M, \bar{g})$ has isometric cone at p_0.

Remark 3.1. We elaborate a bit more on (3.3) and its $O(t)$ remainder term. Since the metric g is close to $\mathrm{d}t^2 + \psi(t)^2 g_{\mathbb{S}^2}$ when $t \to 0^+$, we calculate the expansions with respect to the rescaled metric $t^{-2}g$ when computing for small $t > 0$. This is similar to [Ye91]. Then we rescale back and we obtain (3.3). The term $O(t)$ involves products of $|A_t|$ which is of order t^{-1} with terms of order at most $O(1)$. That is why the remainder is only of order $O(t)$ instead of $O(t^2)$.

Also, the variation of angles give

$$\begin{aligned} & t^{-1}[\langle X_{t,t^2u}, N_{t,t^2u}\rangle - \langle X_t, N_t\rangle] \\ = & -\sin\gamma \tfrac{\partial u}{\partial \nu_t} + t(-\cos\gamma A(t^{-1}\nu_t, t^{-1}\nu_t) + A_{\partial M}(\eta_t,\eta_t))u + O(t^2), \end{aligned} \tag{3.5}$$

where ν_t is the outward unit normal of $\partial\Sigma_t$ in Σ_t with respect to the rescaled induced metric $t^{-2}g|_{\Sigma_t}$ (note that $t^{-1}\nu_t$ is of unit length with respect to g). Other geometric quantities are not rescaled. By the variation of the prescribed angle $\bar{\gamma}$,

$$t^{-1}(\cos\bar{\gamma}_{t,t^2u} - \cos\bar{\gamma}_t) = -\tfrac{tu}{\sin\bar{\gamma}}\partial_{\bar{\eta}_t}\cos\bar{\gamma} + O(t^2).$$

So

$$\begin{aligned} & t^{-1}[(\langle X_{t,t^2u}, N_{t,t^2u}\rangle - \cos\bar{\gamma}_{t,t^2u}) - (\langle X_t, N_t\rangle - \cos\bar{\gamma}_t)] \\ = & -\sin\gamma\tfrac{\partial u}{\partial\nu_t} \\ & + t(-\cos\gamma A(t^{-1}\nu_t, t^{-1}\nu_t) + A_{\partial M}(\eta_t,\eta_t) + \tfrac{1}{\sin\bar{\gamma}}\partial_{\bar{\eta}_t}\cos\bar{\gamma})u + O(t^2). \end{aligned} \tag{3.6}$$

Remark 3.2. The term $A(t^{-1}\nu_t, t^{-1}\nu_t) = O(t^{-1})$, however, we observe that $\lim_{t\to 0}\bar{\gamma}_t = \pi/2$, and $A_{\partial M}(\eta_t, \eta_t) = O(1)$ since $A_{\partial M}(\bar{\eta}_t, \bar{\eta}_t) = O(1)$. Or we can calculate with respect to the rescaling metric as in Remark 3.1.

Since g and $\bar{g}$ has isometric tangent cone at p_0, we see that the limit of the surface $(\Sigma_t, t^{-2}g|_{\Sigma_t})$ as $t \to 0$ is $(\Sigma, a^2 g_{\mathbb{S}^2})$ where Σ is a scaling copy of a geodesic disk of radius $\rho(0) = \lim_{t\to 0}\rho(t) > 0$ in the standard 2-sphere. Consider the spaces

$$\mathcal{Y} = \left\{ u \in C^{2,\alpha}(\Sigma) \cap C^{1,\alpha}(\bar{\Sigma}) : \int_\Sigma u = 0 \right\}$$

and

$$\mathcal{Z} = \left\{ u \in C^{0,\alpha}(\Sigma) : \int_D u = 0 \right\}.$$

Given small $\delta > 0$ and $\varepsilon > 0$, we define the map

$$\Phi : (-\varepsilon, \varepsilon) \times B(0,\delta) \to \mathcal{Z} \times C^{1,\alpha}(\partial\Sigma)$$

given by $\Phi(t,u) = (\Phi_1(t,u), \Phi_2(t,u))$ where Φ_i, $i = 1, 2$ are given by

$$\begin{aligned}\Phi_1(t,u) =& (H_{t,t^2u} - \bar{h}_{t,t^2u}) - \frac{1}{|\Sigma|}\int_\Sigma (H_{t,t^2u} - \bar{h}_{t,t^2u}),\\ \Phi_2(t,u) =& t^{-1}(\langle X_{t,t^2u}, N_{t,t^2u}\rangle - \cos\bar{\gamma}_{t,t^2u})\end{aligned}$$

for $t \neq 0$. Here $B(0,\delta) \subset \mathcal{Y}$ is an open ball with radius δ in the $C^{2,\alpha}$ norm and the integration on Σ is with respect to the metric $g_{\mathbb{S}^2}$. We extend $\Phi(t,u)$ to $t = 0$ by taking limits, that is,

$$\Phi(0,u) = \lim_{t\to 0}\Psi(t,u).$$

We have the following proposition.

Proposition 3.3. *For each $t \in [0,\varepsilon)$ with ε small enough, we can find $u_t = u(\cdot,t) \in C^{2,\alpha}(\Sigma) \cap C^{1,\alpha}(\bar{\Sigma})$ such that $\int_\Sigma u(\cdot,t) = 0$ and*

$$\Phi(t, u_t) = (0,0).$$

In particular, each of the surfaces Σ_{t,t^2u} have constant $\lambda_t := H_{t,t^2u} - \bar{h}_{t,t^2u}$ and prescribed angles $\gamma_{t,t^2u} = \bar{\gamma}_{t,t^2u}$. Moreover, $\lambda_t \leqslant 0$ for all small $t \in [0,\varepsilon)$.

Before proving this proposition, we give a variational lemma.

Lemma 3.4. *Suppose that $(\Omega, \hat{g})$ is a compact manifold with piecewise smooth boundary $\partial\Omega$ and Σ is a relatively open, smooth subset of $\partial\Omega$. Let g_s be a smooth family of metrics indexed by $s \in [0,\varepsilon)$ such that $g_s \to \hat{g}$ as $s \to 0$, let $h_s = g_s - \hat{g}$. We now omit the subscript on h_s. Let ν be the unit outward normal of $\partial\Omega$ in (Ω, g), H_g and A_g be the mean curvatures and the second fundamental form of $\partial\Omega$ in (Ω, g) computed with respect to the unit normal pointing outward of Ω, and γ be the dihedral angles formed by Σ and $\partial\Omega\backslash\Sigma$ with respect to the metric g. We put a hat at appropriate places for the geometric quantities with respect to $\hat{g}$.*

Then

$$\begin{aligned}&2\left[-\int_\Sigma (H_g - H_{\hat{g}}) + \int_{\partial\Sigma} \tfrac{1}{\sin\gamma_{\hat{g}}}(\cos\gamma_{\hat{g}} - \cos\gamma_g)\right]\\ =& \int_\Omega ((R_g - R_{\hat{g}}) + \langle \mathrm{Ric}_{\hat{g}}, h\rangle_{\hat{g}}) + 2\int_{\partial\Omega\backslash\Sigma}(H_g - H_{\hat{g}}) + \int_{\partial\Omega}\langle h, A_{\hat{g}}\rangle + O(s^2).\end{aligned}$$

Here, we have used $O(s^2)$ to denote a remainder term comparable to $|h|_{\hat{g}}^2 + |h|_{\hat{g}}|\hat{\nabla} h|_{\hat{g}} + |\hat{\nabla} h|_{\hat{g}}^2$.

Proof. From the variational formulas of the scalar curvature and the mean curvature, we have

$$R_g - R_{\hat{g}} = -\langle \mathrm{Ric}_{\hat{g}}, h\rangle_{\hat{g}} - \mathrm{div}_{\hat{g}}(\mathrm{d}(\mathrm{tr}_{\hat{g}}\, h) - \mathrm{div}_{\hat{g}}\, h) + O(s^2),$$

and

$$2(H_g - H_{\hat{g}}) = (\mathrm{d}(\mathrm{tr}_{\hat{g}}\, h) - \mathrm{div}_{\hat{g}}\, h)(\hat{\nu}) - \mathrm{div}_{\sigma} Y - \langle h, A_{\hat{g}}\rangle_{\sigma} + O(s^2) \tag{3.7}$$

where Y is the tangential component dual to the 1-form $h(\cdot, \hat{\nu})$. For the explicit form of the remainder terms, refer to [BM11, Proposition 4] and [MP21].

We integrate the variation of the mean curvature (3.7) on the boundary $\partial\Omega$ with respect to the metric $\hat{g}$, we see

$$\int_{\partial\Omega} [(\mathrm{d}(\mathrm{tr}_{\hat{g}}\, h) - \mathrm{div}_{\hat{g}}\, h)(\hat{\nu}) - \mathrm{div}_{\hat{\sigma}} Y - \langle h, A_{\hat{g}}\rangle] = 2\int_{\partial\Omega} (H_g - H_{\hat{g}}) + O(s^2).$$

By the divergence theorem and the variation of the scalar curvature,

$$\int_{\partial\Omega} (\mathrm{d}(\mathrm{tr}_{\hat{g}}\, h) - \mathrm{div}_{\hat{g}}\, h)(\hat{g}) = \int_{\Omega} [-(R_g - R_{\hat{g}}) - \langle \mathrm{Ric}_{\hat{g}}, h\rangle_{\hat{g}}] + O(s^2).$$

For the term $\int_{\partial\Omega} \mathrm{div}_{\hat{\sigma}} Y$, we follow [MP21, (3.18)] and obtain

$$\int_{\partial\Omega} \mathrm{div}_{\hat{\sigma}} Y = \int_{\Sigma} \mathrm{div}_{\hat{g}} Y + \int_{\partial\Omega\setminus\Sigma} \mathrm{div}_{\hat{\sigma}} Y = 2\int_{\partial\Sigma} \tfrac{1}{\sin\hat{\gamma}}(\cos\hat{\gamma} - \cos\gamma) + O(s^2).$$

Collecting all the formulas in the proof proves the lemma. □

Lemma 3.5 implies the following by taking the difference of two families of metrics.

Corollary 3.5. *Assume $(\Omega, \hat{g})$ is the manifold from Lemma 3.5, for two family of metrics $\{g_i\}_{i=1,2}$ close to $\hat{g}$ indexed both by a small parameter s, we have*

$$2\left[-\int_{\Sigma}(H_{g_2} - H_{g_1}) + \int_{\partial\Sigma} \tfrac{1}{\sin\hat{\gamma}}(\cos\gamma_{g_1} - \cos\gamma_{g_2})\right]$$

$$= \int_{\Omega}((R_{g_2} - R_{g_1}) + \langle \mathrm{Ric}_{\hat{g}}, g_2 - g_1\rangle_{\hat{g}}) + 2\int_{\partial\Omega\setminus\Sigma}(H_{g_2} - H_{g_1}) + \int_{\partial\Omega}\langle g_2 - g_1, A_{\hat{g}}\rangle + O(s^2).$$

Now we are ready to prove Proposition 3.3.

Proof of Proposition 3.3. The proof is similar to [CW23]. We bring up only the main differences.

Because that the right hand of both (3.4) and (3.6) converge to Δu and $\frac{\partial u}{\partial \nu}$ (up to a constant) respectively, so we can first follow [CW23, Proposition 4.2] to construct a foliation $\{\Sigma_{t,t^2 u}\}_{t\in[0,\varepsilon)}$ near p_0 with constant $H - \bar{h}$ and $\gamma_{t,t^2 u} = \bar{\gamma}_{t,t^2 u}$ along $\partial\Sigma_{t,t^2 u}$, and then [CW23, Lemma 4.3] to obtain that

$$-\lambda_t |\Sigma_t| = \int_{\Sigma_t}(H_t - \bar{h}_t) + \int_{\partial\Sigma_t} \tfrac{1}{\sin\gamma_t}(\cos\bar{\gamma}_t - \cos\gamma_t) + O(t^3). \tag{3.8}$$

Now we show that $\lim_{t\to 0}\lambda_t \leqslant 0$.

We consider the rescaled set $t^{-1}\Omega_t$ with two rescaled metrics $t^{-2}g$ and $t^{-2}\bar{g}$. Since $\bar{g} = \mathrm{d}t^2 + \psi(t)^2 g_{\mathbb{S}^2}$ and $\psi(t) = at + o(t)$, it is easy to see that $(t^{-1}\Omega_t, t^{-2}\bar{g})$ converges to a truncated metric cone $\Lambda = (0,1]\times D$ with the metric $\varrho := \mathrm{d}s^2 + a^2 s^2 g_{\mathbb{S}^2}$ where $s \in (0,1]$ and $(D, a^2 g_{\mathbb{S}^2})$ is some geodesic disk in a 2-sphere $(\mathbb{S}^2, a^2 g_{\mathbb{S}^2})$. For such notions of convergence, we refer the readers to the text [BBI01, Chapter 8]. We set $D_s = \{s\} \times D$. Since g and $\bar{g}$ has isometric tangent cone at p_0, $(t^{-1}\Omega_t, t^{-2}g)$ converges to (Λ, ϱ) as well. Therefore, we can view $g_1 = t^{-2}g$ and $g_2 = t^{-2}\bar{g}$ (indexed by t) as two metrics on Λ getting closer to ϱ as $t \to 0$. We rescale (3.8) by a factor of t^{-2}, we obtain

$$-\lambda_t |\Sigma_t| t^{-2} = \int_{\Sigma_t} (H_t - \bar{h}_t) t^{-2} + \int_{\partial\Sigma_t} \tfrac{1}{\sin\gamma_t} (\cos\bar{\gamma}_t - \cos\gamma_t) t^{-2} + O(t)$$

which is equivalent to

$$-\lambda_t |D|_{g_1} = \int_D (H_{g_2} - H_{g_1}) + \int_{\partial D} \tfrac{1}{\sin\gamma_t} (\cos\bar{\gamma}_t - \cos\gamma_t) + O(t).$$

In the above the integration done is with respect to the metric g_1 and H_{g_i} are the mean curvature of $\{1\} \times D$ in (Λ, g_i) computed with respect to the normal pointing inside of Λ.

All the comparison in Theorem 1.3 carries over to the rescaled metrics g_1 and g_2 on Λ and it is easy to check that (Λ, ϱ) has nonnegative Ricci curvature (since $a \leqslant 1$), $\partial\Lambda$ has nonnegative second fundamental form computed with respect to the outward unit normal in (Λ, ϱ). We use Corollary 3.5 and arrive that $\lambda_t \leqslant O(t)$, that is,

$$\lim_{t\to 0} \lambda(t) \leqslant 0.$$

Since λ_t satisfies the differential inequality (2.28) and considering the asymptotics $u(\cdot, t) = 1 + O(t)$, $\cot\bar{\gamma} = O(t)$ and $\bar{h} = 2/t + O(1)$ in (2.27), we see that $\lambda_t \leqslant 0$ for all $t \in (0, \varepsilon)$. □

Remark 3.6. The Ricci curvature in Corollary 3.5 blows up near $\{0\} \times D$, however, because we are integrating with respect to the metric ϱ, the volume near $\{0\} \times D$ is small. Also, the difference $g_2 - g_1$ is small. So the blowing up of the Ricci curvature will not cause an issue.

3.3. **Barrier construction with non-isometric tangent cones.** Since $\bar{g} = \mathrm{d}t^2 + \psi(t)^2 g_{\mathbb{S}^2}$, the manifold $(M, \bar{g})$ is topologically a cone near $t = 0$ and it is a point at $t = 0$. According to the assumptions of Theorem 1.3, (M, g) at p_0 also locally resembles a cone, that is,

$$g = \mathrm{d}s^2 + s^2 g_0 + g_1, \tag{3.9}$$

where s is a parameter, g_0 is a metric on a two dimensional disk D and g_1 is small compare to $\mathrm{d}s^2 + s^2 g_0$. In other words, the tangent cone at p_0 is a cone with the metric $\mathrm{d}s^2 + s^2 g_0$.

Now we can also identify M near p_0 as $(0, \varepsilon) \times D$ and t as a function on $(0, \varepsilon) \times D$. Let $(s, x) \in (0, \varepsilon) \times D$, we see that $\tau := s/t$ as a function on M only depends on $x \in D$. So we view τ as a function on D. Since $g \geqslant \bar{g}$ on M, we have that $\tau(x) \geqslant 1$. Now we discuss the case that $\tau(x) \equiv 1$ on D.

Lemma 3.7. *If $\tau \equiv 1$ on D, then $g_0 = a^2 g_{\mathbb{S}^2}$. That is, (M, g) and $(M, \bar{g})$ have isometric tangent cones at p_0.*

Proof. Since $\tau \equiv 1$, so we can rescale $(M, \bar{g})$ and (M, g) by the same scale to obtain a cone $\mathcal{C} = (0, \infty) \times D$ but with two different metrics $\chi_1 = \mathrm{d}t^2 + a^2 t^2 g_{\mathbb{S}^2}$ and $\chi_2 = \mathrm{d}t^2 + t^2 g_0$. For $s > 0$, set $D_s = \{s\} \times D \subset \mathcal{C}$. Since the metric comparison, the mean curvature and the scalar curvature comparison are preserved by rescaling, so $g_0 \geqslant a^2 g_{\mathbb{S}^2}$, the scalar curvature $R_{\chi_2} \geqslant R_{\chi_1}$ and the mean curvature of $\partial\mathcal{C}$ at ∂D_1 satisfies $H_{\chi_2} \geqslant H_{\chi_1}$.

Since both χ_i, $i = 1, 2$ are warped product metrics, the comparison $R_{\chi_2} \geqslant R_{\chi_1}$ reduces to Gaussian curvature comparison $K_2 \geqslant K_1 = a^{-2}$ of (D_1, g_0) and $(D_1, a^2 g_{\mathbb{S}^2})$ by a direct computation of scalar curvature (or Gauss equation). Let κ_i be the geodesic curvatures of ∂D_1 with respect to $\chi_i|_{D_1}$. By direct calculation, the second fundamental form $A^{(i)}_{\partial\mathcal{C}}$ of $\partial\mathcal{C}$ in the direction ∂_t vanishes with respect to both metrics χ_i and the second fundamental form $A^{(i)}_{D_1}$ of D_1 in $\mathcal{C}$ with respect to χ_i agree. It then follows from $H_{\chi_2} \geqslant H_{\chi_1}$ and (2.10) that $\kappa_2 \geqslant \kappa_1$.

To summarize, we have comparisons on D_1 that $g_0 \geqslant a^2 g_{\mathbb{S}^2}$, $K_2 \geqslant K_1$ and $\kappa_2 \geqslant \kappa_1$ along ∂D_1. By Gauss-Bonnet theorem, $g_0 \equiv a^2 g_{\mathbb{S}^2}$ on D_1 and it follows that $\chi_1 \equiv \chi_2$. Therefore, (M, g) and $(M, \bar{g})$ have isometric tangent cones at p_0. □

By the above lemma, the case $\tau \equiv 1$ is the case which implies isometric tangent cones of (M, g) and $(M, \bar{g})$ at p_0. This is the case we have already addressed in Subsection 3.2. Without loss of generality, we assume that $\tau \not\equiv 1$.

We first consider the difference of $H - \bar{h}$ of the perturbation for D_s. We now represent $\bar{h}$ at D_s and its value at the graphical perturbations of D_s by ζ to avoid notational confusion. By the first variation of the mean curvatures,

$$\begin{aligned}&(H_{s,s^2u} - \zeta_{s,s^2u}) - (H_s - \zeta_s)\\ =&-\Delta_s u - s^2(\mathrm{Ric}(N_s) + |A_s|^2 + s^{-2}(\zeta_{s,s^2u} - \zeta_s))u + O(s),\end{aligned}$$

where Δ_s is the Laplacian with respect to the metric $s^{-2} g|_{D_s}$.

Remark 3.8. We have $\{(s^{-1}D_s, s^{-2}g|_{D_s})\}_{s>0}$ converges to (D, g_0) as $s \to 0$ by the metric (3.9) near p_0, and to indicate that the limit carries the metric g_0, we use D_0 instead of D only.

Lemma 3.9. *We have that*

$$s^2(|A_s|^2 - s^{-2}(\zeta_{s,s^2u} - \zeta_s)) = (2 - 2\tau) + O(s).$$

Proof. Since $\{(s^{-1}\Lambda_s, s^{-2}g)\}_{s>0}$ converges to a truncated radial cone and $\{(s^{-1}D_s, s^{-2}g|_{D_s})\}_{s>0}$ converges to the section of the radial cone with unit distance to p_0, so the section has second fundamental form -2 and by rescaling,

$$|A_s|^2 = 2s^{-2} + O(s^{-1})$$

as $s \to 0$.

At a point $p = (s, x) \in D_s$, the value of t is given by $t = s\tau(x)$ where x is the projection of p to the second coordinate. Since τ as a function on M only depends on x, we see that the value of the function t at the graphical perturbation $s + s^2 u$ of D_s is given by $(s + s^2 u)\tau$. Since $\bar{h}(t) = 2t^{-1} + O(1)$, so

$$\zeta_{s,s^2u} - \zeta_s = \tfrac{2}{(s+s^2u)\tau} - \tfrac{2}{s\tau} + O(1) = -\tfrac{2\tau}{s^2}(s^2 u) + O(1).$$

Hence

$$s^2(|A_s|^2 + s^{-2}(\zeta_{s,s^2u} - \zeta_s)) = (2 - 2\tau) + O(s),$$

which proves the lemma. □

Let $f = \lim_{s\to 0} s^2(\mathrm{Ric}(N_s) + |A_s|^2 + s^{-2}(\zeta_{s,s^2u} - \zeta_s))$ which is a function on the limit D_0, so

$$\lim_{s\to 0}[(H_{s,s^2u} - \zeta_{s,s^2u}) - (H_s - \zeta_s)] = -\Delta_0 u - fu,$$

where Δ_0 is the Laplacian of D_0. Recall that $\mathrm{Ric}(N_s) = O(s^{-1})$, so

$$f = 2 - 2\tau \text{ on } D_0.$$

Let α_s be the dihedral angles formed by ∂M and D_s, and α_{s,s^2u} be the angles formed by ∂M and the graphical perturbation of D_s.

Lemma 3.10. *The dihedral angles α_s formed by ∂M and D_s approach $\pi/2$ as $s \to 0$.*

Proof. Since $\{(s^{-1}\Lambda_s, s^{-2}g)\}_{s>0}$ converges to a truncated radial cone, $\{(s^{-1}D_s, s^{-2}g|_{D_s})\}_{s>0}$ converges to the section of the radial cone with unit distance to p_0, and this section is orthogonal to the radial direction in the limit, so the intersection angles of ∂M and D_s approaches $\pi/2$ as $s \to 0$. □

Lemma 3.11. *We have that $A_{\partial M}(\eta, \eta) = O(1)$.*

Proof. The lemma can be deduced from that η is approximately the radial direction ∂_s as $s \to 0$, the scaling property of $A_{\partial M}$ and the following lemma. □

Lemma 3.12. *Let (S, σ) be a 2-surface with boundary and $(C = [0, \infty) \times S, \mathrm{d}s^2 + s^2\sigma)$ be the cone over (S, σ). Then the second fundamental form of ∂C in C in the direction ∂_t vanishes.*

Proof. Let Z be a tangent vector field over Σ, then by direct calculation $\nabla_{\partial_t} Z = \nabla_X \partial_t = s^{-1}Z$. So $\langle \nabla_{\partial_t} Z, \partial_t \rangle = 0$ since on C the metric is $\mathrm{d}t^2 + t^2\sigma$. Due to the same reason, the unit normal vector Z of ∂C in M is tangent to Σ, so the claim is proved. □

We are interested in the difference between α_{s,s^2u} and the value of $\bar{\gamma}$ which to avoid confusion we denote by β_s (β_{s,s^2u}) at (the graphical perturbation s^2u of) D_s. Using the relation of s and t, $\beta = \bar{\gamma}_{s/\tau, s^2u/\tau}$. By the expansion of angles (see (3.5)), we see

$$\cos\alpha_{s,s^2u} - \cos\alpha_s = -\sin\alpha_s \tfrac{\partial u}{\partial \nu_s} + s(-\cos\alpha_s A(s^{-1}\nu_s, s^{-1}\nu_s) + A_{\partial M}(\eta_s, \eta_s))u + O(s^2).$$

And

$$s^{-1}(\cos\beta_{s,s^2u} - \cos\beta_s) = su\tau^{-1}\nabla_{\eta_{s/\tau}} \cos\bar{\gamma}_{s/\tau, s^2u/\tau} + O(s^2)$$

Since each Σ_t is stable capillary minimal surface under the metric $\bar{g}$, so we know that

$$\tfrac{1}{\sin\bar{\gamma}} \nabla_{\eta_t} \cos\bar{\gamma} = -\cos\bar{\gamma} A(\nu_t, \nu_t) + A_{\partial M}(\eta_t, \eta_t).$$

Based on the above asymptotic analysis and Lemmas 3.10 and 3.11, we see

$$\lim_{s\to 0}[s^{-1}(\cos\alpha_{s,s^2u} - \cos\alpha_s) - s^{-1}(\cos\beta_{s,s^2u} - \cos\beta_s)] = -\tfrac{\partial u}{\partial \nu_0}$$

on ∂D_0 where ν_0 is the outward normal of ∂D_0 in D_0. By elliptic strong maximum principle, the operator

$$(-\Delta_0 - f, -\tfrac{\partial}{\partial \nu_0}) : C^{2,\alpha}(D_0) \cap C^{1,\alpha}(\bar{D}_0) \to C^{0,\alpha}(D_0) \times C^{0,\alpha}(\partial D_0)$$

is an isomorphism since $f \leqslant 0$ in D_0 due to Lemma 3.9 and $\tau \gneqq 1$. In other words, we can specify the limits

$$\lim_{s\to 0}[(H_{s,s^2u} - \zeta_{s,s^2u}) - (H_s - \zeta_s)]$$
$$\text{and } \lim_{s\to 0}[s^{-1}(\cos\alpha_{s,s^2u} - \cos\alpha_s) - s^{-1}(\cos\beta_{s,s^2u} - \cos\beta_s)]$$

by choosing a suitable $u \in C^{2,\alpha}(D_0) \cap C^{1,\alpha}(\bar{D}_0)$.

We have these facts: by Lemma 3.10, both α_s and β_s tend to $\pi/2$ as $s \to 0$, so $\lim_{s\to 0} s^{-1}(\alpha_s - \beta_s)$ is a function on ∂D_0; $H_s - \zeta_s = (2-2\tau)s^{-1} + O(1)$;

$$H_{s,s^2u} - \zeta_{s,s^2u} = (2-2\tau)s^{-1} + O(1) \tag{3.10}$$

for small $s > 0$ with a remainder term depending on u. Hence, we can specify a function u to counter-effect the $O(1)$ remainder term in $H_s - \zeta_s$ and make the remainder term in (3.10) strictly negative. That is, we can specify a function u such that

$$\lim_{s\to 0}(H_{s,s^2u} - \zeta_{s,s^2u} - (2-2\tau)s^{-1}) = u_0 \text{ in } D_0,$$
$$\lim_{s\to 0} s^{-1}(\cos\alpha_{s,s^2u} - \cos\beta_{s,s^2u}) < 0 \text{ along } \partial D_0,$$

for some negative function $u_0 \in C^{0,\alpha}(\bar{D}_0)$. Recall the definitions of ζ, τ, β, and by continuity, there exists a surface $\Sigma_- \subset M$ satisfying

$$H - \bar{h} < 0 \text{ in } \Sigma_- \text{ and } \alpha > \bar{\gamma} \text{ along } \partial\Sigma_-.$$

This surface Σ_- is a a lower barrier in the sense of Definition 2.7.

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. Assume that g and $\bar{g}$ do not have isometric tangent cone at p_0, then we can construct a barrier Σ_- such that $H - \bar{h} < 0$ in Σ_- and the angle $\alpha > \bar{\gamma}$ along $\partial\Sigma_-$. But due to Theorem 1.2 (see also Remark 2.8), this is not possible. So g and $\bar{g}$ have isometric tangent cones at p_0, then by the construction of the foliation in Theorem 3.3, again we have a barrier near $t = 0$, but the barrier condition is now not strict. We can extend the rigidity $g = \bar{g}$ in Theorem 1.2 beyond the barrier and to all of M. □

Remark 3.13. By considering only the mean curvature, this provide an alternative proof of Theorem 1.1 in dimension 3. Moreover, we allow conical metrics of $(\mathbb{S}^3, g)$ at two antipodal points.

Remark 3.14. During the construction of barriers in the case of non-isometric cones, the Gauss-Bonnet theorem is only used in Lemma 3.7.

4. Construction of barriers (II)

In this section, we prove Theorem 1.5. Our method is similar to the previous work [CW23].

4.1. **Proof of case *a)* of Theorem 1.5.** Letting Σ_t and Ω_t be given in (3.1) and (3.2), the sequence $\{(t^{-1}M, t^{-2}\bar{g})\}_{t>0}$ converges to some right circular cone $\bar{C}$ in $\mathbb{R}^3$ equipped with a flat metric $g_{\mathbb{R}^3}$ as $t \to 0$. Then $\{(t^{-1}M, t^{-2}g)\}_{t>0}$ converges to the same cone $\bar{C}$ but with a different constant metric g_0. The cone $(\bar{C}, g_0)$ is also a circular cone, which might be oblique if represented in $(\mathbb{R}^3, g_{\mathbb{R}^3})$. To see what $g_{\mathbb{R}^3}$ is, we make use of another coordinate. We write the metric $g_{\mathbb{S}^2}$ of 2-spheres of (1.2) in a conformal form. It is well known that there exists a diffeomorphism $\Phi : \mathbb{R}^2 \cup \{\infty\} \to \mathbb{S}^2$ such that the pull back metric of the round metric $g_{\mathbb{S}^2}$ on $\mathbb{S}^2$ is

$$\Phi^*(g_{\mathbb{S}^2}) = 4|\mathrm{d}y|^2(1+|y|^2)^{-2},\ y \in \mathbb{R}^2.$$

It is easy to see that in this coordinate system that

$$g = \mathrm{d}t^2 + 4\psi(t)^2|\mathrm{d}y|^2(1+|y|^2)^{-2} \tag{4.1}$$

and $g_{\mathbb{R}^3}$ is just $\mathrm{d}t^2 + 4\psi(t_-)^2|\mathrm{d}y|^2$.

We have the existence of a barrier if (M, g) and $(M, \bar{g})$ have non-isometric tangent cones at p_0.

Lemma 4.1. *Let M be given as in case a) of Theorem 1.5. If the tangent cones of (M, g) and $(M, \bar{g})$ at p_0 are not isometric, assume that the mean curvature comparison and the metric comparison hold near p_0, then there exists a surface Σ_- satisfying*

$$H - \bar{h} < 0 \text{ in } \Sigma_- \text{ and } \alpha > \bar{\gamma} \text{ along } \partial\Sigma_-$$

as the above. This surface Σ_- is a barrier in the sense of Definition 2.7.

Proof. First, we note that the mean curvature comparison and the metric comparison (we only need boundary metric comparison) are preserved in the limits. By non-isometry of tangent cones and by the angle comparison of [CW23, Proposition 4.9], there exists a plane P in $\bar{C}$ such that the the dihedral angles formed by $\partial\bar{C}$ and P in the metric g_0 are everywhere larger than $\bar{\gamma}(t_-)$.

We gain a lot of freedom to construct the barrier from the *strict* comparison of angles. The rest of the argument is complete analogous to [CW23, Proposition 4.10]. All is needed is a coordinate system to carry out the construction of Σ_t. The coordinate system (4.1) suffices for our purpose. □

Remark 4.2. Note that the scalar curvature comparison is not needed here.

Proof of case a) of Theorem 1.5. First, the tangent cones of (M, g) and $(M, \bar{g})$ at p_0 must be isometric. Indeed, by Lemma 4.1 and Theorem 1.2, the barrier constructed in Lemma 4.1 cannot have $H - \bar{h} < 0$ in Σ_- and $\alpha < \bar{\gamma}$ hold strictly along $\partial\Sigma_-$.

By following Subsection 3.2, we can construct graphical perturbations Σ_{t,t^2u} of Σ_t which satisfy Proposition 3.3. For every sufficiently small $t > 0$, Σ_{t,t^2u} is a barrier in the sense of Definition 2.7, we conclude that $g = \bar{g}$ for the region bounded by Σ_{t,t^2u} and $P_+ \cap \partial M$ for every $t > 0$ from Theorem 1.2. Hence, case *a)* of Theorem 1.5 is proved. □

4.2. **Proof of case *b)* of Theorem 1.5.** This part is a slightly extension of the argument in Section 5 in our previous paper [CW23]. So we only sketch the key steps here and refer to the previous paper for more details.

Suppose M is given by

$$M = \left\{(t^2, r, \theta) : t \in [0, \varepsilon), r \in [0, \phi(t)), \theta \in \mathbb{S}^1\right\},$$

near $p^- = O$ for some smooth function $\phi(0) = 0$. Note that we need to assume $\psi(0) \neq 0$, otherwise the manifold M will have a cusp at point $O = (0, 0, 0)$.

To better illustrate the situation, we use the stereographic projection to describe the metric on sphere. So the manifold M can be written as

$$M = \left\{(t^2, x, y) : t \in [0, \varepsilon), x^2 + y^2 \leq \tan^2(\phi(t))\right\},$$

and the background metric $\bar{g}$ is given by

$$\bar{g} = dt^2 + \frac{4\psi^2(t)(dx_1^2 + dx_2^2)}{(1 + |x|^2)^2} = dt^2 + 4\psi^2(0)(dx_1^2 + dx_2^2) + O(t) + O(|x|^2),$$

where $|x| = \sqrt{x_1^2 + x_2^2}$. For simplicity, we denote $\bar{g}_0 = dt^2 + \psi^2(0)(dx_1^2 + dx_2^2)$ as the linearised part of $\bar{g}$ at O. After a suitable rotation, we can write $g = g_0 + th + O(t^2)$ for some constant metric g_0 defined as

$$g_0 = a_{33}dt^2 + (a_{11}dx_1^2 + a_{22}dx_2^2) + 2a_{13}dx_1dt + 2a_{23}dx_2dt, \tag{4.2}$$

where the matrix

$$\begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

is positive definite and satisfies $a_{11}, a_{22} \geq 4\psi^2(0)$, $a_{33} \geq 1$. We assume ∂M is given by

$$\partial M = \left\{ (\zeta(|x|^2), x) : |x| \leq \varepsilon \right\}$$

near O for some smooth function ζ with $\zeta(0) = 0$ and $\zeta'(0) > 0$ and $\varepsilon > 0$ small enough. Indeed, ζ is defined as

$$\zeta(r^2) = (\phi^{-1}(\arctan(r)))^2.$$

and we can check ζ is smooth with $\zeta(0) = 0$, $\zeta'(0) > 0$ by the property of ϕ.

We write a^{ij} as the inverse matrix of a_{ij} and consider the function $G_{s,t}$ defined by

$$G_{s,t}(x_1, x_2) = \zeta'(0)x_1^2(\sqrt{a_{11}a^{33}} + s - 1) + \zeta'(0)x_2^2(\sqrt{a_{22}a^{33}} + s - 1),$$

and the surface $\Sigma_{s,t}$ is defined by

$$\Sigma_{s,t} = \left\{ (G_{s,t}(x), x) : x \in \mathbb{R}^2 \text{ and } G_{s,t}(x) \geq \zeta(|x|^2) \right\}.$$

We use an ellipse E_s to parameterize $\Sigma_{s,t}$ where $E_s \subset \mathbb{R}^2$ is given by

$$E_s := \left\{ \hat{x} \in \mathbb{R}^2 : (b_1 + s)\hat{x}_1^2 + (b_2 + s)\hat{x}_2^2 < 1 \right\}.$$

Then, the surface $\Sigma_{s,t}$ can be written as

$$\Sigma_{s,t}(\hat{x}) := (G_{s,t}(\Phi_{s,t}(\hat{x})), \Phi_{s,t}(\hat{x}))$$

where $\Phi_{s,t} : E_s \to \mathbb{R}^2$ satisfies

$$\Phi_{s,t}(\hat{x}) = \frac{t\hat{x}}{\sqrt{\zeta'(0)}} + O(t^3).$$

We also use $\Sigma_t = \Sigma_{0,t}$ for short. We have the following result by the argument in [CW23].

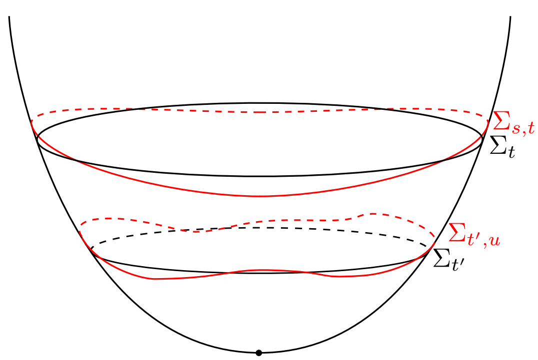

FIGURE 3. Construction of $\Sigma_{s,t}$ and $\Sigma_{t,u}$.

Proposition 4.3. *Suppose the metric g can be written as $g = g_0 + th + O(t^2)$ where g_0 is the constant metric defined in* (4.2) *and h is a bounded symmetric two-tensor. Then, we have*

$$\cos\gamma_{s,t}(\hat{x}) = \cos\bar{\gamma}_{s,t}(\hat{x}) + 2\zeta'(0)\hat{x}_\alpha^2 t^2 \left(\frac{1}{4\psi^2(0)} - \frac{(b_\alpha + s)^2}{a_{\alpha\alpha}a^{33}}\right) + A(\hat{x})t^3 + L(h)t^3 + O(t^4).$$

for any $\hat{x} \in E_s$. Here, $A(\hat{x})$ is a bounded term (not related to t and h) which is also odd symmetric with respect to $\hat{x}$, $L(h)$ is a bounded term (not related to t) relying on h linearly.

Sketch of the proof. We use the same argument for Proposition 5.16 in [CW23] to prove

$$\begin{aligned}
\cos\measuredangle_{g_0}(\Sigma_{s,t}, \partial M) &= 1 - \frac{2\zeta'(0)(b_\alpha + s)^2\hat{x}_\alpha^2}{a_{\alpha\alpha}a^{33}}t^2 + A(\hat{x})t^3 + O(t^4)\\
\cos\bar{\gamma}_{s,t}(\hat{x}) &= 1 - \frac{2\zeta'(0)\hat{x}_\alpha^2 t^2}{4\psi^2(0)} + A(\hat{x})t^3 + O(t^4).
\end{aligned}$$

Together with the Corollary 5.5 in [CW23] (see the proof for Corollary 5.17 in [CW23]), we can establish the result. □

As a corollary, we can easily establish the following result for $\sin\gamma_t$ (cf. Corollary 5.18 in [CW23]):

$$\sin\gamma_t(\hat{x}) = \sin\bar{\gamma}_t(\hat{x}) + O(t^2) = 4\sqrt{\zeta'(0)}\psi(0)|\hat{x}|t + O(t^2), \tag{4.3}$$

Proposition 4.4. *Suppose the conditions in Proposition 4.3 holds. Then, for any $s > 0$, we can find $t_0 > 0$ (might rely on s) such that for any $t < t_0$, we have*

$$\gamma_{s,t}(\hat{x}) > \bar{\gamma}_{s,t}(\hat{x})$$

for any $\hat{x} \in \partial E_s$.

We need to analyze the asymptotic behavior of mean curvature. We define the following mean curvatures:

$$\begin{aligned}
H_{s,t}^g(\hat{x}) &:= \text{Mean curvature of } \Sigma_{s,t} \text{ at } \Sigma_{s,t}(\hat{x}) \text{ under metric } g,\\
H_{s,t,\partial M}^g(\hat{x}) &:= \text{Mean curvature of } \partial M \text{ at } (\varphi(|\Phi_{s,t}(\hat{x})|^2), \Phi_{s,t}(\hat{x})) \text{ under metric } g.
\end{aligned}$$

Proposition 4.5. *Suppose the metric g can be written as $g = g_0 + th + O(t^2)$ where g_0 is a constant metric defined in* (4.2), *and h is a bounded symmetric two-tensor. Then, we have the following formula for the behavior of mean curvature*

$$H_t^g(\hat{x}) = H_{t,\partial M}^g(\hat{x}) - H_{t,\partial M}^{\bar{g}}(\hat{x}) - 2\zeta'(0)\left(\frac{1}{\psi(0)} - \frac{1}{\sqrt{a_{11}}} - \frac{1}{\sqrt{a_{22}}}\right) + tL(\hat{x}) + O(t^2), \tag{4.4}$$

for any $\hat{x} \in E$. Here, we write $H_t^g = H_{0,t}^g$ and $H_{t,\partial M}^g = H_{0,t,\partial M}^g$ for short.

Sketch of the proof. We can establish the following mean curvature relations under metric g_0 (cf. Proposition 5.21 in [CW23], note the sign difference due to the different choice of the normal vector):

$$H_t^{g_0}(\hat{x}) - H_{t,\partial M}^{g_0}(\hat{x}) = \frac{2\zeta'(0)}{\sqrt{a_{11}}} + \frac{2\zeta'(0)}{\sqrt{a_{22}}} + tL(\hat{x}) + O(t^2),$$

Note that by the proof for Corollary 5.9 in [CW23], we can establish the following results:

$$H^g_{t,\partial M}(\hat{x}) - H^{g_0}_{t,\partial M}(\hat{x}) = H^g_t(\hat{x}) - H^{g_0}_t(\hat{x}) + tL(\hat{x}) + O(t^2).$$

In particular, it implies

$$\begin{aligned} H^g_t(\hat{x}) &= H^{g_0}_t(\hat{x}) - H^{g_0}_{t,\partial M}(\hat{x}) + H^g_{t,\partial M}(\hat{x}) + tL(\hat{x}) + O(t^2) \\ &= H^g_{t,\partial M}(\hat{x}) + 2\zeta'(0)\left(\frac{1}{\sqrt{a_{11}}} + \frac{1}{\sqrt{a_{22}}}\right) + tL(\hat{x}) + O(t^2) \\ &= H^g_{t,\partial M}(\hat{x}) - H^{\bar{g}}_{t,\partial M}(\hat{x}) - 2\zeta'(0)\left(\frac{1}{\psi(0)} - \frac{1}{\sqrt{a_{11}}} - \frac{1}{\sqrt{a_{22}}}\right), \end{aligned}$$

where we have used the fact that $H^{\bar{g}_0}_{t,\partial M}(\hat{x}) = -\frac{2\zeta'(0)}{\psi(0)} + \bar{h}(0) + O(t^2)$ by a direct computation. □

Now, we consider

$$H_0 := \lim_{t\to 0} H^g_t(\hat{x}),$$

which is well-defined by (4.4) (the limit does not depend on the choice of $\hat{x}$.)

We have two subcases to consider.

If $H_0 < \bar{h}(0)$, then we can use the continuation of $H^g_{s,t}$ with respect to s and t, together with Proposition 4.4, we can show the following results (cf. Proposition 5.10 in [CW23]).

Proposition 4.6. *Suppose the metric g can be written as $g = g_0 + th + O(t^2)$ where g_0 is a constant metric defined in (4.2), and h is a bounded symmetric two-tensor. If $H_0 < \bar{h}(0)$, we can choose some $s > 0, t > 0$ small such that $H^g_{s,t}(\hat{x}) > \bar{h}(\Sigma_{s,t}(\hat{x}))$ for any $\hat{x} \in \partial E_s$ and $\gamma_{s,t}(\hat{x}) < \bar{\gamma}_{s,t}(\hat{x})$ for each $\hat{x} \in \partial E_s$.*

Now, we focus on the case $H_0 = \bar{h}(0)$. In particular, it implies $a_{11} = a_{22} = 2\psi(0)$ and $H^g_{\partial M}(O) = H^{\bar{g}}_{\partial M}(O)$.

Then, we need to construct a foliation near O. We define the vector field $Y_t(\hat{x}) := \frac{\partial}{\partial t}\Sigma_t(\hat{x})$. Given $u \in C^{1,\alpha}(\bar{E}) \cap C^{2,\alpha}(E)$ where $E = E_0$, we can define the perturbation surface $\Sigma_{t,u}$ by

$$\Sigma_{t,u} := \left\{\Sigma_{t+\frac{u}{\langle Y_t(\hat{x}), N_t(\hat{x})\rangle}}(\hat{x}) : \hat{x} \in E\right\}$$

where $N_t(\hat{x})$ is the unit normal vector field of Σ_t.

Recall that

$$E = \left\{\hat{x} : 2\psi(0)a^{33}|\hat{x}|^2 < 1\right\}.$$

Replacing u by t^3u and assuming that $u = O(1)$, we have

$$\begin{aligned} \frac{H_{t,t^3u} - \bar{h}_{t,t^3u}}{t} &= -\Delta^E_t u + \frac{H_t - \bar{h}_t}{t} + O(t), \\ \frac{\cos\gamma_{t,t^3u} - \cos\bar{\gamma}_{t,t^3u}}{t^3} &= -4\sqrt{\zeta'(0)}\psi(0)|\hat{x}|\frac{\partial u}{\partial \nu^E_t} + (A_{\partial M}(\eta_t,\eta_t) - \cos\gamma_t A(\nu_t,\nu_t) \\ &\quad - \bar{A}_{\partial M}(\bar{\eta}_t,\bar{\eta}_t))u + \frac{\cos\gamma_t - \cos\bar{\gamma}_t}{t^3} + O(t), \end{aligned}$$

where Δ^E_t denotes the Laplacian-Beltrami operator on E under the metric $\frac{1}{t^2}\Sigma^*_t(g)$, and ν^E_t is the unit normal vector field of ∂E under the metric $\frac{1}{t^2}\Sigma^*_t(g)$. Here, we have used (4.3).

By using the same argument for Proposition 5.27 in [CW23], together with the asymptotic behavior of mean curvature, for each $t \in (0, \varepsilon)$ sufficiently small, we can find $u_t(\cdot) = u(\cdot, t)$ such that the mean curvature $H_{t,t^3 u_t}$ is $\bar{h}_{t,t^3 u} + t\lambda(t)$ where $\lambda(t)$ is a function only depends on t, the contact angle $\gamma_{t,t^3 u_t} = \bar{\gamma}_{t,t^3 u_t}$, and u satisfies the following

$$\lim_{t\to 0}(u(\hat{x}, t) + u(-\hat{x}, t)) = 0$$

for any $\hat{x} \in E$. A finer analysis of λ_t will give $\lambda_t < 0$ for t sufficiently small (cf. Proposition 5.28 in [CW23]), and it leads to the following.

Proposition 4.7. *We can construct a surface Σ_- near O such that the mean curvature of Σ_- is not greater than $\bar{h}$ and it has prescribed contact angle $\bar{\gamma}$ with ∂M.*

Proof of case b) of Theorem 1.5. If $\rho(t)$ satisfies b) in Theorem 1.5, then we can use Proposition 4.6 or Proposition 4.7 depending on the value of H_0 to construct a barrier surface Σ_- with mean curvature not greater than $\bar{h}$ and prescribed contact angle $\bar{\gamma}$ with ∂M. Then, we can use Theorem 1.2 to extend the rigidity to all of M. $\square$

References

[Amb15] Lucas C. Ambrozio. Rigidity of area-minimizing free boundary surfaces in mean convex three-manifolds. *J. Geom. Anal.*, 25(2):1001–1017, 2015.

[BBHW24] Christian Bär, Simon Brendle, Bernhard Hanke, and Yipeng Wang. Scalar curvature rigidity of warped product metrics. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 20:Paper No. 035, 26, 2024.

[BBI01] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in Metric Geometry*, volume 33 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, Rhode Island, 2001.

[BBN10] Hubert Bray, Simon Brendle, and Andre Neves. Rigidity of area-minimizing two-spheres in three-manifolds. *Comm. Anal. Geom.*, 18(4):821–830, 2010.

[BM11] Simon Brendle and Fernando C. Marcques. Scalar Curvature Rigidity of Geodesic Balls in s^n. *Journal of Differential Geometry*, 88(3):379–394, 2011.

[CLZ24] Jianchun Chu, Man-Chun Lee, and Jintian Zhu. Llarull's theorem on punctured sphere with $L\infty$ metric. *arXiv: 2405.19724*, 2024.

[CW23] Xiaoxiang Chai and Gaoming Wang. Scalar curvature comparison of rotationally symmetric sets. *arXiv/2304.13152*, 2023.

[CW24] Xiaoxiang Chai and Xueyuan Wan. Scalar curvature rigidity of domains in a warped product. *arXiv:2407.10212 [math]*, 2024.

[CWXZ24] Simone Cecchini, Jinmin Wang, Zhizhang Xie, and Bo Zhu. Scalar curvature rigidity of the four-dimensional sphere. *arXiv: 2402.12633v2*, 2024.

[CZ24] Simone Cecchini and Rudolf Zeidler. Scalar and mean curvature comparison via the Dirac operator. *Geom. Topol.*, 28(3):1167–1212, 2024.

[DPM15] G. De Philippis and F. Maggi. Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law. *Arch. Ration. Mech. Anal.*, 216(2):473–568, 2015.

[FCS80] Doris Fischer-Colbrie and Richard Schoen. The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. *Communications on Pure and Applied Mathematics*, 33(2):199–211, 1980.

[Gro21] Misha Gromov. Four Lectures on Scalar Curvature. *arXiv:1908.10612 [math]*, 2021.

[HKKZ] Sven Hirsch, Demetre Kazaras, Marcus Khuri, and Yiyue Zhang. Rigid comparison geometry for riemannian bands and open incomplete manifolds. (arXiv:2209.12857).

[HLS23] Yuhao Hu, Peng Liu, and Yuguang Shi. Rigidity of 3D spherical caps via μ-bubbles. *Pacific J. Math.*, 323(1):89–114, 2023.

[KY24] Dongyeong Ko and Xuan Yao. Scalar curvature comparison and rigidity of 3-dimensional weakly convex domains. *arXiv:2410.20548*, 2024.

[Li20] Chao Li. A polyhedron comparison theorem for 3-manifolds with positive scalar curvature. *Invent. Math.*, 219(1):1–37, 2020.

[Lis10] Mario Listing. Scalar curvature on compact symmetric spaces. *arXiv:1007.1832 [math]*, 2010.

[Lla98] Marcelo Llarull. Sharp estimates and the Dirac operator. *Math. Ann.*, 310(1):55–71, 1998.

[Lot21] John Lott. Index theory for scalar curvature on manifolds with boundary. *Proc. Amer. Math. Soc.*, 149(10):4451–4459, 2021.

[MO89] Maung Min-Oo. Scalar curvature rigidity of asymptotically hyperbolic spin manifolds. *Math. Ann.*, 285(4):527–539, 1989.

[MP21] Pengzi Miao and Annachiara Piubello. Mass and Riemannian Polyhedra. *arXiv:2101.02693 [gr-qc]*, 2021.

[RS97] Antonio Ros and Rabah Souam. On stability of capillary surfaces in a ball. *Pacific J. Math.*, 178(2):345–361, 1997.

[SY79a] R. Schoen and Shing Tung Yau. Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. *Ann. of Math. (2)*, 110(1):127–142, 1979.

[SY79b] Richard Schoen and Shing Tung Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65(1):45–76, 1979.

[WWZ24] Jinmin Wang, Zhichao Wang, and Bo Zhu. Scalar-mean rigidity theorem for compact manifolds with boundary. *arXiv: 2409.14503*, 2024.

[WX23a] Jinmin Wang and Zhizhang Xie. Dihedral rigidity for submanifolds of warped product manifolds. *arXiv: 2303.13492*, 2023.

[WX23b] Jinmin Wang and Zhizhang Xie. Scalar curvature rigidity of degenerate warped product spaces. *arXiv: 2306.05413*, 2023.

[Ye91] Rugang Ye. Foliation by constant mean curvature spheres. *Pacific Journal of Mathematics*, 147(2):381–396, 1991.

Department of Mathematics, POSTECH, Pohang, Gyeongbuk, South Korea
Email address: xxchai@kias.re.kr, xxchai@postech.ac.kr

Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, China
Email address: gmwang@tsinghua.edu.cn