SCALAR CURVATURE COMPARISON THEOREM OF WEAKLY CONVEX ROTATIONALLY SYMMETRIC SETS

Abstract. Let (M, g) be a compact 3-manifold with nonnegative scalar curvature $R_g \geqslant 0$. The boundary ∂M is diffeomorphic to the boundary of a rotationally symmetric and weakly convex body $\bar{M}$ in $\mathbb{R}^3$. We call $(\bar{M}, \delta)$ a model or a reference. Let $H_{\partial M}$ and $\bar{H}_{\partial M}$ be respectively the mean curvatures of ∂M in (M, g) and ∂M in $(\bar{M}, \delta)$, σ and $\bar{\sigma}$ be the induced metric from g and δ. We show that for some classes of ∂M, if $H_{\partial M} \geqslant \bar{H}_{\partial M}$ and $\sigma \geqslant \bar{\sigma}$, then M is flat. We also generalize this result to the hyperbolic case. Our approach is inspired by Gromov.

1. Introduction

Miao [Mia02, Corollary 1] showed the following via a smoothing technique and a reduction to Schoen and Yau's positive mass theorem [SY79].

Theorem 1.1. *Let (M^3, g) be a compact Riemannian 3-manifold with nonnegative scalar curvature R_g. Assume its boundary ∂M is isometrically embedded in $\mathbb{R}^3$ and if the mean curvature $H_{\partial M}$ of ∂M in M computed with respect to g is no less than the mean curvature $\bar{H}_{\partial M}$ of ∂M in M computed with respect to the Euclidean metric, then (M, g) is flat.*

This result was also obtained by Shi and Tam [ST02, Theorem 1]. In fact, their result is a stronger version which asserted the positivity and rigidity of the Brown-York mass (see [BY93])

$$m_{\mathrm{BY}}(M, g) = \int_{\partial M} (\bar{H}_{\partial M} - H_{\partial M})$$

under the assumptions $R_g \geqslant 0$, $H_{\partial M} > 0$ and ∂M has positive Gaussian curvature. See also [BQ08, ST07] for the hyperbolic analog.

Let σ and $\bar{\sigma}$ be the induced metric of ∂M from g and δ, Gromov (see [Gro21, Section 5.8.1]; *spin extremality of doubly punctured balls*) generalized Miao's result (Theorem 1.1) for standard unit ball by allowing the induced metric $\sigma \geqslant \bar{\sigma}$.

Theorem 1.2. *Assume that (M, g) satisfies the assumptions in Theorem 1.1 except that we assume ∂M is diffeomorphic to a standard 2-sphere and the induced metric satisfies $\sigma \geqslant \bar{\sigma}$, then (M, g) is flat.*

We view Theorem 1.2 as a scalar curvature rigidity result of compact 3-manifolds bounded by a standard 2-sphere, it is natural to ask the same questions manifolds bounded by other surfaces. We consider compact 3-manifolds which are bounded by a weakly convex surface which is also rotationally symmetric with respect to the x^3-coordinate axis in $\mathbb{R}^3$. Without loss of generality, we assume that the surface ∂M lies between the two coordinate planes

$$P_{\pm} = \{x \in \mathbb{R}^3 : x^3 = \pm 1\}$$

and $\partial M \cap P_\pm$ are nonempty. We fix the points $p_\pm = (0, 0, \pm 1)$ and we call p_+ (p_-) north (south) pole. We call $\partial M \cap \{x^3 = s\}$ a boundary s-level set.

We have three cases depending on the geometry of $\partial M \cap P_\pm$:

1. The set $\partial M \cap P_\pm$ is a disk;
2. The set $\partial M \cap P_\pm$ conains only $p_\pm$ and ∂M is conical at $p_\pm$;
3. The set $\partial M \cap P_\pm$ contains only $p_\pm$ and ∂M is smooth at $p_\pm$.

It might happen that at p_+ and p_- have different geometries.

FIGURE 1. Various types of p_+.

We establish the following scalar curvature rigidity.

Theorem 1.3. *Let (M^3, g) be a compact 3-manifold with nonnegative scalar curvature such that its boundary ∂M is diffeomorphic to a weakly convex rotationally symmetric surface in $\mathbb{R}^3$. The boundary ∂M bounds a region $\bar{M}$ (which we call a model or a reference) in $\mathbb{R}^3$, let the induced metric of the flat metric be $\bar{\sigma}$ and the induced metric of g on ∂M be σ. We assume that $\sigma \geqslant \bar{\sigma}$ and $H_{\partial M} \geqslant \bar{H}_{\partial M}$ on $\partial M \cap \{x \in \mathbb{R}^3 : -1 < x^3 < 1\}$.*

(1) *If $\partial M \cap P_\pm$ is a disk, we further assume that $H_{\partial M} \geqslant 0$ at $\partial M \cap P_\pm$ and the dihedral angles forming by $P_\pm$ and $\partial M \backslash (P_+ \cup P_-)$ are no greater than the Euclidean reference.*

(2) *If ∂M is conical at $p_\pm$, we further assume that $\sigma = \bar{\sigma}$ at $p_\pm$.*

Then (M, g) is flat.

When the model $\bar{M}$ is a standard unit ball, Gromov proposed the use of minimal capillary surface. We follow Gromov's argument and make adaptations to handle various geometries of $\partial M \cap P_\pm$. When $\partial M \cap P_\pm$ is a slab, we know easily they are natural barriers for the existence of minimal surfaces. We construct a constant mean curvature foliation near the $p_\pm$ if ∂M is conical there. Interestingly, this approach no longer applies when ∂M is smooth at $p_\pm$.

We are also able to generalize to the hyperbolic case using the upper half space model.

We also remark that Gromov mentioned the cases when $\bar{M}$ is a geodesic ball in hyperbolic space (see [Gro21, Section 3.5]; *on non-spin manifolds and on $\sigma < 0$*) and truncated round cone (see [Gro21, Section 5.9]).

The article is organized as follows:

In Section 2, we introduce the variational problem whose minimiser is vital to our proof. In Section 3, we give the proof of Case 1 of Theorem 1.3 by studying the minimiser of the variational problem introduced in Section 2. In Section 4, we give a proof of the Case 2 of Theorem 1.3 by constructing a constant mean curvature foliation near the conical point.

2. THE VARIATIONAL PROBLEM

2.1. **The functional and its stability.** Let $\bar{\gamma} \in [0, \pi]$ be the contact angle formed by the coordinate planes $\{x^3 = s\}$ with the boundary ∂M under the flat metric, that is,

$$\cos \bar{\gamma} = \langle \bar{X}, \tfrac{\partial}{\partial x^3} \rangle_\delta. \tag{2.1}$$

Obviously, $\cos \bar{\gamma}$ is a function on ∂M to $\mathbb{R}$.

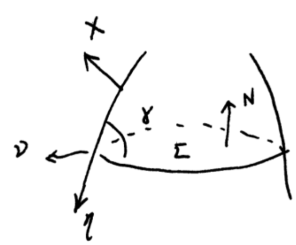

FIGURE 2. Naming of various vectors.

Let E be an open set with its boundary containing the north pole p_+. We define

$$I(E) = |\partial^* E \cap \operatorname{int} M| - \int_{\partial^* E \cap \partial M} \cos \bar{\gamma}. \tag{2.2}$$

We define $I(\emptyset) = 0$ and $I(\{N\}) = 0$ which makes sense because we can take a subset E of M arbitrary small and approaching the pole p_+, such $I(E)$ would approach zero.

Let Σ be a surface with boundary $\partial \Sigma \subset \partial M$ away from $P_\pm$. Then $\partial \Sigma$ separates ∂M into two components, we fix the component closer to the north pole p_+ to be Ω. And Σ separates M into two componets, we fix the component closer to the north pole p_+ to be E.

We define

$$F(\Sigma) := I(E) = |\Sigma| - \int_\Omega \cos \bar{\gamma}. \tag{2.3}$$

Let ϕ_t be a family of diffeomorphisms $\phi_t : \Sigma \to M$ such that $\phi_t(\partial \Sigma) \subset \partial M$ and $\phi_0(\Sigma) = \Sigma$. Let Σ_t be $\phi_t(\Sigma)$ and E_t be the corresponding component separated by Σ_t. Let Y be the vector $\frac{\partial \phi_t}{\partial t}$. Define $\mathcal{A}(t)$ by

$$A(t) = F(\Sigma_t).$$

Letting $f = \langle Y, N\rangle$, by first variation formula,

$$\mathcal{A}'(0) = \int_\Sigma Hf + \int_{\partial\Sigma} \langle Y, \nu - \eta\cos\bar\gamma\rangle. \tag{2.4}$$

We know that Σ is a critical point of (2.1) if and only if $H \equiv 0$ on Σ and $\nu - \eta\cos\bar\gamma$ is normal to ∂M (that is, the angle formed by the vectors N and X is $\bar\gamma$). We call Σ a *minimal capillary* surface if Σ is a critical point of (2.1).

Assume that Σ is minimal capillary, we have the second variation formula

$$\mathcal{A}''(0) = Q(f,f) := -\int_\Sigma (f\Delta f + (|A|^2 + \mathrm{Ric}(N))f^2) + \int_{\partial\Sigma} f(\tfrac{\partial f}{\partial\nu} - qf), \tag{2.5}$$

where q is defined to be

$$q = \tfrac{1}{\sin\bar\gamma} A_{\partial M}(\eta,\eta) - \cot\bar\gamma A(\nu,\nu) + \tfrac{1}{\sin^2\bar\gamma}\partial_\eta\cos\bar\gamma. \tag{2.6}$$

We call Σ *stable minimal capillary* if $\mathcal{A}''(0) \geqslant 0$ for all $f \in C^\infty(\Sigma)$. Define

$$Lf = -\Delta f - (\mathrm{Ric}(N) + |A|^2)f,$$

and

$$Bf = \tfrac{\partial f}{\partial\nu} - qf.$$

We call L the *stability* (or Jacobi) *operator* and B the *boundary stability operator.*

Proof of (2.1). Let Y be a vector field tangent to ∂M, moreover, without loss of generality, we assume that Y is normal to $\partial\Sigma$ in ∂M. The first variation of the mean curvature is given by

$$\delta_Y H = \nabla_{Y^\top} H - \Delta f - (\mathrm{Ric}(N) + |A|^2)f.$$

Note that

$$\delta_Y(\langle X, N\rangle - \cos\bar\gamma) = -\sin\bar\gamma\tfrac{\partial f}{\partial\nu} + \sin\bar\gamma q f + \nabla_{Y^\top}(\langle X, N\rangle - \cos\bar\gamma).$$

Indeed, the term

$$\delta_Y\langle X, N\rangle = -\sin\bar\gamma\tfrac{\partial f}{\partial\nu} + [A_{\partial M}(\eta,\eta) - \cos\bar\gamma A(\nu,\nu)]f + \nabla_{Y^\top}\langle X, N\rangle \tag{2.7}$$

is already computed in [RS97, Appendix], we only have to compute $\delta_Y\cos\bar\gamma$. We have $Y = -\frac{f}{\sin\bar\gamma}\bar\nu$.

$$\delta_Y\cos\bar\gamma = -\tfrac{f}{\sin\bar\gamma}\partial_\eta\cos\bar\gamma + \nabla_{Y^\top}\cos\bar\gamma. \tag{2.8}$$

So we obtained the expression of our q. □

2.2. **Rewrite of the second variation.** Using Schoen-Yau rewrite (essentially Gauss equation, see [SY79]),

$$|A|^2 + \mathrm{Ric}(\nu) = \tfrac{1}{2}R_g - K + \tfrac{1}{2}|A|^2 + \tfrac{1}{2}H^2 \tag{2.9}$$

where K is the Gauss curvature of Σ.

Lemma 2.1 ([Li20, (4.13)]). *Along the boundary $\partial\Sigma$, we have that*

$$\tfrac{1}{\sin\bar\gamma}A_{\partial M}(\eta,\eta) - \cot\bar\gamma A(\nu,\nu) = -H\cot\bar\gamma + \tfrac{H_{\partial M}}{\sin\bar\gamma} - \kappa, \tag{2.10}$$

where κ is the geodesic curvature of $\partial\Sigma$ in Σ.

Proof. We show by direct calculation. Since Σ is of mean curvature H, so $A(\nu,\nu) = H - A(T,T)$. So

$$
\begin{aligned}
& -\cos\bar{\gamma} A(\nu,\nu) + \kappa \sin\bar{\gamma} \\
= & - H\cos\bar{\gamma} + \cos\bar{\gamma} A(T,T) + \langle \nabla_T \nu, T\rangle \sin\bar{\gamma} \\
= & - H\cos\bar{\gamma} - \langle \nabla_T T, \sin\bar{\gamma}\nu + \cos\bar{\gamma} N\rangle \\
= & - H\cos\bar{\gamma} - \langle \nabla_T T, X\rangle \\
= & - H\cos\bar{\gamma} + A_{\partial M}(T,T).
\end{aligned}
$$

Since T and η form an orthonormal basis of ∂M, we have that

$$A_{\partial M}(T,T) = H_{\partial M} - A_{\partial M}(\eta,\eta)$$

and hence

$$-\cos\bar{\gamma} A(\nu,\nu) + \kappa\sin\bar{\gamma} = -H\cos\bar{\gamma} + H_{\partial M} - A_{\partial M}(\eta,\eta).$$

Dividing both sides by $\sin\bar{\gamma}$ finishes the proof of the lemma. □

3. Convex sets between coordinate planes

In this section, we deal with Case 1 of Theorem 1.3. This is the easiest case since the boundary $\partial M \cap P_\pm$ provide natural barriers for minimal capillary surface. Provided the existence of a minimiser to (2.1), it is therefore quite direct to obtain rigidity via constructing CMC capillary foliation near an infinitesimally rigid surface.

3.1. **Minimiser.**

Since $H_{\partial M} \geqslant 0$ at $\partial M \cap P_\pm$ and the dihedral angles forming by $P_\pm$ and $\partial M \backslash (P_+ \cup P_-)$ are no greater than the Euclidean reference, there exists a minimiser E to the problem (2.1). By the classical interior maximum principle and [Li20, Proposition 2.2], ∂E either is either P_+, P_- or lies away from $P_\pm$.

Let $\Sigma = \partial E \cap \operatorname{int} M$, we know that Σ is stable minimal capillary. By [DM15, Theorem 1.3], Σ is free of singularities when the dimension of M is 3. Taking f to be 1 in (2.1), and using (2.2) and (2.1), we have that

$$\int_\Sigma K + \int_{\partial\Sigma} \kappa \geqslant \int_{\partial\Sigma} \tfrac{H_{\partial M}}{\sin\bar{\gamma}} + \tfrac{1}{\sin^2\bar{\gamma}} \partial_\eta \cos\bar{\gamma} + \tfrac{1}{2}\int_\Sigma R_g + |A|^2.$$

Using the bounds $R_g + |A|^2 \geqslant 0$, $H_{\partial M} \geqslant \bar{H}_{\partial M}$ and the Gauss-Bonnet theorem,

$$2\pi\chi(\Sigma) \geqslant \int_{\partial\Sigma} \left(\tfrac{H_{\partial M}}{\sin\bar{\gamma}} + \tfrac{1}{\sin^2\bar{\gamma}} \partial_\eta \cos\bar{\gamma} \right) \mathrm{d}\lambda. \tag{3.1}$$

In the above, we have made the line element $\mathrm{d}\lambda$ under the metric g explicit, because we have to also deal with $\mathrm{d}\bar{\lambda}$ under the flat metric as well.

3.2. **Comparison of σ and $\bar{\sigma}$.**

We show the left hand side of (3.1) has a favorable lower bound.

Lemma 3.1. *If $\partial\Sigma$ is any closed curve separating $\partial M \cap P_\pm$, then*

$$\int_{\partial\Sigma} \left(\frac{H_{\partial M}}{\sin\bar{\gamma}} - \frac{1}{\sin\bar{\gamma}} \frac{\partial\bar{\gamma}}{\partial\eta} \right) \mathrm{d}\lambda \geqslant 2\pi. \tag{3.2}$$

Proof. First, since $\bar\gamma \in (0,\pi)$, we have

$$\tfrac{H_{\partial M}}{\sin\bar\gamma} \geqslant \tfrac{\bar H_{\partial M}}{\sin\bar\gamma}.$$

Let ℓ_s be the curve $\partial M \cap \{x^3 = s\}$ and $\bar\eta$ unit normal of ℓ_s in ∂M pointing in the direction of $-\frac{\partial}{\partial x^3}$. Since M is rotationally symmetric, ℓ_s is a circle and $\bar\gamma$ is constant on ℓ_s, so if $\frac{\partial\bar\gamma}{\partial\bar\eta} = 0$, we obviously have $\frac{\partial\bar\gamma}{\partial\eta} = \frac{\partial\bar\gamma}{\partial\bar\eta} = 0$. If $\frac{\partial\bar\gamma}{\partial\bar\eta} \neq 0$, then by weak convexity, we have that $\frac{\partial\bar\gamma}{\partial\bar\eta} > 0$ and the gradient of $\bar\gamma$ under $\bar\sigma$ is parallel to $\bar\eta$. Since

$$|\eta|_{\bar\sigma} \leqslant |\eta|_\sigma = 1,$$

so

$$\tfrac{\partial\bar\gamma}{\partial\bar\eta} \geqslant \tfrac{\partial\bar\gamma}{\partial\eta}$$

and

$$\int_{\partial\Sigma}\left(\frac{H_{\partial M}}{\sin\bar\gamma} - \frac{1}{\sin\bar\gamma}\frac{\partial\bar\gamma}{\partial\eta}\right)\mathrm{d}\lambda \geqslant \int_{\partial\Sigma}\left(\frac{\bar H_{\partial M}}{\sin\bar\gamma} - \frac{1}{\sin\bar\gamma}\frac{\partial\bar\gamma}{\partial\bar\eta}\right)\mathrm{d}\lambda. \tag{3.3}$$

Now we work with the flat metric. Let e_1 be the unit tangent vector field of ℓ_s, e_2 the unit normal vector field of ℓ_s pointing outside of the region bounded by ℓ_s in the plane $\{x^3 = s\}$ and e_3 be the third positive coordinate direction. Now we can calculate explicitly $\bar X = \cos\bar\gamma e_3 + \sin\bar\gamma e_2$, normal vector field of ∂B, $\bar\eta = -\sin\bar\gamma e_3 + \cos\bar\gamma e_2$. We use D to denote the covariant derivative in $\mathbb{R}^3$. So

$$\begin{aligned}\bar H_{\partial M} =& \langle D_{e_1}e_1, -\bar X\rangle + \langle D_{\bar\eta}\bar\eta, -\bar X\rangle\\ =& \sin\bar\gamma\langle D_{e_1}e_1, -e_2\rangle + \cos\bar\gamma\langle D_{e_1}e_1, -e_3\rangle + \langle D_{\bar\eta}\bar\eta, -\bar X\rangle\\ =& \sin\bar\gamma\langle D_{e_1}e_1, -e_2\rangle + \langle D_{\bar\eta}\bar\eta, -\bar X\rangle,\end{aligned}$$

where $\langle D_{e_1}e_1, -e_3\rangle$ vanishes because ℓ_s is planar. Meanwhile, by (2.1),

$$\begin{aligned}-\langle D_{\bar\eta}\bar\eta, \bar X\rangle - \frac{\partial\bar\gamma}{\partial\bar\eta} =& \langle D_{\bar\eta}\bar X, \bar\eta\rangle - \frac{\partial\bar\gamma}{\partial\bar\eta}\\ =& \langle D_{\bar\eta}\bar X, \bar\eta\rangle + \frac{1}{\sin\bar\gamma}\frac{\partial}{\partial\bar\eta}\langle\bar X, e_3\rangle\\ =& \frac{1}{\sin\bar\gamma}\langle D_{\bar\eta}\bar X, -\sin^2\bar\gamma e_3 + \sin\bar\gamma\cos\bar\gamma e_2 + e_3\rangle\\ =& \frac{1}{\sin\bar\gamma}\langle D_{\bar\eta}\bar X, \cos\bar\gamma\bar X\rangle\\ =& \cot\bar\gamma\langle D_{\bar\eta}\bar X, \bar X\rangle\\ =& 0.\end{aligned}$$

Hence, we have

$$\frac{\bar H_{\partial M}}{\sin\bar\gamma} - \frac{1}{\sin\bar\gamma}\frac{\partial\bar\gamma}{\partial\bar\eta} = \langle D_{e_1}e_1, -e_2\rangle.$$

The right hand side is the geodesic curvature of the circle ℓ_s on the plane $\{x^3 > 0\}$ and so it is positive. Considering $\mathrm{d}\lambda \geqslant \mathrm{d}\bar{\lambda}$ in (3.2), we have

$$\begin{aligned} & \int_{\partial\Sigma} \left(\frac{H_{\partial M}}{\sin\bar{\gamma}} - \frac{1}{\sin\bar{\gamma}} \frac{\partial\bar{\gamma}}{\partial\eta} \right) \mathrm{d}\lambda \\ \geqslant & \int_{\partial\Sigma} \langle D_{e_1} e_1, -e_2 \rangle_\delta \mathrm{d}\lambda \\ \geqslant & \int_{\partial\Sigma} \langle D_{e_1} e_1, -e_2 \rangle_\delta \mathrm{d}\bar{\lambda}. \end{aligned}$$

We conclude our proof by invoking the following lemma. □

Lemma 3.2. *For any rectifiable curve ℓ separating $p_\pm$ ($P_\pm$ if we are in Case 1 of Theorem 1.3)*

$$\int_\ell \langle D_{e_1} e_1, -e_2 \rangle_\delta \mathrm{d}\bar{\lambda} \geqslant 2\pi.$$

and equality holds if and only if ℓ is at the same height of ∂M.

Proof. It suffices to prove for such ℓ which is C^1. Since every height of ∂M is a circle, so we parametrize ∂M by

$$(r(s)\cos\theta, r(s)\sin\theta, s) \in \mathbb{R}^3,\ \ s \in [-1,1], \theta \in \mathbb{S}^1.$$

Then $\langle D_{e_1} e_1, -e_2 \rangle = \frac{1}{r(s)}$ if the third coordinate is s. We assume that the curve ℓ be parametrized by $t \in [0, t_0]$ and we write ℓ as

$$(r(s(t))\cos\theta(t), r(s(t))\sin\theta(t), s(t)).$$

The tangent vector

$$\partial_t = \tfrac{\mathrm{d}s}{\mathrm{d}t}(r'\cos\theta, r'\sin\theta, 1) + \tfrac{\mathrm{d}\theta}{\mathrm{d}t}(-r\sin\theta, r\cos\theta, 0),$$

with length

$$|\partial_t|^2 = |\tfrac{\mathrm{d}s}{\mathrm{d}t}|^2(1 + (r')^2) + (\tfrac{\mathrm{d}\theta}{\mathrm{d}t})^2 r^2.$$

The line element is $\mathrm{d}\bar{\lambda} \geqslant |\frac{\mathrm{d}\theta}{\mathrm{d}t}| r$. Then

$$\int_\ell \langle D_{e_1} e_1, -e_2 \rangle_\delta \geqslant \int_0^{t_0} |\tfrac{\mathrm{d}\theta}{\mathrm{d}t}| \mathrm{d}t \geqslant \int_0^{t_0} \tfrac{\mathrm{d}\theta}{\mathrm{d}t} \mathrm{d}t = \int_{\mathbb{S}^1} \mathrm{d}\theta = 2\pi.$$

By tracing back the proof, we see that the equality holds if and only if ℓ is at the same height of ∂M. □

Remark 3.3. Observe that if we assume that $\sigma = \bar{\sigma}$ on ∂M, then we can obtain the same bound (3.1) without the assumption that ∂M is weakly convex. This in turn could be used to establish a version of Theorem 1.3.

3.3. **Infinitesimally rigid surface.** Since Σ has at least one boundary component, so $\chi(\Sigma) \leqslant 1$ and by Lemma 3.1,

$$\begin{aligned} 2\pi & \geqslant 2\pi\chi(\Sigma) \\ & \geqslant \int_{\partial\Sigma} \left(\frac{H_{\partial M}}{\sin\bar{\gamma}} - \frac{1}{\sin\bar{\gamma}} \frac{\partial\bar{\gamma}}{\partial\eta} \right) \mathrm{d}\lambda \\ & \geqslant \int_{\partial\Sigma} \left(\frac{\bar{H}_{\partial M}}{\sin\bar{\gamma}} - \frac{1}{\sin\bar{\gamma}} \frac{\partial\bar{\gamma}}{\partial\bar{\eta}} \right) \mathrm{d}\bar{\lambda} \\ & \geqslant 2\pi. \end{aligned}$$

All inequalities are equalities, by tracing the proof, and

$$R_g = |A| = 0 \text{ in } \Sigma \tag{3.4}$$

and

$$H_{\partial M} = \bar{H}_{\partial M},\ \langle X, N\rangle = \cos\bar\gamma, \sigma = \bar\sigma \text{ along } \partial\Sigma \tag{3.5}$$

for some constant angle $\bar\gamma \in (0,\pi)$. Because Σ is stable, then eigenvalue problem

$$\begin{cases} Lf &= \mu f \text{ in } \Sigma \\ Bf &= 0 \text{ on } \partial\Sigma \end{cases} \tag{3.6}$$

has a nonnegative first eigenvalue $\mu_1 \geqslant 0$. The analysis now is similar to [FS80]. Letting $f \equiv 1$ in (2.1), and using both (2.2), (2.1), (3.3) and (3.3) we have

$$Q(1,1) = \int_\Sigma K + \int_{\partial\Sigma}\kappa - \int_{\partial\Sigma}\left(\tfrac{H_{\partial M}}{\sin\bar\gamma} - \tfrac{1}{\sin\bar\gamma}\tfrac{\partial\bar\gamma}{\partial\eta}\right) = 0. \tag{3.7}$$

So $\mu_1 = 0$ and the constant 1 is its corresponding eigenfunction. By (3.3) and (2.2), the stability operator reduces to $L = -\Delta + K$; by (2.1) and (3.3), the boundary stability operator reduces to $B = \partial_\nu - (\frac{H_{\partial M}}{\sin\bar\gamma} - \frac{1}{\sin\bar\gamma}\frac{\partial\bar\gamma}{\partial\eta} - \kappa)$. So

$$K = 0 \text{ in } \Sigma,\ \kappa = \tfrac{H_{\partial M}}{\sin\bar\gamma} - \tfrac{1}{\sin\bar\gamma}\tfrac{\partial\bar\gamma}{\partial\eta}. \tag{3.8}$$

We call Σ satisfying (3.3), (3.3) and (3.3) an *infinitesimally rigid surface.*

Remark 3.4. It is direct to see that

$$\kappa = \langle D_{e_1}e_1, -e_2\rangle_\delta$$

and Σ has to be a flat disk of radius $\langle D_{e_1}e_1, -e_2\rangle_\delta^{-1}$.

3.4. **CMC capillary foliation.** See for instance the works [Ye91], [BBN10] and [Amb15] on constructing CMC foliations.

We obtain a CMC foliation with prescribed angles. Let $\phi(x,t)$ be a local flow of a vector field Y which is tangent to ∂M and trasverse to Σ and that $\langle Y, N\rangle = 1$. In the following theorem, we only require that the scalar curvature of (M,g) and the mean curvature of ∂M are bounded below.

Theorem 3.5. *Suppose (M,g) is a three manifold with boundary, if Σ is an infinitesimally rigid surface, then there exists $\varepsilon > 0$ and a function $w(x,t)$ on $\Sigma \times (-\varepsilon,\varepsilon)$ such that for each $t \in (-\varepsilon,\varepsilon)$, the surface*

$$\Sigma_t = \{\phi(x, w(x,t)) : x \in \Sigma\}$$

is a constant mean curvature surface intersecting ∂M with prescribed angle $\bar\gamma$. Moreover, for every $x \in \Sigma$ and every $t \in (-\varepsilon,\varepsilon)$,

$$w(x,0) = 0,\ \int_\Sigma (w(x,t) - t) = 0 \text{ and } \tfrac{\partial}{\partial t}w(x,t)|_{t=0} = 1.$$

Proof. Given a function in the Hölder space $C^{2,\alpha}(\Sigma)$ $(0<\alpha<1)$, we consider

$$\Sigma_u = \{\phi(x, u(x)) : x \in \Sigma\},$$

which is a properly embedded surface if the norm of u is small enough. We use the subscript u to denote the quantities associated with Σ_u.

Consider the space

$$\mathcal{Y} = \{u \in C^{2,\alpha}(\Sigma) : \int_\Sigma u = 0\}$$

and
$$\mathcal{Z} = \{u \in C^{0,\alpha}(\Sigma) : \int_\Sigma u = 0\}.$$
Given small $\delta > 0$ and $\varepsilon > 0$, we define the map
$$\Phi : (-\varepsilon, \varepsilon) \times B(0, \delta) \to \mathcal{Z} \times C^{1,\alpha}(\partial\Sigma)$$
given by
$$\Phi(t, u) = \left(H_{t+u} - \tfrac{1}{|\Sigma|} \int_\Sigma H_{t+u}, \langle X_{t+u}, N_{t+u} \rangle - \cos \bar{\gamma}_{t+u} \right).$$
For each $v \in \Sigma$, the map
$$f : (x, s) \in \Sigma \times (-\varepsilon, \varepsilon) \to \phi(x, sv(x)) \in M$$
gives a variation with
$$\tfrac{\partial f}{\partial s}|_{s=0} = \tfrac{\partial}{\partial s} \phi(x, sv(x))|_{s=0} = vN.$$
Since Σ is infinitesimally rigid, we obtain that using (2.1) and (2.1),
$$D\Phi_{(0,0)}(0, v) = \tfrac{\mathrm{d}}{\mathrm{d}s} \Phi(0, sv)|_{s=0} = \left(-\Delta v + \tfrac{1}{|\Sigma|} \int_{\partial\Sigma} \Delta v, -\sin \bar{\gamma} \tfrac{\partial v}{\partial \nu} \right).$$
It follows from the elliptic theory for the Laplace operator with Neumann type boundary conditions that $D\Phi(0, 0)$ is an isomorphism when restricted to $0 \times \mathcal{Y}$.

Now we apply the implicit function theorem: For some smaller ε, there exists a function $u(t) \in B(0, \delta) \subset \mathcal{X}$, $t \in (-\varepsilon, \varepsilon)$ such that $u(0) = 0$ and $\Phi(t, u(t)) = \Phi(0, 0) = (0, 0)$ for every t. In other words, the surfaces
$$\Sigma_{t+u(t)} = \{\phi(x, t + u(t)) : x \in \Sigma\}$$
are constant mean curvature surfaces with prescribed angles $\bar{\gamma}$.

Let $w(x, t) = t + u(t)(x)$ where $(x, t) \in \Sigma \times (-\varepsilon, \varepsilon)$. By definition, $w(x, 0) = 0$ for every $x \in \Sigma$ and $w(\cdot, t) - t = u(t) \in B(0, \delta) \subset \mathcal{X}$ for every $t \in (-\varepsilon, \varepsilon)$. Observe that the map $s \mapsto \phi(x, w(x, s))$ gives a variation of Σ with variational vector field given by
$$\tfrac{\partial \phi}{\partial t} \tfrac{\partial w}{\partial s}|_{s=0} = \tfrac{\partial w}{\partial s}|_{s=0} Y.$$
Since for every t we have that
$$0 = \Phi(t, u(t)) = \left(H_{w(\cdot,t)} - \tfrac{1}{|\Sigma|} \int_\Sigma H_{w(\cdot,t)}, \langle X_{t+u}, N_{t+u} \rangle - \cos \bar{\gamma}_{t+u} \right),$$
by taking the derivative at $t = 0$ we conclude that
$$\langle \tfrac{\partial w}{\partial t}|_{t=0} Y, N \rangle = \tfrac{\partial w}{\partial t}|_{t=0}$$
satisfies the homogeneous Neumann proble. Therefore, it is constant on Σ. Since
$$\int_\Sigma (w(x, t) - t) = \int_\Sigma u(x, t) = 0$$
for every t, by taking derivatives at $t = 0$ again, we conclude that
$$\int_\Sigma \tfrac{\partial w}{\partial t}|_{t=0} = |\Sigma|.$$
Hence, $\frac{\partial w}{\partial t}|_{t=0} = 1$. Taking ε small, we see that $\phi(x, w(x, t))$ parametrizes a foliation near Σ. $\square$

Theorem 3.6. *There exists a continuous function $\Psi(\rho)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\exp(-\int_0^\rho \Psi(\tau)\mathrm{d}\tau)H\right) \leqslant 0.$$

Proof. Let $\psi : \Sigma \times I \to M$ parametrizes the foliation, $Y = \frac{\partial \psi}{\partial t}$, $v_\rho = \langle Y, N_\rho\rangle$. Then

$$-\frac{\mathrm{d}}{\mathrm{d}\rho}H(s) = \Delta_\rho v_\rho + (\mathrm{Ric}(N_\rho) + |A_\rho|^2)v_\rho \text{ in } \Sigma_\rho, \tag{3.9}$$

and

$$\frac{\partial v_\rho}{\partial \rho} = [-\cot\bar{\gamma}A_\rho(\nu_\rho,\nu_\rho) + \frac{1}{\sin\bar{\gamma}}A_{\partial M}(\eta_\rho,\eta_\rho) + \frac{1}{\sin^2\bar{\gamma}}\nabla_{\eta_\rho}\cos\bar{\gamma}]v_\rho.$$

By shrinking the interval if indeeded, we assume that $v_\rho > 0$ for $\rho \in I$. By multiplying of (3.4) and integrate on Σ_ρ, we deduce that

$$\begin{aligned}
& - H'(s)\int_{\Sigma_\rho}\frac{1}{v_\rho} \\
= & \int_{\Sigma_\rho}\frac{\Delta_\rho v_\rho}{v_\rho} + (\mathrm{Ric}(N_\rho) + |A_\rho|^2) \\
= & \int_{\partial\Sigma_\rho}\frac{1}{v_\rho}\frac{\partial v_\rho}{\partial \nu_\rho} + \frac{1}{2}\int_{\Sigma_\rho}(R_g + |A_\rho|^2 + H(\rho)^2) - \int_{\Sigma_\rho}K_{\Sigma_\rho} \\
\geqslant & \int_{\partial\Sigma_\rho}[-\cot\bar{\gamma}A_\rho(\nu_\rho,\nu_\rho) + \frac{1}{\sin\bar{\gamma}}A_{\partial M}(\eta_\rho,\eta_\rho) + \frac{1}{\sin^2\bar{\gamma}}\nabla_{\eta_\rho}\cos\bar{\gamma}] - \int_{\Sigma_\rho}K_{\Sigma_\rho} \\
\geqslant & -\left[\int_{\partial\Sigma_\rho}\kappa_{\partial\Sigma_\rho} + \int_{\Sigma_\rho}K_{\Sigma_\rho}\right] - \int_{\partial\Sigma_\rho}H(\rho)\cot\bar{\gamma} + \int_{\partial\Sigma_\rho}\frac{H_{\partial M}}{\sin\bar{\gamma}} + \frac{1}{\sin^2\bar{\gamma}}\nabla_{\eta_\rho}\cos\bar{\gamma}
\end{aligned}$$

where in the last line we have used the following version of (2.1)

$$\kappa_{\partial\Sigma_\rho} - \cot\bar{\gamma}A(\nu_\rho,\nu_\rho) + \frac{1}{\sin\bar{\gamma}}A_{\partial M}(\eta_\rho,\eta_\rho) = -H(\rho)\cot\bar{\gamma} + \frac{1}{\sin\bar{\gamma}}H_{\partial M}.$$

By the Gauss-Bonnet theorem and by Lemma 3.1, we have

$$-H'(\rho)\int_{\Sigma_\rho}\frac{1}{v_\rho} \geqslant -H(\rho)\int_{\partial\Sigma_\rho}\cot\bar{\gamma}.$$

Let

$$\Psi(\rho) = \left(\int_{\Sigma_\rho}\frac{1}{v_\rho}\right)^{-1}\int_{\partial\Sigma_\rho}\cot\bar{\gamma},$$

then note that we have assume that $v_\rho > 0$ near $\rho = 0$, so $H(\rho)$ satisfies the ordinary differential inequality

$$H' - \Psi(\rho)H \leqslant 0. \tag{3.10}$$

We see then

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\exp\left(-\int_0^\rho \Psi(\tau)\mathrm{d}\tau\right)H\right) \leqslant 0.$$

So the function $\mathrm{e}^{-\int_0^\rho \Psi(\tau)\mathrm{d}\tau}H(\rho)$ is nonincreasing. ∎

3.5. **From local foliation to rigidity.** Let Σ_ρ be the constant mean curvature surfaces with prescribed contact angles $\bar{\gamma}$ with ∂M.

Proposition 3.7. *Every Σ_ρ constructed in Theorem 3.5 is infinitesimally rigid.*

Proof. We abuse the notation and let

$$F(\rho) = |\Sigma_\rho| - \int_{\partial\Omega_\rho} \cos\bar{\gamma}.$$

By the first variation formula (2.1),

$$F(\rho_2) - F(\rho_1) = \int_{\rho_1}^{\rho_2} \mathrm{d}\rho \int_{\Sigma_\rho} H(\rho) v_\rho.$$

By Theorem 3.6,

$$H(\rho) \leqslant 0 \text{ if } \rho \geqslant 0;\ H(\rho) \geqslant 0 \text{ if } \rho \leqslant 0,$$

which in turn implies that

$$F(\rho) \leqslant 0 \text{ if } \rho \geqslant 0;\ F(\rho) \leqslant 0 \text{ if } \rho \leqslant 0.$$

However, Ω is a minimiser to the functional (2.1), hence

$$F(\rho) \equiv F(0).$$

It then follows every Σ_ρ is a minimiser, hence infinitesimally rigid. □

Now we finish the proof of Case 1 of Theorem 1.3 by using Theorem 3.6.

Proof of Case 1 of Theorem 1.3. Let $Y = \frac{\mathrm{d}}{\mathrm{d}t}\phi(x, w(x,t))$ where ϕ and w are as Theorem 3.5, we show first that N_ρ is a parallel vector field. Since every Σ_ρ is infinitesimally rigid, from (3.3) and (3.3), we know that $\langle Y, N_\rho\rangle$ is a constant. Let ∂_i, $i = 1, 2$ are vector fields induced by local coordinates on Σ, then $\nabla_{\partial_i}\langle Y, N\rangle = 0$. Note that Σ_ρ are totally geodesic, so $\nabla_{\partial_i} N \equiv 0$ and

$$0 = \nabla_{\partial_i}\langle Y, N\rangle = \langle \nabla_{\partial_i} Y, N\rangle + \langle Y, \nabla_{\partial_i} N\rangle = \langle \nabla_{\partial_i} Y, N\rangle,$$

and

$$0 = \langle \nabla_{\partial_i} Y, N\rangle = \langle \nabla_Y \partial_i, N\rangle = \langle \nabla_{Y^\perp}\partial_i, N\rangle.$$

So

$$0 = \langle \nabla_{Y^\perp}\partial_i, N\rangle = Y^\perp\langle \partial_i, N\rangle - \langle \partial_i, \nabla_{Y^\perp} N\rangle = -\langle \partial_i, \nabla_{Y^\perp} N\rangle.$$

We conclude that N is a parallel vector field, and since every Σ_ρ is flat, then $\cup_\rho \Sigma_\rho$ foliates a subset of the Euclidean space $\mathbb{R}^3$ with the flat metric. So Σ_ρ is a family of parallel disks in $\mathbb{R}^3$ and we can parametrize $\partial M \cap (\cup_\rho \partial\Sigma_\rho)$ locally by

$$\vec{x}(\rho, \theta) = (\psi(\rho)\cos\theta + a_1(\rho), \psi(\rho)\sin\theta + a_2(\rho), -\rho).$$

By translation invariance, we can set $a_1(0) = a_2(0) = 0$. Then the tangent vectors are

$$\vec{x}_\rho = (\psi'\cos\theta + a_1', \psi'\sin\theta + a_2', -1), \vec{x}_\theta = (-\psi\sin\theta, \psi\cos\theta).$$

It is easy to see that $\tilde{X} = (\cos\theta, \sin\theta, \psi' + a_1'\cos\theta + a_2'\sin\theta)$ is normal to $\partial M \cap (\cup_\rho \partial\Sigma_\rho)$. Since $\tilde{X}$ forms an angle with $N = (0,0,1)$ independent of θ by (3.3), then

$$\tfrac{\partial}{\partial\theta}\tfrac{\langle \tilde{X}, N\rangle_\delta}{|\tilde{X}|_\delta} = 0.$$

By an easy calculation,

$$-a_1'\sin\theta + a_2'\cos\theta = 0$$

which requires that $a_1' = a_2' = 0$ and so $a_1 = a_2 = 0$. So $\partial M \cap (\cup_\rho \partial \Sigma_\rho)$ is rotationally symmetric. Since M is connected, we conclude that M is a rotationally symmetric set in $\mathbb{R}^3$ with the flat metric. $\square$

4. Conical pole

In this section, we deal with the conical case of Theorem 1.3. Under the extra assumption that the metric at the vertex is isometric to the model, we construct a constant mean curvature foliation with prescribed angles near the vertex. We derive a lower bound on the mean curvature of the leaf such that each leaf is a barrier, which reduces the problem to the previous section.

4.1. **Foliation near the conical point.** We adapt the results in Section 3.4 to construct CMC foliations near the conical point p_+.

We assume that $g = \delta$ at p_+ and p_+ is conical. Let Σ_ρ be the disk given by

$$\Sigma_\rho = (\psi(\rho)\hat{x}, -\rho).$$

Since ∂M at p_+ is conical, so $\psi(\rho) = \psi'(0)\rho + O(\rho^2)$. Let D be the unit disc, assume that $|\hat{u}(\cdot, \rho)|_{C^{2,\alpha}(D)} = O(\rho^2)$, define

$$\Sigma_{\rho,u} = (\psi(\rho + \hat{u})\hat{x}, -\rho - u),$$

where $\hat{x} = (x_1, x_2) \in D$.

then

$$\Sigma_{\rho,\hat{u}} = \Sigma_\rho + \hat{u}(\psi'(\rho)x_1, \psi'(\rho)x_2, -1) + O(u^2)$$

by the Taylor expansion. The normal to Σ_ρ is

$$N_\rho = g^{3j} e_j / \sqrt{g^{33}},$$

where $\{e_i\}_{i=0}$ are the standard coordinates of $\mathbb{R}^3$.

Note that $\Sigma_{\rho,\hat{u}}$ is approximately a graph of a function over Σ_ρ, it is given by

$$\begin{aligned} & U \\ =& \hat{u}\langle(\psi' x, \psi' y, -1), g^{3j} e_j/\sqrt{g^{33}}\rangle + O(\hat{u}^2) \\ =& -\frac{\hat{u}}{\sqrt{g^{33}}} + O(\hat{u}^2) \\ =& -\frac{\hat{u}}{\sqrt{g^{33}}} + O(\rho^3) \\ =& -\hat{u} + O(\rho^3). \end{aligned}$$

because $g^{33} = g^{33}(\psi x, \psi y, -\rho)$, we are computing Δ_ρ and ν_ρ with respect to only the first two variables.

Note that

$$\Delta_\rho U = -\Delta_\rho \hat{u} + O(\rho^3).$$

Consider the space

$$\mathcal{Y} = \left\{ u \in C^{2,\alpha}(D) : \int_D u = 0 \right\}$$

and

$$\mathcal{Z} = \left\{ u \in C^{0,\alpha}(D) : \int_D u = 0 \right\}.$$

Given small $\delta > 0$ and $\varepsilon > 0$, we define the map

$$\Phi : (-\varepsilon, \varepsilon) \times B(0, \delta) \to \mathcal{Z} \times C^{1,\alpha}(\partial\Sigma)$$

given by

$$\Psi(\rho, u) = \left(H_{\rho,\rho^2 u} - \frac{1}{|D|} \int_D H_{\rho,\rho^2 u}, \rho^{-1}(\langle X_{\rho,\rho^2 u}, N_{\rho,\rho^2 u}\rangle - \cos\bar{\gamma}_{\rho,\rho^2 u}) \right)$$

for $\rho \neq 0$. Here the integration on D is calculated with respect to the flat metric.

We extend $\Psi(\rho, u)$ to $\rho = 0$ by taking limits, that is,

$$\Psi(0, u) = \lim_{\rho \to 0} \Psi(\rho, u).$$

Using the variational formulas of the mean curvature and the angles, we obtain

$$H_{\rho,\rho^2 u} - H_\rho = \tfrac{\rho^2}{\psi(\rho)^2} \Delta_\rho u + \rho^2(\mathrm{Ric}(N_\rho) + |A_\rho|^2)u + O(\rho^3). \tag{4.1}$$

and

$$\begin{aligned} &\rho^{-1}[\langle X_{\rho,\rho^2 u}, N_{\rho,\rho^2 u}\rangle - \langle X_\rho, N_\rho\rangle] \\ =&\tfrac{\rho \sin\gamma}{\psi(\rho)} \tfrac{\partial u}{\partial \nu_\rho} - \rho(-\cos\gamma A(\nu_\rho, \nu_\rho) + A_{\partial M}(\eta_\rho, \eta_\rho))u + O(\rho^2). \end{aligned} \tag{4.2}$$

Lemma 4.1. *There exists $u_0 \in C^{2,\alpha}(D)$ such that $\int_D u_0 = 0$ and*

$$\Psi(0, u_0) = (0, 0). \tag{4.3}$$

Proof. First, we note

$$\lim_{\rho \to 0} H_{\rho,\rho^2 u} = \lim_{\rho \to 0} H_\rho + \tfrac{1}{\psi'(0)^2} \Delta u$$

from (4.1). Since $\lim_{\rho\to 0} H_\rho$ is a constant, we obtain that the first component $\Psi_1(\rho, u)$ of $\Psi(\rho, u)$ at $(0, u)$ is

$$\Psi_1(0, u) = \tfrac{1}{\psi'(0)^2} \left(\Delta u - \tfrac{1}{|D|} \int_D \Delta u \right).$$

Since

$$\begin{aligned} &\rho^{-1}[\langle X_{\rho,\rho^2 u}, N_{\rho,\rho^2 u}\rangle - \cos\bar{\gamma}_{\rho,\rho^2 u}] \\ =&\rho^{-1}[(\langle X_{\rho,\rho^2 u}, N_{\rho,\rho^2 u}\rangle - \langle X_\rho, N_\rho\rangle) - (\cos\bar{\gamma}_{\rho,\rho^2 u} - \langle \bar{X}_\rho, \bar{N}_\rho\rangle_\delta)] \\ &+ \rho^{-1}(\langle X_\rho, N_\rho\rangle - \langle \bar{X}_\rho, \bar{N}_\rho\rangle), \end{aligned}$$

and (A), (4.1), the second component $\Psi_2(\rho, u)$ of $\Psi(\rho, u)$ at $(0, u)$ is

$$\Psi_2(0, u) = \tfrac{\sin\bar{\gamma}}{\psi'(0)} \tfrac{\partial u}{\partial \nu} + \lim_{\rho \to 0} \frac{\cos\gamma_\rho - \cos\bar{\gamma}_\rho}{\rho}.$$

Note that $\lim_{\rho\to 0} \frac{\cos\gamma - \cos\bar{\gamma}}{\rho}$ exists for every point of ∂D and is a function on ∂D. Then $\Psi(0, u) = (0, 0)$ is equivalent to the elliptic boundary value problem

$$\begin{aligned} \Delta u &= \tfrac{1}{|D|} \int_D \Delta u \text{ in } D \\ \tfrac{\partial u}{\partial \nu} &= -\tfrac{\psi'(0)}{\sin\bar{\gamma}_0} \lim_{\rho \to 0} \tfrac{\cos\gamma - \cos\bar{\gamma}}{\rho} \text{ on } \partial D. \end{aligned}$$

It is well known that the above has a unique solution u_0 in $C^{2,\alpha}(D)$ with $\int_D u_0 = 0$. □

We compute

$$D\Psi_{(0,u_0)}(0,v)=\tfrac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\Psi(0,u_0+sv).$$

In particular,

$$\begin{aligned}&D\Psi|_{(0,u_0)}(0,v)\\=&\tfrac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\Psi(0,u_0+sv)\\=&\left(\tfrac{1}{\psi'(0)^2}\left[-\Delta v+\tfrac{1}{|D|}\int_D\Delta v\right],\tfrac{\sin\bar\gamma}{\psi'(0)}\tfrac{\partial v}{\partial\nu}\right)\end{aligned}$$

because u_0 satisfies (4.1). It follows from the elliptic theory for the Laplace operator with Neumann type boundary conditions that $D\Phi|_{(0,u_0)}$ is an isomorphism when restricted to $0\times\mathcal{Y}$.

Now we apply the implicit function theorem: For some smaller ε, there exists a function $u(\cdot,\rho)\in B(0,\delta)\subset\mathcal{X}$, $\rho\in(-\varepsilon,\varepsilon)$ such that $u(\cdot,0)=u_0$ and

$$\Phi(\rho,u(\cdot,\rho))=\Phi(0,u_0)=(0,0)$$

for every ρ. In other words, the surfaces Σ_{ρ,ρ^2u} are constant mean curvature surfaces with prescribed angles $\gamma=\bar\gamma$ at $\partial\Sigma$.

4.2. **Behavior of the mean curvature of the leaf.** Let $u=u(\hat x,\rho)$ be as constructed as above, let $\lambda_\rho=H_{\rho,\rho^2u}$, then λ_ρ is a constant. Similar to [Li20] and [CW22], we obtain the limiting behavior of λ_ρ as $\rho\to0$.

Lemma 4.2. *For the foliation constructed in the above,*

$$\lambda_\rho|D_\rho|=\int_{D_\rho}H_\rho+\int_{\partial D_\rho}\tfrac{1}{\sin\gamma}(\cos\bar\gamma_\rho-\cos\gamma_\rho)+O(\rho^3).\tag{4.4}$$

Proof. Let $\sigma^\rho_{ij}(\hat x)=g_{ij}((\psi(\rho)\hat x,-\rho))$ where i, j ranges in $\{1,2\}$. We integrate (4.1) with respect to the metric σ^ρ (we omit the area element and line element) and using the divergence theorem and applying (4.1),

$$\begin{aligned}&\lambda_\rho|D|_{\sigma^\rho}-\int_D H_\rho\\=&\tfrac{\rho^2}{\psi^2(\rho)}\int_D\Delta_\rho u+\rho^2\int_D(\mathrm{Ric}(N_\rho)+|A_\rho|^2)u+O(\rho^3)\\=&\tfrac{\rho^2}{\psi^2(\rho)}\int_{\partial D}\tfrac{\partial u}{\partial\nu_\rho}+O(\rho^2)\\=&\tfrac{\rho}{\psi(\rho)}\left[\int_{\partial D}\tfrac{\rho}{\sin\gamma}(-\cos\gamma A(\nu_\rho,\nu_\rho)+A_{\partial M}(\eta_\rho,\eta_\rho))u+O(\rho^2)\right]\\&+\tfrac{\rho}{\psi(\rho)}\int_{\partial D}\frac{\langle X_{\rho,\rho^2u},N_{\rho,\rho^2u}\rangle-\langle X_\rho,N_\rho\rangle}{\rho\sin\gamma}+O(\rho^2)\\=&\tfrac{\rho}{\psi(\rho)}\int_{\partial D}\frac{\cos\gamma_{\rho,\rho^2u}-\cos\gamma_\rho}{\rho\sin\gamma}+O(\rho).\end{aligned}$$

It follows from (A) that

$$\begin{aligned}&\cos\gamma_{\rho,\rho^2u}-\cos\gamma_\rho\\=&(\cos\gamma_{\rho,\rho^2u}-\cos\bar\gamma_{\rho,\rho^2u})+(\cos\bar\gamma_{\rho,\rho^2u}-\cos\bar\gamma_\rho)+(\cos\bar\gamma_\rho-\cos\gamma_\rho)\\=&0+(-\rho^2\bar A_{\partial M}(\bar\eta_\rho,\bar\eta_\rho)u+O(\rho^3))+(\cos\bar\gamma_\rho-\cos\gamma_\rho).\end{aligned}$$

Hence,

$$\lambda_\rho |D|_{\sigma^\rho} - \int_D H_\rho = \tfrac{1}{\psi(\rho)} \int_{\partial D} \frac{\cos\bar{\gamma}_\rho - \cos\gamma_\rho}{\sin\gamma} + O(\rho),$$

and a rescaling proves the lemma. □

Now we recall a result [MP21, Proposition 2.1] interpretated slightly differently.

Proposition 4.3. *Let M be a manifold with two metrics g and $\bar{g}$. We use bar to denote geometric quantities computed with respect to the metric $\bar{g}$. Let S be a hypersurface in M, X be a chosen unit normal and at a point $p \in S$. Let H and A denote respectively the mean curvature and the second fundamental form of S in M. If at some point $p \in S$, the two metrics agree $g(p) = \bar{g}(p)$, then near p,*

$$\begin{aligned} 2(H - \bar{H}) =& (\mathrm{d}\,\mathrm{tr}_{\bar{g}}\, h - \mathrm{div}_{\bar{g}}\, h)(\bar{X}) - \mathrm{div}_{\bar{\sigma}}\, W - \langle h, \bar{A}\rangle_{\bar{\sigma}} \\ & + |\bar{A}|_{\bar{g}} O(|h|^2_{\bar{g}}) + O(|\bar{\nabla}^M h|_{\bar{g}} |h|_{\bar{g}}), \end{aligned} \tag{4.5}$$

where $h = g - \bar{g}$ and W is the dual vector field on S of the 1-form $h(\cdot, \bar{X})$.

Proof. While following [MP21, Proposition 2.1], we only have to note that $|h|_{\bar{g}}$ is small and $|\bar{\nabla}^M h|$ is not small when getting closer to p. □

Using the above lemma, we are able to show the following.

Lemma 4.4. *For the foliation constructed in the above,*

$$\lim_{\rho\to 0} \lambda_\rho \geqslant 0.$$

Proof. First, we show that

$$\lim_{\rho\to 0} \lambda_\rho \geqslant 0.$$

We integrate both hands of (4.3) on Ω_ρ, we obtain

$$\int_{\partial\Omega_\rho} 2(H - \bar{H}) + \mathrm{div}_{\bar{\sigma}}\, W + \langle h, \bar{A}\rangle_{\bar{\sigma}} = \int_{\partial\Omega_\rho} (\mathrm{d}\,\mathrm{tr}_{\bar{g}}\, h - \mathrm{div}_{\bar{g}}\, h)(\bar{X}) + O(\rho^3).$$

The $O(\rho^3)$ remainder terms follows from that $|h|_{\bar{g}} = O(\rho)$, $|\nabla^M h|_{\bar{g}} = O(1)$ and $|\bar{A}|_{\bar{g}} = O(\rho^{-1})$. Applying the divergence theorem on the right,

$$\int_{\partial\Omega_\rho} (\mathrm{d}\,\mathrm{tr}_{\bar{g}}\, h - \mathrm{div}_{\bar{g}}\, h)(\bar{X}) = \int_{\Omega_\rho} \bar{\nabla}^M_i (\bar{\nabla}^M_i \mathrm{tr}_{\bar{g}}\, h - \bar{\nabla}^M_j h_{ij}) = O(\rho^3).$$

Here i, j denotes the indices of a local $\bar{g}$-orthonormal frame at the tangent space of Ω_ρ. Hence, we have obtained that

$$\int_{\partial\Omega_\rho} 2(H - \bar{H}) + \mathrm{div}_{\bar{\sigma}}\, W + \langle h, \bar{A}\rangle_{\bar{\sigma}} = O(\rho^3).$$

On Σ_ρ, $H = -H_\rho$ and $\bar{H} = 0$, $\bar{A} = 0$, we obtain that

$$2\int_{\Sigma_\rho} H_\rho - \int_{\partial\Omega_\rho} \mathrm{div}_{\bar{\sigma}}\, W = \int_{\partial\Omega_\rho \cap \partial M} 2(H_{\partial M} - \bar{H}_{\partial M}) + \langle h, \bar{A}_{\partial M}\rangle_{\bar{\sigma}} + O(\rho^3).$$

From the assumptions of Theorem 1.3 we have

$$2\int_{\Sigma_\rho} H_\rho - \int_{\Sigma_\rho \cup (\partial\Omega_\rho \cap \partial M)} \mathrm{div}_{\bar{\sigma}}\, W \geqslant O(\rho^3).$$

It follows the same lines of [MP21, (3.18)] that

$$\begin{aligned} & \int_{\Sigma_\rho} \operatorname{div}_{\bar{\sigma}} W + \int_{\partial\Omega_\rho \cap \partial M} \operatorname{div}_{\bar{\sigma}} W \\ = & \int_{\partial\Sigma_\rho} g(\bar{X}, \bar{\eta}) + \int_{\partial\Sigma_\rho} g(-\bar{N}, \bar{\nu}) \\ = & -\int_{\partial\Sigma_\rho} \tfrac{2}{\sin\bar{\gamma}_\rho} (\cos\bar{\gamma}_\rho - \cos\gamma_\rho), \end{aligned}$$

which in turn implies that

$$2\int_{\Sigma_\rho} H_\rho + 2\int_{\partial\Sigma_\rho} \tfrac{1}{\sin\bar{\gamma}_\rho} (\cos\bar{\gamma}_\rho - \cos\gamma_\rho) \geqslant O(\rho^3).$$

From (4.2), we arrive

$$\lambda_\rho |D_\rho| \geqslant O(\rho^3).$$

Since $|D_\rho| = \pi\psi(\rho)^2 + O(\rho^3)$, so $\lambda_\rho \geqslant O(\rho)$. By taking limits, we have that $\lim_{\rho\to 0} \lambda_\rho \geqslant 0$. □

Remark 4.5. This lemma and (4.2) are quite natural in the sense that the limit of λ_ρ can be regarded as an averaged mean curvature near the conical point.

In fact, we have a stronger result which asserts that every leaf has nonnegative mean curvature.

Lemma 4.6. *For the foliation constructed in the above, for each $\rho \in (0, \varepsilon)$,*

$$\lambda_\rho \geqslant 0.$$

Proof. We rename $\Sigma_{\rho,\rho^2 u}$ constructed earlier to Σ_ρ. We use the subscript ρ on the geometric quantities of Σ_ρ. Note that the chosen unit normal N_ρ points in the direction that ρ decreases, so the ordinary differential inequality (3.4) established in Theorem 3.6 is valid but with a reversed sign for the foliations near the conical point, that is,

$$\lambda' - \Psi(\rho)\lambda \geqslant 0 \tag{4.6}$$

where $\lambda(\rho) = \lambda_\rho$ and

$$\Psi(\rho) = \left(\int_{\Sigma_\rho} \tfrac{1}{v_\rho}\right)^{-1} \int_{\partial\Sigma_\rho} \cot\bar{\gamma}.$$

Since Σ_ρ is constructed via a lower order perturbation, we see that $v_\rho = 1 + O(\rho)$,

$$\int_{\Sigma_\rho} \tfrac{1}{v_\rho} = \pi\psi(\rho)^2 + O(\rho^3), \quad \int_{\partial\Sigma_\rho} \cot\bar{\gamma} = 2\pi\rho\psi'(0) + O(\rho^2).$$

Therefore,

$$\Psi(\rho) = \tfrac{2}{\psi'(0)}\rho^{-1} + C_1(\rho)$$

where $C_1(\rho)$ is a continuous function of order $O(1)$. So it follows from (4.2) that λ_ρ satisfies the ordinary differential inequality

$$\tfrac{\mathrm{d}}{\mathrm{d}\rho}\left(\exp\left(-\int_0^\rho \tfrac{\psi'(0)C_1(s)}{2}\mathrm{d}s\right)\rho\lambda(\rho)\right) \geqslant 0$$

and the lemma now follows combining with Lemma 4.4. □

Now we can finish the proof of the conical case.

Proof of case 2 of Theorem 1.3. By Lemma 4.6, we fix $\rho_1 > 0$ small, then Σ_{ρ_1} is of nonnegative mean curvature and meets the boundary ∂M at prescribed angles $\bar\gamma$. So Σ_{ρ_1} is a barrier for the existance of the minimiser of (2.1) and we can find a minimiser of (2.1) in between Σ_{ρ_1}. The rest of the proof is the same as Section 3. □

5. Generalizations to the hyperbolic case

Hyperbolic case

Let e_1 be the unit tangent vector of $\bar\beta_s$, e_2 be the unit outward normal pointing to the outward in the plane $\{x^3 = s\}$, $e^3 = x^3 \frac{\partial}{\partial x^3}$.

$$\bar H = \langle D_{e_1} e_1 + D_{\bar\eta}\bar\eta, -X\rangle.$$

We have that $X = e_3 \cos\bar\gamma + e_2 \sin\bar\gamma$, $\bar\eta = -e_3 \sin\bar\gamma + \cos\bar\gamma e_2$, so

$$\langle D_{e_1} e_1, -X\rangle = -\langle D_{e_1} e_1, e_3 \cos\bar\gamma + e_2 \sin\bar\gamma\rangle = \sin\bar\gamma \langle D_{e_1} e_1, -e_2\rangle - \cos\bar\gamma$$

because e_1 is tangent to the horosphere and e_3 is normal to the horosphere.

$$\begin{aligned}
&\langle D_{\bar\eta} X, \bar\eta\rangle - \tfrac{\partial\bar\gamma}{\partial\bar\eta}\\
=&\langle D_{\bar\eta} X, \bar\eta\rangle + \tfrac{1}{\sin\bar\gamma} D_{\bar\eta}\langle X, e_3\rangle\\
=&\langle D_{\bar\eta} X, \bar\eta\rangle + \tfrac{1}{\sin\bar\gamma}\langle D_{\bar\eta} X, e_3\rangle + \tfrac{1}{\sin\bar\gamma}\langle X, D_{\bar\eta} e_3\rangle\\
=&\cot\bar\gamma\langle D_{\bar\eta} X, X\rangle + \tfrac{1}{\sin\bar\gamma}\langle X, D_{\bar\eta} e_3\rangle\\
=&\tfrac{1}{\sin\bar\gamma}\langle X, D_{\bar\eta} e_3\rangle\\
=&-\cos\bar\gamma.
\end{aligned}$$

where in the last line we have used upper half space model to get

$$D\tfrac{\partial}{\partial x^3} = -\tfrac{1}{x^3}\mathrm{d}x^3 \otimes \partial_{x^3} + \sum_{i,j=1}^{2} \tfrac{1}{x^3} e^i \otimes e_j.$$

So

$$\bar H + \tfrac{\partial\bar\gamma}{\partial\bar\eta} = \sin\bar\gamma\langle D_{e_1} e_1, -e_2\rangle$$

Appendix A. Rotationally symmetric surface in $\mathbb{R}^3$

Let $\vec{x} : [0, \varepsilon) \times \mathbb{S}^1 \to \mathbb{R}^3$ be a rotationally symmetric surface S given by

$$\vec{x}(\rho, \theta) = (\psi(\rho)\cos\theta, \psi(\rho)\sin\theta, -\rho).$$

Then the tangent vectors are

$$\vec{x}_\rho = (\psi'\cos\theta, \psi'\sin\theta, -1), \vec{x}_\theta = (-\psi\sin\theta, \psi\cos\theta).$$

The unit normal to S is

$$X = (\cos\theta, \sin\theta, \psi')/\sqrt{1 + (\psi')^2}.$$

The angle $\bar\gamma$ is given by

$$\cos\bar\gamma = \tfrac{\psi'}{\sqrt{1+(\psi')^2}}.$$

And

$$\bar{\eta} = \vec{x}_\rho / \sqrt{1 + (\psi')^2}.$$

Let A_S denote the second fundamental form, then

$$A_S(\vec{x}_\rho, \vec{x}_\rho) = -\frac{\psi''}{\sqrt{1+(\psi')^2}}, A_S(\vec{x}_\theta, \vec{x}_\theta) = \frac{1}{\sqrt{1+(\psi')^2}}\psi. \tag{A.1}$$

Let $\Sigma_\rho = \{(\psi(\rho)\hat{x}, -\rho) : \hat{x} \in D\}$. We use the subscript ρ on every geometric quantities on Σ_ρ, for example, γ_ρ denotes the contact angles of Σ_ρ with ∂M under the metric g and $\bar{\gamma}_\rho$ denotes the contact angles of Σ_ρ with ∂M under the flat metric.

Let

$$\Sigma_{\rho,\rho^2 u} = \{(\psi(\rho + \rho^2 u)\hat{x}, -(\rho + \rho^2 u)) : \hat{x} \in D\}, u = u(\cdot, \rho) \in C^{2,\alpha}(D)$$

be a small perturbation of Σ_ρ, we use the subscript $\rho, \rho^2 u$ on every geometric quantites on $\Sigma_{\rho,\rho^2 u}$ except that $\bar{\gamma}_{\rho,\rho^2 u}$ denotes the value of $\bar{\gamma}_\rho$ at $\Sigma_{\rho,\rho^2 u}$. From the definition of $\bar{\gamma}_{\rho,\rho^2 u}$,

$$\cos \bar{\gamma}_{\rho,\rho^2 u} = \frac{\psi'(\rho+\rho^2 u)}{\sqrt{1+(\psi'(\rho+\rho^2 u))^2}}, \hat{x} \in \partial D$$

It is easy to see that for each $\hat{x} \in \partial D$,

$$\cos \bar{\gamma}_{\rho,\rho^2 u} - \cos \bar{\gamma}_\rho = \frac{1}{\sqrt{1+(\psi'(\rho))^2}} \frac{1}{1+(\psi'(\rho))^2} \psi''(\rho)\rho^2 u(\hat{x}, \rho) + O(\rho^3).$$

In fact it follows from (A) that

$$\cos \bar{\gamma}_{\rho,\rho^2 u} - \cos \bar{\gamma}_\rho = -\bar{A}_{\partial M}(\bar{\eta}, \bar{\eta})\rho^2 u + O(\rho^3). \tag{A.2}$$

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