A MIXED BOUNDARY VALUE PROBLEM FOR JANG'S EQUATION AND THE EXISTENCE OF FREE BOUNDARY MARGINALLY OUTER TRAPPED SURFACES

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Abstract. We use PDE methods to find marginally outer trapped surface with a free boundary. In particular, we establish a priori estimates for the Jang equation with mixed boundary conditions and develop a Perron method following Eichmair. We use the theory of Λ-minimizing currents with a free boundary to study the regularity.

Contents

1. Introduction

Schoen-Yau [32, 31] utilized the minimal surface theory to study the scalar curvature geometry. In particular, they gave the first proof of the time-symmetric case positive mass theorem in [32]. See Schoen [34] for generalization to dimensions up to seven. Two key ingredients of the proof are the existence of certain minimal hypersurfaces and the stability properties.

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Later, Schoen and Yau [33] used the Jang equation [21] to settle the positive energy conjecture. Given an initial data set (M, g, p), the Jang equation is

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + (g^{ij} - \frac{D^i u D^j u}{1+|Du|^2})p_{ij} = 0.$$

The Jang equation is a type of prescribed mean curvature equations. They found via adding a capillarity term that the solution blows up at open sets whose boundaries are so called *marginally outer/inner trapped surfaces* (abbreviated as MOTS/MITS).

The work [6] discussed the stability properties of the MOTS and the existence theory were developed in [12, 7]. This theory lead to a recent proof of the spacetime positive mass theorem [13].

A recent study of the asymptotically flat manifold with a noncompact boundary [3], [**?**] and [4] suggests that we look at the free boundary analogue of the marginally outer trapped surfaces.

We are going to extend the existence theory of Eichmair [12] by studying the Jang equation with mixed boundary conditions in this work.

We assume that (M, g) is an n-dimensional manifold, the boundary ∂M is composed two parts $\mathcal{N}$ and $\mathcal{D}$. Both $\mathcal{N}$ and $\mathcal{D}$ are relatively open in ∂M, $\mathcal{D}$ and $\mathcal{N}$ intersects at an $n-2$ dimensional submanifold $I = \bar{\mathcal{D}} \cap \bar{\mathcal{N}}$. We refer I as the *intersection place*. Let the outward pointing normal of $\mathcal{N}$ and $\mathcal{D}$ be respectively ν and γ. Both ν and γ extend continuously to I. We assume that $\langle \nu, \gamma \rangle < 0$ along I. This is equivalent to say that $\mathcal{N}$ and $\mathcal{D}$ intersect at acute angles. The Jang equation with mixed boundary equations are the following:

$$\begin{aligned} v^{-1}\bar{g}^{ij}u_{ij} &= \bar{g}^{ij}p_{ij} + tu \quad \text{ in } M,\\ \langle Du, \nu \rangle &= 0 \quad \text{ on } \mathcal{N},\\ u &= \varphi \text{ on } \mathcal{D}. \end{aligned}$$

Here $v = \sqrt{1+|Du|^2}$ and $\bar{g}^{ij} = g^{ij} - \frac{D^i u D^j u}{1+|Du|^2}$. We shall refer to $\mathcal{N}$ as the Neumann boundary and $\mathcal{D}$ as the Dirichlet boundary. Normally, the case $t = 0$ is called Jang equation. Here, we abuse this name to allow $t > 0$. The free boundary marginally outer trapped surface is defined as follows:

Definition 1.1 ([1])**.** Assume that (M, g, p) is an initial data set with $\partial M \neq \emptyset$, we say that Σ is a marginally outer/inner trapped surface with a free boundary if

$$H_\Sigma \pm \operatorname{tr}_\Sigma p = 0,$$

and Σ intersects ∂M orthogonally along $\partial\Sigma$.

Our main result is the following existence theorem.

$$\mu := \frac{1}{2}(R_M - |k|^2_M + (\operatorname{Tr}_M k)^2) J := \operatorname{div}_M(k - (\operatorname{Tr}_M k)g)$$

Our first main result is the following totally free boundary problem for MOTS.

Theorem 1.2. *Let (M^n, g, k) be an initial data set with smooth non-empty manifold boundary ∂M, and $\Omega \subset M^n$ be a connected bounded relatively open subset with topological boundary $\partial\Omega$ which is smoothly embedded in M. Suppose this boundary $\partial\Omega$ consists of two non-empty smooth compact hypersurfaces $\partial_1\Omega$ and $\partial_2\Omega$ with boundary lying on ∂M so that*

(i) $H_{\partial_1\Omega} + \operatorname{Tr}_{\partial_1\Omega} k > 0$ *and* $H_{\partial_2\Omega} - \operatorname{Tr}_{\partial_2\Omega} k > 0$*;*

(ii) *For $i = 1, 2$, along the intersection $I_i := \partial_i\Omega \cap \partial M$, the angle θ_i between $\partial_i\Omega$ and ∂M, measured with respect to Ω, lies in the range $(0, \pi/2)$.*

Here, the mean curvature scalar is computed as the tangential divergence of the unit normal vector field pointing out of Ω. Denote the manifold boundary of Ω as $T := \Omega \cap \partial M$.

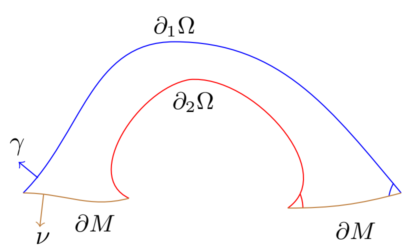
Then, there exists a compact embedded hypersurface $\Sigma^{n-1} \subset \Omega$ with manifold boundary $\partial\Sigma \subset T$ such that the following hold:

(a) *Σ is homologous to $\partial_1\Omega$ relative to T in Ω, i.e. there exists some relatively open subset $E \subset \Omega$ whose topological boundary $\partial E = \partial_1\Omega \cup \Sigma$;*
(b) *Σ is "C-almost minimizing in Ω relative to T" for an explicit constant $C = C(|k|_{C^0(\overline{\Omega})})$;*
(c) *Σ is smooth away from a singular set of Hausdorff dimension $\leq n-8$ and satisfies, in the distributional sense, the marginally outer trapped surface equation $H_\Sigma + \mathrm{Tr}_\Sigma\, k = 0$ and Σ meets T orthogonally along its free boundary $\partial\Sigma \subset T$.*

In particular, if $2 \leq n \leq 7$, then Σ^{n-1} is a non-empty smooth embedded compact marginally outer trapped surface in Ω meeting T orthogonally along its free boundary $\partial\Sigma$. If $3 \leq n \leq 7$, and furthermore the initial data set (M^n, g, k) satisfies the dominant energy conditions, i.e.

$$\mu \geq |J| \qquad \text{and} \qquad H_{\partial M} \geq |j|,$$

then Σ^{n-1} has "non-negative Yamabe type".



Our second main result is the following partially free boundary problem for MOTS.

Theorem 1.3. *Let (M^n, g, k) be an initial data set with smooth non-empty manifold boundary ∂M, and $\Omega \subset M^n$ be a connected bounded relatively open subset with topological boundary $\partial\Omega$ which is smoothly embedded in M. Let $\Gamma^{n-2} \subset \partial\Omega$ be a non-empty, smooth, compact, embedded submanifold with boundary lying inside $\partial\Omega \cap \partial M$ such that $\partial\Omega \setminus \Gamma$ is a disjoint union of relatively open subsets $\partial_1\Omega, \partial_2\Omega$ of $\partial\Omega$. Suppose that*

(a) *$H_{\partial\Omega} + \mathrm{Tr}_{\partial\Omega}\, k > 0$ near $\partial_1\Omega$ and $H_{\partial\Omega} - \mathrm{Tr}_{\partial\Omega}\, k > 0$ near $\partial_2\Omega$;*
(b) *Along the intersection $I := \partial\Omega \cap \partial M$, the angle between $\partial\Omega$ and ∂M, measured with respect to Ω, lies in the range $(0, \pi/2)$;*
(c) *Γ intersections I transversely inside $\partial\Omega$.*

Here, the mean curvature scalar is computed as the tangential divergence of the unit normal vector field pointing out of Ω. Denote the manifold boundary of Ω as $T := \Omega \cap \partial M$.

Then, there exists a a compact embedded hypersurface $\Sigma^{n-1} \subset \Omega$ *with manifold boundary* $\partial\Sigma$ *consisting of the fixed boundary* $\partial_D \Sigma = \Gamma$ *and the free boundary* $\partial_N \Sigma = \partial\Sigma \cap T$ *such that the following hold:*

- (i) Σ *is homologous to* $\partial_1 \Omega$ *relative to* T *in* Ω*, i.e. there exists some relatively open subset* $E \subset \Omega$ *whose topological boundary* $\partial E = \partial_1 \Omega \cup \Sigma$*;*
- (ii) Σ *is "*C*-almost minimizing in* Ω *relative to* T*' with fixed boundary* Γ*' for an explicit constant* C*;*
- (iii) Σ *is smooth away from a singular set of Hausdorff dimension* $\leq n-8$ *which is contained strictly inside* Ω*, so that* Σ *satisfies, in the distributional sense, the marginally outer trapped surface equation* $H_\Sigma + \mathrm{Tr}_\Sigma k = 0$ *and* Σ *meets* T *orthogonally along its free boundary* $\partial_N \Sigma$*.*
- (iv) Σ *is a smooth hypersurface near* Γ *with piecewise boundary* Γ *and* $\partial_N \Sigma$ *and Hölder continuous up to the corner* $\Gamma \cap \partial_N \Sigma$*.*

Smooth means C^∞ unless otherwise stated.

The article is organized as follows:

In Section **??**, we prove various a priori estimates of the Jang equation needed in the existence of the Jang equation. In Section 3, we extend the Perron method of [12] and [25] to show the existence of marginally outer trapped surfaces with a free boundary. In Section 5, we use the Λ-minimizing currents with a free boundary to address the regularity of the MOTS obtained from the previous section.

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2. A-priori estimates for the regularized Jang's equation

Assume $n \geq 3$. Let (M^n, g, k) be a *bordered initial data set* which consists of a smooth n-dimensional Riemannian manifold (M^n, g) with non-empty boundary ∂M and a symmetric $(0,2)$-tensor k on M smoothly defined up to the boundary. We shall use $x^1, \cdots, x^n$ to denote any local coordinates on M with metric components g_{ij} and inverse g^{ij}. The covariant derivative on (M, g) is denoted by D.

Throughout this paper, we will always assume that (M^n, g) (and its extension $(\tilde{M}, g)$ c.f. Appendix A) has a lower bound on Ricci curvature, i.e., there exists some $\kappa \geq 0$ such that

$$\mathrm{Ric} \geq -(n-1)\kappa g; \tag{2.1}$$

a lower bound of the injectivity radius, i.e. there exists some $i_0 > 0$ such that for all $x \in M$,

$$\mathrm{inj}_M(x) \geq i_0 > 0, \tag{2.2}$$

and ∂M has a upper bound on the interior ball curvature, i.e. there exists some $K > 0$ such that

$$\mu \leq K \tag{2.3}$$

where $\mu(x)$ is the interior ball curvature at $x \in \partial M$ as defined in [9].

Suppose $\Omega \subset M$ is a regular domain of M as defined in Definition A.2. From now on, we adopt the notations used in Appendix A. For any $u \in C^2(\Omega^\circ)$ what is Ω°?, we denote

$$H(u) := \left(g^{ij} - \frac{u^i u^j}{1+|Du|^2}\right) \frac{D_i D_j u}{\sqrt{1+|Du|^2}} \tag{2.4}$$

and

$$\mathrm{Tr}(k)(u) := \left(g^{ij} - \frac{u^i u^j}{1+|Du|^2}\right) k_{ij} \tag{2.5}$$

where $u_i = \frac{\partial u}{\partial x^i}$ and $|Du|^2 = u^i u_i$. We use the metric g to raise and lower indices and adopt Einstein's summation convention throughout this paper. It is well-known that the quantities (2.4) and (2.5) have the following geometric interpretation. As in [12], we denote $\Sigma^n := \mathrm{graph}(u) \subset M^n \times \mathbb{R}$ and endow this hypersurface with the induced metric from $M^n \times \mathbb{R}$ equipped with the metric $\overline{g} = g + dt^2$. Any local coordinates $x^1, \cdots, x^n$ on M induces a coordinate system $(x^1, \cdots, x^n) \mapsto (x^1, \cdots, x^n, u(x^1, \cdots, x^n))$ on Σ. Under this coordinate system, the metric components on Σ and its inverse is given respectively by

$$\overline{g}_{ij} = g_{ij} + u_i u_j \qquad \text{and} \qquad \overline{g}^{ij} = g^{ij} - \frac{u^i u^j}{1+|Du|^2}. \tag{2.6}$$

The downward pointing unit normal of Σ in $M \times \mathbb{R}$ is given by

$$\nu = \frac{u^i}{\sqrt{1+|Du|^2}} \frac{\partial}{\partial x^i} - \frac{1}{\sqrt{1+|Du|^2}} \frac{\partial}{\partial t}$$

and the second fundamental form of Σ (with respect to ν) is

$$h_{ij} := \overline{g}\left(\overline{D}_{\frac{\partial}{\partial x^i}} \nu, \frac{\partial}{\partial x^j}\right) = \frac{D_i D_j u}{\sqrt{1+|Du|^2}}$$

Should it be $\partial_i + u_i \partial_t$ in the definition of h_{ij}? where $\overline{D}$ is the covariant derivatives with respect to $\overline{g}$ and $D_i D_j u$ is the Hessian of u with respect to g. From this we see that $H(u) = \overline{g}^{ij} h_{ij}$ is the mean curvature H_Σ of Σ inside $M \times \mathbb{R}$ and $\mathrm{Tr}(k)(u) = \overline{g}^{ij} k_{ij}$ is the partial trace $\mathrm{Tr}_\Sigma k$ of the symmetric $(0,2)$-tensor k over $T\Sigma$. Here, k has been extended to $M \times \mathbb{R}$ trivially along the $\mathbb{R}$-factor. Using these notations, the Jang's equation is simply

$$H(u) + \mathrm{Tr}(k)(u) = 0, \tag{2.7}$$

a second order quasli-linear elliptic equation which is not uniformly elliptic unless $|Du|$ is bounded. For our purpose we will consider the linear homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \eta} = 0 \text{ on } \partial M \tag{2.8}$$

where $\eta = \eta_{\partial M}$ is the outward unit normal of ∂M with respect to (M, g). Note that (2.8) ensures that Σ meets $\partial M \times \mathbb{R}$ orthogonally.

As observed in [33], the existence of solutions to (2.7) and (2.8) is hampered by the lack of sub- and super-solutions for a-priori estimates. They introduced a capillary term to (2.7) and consider the regularized Jang's equation

$$H(u) + \mathrm{Tr}(k)(u) = \tau u \tag{2.9}$$

where $\tau > 0$ is a regularization parameter. We will assume $\tau \in (0,1)$. To carry out the Perron process, we consider a local mixed boundary value problem for (2.9) subject to (2.8). More precisely, given $\varphi \in C^0(\partial_D\Omega)$, we look for a solution $u \in C^2(\Omega^\circ) \cap C^1(\Omega) \cap C^0(\overline{\Omega})$ what is Ω°? Is $C^1(\Omega)$ the same as $C^1(\Omega^\circ)$? to the mixed boundary value problem

$$\begin{cases} H(u) + \mathrm{Tr}(k)(u) = \tau u & \text{in } \Omega^\circ \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial_N\Omega \\ u = \varphi & \text{on } \partial_D\Omega \end{cases} \tag{2.10}$$

The rest of this section is devoted to establishing a-priori estimates for (2.10), which is a quasi-linear non-uniformly elliptic equation supplemented by mixed linear boundary conditions. Since the PDE theory for such equations are not readily available in the literature, we collect the required elliptic theory for such problems in Appendix A.

2.1. C^0**-estimates.** The a-priori C^0-estimate to (2.10) can be established by maximum and comparison principles. As compared to [33, (4.4)], our result below gives the explicit estimate in terms of $\mathrm{Tr}_M k$.

Proposition 2.1 (C^0-estimate)**.** Let $u \in C^2(\Omega^\circ) \cap C^1(\Omega) \cap C^0(\overline{\Omega})$ be a solution to (2.10). Then we have

$$\sup_{\overline{\Omega}} |u| \leq \tau^{-1} |\mathrm{Tr}_M k|_{C^0(\overline{\Omega})} + \sup_{\partial_D\Omega} |u|.$$

Proof. Note that if we let $Qu := H(u) + Tr(k)(u) - \tau u$ and $N(u) = -\frac{\partial u}{\partial \eta}$, then the conditions (i)-(vi) in Theorem A.17 are satisfied. The constant function

$$v = \tau^{-1} |\mathrm{Tr}_M k|_{C^0(\overline{\Omega})} + \sup_{\partial_D\Omega} u^+,$$

where $u^+ = \max\{u, 0\}$, satisfies $Qv \leq 0$ in Ω°, $u \leq v$ on $\partial_D\Omega$ and $\frac{\partial v}{\partial \eta} = 0$ on $\partial_N\Omega$. By Theorem A.17, we have $u \leq v$ in $\overline{\Omega}$ which is our desired upper bound. The lower bound can be obtained similarly by taking $v = -\tau^{-1} |\mathrm{Tr}_M k|_{C^0(\overline{\Omega})} - \sup_{\partial_D\Omega} u^-$, where $u^- = \max\{-u, 0\}$. $\square$

2.2. C^1**-estimates.** We now proceed with a-priori estimates for the gradient of the solutions to (2.10) up to the boundary. More precisely, we will bound $|Du|$ on $\overline{\Omega}$ in terms of $\sup_{\overline{\Omega}} |u|$, $\sup_{\partial_D\Omega} |Du|$ and the geometry of the initial data set (M, g, k).

First, we apply a maximum principle argument to a suitable auxiliary function to reduce the global gradient estimate on $\overline{\Omega}$ to the gradient estimate on the Dirichlet boundary $\partial_D\Omega$.

Proposition 2.2. There exists a dimensional constant $C = C(n) > 0$ such that if $u \in C^3(\Omega^\circ) \cap C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution to (2.10) with $\varphi \in C^1(\partial_D\Omega)$, then we have

$$\tau \sup_{\overline{\Omega}} |Du| \leq \max\left\{ \tau \sup_{\partial_D\Omega} |Du|, C\left[K^2 + \kappa + K\left(\tau|\varphi|_{C^0(\partial_D\Omega)} + |k|_{C^0(\overline{\Omega})} \right) + |Dk|_{C^0(\overline{\Omega})} \right] \right\}$$

where κ and K are the Ricci curvature lower bound of (M, g) and the upper bound of the interior ball curvature of ∂M as in (2.1) and (2.3).

Proof. We shall apply a maximum principle argument to the auxiliary function

$$\Phi := \log |Du|^2 + \alpha_0 d'$$

where $d' \in C^\infty(\overline{\Omega})$ is the regularized distance function from $\partial_N \Omega$ in Lemma A.6, and $\alpha_0 > 0$ is a sufficiently large constant to be determined later.

By a well-known formula (see e.g. [38, Lemma 3.1]), the Neumann condition (2.8) implies $\frac{\partial}{\partial \eta}|Du|^2 = 2A_{\partial M}(Du, Du)$ along $\partial_N \Omega$. Therefore, at any point on $\partial_N \Omega$ where $Du \neq 0$, we have the following inequality

$$\frac{\partial \Phi}{\partial \eta} = 2A_{\partial M}\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right) - \alpha_0 < 0$$

provided that $\alpha_0 > 2K$ where K is an upper bound of the interior ball curvature of ∂M on $\partial_N \Omega$. For simplicity, we choose $\alpha_0 = 4K$. Therefore, Φ cannot achieve its maximum along $\partial_N \Omega$.

Suppose that $\Phi(x_0) = \sup_{\overline{\Omega}} \Phi$. From the above, we have $x_0 \notin \partial_N \Omega$. If $x_0 \in \partial_D \Omega$, then the assertion follows trivially. Therefore, we can assume that $x_0 \in \Omega^\circ$ is an interior maximum. From $D\Phi(x_0) = 0$, we obtain at x_0 that

$$D(|Du|^2) = -4K|Du|^2 Dd'. \tag{2.11}$$

From $D_i D_j \Phi(x_0) \leq 0$, we obtain at x_0 that

$$0 \geq D_i D_j \Phi = \frac{D_i D_j(|Du|^2)}{|Du|^2} - \frac{D_i(|Du|^2) D_j(|Du|^2)}{|Du|^4} + 4K D_i D_j d'.$$

Substituting (2.11) into the above, we obtain at x_0 that

$$0 \geq \frac{D_i D_j(|Du|^2)}{|Du|^2} - 16K^2 D_i d' D_j d' + 4K D_i D_j d'.$$

Multiplying the above to the positive definite $\overline{g}^{ij}$ in (2.6) and take trace, we obtain at x_0 that

$$0 \geq \overline{g}^{ij} D_i D_j \Phi = \frac{\overline{g}^{ij} D_i D_j(|Du|^2)}{|Du|^2} - 16K^2 \overline{g}^{ij} D_i d' D_j d' + 4K \overline{g}^{ij} D_i D_j d'. \tag{2.12}$$

By the definition of $\overline{g}^{ij}$ in (2.6), we have

$$\overline{g}^{ij} D_i d' D_j d' = |Dd'|^2 - \frac{\langle Du, Dd' \rangle^2}{1 + |Du|^2},$$

$$\overline{g}^{ij} D_i D_j d' = \Delta d' - \frac{u^i u^j D_i D_j d'}{1 + |Du|^2}.$$

From (2.12) and Lemma A.6, at x_0 we have

$$\overline{g}^{ij} D_i D_j(|Du|^2) \leq 5000 K^2 |Du|^2. \tag{2.13}$$

Observe that $D_j(|Du|^2) = 2u^k u_{kj}$. Thus we have

$$\begin{aligned} D_i D_j(|Du|^2) &= 2u^k{}_i u_{kj} + 2u^k u_{kji} \\ &= 2u^k{}_i u_{kj} + 2u^k u_{ijk} + 2R_{ikj\ell} u^k u^\ell \end{aligned} \tag{2.14}$$

where we have used the Ricci identity in the last equality. Next, we rewrite the third order term u_{ijk} using the regularized Jang's equation. In terms of $\overline{g}^{ij}$, (2.9) reads

$$\frac{\overline{g}^{ij} u_{ij}}{\sqrt{1 + |Du|^2}} - \overline{g}^{ij} k_{ij} = \tau u.$$

Differentiating the above along the direction of Du, we obtain

$$u^k D_k \left(\frac{\overline{g}^{ij} u_{ij}}{\sqrt{1+|Du|^2}} \right) - u^k (D_k \overline{g}^{ij}) k_{ij} - \overline{g}^{ij} u^k D_k k_{ij} = \tau |Du|^2. \tag{2.15}$$

We will show that each term on the left hand side of (2.15) is bounded from above by $C|Du|$ at x_0.

Claim 1: $|u^k D_k \overline{g}^{ij}| \leq 200K|Du|$ *at* x_0.

Proof of Claim 1: By (2.6) and (2.11), we have at x_0 that

$$\begin{aligned} u^k D_k \overline{g}^{ij} &= -u^k \frac{u^i{}_k u^j + u^i u^j{}_k}{1+|Du|^2} + \frac{u^i u^j u^k D_k(|Du|^2)}{(1+|Du|^2)^2} \\ &= -\frac{u^j D^i(|Du|^2) + u^i D^j(|Du|^2)}{1+|Du|^2} + \frac{u^i u^j u^k D_k(|Du|^2)}{(1+|Du|^2)^2} \\ &= 4K \frac{|Du|^2}{1+|Du|^2} \left(u^j D^i d' + u^i D^j d' - \frac{u^i u^j}{1+|Du|^2} u^k D_k d' \right). \end{aligned}$$

The claim then follows from Lemma A.6.

Claim 2: $|(u^k D_k \overline{g}^{ij}) u_{ij}| \leq 2000K^2 |Du|^2$ *at* x_0.

Proof of Claim 2: Using the last expression of $u^k D_k \overline{g}^{ij}$ in the proof of Claim 1 and (2.11), we have at x_0 that

$$(u^k D_k \overline{g}^{ij}) u_{ij} = -8K^2 \frac{|Du|^4}{1+|Du|^2} \left(2|Dd'|^2 - \frac{(u^k D_k d')^2}{1+|Du|^2} \right).$$

The claim then follows again from Lemma A.6.

Claim 3: $\overline{g}^{ij} u^k u_{ijk} \leq (5000K^2 + (n-1)\kappa)|Du|^2$ *at* x_0.

Proof of Claim 3: From (2.14), we have

$$\overline{g}^{ij} u^k u_{ijk} = \frac{1}{2} \overline{g}^{ij} D_i D_j (|Du|^2) - \overline{g}^{ij} u^k{}_i u_{kj} - \overline{g}^{ij} R_{ikj\ell} u^k u^\ell.$$

Note that

$$\overline{g}^{ij} u^k{}_i u_{kj} = |\mathrm{Hess}(u)|^2 - \frac{|Du|^2}{1+|Du|^2} \left| \mathrm{Hess}(u) \left(\frac{Du}{|Du|}, \cdot \right) \right|^2 \geq 0$$

and

$$\overline{g}^{ij} R_{ikj\ell} u^k u^\ell = \mathrm{Ric}(Du, Du) \geq -(n-1)\kappa |Du|^2.$$

Combining the two inequalities above with (2.13) completes the proof of Claim 3.

We now proceed to estimate the terms on the left hand side of (2.15) from above. For the first term,

$$u^k D_k \left(\frac{\overline{g}^{ij} u_{ij}}{\sqrt{1+|Du|^2}} \right) = \frac{(u^k D_k \overline{g}^{ij}) u_{ij} + \overline{g}^{ij} u^k u_{ijk}}{\sqrt{1+|Du|^2}} - \frac{\overline{g}^{ij} u_{ij} u^k D_k(|Du|^2)}{2(1+|Du|^2)^{\frac{3}{2}}}.$$

By Claim 2 and Claim 3, at x_0 the first term on the right hand side above satisfies

$$\frac{(u^k D_k \overline{g}^{ij}) u_{ij} + \overline{g}^{ij} u^k u_{ijk}}{\sqrt{1+|Du|^2}} \leq (10000K^2 + (n-1)\kappa) \frac{|Du|^2}{\sqrt{1+|Du|^2}}.$$

To estimate the second term, we make use of the regularized Jang's equation (2.9) and (2.11) that at x_0,

$$\begin{aligned}
-\frac{\overline{g}^{ij}u_{ij}u^k D_k(|Du|^2)}{2(1+|Du|^2)^{\frac{3}{2}}} &= \frac{2K(\tau u-\overline{g}^{ij}k_{ij})|Du|^2\langle Du, Dd'\rangle}{(1+|Du|^2)} \\
&\le 18K\left(\sup_{\overline{\Omega}}\tau|u|+n|k|_{C^0(\overline{\Omega})}\right)\frac{|Du|^3}{1+|Du|^2} \\
&\le 18K\left(\tau|\varphi|_{C^0(\partial_D\Omega)}+2n|k|_{C^0(\overline{\Omega})}\right)\frac{|Du|^3}{1+|Du|^2}
\end{aligned}$$

Putting all of these together, we have the following estimate at x_0:

$$u^k D_k\left(\frac{\overline{g}^{ij}u_{ij}}{\sqrt{1+|Du|^2}}\right)\le C(n)\left[K^2+\kappa+K\left(\tau|\varphi|_{C^0(\partial_D\Omega)}+|k|_{C^0(\overline{\Omega})}\right)\right]|Du|.$$

Using Claim 1, we can estimate the second term on the left hand side of (2.15) at x_0 that

$$-u^k(D_k\overline{g}^{ij})k_{ij}\le 200K|k|_{C^0(\overline{\Omega})}|Du|.$$

Finally, it is immediate that

$$-\overline{g}^{ij}u^k D_k k_{ij}\le 2|Dk|_{C^0(\overline{\Omega})}|Du|.$$

Combining all the three estimates above and plug into (2.15), we then obtain

$$\tau|Du|(x_0)\le C(n)\left[K^2+\kappa+K\left(\tau|\varphi|_{C^0(\partial_D\Omega)}+|k|_{C^0(\overline{\Omega})}\right)+|Dk|_{C^0(\overline{\Omega})}\right].$$

This proves the proposition as Φ achieves it maximum at x_0 and $|4Kd'|\le 2$ by Lemma A.6. □

From Proposition 2.2, to obtain the global gradient bound on $\overline{\Omega}$ it suffices to prove a boundary gradient estimate on $\partial_D\Omega$. As the boundary gradient estimates along the interior part of $\partial_D\Omega$ can be established using the standard barrier techniques of Serrin [36] as in [12], our major contribution here is to show that boundary gradient estimates on $\partial_D\Omega$ continue to hold up to the edge E, subject to an extra angle condition.

Proposition 2.3 (C^1-estimate along $\partial_D\Omega$). Let $\varphi\in C^2(\partial_D\Omega)$. Suppose there exist positive constants $\delta_H,\delta_\theta>0$ that the following are satisfied:

(i) (Serrin condition) At every point on $\partial_D\Omega$, we have

$$H_{\partial_D\Omega}-|\operatorname{Tr}_{\partial_D\Omega}k-\tau\varphi|\ge\delta_H>0 \tag{2.16}$$

where $H_{\partial_D\Omega}$ is computed as the tangential divergence of the outward unit normal of $\partial_D\Omega$;

(ii) (Angle condition) At every point along the edge $E=\partial_D\Omega\cap\overline{\partial_N\Omega}$, we have

$$0<\theta\le\frac{\pi}{2}-\delta_\theta<\frac{\pi}{2} \tag{2.17}$$

where θ is the angle between $\partial_D\Omega$ and $\partial_N\Omega$ as defined in Definition A.2.

Then for any $u\in C^2(\Omega^\circ)\cap C^1(\overline{\Omega})$ which is a solution to (2.10),we have

$$\tau\sup_{\partial_D\Omega}|Du|\le C$$

where $C>0$ is a constant depending only on δ_H, δ_θ, n, κ, K, $|k|_{C^1(\overline{\Omega})}$, $|\varphi|_{C^2(\partial_D\Omega)}$ and the upper bound on the exterior ball curvature of $\partial_D\Omega$.

Proof. According to the arguments in Appendix A, it suffices to construct an upper and lower barrier at any point $x_0 \in \partial_D\Omega$ as defined in Definition A.19 with bounded gradient. We first treat the case with zero Dirichlet boundary value, i.e. $\varphi \equiv 0$. Consider the quasilinear operator

$$Qu := (1+|Du|^2)^{\frac{3}{2}}(H(u) + \mathrm{Tr}(k)(u) - \tau u)$$

which can be decomposed as in [15, (14.43)]

$$Qu = a^{ij}(x, Du, u)D_iD_ju + a(x, Du, u)$$

with

$$\begin{aligned} a^{ij}(x,p,z) &= \Lambda(x,p,z)a_\infty^{ij}(x,p,z) + a_0^{ij}(x,p,z), \\ a(x,p,z) &= |p|\Lambda(x,p,z)a_\infty(x,p,z) + a_0(x,p,z). \end{aligned}$$

where (cf. [12, Lemma 2.2], noting that D_iD_ju denotes the Riemannian Hessian of u hence we do not have the terms involving the Christoffel symbols Γ_{ij}^k)

$$\begin{aligned} \Lambda(x,p,z) &= 1+|p|^2, \\ a_\infty^{ij}(x,p,z) &= a_\infty^{ij}\left(x, \frac{p}{|p|}\right) = g^{ij} - \frac{p^ip^j}{|p|^2}, \\ a_0^{ij}(x,p,z) &= \frac{p^ip^j}{|p|^2}, \\ a_\infty(x,p,z) &= a_\infty\left(x, \frac{p}{|p|}, z\right) = \left(g^{ij} - \frac{p^ip^j}{|p|^2}\right)k_{ij} - \tau z, \\ a_0(x,p,z) &= -\tau z\frac{\Lambda}{\Lambda^{1/2}+|p|} + k_{ij}\left(g^{ij}\frac{\Lambda}{\Lambda^{1/2}+|p|} + \frac{p^ip^j}{|p|}\frac{\Lambda^{1/2}}{\Lambda^{1/2}+|p|}\right). \end{aligned}$$

Note that both a_0 and a_∞ are non-increasing in z. Moreover, we have

$$\left|\frac{a_0^{ij}}{\Lambda}\right| \le \frac{1}{|p|^2} \quad \text{and} \quad \left|\frac{a_0}{\Lambda}\right| \le (|\tau z| + (n+1)|k|_{C^0})\frac{1}{|p|} \tag{2.18}$$

and hence the structural conditions (14.59) in [15] are satisfied.

Define $w = C\tau^{-1}d$, where d is the regularized distance function from $\partial_D\Omega$ as in Lemma A.5 and $C > 0$ is a large constant to be chosen later. Using Serrin condition (2.16) and $\varphi = 0$, $H_{\partial_D\Omega} - |\operatorname{Tr}_{\partial_D\Omega} k| \ge \delta_H > 0$. This implies that the boundary curvature conditions [15, (14.60)] are satisfied at every $y \in \partial_D\Omega$ (cf. [12, Lemma 2.2]):

$$\begin{aligned} \mathcal{K}^+ - a_\infty(y, -\eta_{\partial_D\Omega}, 0) &= H_{\partial_D\Omega} - \operatorname{Tr}_{\partial_D\Omega} k \ge \delta_H > 0, \\ \mathcal{K}^- + a_\infty(y, \eta_{\partial_D\Omega}, 0) &= H_{\partial_D\Omega} + \operatorname{Tr}_{\partial_D\Omega} k \ge \delta_H > 0. \end{aligned}$$

Using the first inequality, together with Lemma A.5, [15, (14.45)], and the expressions of a_∞^{ij} and a_∞, we can infer by the Riccati equation, $\tau w \ge 0$ and $k \in C^1$ that if we choose $a \in (0, 10^{-2}K^{-1})$ such that

$$a \cdot \Big((n-1)\kappa + n|Dk|_{C^0(\overline{\Omega})}\Big) < \frac{\delta_H}{2} \tag{2.19}$$

then

$$a_\infty^{ij}(x, Dd)D_iD_jd + a_\infty(x, Dd, w) \le -\frac{\delta_H}{2} < 0 \tag{2.20}$$

inside the region $\{x \in \Omega : d(x) < a\}$. As in p. 346 of [15, Section 14.3], we have from Lemma A.5 inside the same region that $a_0^{ij}(x, u, Dw)D_iD_jd = 0$, $\Lambda(x, u, Dw) = 1 + \tau^{-2}C^2$ and hence by (2.18) and (2.20),

$$\begin{aligned}\overline{\overline{Q}}w &:= a^{ij}(x, Dw, u)D_iD_jw + |Dw|\Lambda(x, Dw, u)a_\infty(x, Dw, w) + a_0(x, Dw, u) \\ &= \Lambda\big[C \cdot \big(a_\infty^{ij}(x, Dd)D_iD_jd + a_\infty(x, Dd, w)\big) + \Lambda^{-1}a_0(x, C\tau^{-1}Dd, u)\big] \\ &\leq \Lambda\left[-C\frac{\delta_H}{2} + (\tau|u|_{C^0(\overline{\Omega})} + (n+1)|k|_{C^0})\frac{\tau}{C}\right].\end{aligned}$$

Therefore, using the C^0-estimate in Proposition 2.1 with $\varphi \equiv 0$, we have $\overline{\overline{Q}}w < 0$ inside $\{x \in \Omega : d(x) < a\}$ provided that $C > 0$ is chosen sufficiently large so that

$$C^2 > 4(n+1)\delta_H^{-1}|k|_{C^0(\overline{\Omega})}. \tag{2.21}$$

Let $W = \{x \in \Omega : d(x) < a\}$ be a relatively open neighborhood of $\partial_D\Omega$ in $\overline{\Omega}$ and we have $w > u$ along ∂W by the C^0-estimate in Proposition 2.1 provided that $C > 0$ is sufficiently large such that

$$Ca \geq n|k|_{C^0(\overline{\Omega})}. \tag{2.22}$$

In other words, $w^+ = w$ satisfies (i)-(iii) of Definition A.19 for the operator $\overline{\overline{Q}}$ provided that (2.19), (2.21) and (2.22) hold. Finally, it remains to check that (iv) in Definition A.19 holds with respect to $N = -\frac{\partial}{\partial\eta}$. By the angle condition (2.17) and a Riccati-type argument for the angle (c.f. [22, Section 2]), by requiring $a > 0$ sufficiently small, depending only on δ_θ, K and the upper bound on the exterior ball curvature of $\partial_D\Omega$, such that

$$Nw = -C\tau^{-1}\frac{\partial d}{\partial\eta} \leq -C\tau^{-1}\cos\left(\frac{\pi}{2} - \frac{\delta_\theta}{2}\right) < 0 \tag{2.23}$$

along $W \cap \partial_N\Omega$. Therefore, if we first choose $a \in (0, 10^{-2}K^{-1})$ sufficiently small (depending on K, n, κ, $|Dk|_{C^0(\overline{\Omega})}$, δ_θ and the upper bound on the exterior ball curvature of $\partial_D\Omega$) such that (2.19) and (2.23) are satisfied, then we choose accordingly $C > 0$ sufficiently large (depending on a, n, δ_H and $|k|_{C^0(\overline{\Omega})}$) such that (2.21) and (2.22) are satisfied. Altogether, we have an upper barrier $w = C\tau^{-1}d$ for u at every $x_0 \in \partial_D\Omega$ defined in the tubular neighborhood $\{x \in \Omega : d(x) < a\}$ with $|Dw| = C\tau^{-1}$, where $C > 0$ is a constant depending only on δ_H, δ_θ, n, κ, K, $|k|_{C^1(\overline{\Omega})}$ and the upper bound on the exterior ball curvature of $\partial_D\Omega$. A similar argument gives a lower barrier by considering $w = -C\tau^{-1}d$.

Finally, in order to extend the previous arguments to general non-zero boundary values φ, we proceed as in [15, Chapter 14] by replacement of u by $v := u - \varphi$, where φ has been extended to $\overline{\Omega}$ by Lemma A.7. Consider the quasi-linear operator (c.f. [15, (14.5)])

$$\tilde{Q}v := Qu = \tilde{a}^{ij}(x, Dv, v)D_iD_jv + \tilde{a}(x, Dv, v)$$

where

$$\tilde{a}^{ij}(x, p, z) = a^{ij}(x, p + D\varphi, z + \varphi),$$
$$\tilde{a}(x, z, p) = a^{ij}(x, p + D\varphi, z + \varphi)D_iD_j\varphi + a(x, p + D\varphi, z + \varphi).$$

On the other hand, we have $Nv = -N\varphi$ along $\partial_N\Omega$ and $v = 0$ on $\partial_D\Omega$. We then apply the same argument as before to obtain a barrier w for v with respect to the operator $\tilde{Q}$ and N, where the bound also depends on $|\varphi|_{C^2(\partial_D\Omega)}$. □

Combining Proposition 2.2 and 2.3, we then have the following global C^1-bound.

Proposition 2.4 (Global C^1-estimate)**.** Suppose $u \in C^3(\Omega) \cap C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution to (2.10) with $\varphi \in C^2(\partial_D \Omega)$ and the Serrin condition and angle condition in Proposition 2.3 are satisfied. Then, we have

$$\sup_{\overline{\Omega}} |Du| \leq C\tau^{-1}$$

where $C > 0$ is a constant depending only on δ_H, δ_θ, n, κ, K, $|k|_{C^1(\overline{\Omega})}$, $|\varphi|_{C^2(\partial_D \Omega)}$ and the upper bound on the exterior ball curvature of $\partial_D \Omega$.

2.3. **Interior gradient estimates.** When establishing the gradient bounds in Section 2.2, we assume that the solution u to (2.10) is C^1 up to the Dirichlet boundary. In fact, one can prove interior gradient estimates using the arguments of Korevaar-Simon in [23]. The interior gradient estimates for the regularized Jang's equation (2.9) has been established in [12, Lemma 2.1], which applies to balls that are contained in the interior of M, i.e. away from the boundary ∂M. We now show that the estimate continue to hold near ∂M. The bound would depend more on the geometry of (M, g) and ∂M but the argument is local and we can derive the gradient estimate merely from a one-sided oscillation bound of u.

Proposition 2.5 (Boundary interior gradient estimate)**.** Let $\tilde{B}(x_0, \rho)$ be an open geodesic ball contained in the interior of $\tilde{M}$ centered at $x_0 \in M$ with radius ρ less than a half of the injectivity radius of $(\tilde{M}, g)$ at x_0. Suppose that $\tilde{B}(x_0, \rho) \cap \partial M \neq \emptyset$, $\rho < K^{-1}/4$ and $\Omega := \tilde{B}(x_0, \rho) \cap M$ is a regular domain of M. Suppose that $u \in C^3(\Omega^\circ) \cap C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution to (2.9) and (2.8) satisfying a one-sided oscillation bound, i.e. there exists some $T > 0$ such that

$$\begin{cases} \text{either} \quad u(x) \leq u(x_0) + T \text{ for all } x \in \Omega, \\ \quad\text{or} \quad\;\; u(x) \geq u(x_0) - T \text{ for all } x \in \Omega. \end{cases}$$

Then, we have

$$|Du|(x_0) \leq C$$

where $C > 0$ is a constant depending only on T, ρ, n, and the bounds on Ω for $|k|_{C^1}$, $\tau |u|_{C^0}$, the sectional curvatures of (M, g) and the second fundamental form of ∂M.

Proof. We only consider $u(x) \leq u(x_0) + T$. Following [12, Lemma 2.1], we define

$$\phi(x) := -u(x_0) + u(x) + \rho - \frac{T + \rho}{\rho^2} \mathrm{dist}_g(x_0, x)^2$$

and consider the following test function

$$w := (\mathrm{e}^{C\phi + C\alpha d'} - 1)v$$

where $C \geq 1$ and $\alpha > 0$ are constants to be determined, $v := \sqrt{1 + |Du|^2}$, and d' is the regularized distance function from ∂M as in Lemma A.6. Denote $\Omega_+ := \{x \in \Omega : \phi > 0\}$. Note that $x_0 \in \Omega$ and $\phi = 0$ on $\partial \Omega_+$. Suppose that the function w achieves its maximum in $\overline{\Omega}_+$ at a point p.

We first show that by choosing $\alpha > 0$ sufficiently large, we can make $\frac{\partial w}{\partial \eta} < 0$ everywhere on $\Omega_+ \cap \partial M$ and hence $p \notin \partial M$. As in the proof of Proposition 2.2, we have from (2.8) that $\frac{\partial}{\partial\eta}|Du|^2 = 2A_{\partial M}(Du, Du)$ and thus $\frac{\partial v}{\partial\eta} = v^{-1}A_{\partial M}(Du, Du)$. On the other hand,

$$\frac{\partial\phi}{\partial\eta} = \frac{\partial u}{\partial\eta} - \frac{2(T+\rho)\mathrm{dist}_g(x_0, x)}{\rho^2}\frac{\partial\mathrm{dist}_g(x_0, x)}{\partial\eta} \le \frac{2(T+\rho)}{\rho}.$$

Combining these estimates, we have

$$\begin{aligned}\frac{\partial w}{\partial\eta} &\le \left(\frac{2(T+\rho)}{\rho}C - \alpha C\right)e^{C\phi}v + (e^{C\phi}-1)A_{\partial M}(Du, Du)v^{-1}\\ &\le \left(\frac{2(T+\rho)}{\rho}C - \alpha C + K\right)e^{C\phi}v.\end{aligned}$$

Therefore, by taking $\alpha > 0$ sufficiently large depending only on T, ρ and K, we have $\frac{\partial w}{\partial\eta} < 0$ along $\partial_N\Omega$.

It remains to treat the case that $p \in \Omega$ is an interior maximum point of w. Let Σ be the graph of u inside $M \times \mathbb{R}$. As in [12, Section 2], there exist a constant $\beta \ge 0$, depending only on n, κ and $|k|_{C^1}$ such that

$$\Delta_\Sigma\left(\frac{1}{v}\right) \le \frac{\beta}{v}.$$

By the same calculations as in [12, Lemma 2.1], we have at p the following

$$0 \ge \frac{1}{v}\Delta_\Sigma w \ge e^{C\phi + C\alpha d'}\left(C\Delta_\Sigma\phi + C\alpha\Delta_\Sigma d' + C^2|\nabla_\Sigma\phi + \alpha\nabla_\Sigma d'|^2 - \beta\right).$$

We have as in [12, Lemma 2.1] (recall that $\Delta_\Sigma u = v^{-1}H_\Sigma$) that

$$\begin{aligned}&C\left|\nabla_\Sigma u - \frac{T+\rho}{\rho^2}\nabla_\Sigma\mathrm{dist}_g^2(x_0,\cdot) + \alpha\nabla_\Sigma d'\right|^2\\ &\qquad\le \left|v^{-1}H_\Sigma - \frac{T+\rho}{\rho^2}\Delta_\Sigma\mathrm{dist}_g^2(x_0,\cdot) - \beta + \alpha\Delta_\Sigma d'\right|.\end{aligned}$$

The right hand side of the inequality above is a-priori bounded as follow. Using the regularized Jang's equation (2.9), H_Σ is bounded by $\tau|u|_{C^0}$ and $n|k|_{C^0}$. By [5, Lemma 1.4], $\Delta_\Sigma\mathrm{dist}_g^2(x_0,\cdot)$ is bounded in terms of ρ, H_Σ and the sectional curvatures of (M, g) by Hessian comparision theorem. Finally, $\Delta_\Sigma d$ is bounded by ρ, H_Σ and the second fundamental form of ∂M by Rauch comparison theorem. The rest follows by the same argument of [12, Lemma 2.1]. □

2.4. $C^{1,\alpha}$**-estimates.** We now proceed to establish the Hölder gradient estimates for solutions to the regularized Jang's equation (2.10). The idea is to apply De Giorgi-Nash-Moser theory adapted to certain mixed boundary value problems. We also obtain estimates in terms of the geometry of (M, g) by expressing the equations in suitable harmonic coordinates.

We first recall the following result of Cheeger-Anderson [5] which control the interior harmonic radius in terms of the Ricci curvature lower bound. Let us recall the definition of $W^{1,p}$-harmonic radius (which is called $L^{1,p}$-harmonic radius in [5]).

Definition 2.6 ($W^{1,p}$-harmonic radius)**.** Let $p \in (n, \infty)$ and $Q > 1$. The $W^{1,p}$-harmonic radius of an n-dimensional Riemannian manifold (M^n, g) without boundary is the largest number $r_H = r_H(p, Q) > 0$ such that on any geodesic ball

$B(x, r_H)$ of radius r_H in (M, g), there is a harmonic coordinate chart $(y^1, \cdots, y^n)$: $B(x, r_H) \to \mathbb{R}^n$ such that the metric components $g_{ij} := g(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$ satisfy

$$Q^{-1}\delta_{ij} \leq g_{ij} \leq Q\delta_{ij} \quad \text{and} \quad (r_H)^{1-\frac{n}{p}} |\partial g_{ij}|_{L^p} \leq Q - 1 \tag{2.24}$$

where the norms are taken with respect to the harmonic coordinates y^i on $B(x, r_H)$.

Lemma 2.7 (Estimate for $W^{1,p}$-harmonic radius). *Suppose (M^n, g) has a lower Ricci curvature bound* (2.1) *and an injectivity radius lower bound* (2.2)*. Then, there exists a constant $r_0 > 0$, depending only on Q, n, p, κ and i_0, such that*

$$r_H \geq r_0$$

where r_H is the $W^{1,p}$-harmonic radius of (M, g) as defined in Definition 2.6.

Proof. It follows from [5, Theorem 0.3], noting that $s_M(x) \geq \mathrm{inj}_M(x)$ by definition (0.6) in [5]. □

We now prove the interior Hölder gradient estimate for solutions to the regularized Jang's equation (2.9).

Proposition 2.8 (Interior Hölder gradient estimates)**.** Let $p \in (n, \infty)$ and $Q > 1$. Suppose $u \in C^2(B(x, 4r))$ is a solution to the regularized Jang's equation (2.9) on a geodesic ball $B(x, 4r)$ of (M, g) such that $r < \frac{1}{8}\min\{r_H, \mathrm{dist}_g(x, \partial M)\}$. Then, there exists some $\alpha > 0$, depending only on n, p, Q and $\sup_{\overline{B}(x,2r)} |Du|$, such that

$$|Du|_{C^{0,\alpha}(B(x,r))} \leq C$$

where $C > 0$ is a constant depending only on n, p, Q, r, $\sup_{\overline{B}(x,2r)} |Du|$, $|k|_{C^0(\overline{B}(x,2r))}$ and $\sup_{\overline{B}(x,2r)} \tau|u|$.

Proof. The regularized Jang's equation (2.9) can be expressed in divergence form as (recall (2.6))

$$\mathrm{div}_g \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) + \overline{g}^{ij} k_{ij} - \tau u = 0.$$

Integrating by part, we have for any $\xi \in C_0^1(B(x, 2r))$,

$$\int_{B(x,2r)} \left(\frac{\langle Du, D\xi \rangle}{\sqrt{1 + |Du|^2}} - (\overline{g}^{ij} k_{ij} - \tau u)\xi \right) d\mathrm{Vol}_g = 0. \tag{2.25}$$

Let $y^1, \cdots, y^n$ be the harmonic coordinates on $B(x, 2r)$, which exist by Lemma 2.7, and denote $\partial_k = \frac{\partial}{\partial y^k}$. In such coordinates, the metric components are $g_{ij} = g(\partial_i, \partial_j)$ and the volume form is $d\mathrm{Vol}_g = \sqrt{g}dy$ where $g = \det(g_{ij})$. Taking $\xi = \partial_k \zeta$, $k = 1, \cdots, n$, in (2.25) where $\zeta \in C_0^1(B(x, 2r))$, we have

$$\int_{B(x,2r)} \left(\frac{g^{ij}\partial_j u}{\sqrt{1 + |Du|^2}} \partial_i \partial_k \zeta - (\overline{g}^{ij} k_{ij} - \tau u)\partial_k \zeta \right) \sqrt{g}\, dy = 0.$$

Since $\partial_i \partial_k \zeta = \partial_k \partial_i \zeta$, we can integrate by part and obtain

$$\int_{B(x,2r)} \left[\partial_k \left(\frac{\sqrt{g} g^{ij} \partial_j u}{\sqrt{1 + |Du|^2}} \right) + \delta_k^i \sqrt{g}(\overline{g}^{pq} k_{pq} - \tau u) \right] \partial_i \zeta\, dy = 0.$$

Note that $\partial_k g^{ij} = -g^{pi}g^{qj}\partial_k g_{pq}$ and $\partial_k\sqrt{g} = \frac{1}{2}\sqrt{g}g^{ij}\partial_k g_{ij}$. Therefore, if we define $w = \partial_k u$, then we have for all $\zeta \in C_0^1(B(x,2r))$,

$$\int_{B(x,2r)} (a^{ij}\partial_j w + f_k^i)\partial_i\zeta \, dy = 0$$

where

$$a^{ij} := \frac{\sqrt{g}}{\sqrt{1+|Du|^2}}\overline{g}^{ij},$$

$$f_k^i := \frac{\sqrt{g}}{2}\left(g^{mn} + \frac{u^m u^n}{1+|Du|^2}\right)\frac{u^i}{\sqrt{1+|Du|^2}}\partial_k g_{mn}$$
$$-\sqrt{g}\frac{u^q}{\sqrt{1+|Du|^2}}g^{pi}\partial_k g_{pq} + \delta_k^i\sqrt{g}(\overline{g}^{pq}k_{pq} - \tau u).$$

Hence, $w = \partial_k u \in C^1(B(x,4r))$ is a generalized $W^{1,2}$-solution of the linear elliptic equation on the Euclidean ball $B_{2r} \subset \mathbb{R}^n$ under the harmonic coordinate chart:

$$\partial_i(a^{ij}\partial_j w) = -\partial_i f_k^i.$$

Since $u \in C^2(B(x,4r))$, $|Du|$ must be bounded on $\overline{B}(x,2r)$. From (2.24), $a^{ij} \in C^1(\overline{B}_{2r})$ is bounded and uniformly elliptic on $\overline{B}_{2r}$ with the ellipticity ratio bounded in terms of Q and $\sup_{\overline{B}_{2r}}|Du|$. Moreover, $f_k^i \in C^1(\overline{B}_{2r})$ and by (2.24) its L^p-norm (with respect to the Euclidean metric) is bounded in terms of Q, p, n, r_H, $|k|_{C^0}$ and $\tau\sup_{\overline{B}_{2r}}|u|$. By De Giorgi-Nash-Moser theory (c.f. [15, Theorem 8.22]), we then have

$$|w|_{C^{0,\alpha}(B_r)} \le Cr^{-\alpha}$$

where $\alpha \in (0,1)$ is a constant depending only on n, p, Q and $\sup_{\overline{B}_{2r}}|Du|$; $C > 0$ is a constant depending only on n, p, Q, r, $\sup_{\overline{B}_{2r}}|Du|$, $|k|_{C^0}$ and $\tau\sup_{\overline{B}_{2r}}|u|$. This proves the proposition as the relevant norms are comparable for the Euclidean metric and g by (2.24). □

By a standard covering argument, Proposition 2.8 gives a uniform Hölder gradient estimate on any compact subsets $\Omega' \subset\subset \Omega$ which stays a fixed distance away from the boundary $\partial_D\Omega \cup \partial_N\Omega$. To extend these estimates up to the boundary, we need a notion of harmonic coordinates which adapts well to the boundary under considerations. Recall from Appendix A that we always regard (M,g) as a closed subset contained in some larger manifold $(\tilde{M},g)$.

Definition 2.9 ($W^{1,p}$-Fermi harmonic radius)**.** Let $p \in (n,\infty)$ and $Q > 1$. The $W^{1,p}$-Fermi harmonic radius of an n-dimensional Riemannian manifold (M^n,g) with boundary is the largest number $r_{FH} = r_{FH}(p,Q) > 0$ such that on any geodesic ball $B(x,r_{FH})$ of $(\tilde{M},g)$ centered at a point $x \in \partial M$ with radius r_{FH}, there is a harmonic coordinate chart $(y^1,\cdots,y^n) : B(x,r_{FH}) \cap M \to \mathbb{R}^n_+$ such that the metric components $g_{ij} := g(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})$ satisfy (2.24) and for $i = 1,\cdots,n-1$,

$$g_{in} \equiv 0 \text{ along } B(x,r_{FH}) \cap \partial M. \tag{2.26}$$

We call such a coordinate system $(y^1,\cdots,y^n)$ *Fermi harmonic coordinates.*

The lemme below establish the local existence of Fermi harmonic coordinates centered at any boundary point $x \in \partial M$.

Lemma 2.10 (Local existence of Fermi harmonic coordinates)**.** *Suppose (M^n, g) is a smooth Riemannian manifold with boundary. For any $x \in \partial M$, there exists some $r = r(x) > 0$ such that there exists a Fermi harmonic coordinate system $(y^1, \cdots, y^n)$ on $B(x, r) \cap M$ as in Definition 2.9.*

Proof. Let $(x^1, \cdots, x^n)$ be a standard Fermi coordinate system centered at $x \in \partial M$ (see e.g. [24, Appendix A]). In such coordinate chart, we have these coordinate functions $x^1, \cdots, x^n$ defined on the Euclidean half-ball $B_R^+(0) \subset \mathbb{R}_+^n$ of Euclidean radius $R > 0$ centered at the origin. We then construct the Fermi harmonic coordinates $(y^1, \cdots, y^n)$ on a smaller half-ball by requiring the coordinates y^i, $i = 1, \cdots, n-1$ to satisfy a mixed linear boundary value problem

$$\left\{ \begin{array}{ll} \Delta_g y^i = 0 & \text{in } B_R^+(0) \\ y^i = x^i & \text{along } \partial B_R(0) \cap \mathbb{R}_+^n \\ \frac{\partial y^i}{\partial x^n} = 0 & \text{along } B_R(0) \cap \{x^n = 0\} \end{array} \right. \tag{2.27}$$

and the last coordinate y^n to satisfy a Dirichlet linear boundary value problem

$$\left\{ \begin{array}{ll} \Delta_g y^n = 0 & \text{in } B_R^+(0) \\ y^n = x^n & \text{along } \partial B_R(0) \cap \mathbb{R}_+^n \\ y^n = 0 & \text{along } B_R(0) \cap \{x^n = 0\}. \end{array} \right. \tag{2.28}$$

By the Schauder estimates in [27], [15] and [14, Section 7], the boundary value problems (2.27) and (2.28) are solvable on $B_R^+(0)$ and forms a coordinate system on $B_{R/2}^+(0)$ when R is small enough. Note that the choice of the boundary conditions along $B_R(0) \cap \{x^n = 0\}$ guarantees that the Jacobian matrix $\left(\frac{\partial y^r}{\partial x^s}\right)_{1 \le r, s \le n}$ is in block form, i.e. $\frac{\partial y^n}{\partial x^i} = \frac{\partial y^i}{\partial x^n} = 0$ along $B_R(0) \cap \{x^n = 0\}$ for $i = 1, \cdots, n-1$.

It remains to check that (2.26) holds for the new coordinate system $(y^1, \cdots, y^n)$. Let $\tilde{g}_{ij}$ be the metric components of g with respect to the coordinate system $(x^1, \cdots, x^n)$. Since $(x^1, \cdots, x^n)$ is a Fermi coordinate system, for $i = 1, \cdots, n-1$,

$$g_{in} := g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^n}\right) = \sum_{p,q=1}^{n-1} \tilde{g}_{pq} \frac{\partial x^p}{\partial y^i} \frac{\partial x^q}{\partial y^n} + \frac{\partial x^n}{\partial y^i} \frac{\partial x^n}{\partial y^n} = 0$$

where the last equality holds because $\frac{\partial x^q}{\partial y^n} = \frac{\partial x^n}{\partial y^i} = 0$ along $B_R(0) \cap \{x^n = 0\}$ from (2.27) and (2.28). □

We need to prove a lower bound of the $W^{1,p}$-Fermi harmonic radius similar to the one in Lemma 2.7.

Lemma 2.11 (Estimate for $W^{1,p}$-Fermi harmonic radius)**.** *Suppose (M^n, g) has a lower Ricci curvature bound* (2.1)*, an injectivity radius lower bound* (2.2) *and an interior ball curvature upper bound 2.3. Then, there exists a constant $r_0 > 0$, depending only on Q, n, p, κ, i_0 and K, such that*

$$r_{FH} \ge r_0$$

where r_{FH} is the $W^{1,p}$-Fermi harmonic radius of (M, g) as defined in Definition 2.9.

Proof. Our proof is based on the blow-up argument as in [5]. First, we define a new geometric quantity

$$\tilde{s}_M(x) := \sup_{r>0} \{\min(r, \inf_{y \in B(x,r) \cap M})\}$$

□

Proposition 2.12 (Hölder gradient estimates near Neumann boundary)**.** Let $p \in (n, \infty)$ and $Q > 1$. Let $B(x, 4r)$ be a geodesic ball of $(\tilde{M}, g)$ centered at a point $x \in \partial M$ with radius $4r < r_{FH}/2$. Denote $\Omega = B(x, 4r) \cap M$. Suppose $u \in C^2(\Omega)$ is a solution to the regularized Jang's equation (2.9) on Ω° with the Neumann boundary condition (2.8) satisfied on $\partial_N \Omega$. Then, there exists some $\alpha > 0$, depending only on n, p, Q and $\sup_{\overline{B}(x,2r)\cap M} |Du|$, such that

$$|Du|_{C^{0,\alpha}(B(x,r)\cap M)} \leq C$$

where $C > 0$ is a constant depending only on n, p, Q, r, $\sup_{\overline{B}(x,2r)\cap M} |Du|$, $|k|_{C^0(\overline{B}(x,2r)\cap M)}$ and $\sup_{\overline{B}(x,2r)\cap M} \tau|u|$.

Proof. The proof is similar to the one for Proposition 2.8, except that we are using the $W^{1,p}$-Fermi harmonic coordinates. In such coordinates, the tangential derivatives $w = \partial_k u$, $k = 1, \cdots, n-1$, are generalized $W^{1,2}$-solutions of the linear oblique elliptic boundary value problem of conormal type

$$\begin{cases} \partial_i(a^{ij}\partial_j w) &= -\partial_i f^i_k & \text{in } B^+_{2r} \\ a^{nj}\partial_j w &= 0 & \text{along } B^+_{2r} \cap \{y^n = 0\} \end{cases}$$

and the result then follows from [29, Theorem 5.45]. Finally, the normal derivative $w = \partial_n u$ is a generalized $W^{1,2}$-solutions of the linear elliptic Dirichlet boundary value problem

$$\begin{cases} \partial_i(a^{ij}\partial_j w) &= -\partial_i f^i_k & \text{in } B^+_{2r} \\ w &= 0 & \text{along } B^+_{2r} \cap \{y^n = 0\} \end{cases}$$

and the result follows from [15, Theorem 8.29]. □

2.5. **Local solvability for the regularized Jang's equation.** Using the a-priori estimates in the previous subsections, we can now proceed to study the local solvability of the regularized Jang's equation with mixed boundary conditions. In order to study the regularity up to the edge, we need to introduce some weighted Hölder spaces as for example in [27].

Let Ω be a regular domain in M as in Definition A.2 with regularized distance function $d_{\partial\Omega}$ in Lemma A.5. For any $\delta > 0$, we denote

$$\Omega_\delta := \{x \in \Omega \ : \ d_{\partial\Omega}(x) > \delta\}.$$

For any constants $a > 0$ and $b \geq -a$, we define the weighted Hölder norm

$$|u|^{(b)}_{a;\Omega} := \sup_{\delta > 0} \delta^{a+b} |u|_{C^a(\overline{\Omega}_\delta)}$$

and the weighted Hölder space $H^{(b)}_a(\Omega)$ consisting of all functions u on Ω such that $|u|^{(b)}_{a;\Omega}$ is finite.

Theorem 2.13 (Local solvability of mixed boundary value problem for regularized Jang's equation)**.** *Let $\Omega \subset M$ be a regular domain as in Definition A.2, $\tau \in (0, 1)$ and $\varphi \in C^0(\partial\Omega)$. Suppose that the following conditions are satisfied:*

(i) *(Serrin condition) $H_{\partial\Omega} > |\operatorname{Tr}_{\partial\Omega}(k) - \tau\varphi|$ everywhere on $\partial\Omega$;*
(ii) *(Angle condition) $0 < \theta < \frac{\pi}{2}$ everywhere on $\overline{T} \cap \partial\Omega$.*

Then, there exists $u \in C^3(\Omega) \cap C^0(\overline{\Omega})$ which is the unique solution to (2.7).

Proof. Step 1: Reduction to the case $\varphi \in C^{2,\alpha}(\partial\Omega)$.

Suppose that the theorem holds when $\varphi \in C^{2,\alpha}(\partial\Omega)$ for some $\alpha \in (0,1)$, we show that it also holds when $\varphi \in C^0(\partial\Omega)$ by an approximation argument together with maximum principle. Let $\varphi \in C^0(\partial\Omega)$ and $\{\varphi_m\}_{m\in\mathbb{N}}$ be a sequence in $C^{2,\alpha}(\partial\Omega)$ converging uniformly to φ on $\partial\Omega$. By assumption, (2.7) has a unique solution $u_m \in C^3(\Omega) \cap C^0(\Omega)$ with $u_m|_{\partial\Omega} = \varphi_m$. By the comparison principle, i.e. Lemma **??**, taking $v_1 = u_m$ and $v_2 = u_n + \sup_{\partial\Omega} |\varphi_m - \varphi_n|$, we have for any $m, n \in \mathbb{N}$,

$$\sup_{\overline{\Omega}} |u_m - u_n| \leq \sup_{\partial\Omega} |\varphi_m - \varphi_n|.$$

Therefore, the sequence $\{u_m\}$ converges uniformly on $\overline{\Omega}$ to some $u \in C^0(\overline{\Omega})$ with $u|_{\partial\Omega} = \varphi$. Using the interior derivative estimates (Proposition ?), u_m has locally uniformly bounded $C^{2,\alpha}$-norms inside Ω. By the Arzela-Ascoli theorem, we have $u \in C^{2,\alpha'}(\Omega) \cap C^0(\overline{\Omega})$ for any $0 < \alpha' < \alpha$. Finally, we can conclude that $u \in C^3(\Omega)$ by Schauder theory for second order linear elliptic equations with oblique derivative boundary conditions (Theorem 2.28 and Exercise 2.5 in [29]). The uniqueness part follows directly from Lemma **??**.

Step 2: Method of continuity.

We use the linear theory from [28] and the method of continuity to solve the equation. We use the continuity method to solve the Jang equation. We define a family of mixed boundary value problems parametrized by $s \in [0,1]$,

$$\begin{aligned} v^{-1}\bar{g}^{ij}u_{ij} &= s\bar{g}^{ij}p_{ij} + tu \quad \text{ in } M, \\ \langle Du, \nu\rangle &= 0 \quad \text{ on } \mathcal{N}, \\ u &= s\varphi \text{ on } \mathcal{D}. \end{aligned} \tag{2.29}$$

Define the set I to be the set of values of $s \in [0,1]$ such that there is a solution u to the above problem in $H^{(-1-\mu)}_{2+\mu}$ with μ determined later. We clearly have that when $s = 0$, from maximum principle $u = 0$ is the unique solution i.e. $0 \in I$. Due to (**??**), we have the a priori estimate

$$\sup_{\bar{M}} |u| + |Du| \leqslant \tfrac{C}{t}.$$

where $C > 0$ is independent of s. From the Hölder gradient estimate from [37, Chapter 6], we have that

$$|Du|_{\alpha',M} \leqslant C$$

for some C independent of s and some $\alpha' \in (0,1)$. The coefficients of the Jang equation is then Hölder continuous, then from the Schauder estimates [28] (see also [27, 25]), we have

$$\|u\|^{(-\lambda)}_{2,\alpha'} \leqslant C \tag{2.30}$$

where $\lambda \in (1,2)$. The exponent λ here depends on the contact angles and the Hölder exponents of Dirichlet boundary value φ due to Theorem 4 and Lemma 4.1 of [28]. We adust the exponents and pick a $\mu \in (0,1)$ depending on λ and α'. We know that from (2.30) that

$$\|u\|^{(-1-\mu)}_{2,\mu} \leqslant C.$$

We see then I is closed. The linearization of the Jang equation (2.29) at u is given by

$$L_u\varphi = \operatorname{div}\left(\frac{D\varphi}{\sqrt{1+|Du|^2}} - \frac{Du\langle Du, D\varphi\rangle}{(1+|Du|^2)^{3/2}}\right) - t\varphi.$$

And the linearization at u at the boundary $\mathcal{N}$ is

$$B_u\varphi = \langle D\varphi, \nu\rangle,$$

we see that B_u is a linear operator along $\mathcal{N}$. Suppose that the boundary value problem is solvable at $s_0 \in [0,1]$, then from Fredholm alternative of mixed linear boundary value problems and inverse function theorem, the boundary value problem is solvable in a neighborhood of s_0. We get that I is open.

We know from $\|u\|_{2,\mu}^{(-1-\mu)} < \infty$ that $u \in C^{2,\mu}(M)$ and therefore $u \in C^{1,\mu}(\bar{M})$ due to the obvious relation $\|u\|_{1,\mu}^{(-1-\mu)} \leqslant C\|u\|_{2,\mu}^{(-1-\mu)}$ of the unweighted Hölder norms. □

2.6. **Harnack principle.** It follows immediately from (**??**) that

$$\frac{\partial}{\partial\nu}(v^{-1}) = v^{-1}\frac{B(Du, Du)}{1+|Du|^2}. \tag{2.31}$$

Since $(1+|Du|^2)^{-1}B(Du,Du)$ is bounded due to (**??**), further by combining with [12, Lemma 2.3] and standard elliptic theory, we have a similar Harnack principle.

Lemma 2.14. *(Harnack principle) Let $f_k : M \to \mathbb{R}$, $k = 1, 2, \cdots$, be a family of C^3 functions so that for some constants $\alpha > 0$ and $\beta > 0$, it is true that $\Delta_{G_k}\frac{1}{v_k} \leqslant \frac{\beta}{v_k}$ and $\partial_\nu \frac{1}{v_k} \geqslant -\alpha\frac{1}{v_k}$. If for some open set $U \subset M \times \mathbb{R}$ the graphs $\{G_k\}$ converge in C^3 to a submanifold $G \subset U$ as $k \to +\infty$, then every connected component of G is either a vertical cylinder (with free boundary) or is itself a graph over an open set of M.*

Proof. We orient G_k by taking their normal in the upward direction. Since $G_k \to G$ in the open set U, it follows that the vertical component w of the normal vector of G is the limit $\lim_k \frac{1}{v_k}$. Clearly, $w \geqslant 0$ and $\Delta_G w - \beta w \leqslant 0$. By (2.31), also $\frac{\partial}{\partial\nu}w \geqslant -\alpha w$ along $\bar{G} \cap (\partial M \times \mathbb{R})$, it follows then from Hopf maximum principle that either w vanishes identically in U or is everywhere positive on each connected component. In the first case, the component is cylindrical whose boundary is orthogonal to $\partial M \times \mathbb{R}$. The second says that the connected component is a graph. □

2.7. **Topological properties of the cylindrical limit.** By the Harnack principle of Lemma 2.14, the limit submanifold has a limit with some components of the form $\Sigma \times \mathbb{R}$ as $t \to 0$. Denote by η the outward normal of ∂M in M. Since Σ meets ∂M orthogonally, also η is normal to $\partial\Sigma$ in Σ.

Definition 2.15. We say that dominant energy condition is satisfied if

$$\mu \geqslant |J|, H_{\partial M} \geqslant |j|, \tag{2.32}$$

where $j = p(\eta, \cdot)_{|\partial M}$.

The boundary condition $H_{\partial M} \geqslant |j|$ is introduced [4]. We have the following theorem,

Theorem 2.16. *Assume the dominant energy condition on M, if one of the connected component is of the form $\Sigma\times\mathbb{R}$, Σ meets ∂M along $\partial\Sigma\subset\partial M$ orthogonally, meet then Σ has a positive Yamabe type. In particular, when Σ is of dimension two with non-empty boundary, Σ has to be an annulus or a disk.*

Proof. We recall Schoen-Yau inequality [33, (2.29)],

$$
\begin{aligned}
&0\leqslant 2(\mu-|J|)\\
&\quad\leqslant \bar{R}-\sum_{i,j}(h_{ij}-p_{ij})^2-2\sum_i(h_{i0}-p_{i0})^2+2\sum_i\bar{D}_i(h_{i0}-p_{i0}).
\end{aligned}
$$

We have extended p,μ,J parallel along the $\mathbb{R}$ factor. Here we denote the unit normal (with direction inherited from the graph of u_t) of $\Sigma\times\mathbb{R}$ by the index 0. This inequality is satisfied on $\Sigma\times\mathbb{R}$ and now we reduce it to an inequality for Σ. We take a function φ with compact support in $S:=\Sigma\times\mathbb{R}$. We have

$$
\begin{aligned}
0\leqslant&\int_S(\bar{R}-2(h_{i0}-p_{i0})^2+2\bar{D}_i(h_{i0}-p_{i0}))\varphi^2\\
\leqslant&\int_S(\bar{R}-2(h_{i0}-p_{i0})^2)\varphi^2\\
&+2\int_{\partial S}\nu_i(h_{i0}-p_{i0})\varphi^2-4\int_{\Sigma'}(h_{i0}-p_{i0})\varphi\bar{D}_i\varphi\\
\leqslant&\int_S\bar{R}\varphi^2+2\int_{\partial S}\nu_i(h_{i0}-p_{i0})\varphi^2+2\int_S|\bar{D}\varphi|^2.
\end{aligned}
$$

We note that

$$
2\nu_i(h_{i0}-p_{i0})=2\bar{H}-2H_{\partial M,M}-2\nu_ip_{i0},
$$

where $\bar{H}=H_{\partial S,S}$. By the second part to of dominant energy condition (2.32),

$$
0\leqslant\int_S\bar{R}\varphi^2+2\int_{\partial S}\bar{H}\varphi^2+2\int_S|\bar{D}\varphi|^2.
$$

Now we follow a cutoff trick closely from [33, Proposition 4].

Take $\varphi=\chi(x^0)\zeta$ where ζ is any function on Σ such that $\chi(x^0)=1$ for $|x^0|\leqslant T$, $\chi(x^0)=0$ if $|x^0|\geqslant T+1$ and that $|\chi'|\leqslant 2$. Since $S=\Sigma\times\mathbb{R}$, then

$$
\begin{aligned}
0\leqslant&[\int_\Sigma R\zeta^2+\int_{\partial\Sigma}H\zeta^2]\int_{\mathbb{R}}\chi^2\mathrm{d}x^0+2\int_{\mathbb{R}}\int_\Sigma[|\nabla\varphi|^2+(\frac{\partial\varphi}{\partial x^0})^2]\mathrm{d}\sigma\mathrm{d}x^0.\\
=&[\int_\Sigma R\zeta^2+\int_{\partial\Sigma}H\zeta^2]\int_{\mathbb{R}}\chi^2\mathrm{d}x^0+2\int_{\mathbb{R}}\int_\Sigma[\chi^2|\nabla\zeta|^2+\zeta^2(\chi')^2]\mathrm{d}\sigma\mathrm{d}x^0\\
\leqslant&[\int_\Sigma R\zeta^2+2|\nabla\zeta|^2+\int_{\partial\Sigma}H\zeta^2]\int_{\mathbb{R}}\chi^2\mathrm{d}x^0+16\int_\Sigma\zeta^2.
\end{aligned}
$$

Divide both sides by $\int_{\mathbb{R}}\chi^2$ and letting $T\to+\infty$, we have

$$
0\leqslant\int_\Sigma R\zeta^2+2|\nabla\zeta|^2+2\int_{\partial\Sigma}H\zeta^2.
$$

The above is a topological condition and says that Σ (with or without boundary) is of positive Yamabe type. In particular, we analyze the case when Σ is two dimensional. If Σ is closed, then Σ has to be a sphere or a torus recovering the standard case. If $\partial\Sigma\neq\emptyset$ and letting $\zeta\equiv 1$, then by Gauss-Bonnet theorem,

$$
2-2g-b=\chi_\Sigma\geqslant 0,
$$

where b is the number of boundary components of Σ, g is the genus of the surface Σ and χ_Σ is the Euler characteristic of Σ. Since $\partial\Sigma \neq \emptyset$, so the genus g has to be zero and that $b = 1$ or $b = 2$. Hence Σ is either a disk or an annulus. If the dominant energy condition $\mu \geqslant |J|$ or $H_{\partial M} \geqslant |p_{|\partial M}|$ holds strictly everywhere, the second alternative $b = 2$ cannot happen. If $b = 2$, then $\mu + J(\nu) = 0$ and $H_{\partial M} + p(\eta, \nu) = 0$. □

3. Perron's method

In this section, we collect some basics of Perron's method for mixed boundary value problems. This section is mostly the adaptation of Lieberman's works [27, 25] and Eichmair [12].

Definition 3.1. (Admissible domain) We say that $W \subset M$ is admissible for Jang equation with mixed boundary value problems if $W \cap \mathcal{D} = \emptyset$, $\partial W \cap M$ and $\partial W \cap \mathcal{N}$ intersects at acute angles.

If $y \in M$, we just take a geodesic ball with sufficiently small radius which lies in M. In fact, for every point $y \in N$, it is also easy to find an admissible neighborhood. We take a small number $r > 0$ and consider any geodesic ball $B := B(x, r)$ in $\tilde{M}$ with $y \in B$ and $B \cap \bar{\mathcal{D}} = \emptyset$. Moving the center of the ball towards the outside of M while fixing the radius , since the Neumann boundary N will resemble a flat Euclidean hyperplane in a small scale and ∂B will resemble a Euclidean sphere, at some point $\partial B \cap M$ will intersect $\mathcal{N}$ forming acute angles. The a sufficiently small $r > 0$ will make the mean curvature $H_{\partial B \cap M}$ of $\partial B \cap M$ in M big enough to satisfy (**??**).

Definition 3.2. (Local solvability, [27, 12]) We say that the Jang equation with mixed boundary condition is locally solvable if for every point $y \in M \cup \mathcal{N}$, there exists an admissible neighborhood $W := W_y$ of y such that

$$L_t v = 0 \text{ in } W \cap M, \partial_\nu v = 0 \text{ on } W \cap \mathcal{N}, v = h \text{ in } \partial' W$$

has a unique solution $v \in C(W)$. Here, $\partial' W = \partial W \cap M$.

Definition 3.3. A Perron sub solution of L_t on Ω is a function $\underline{u} \in C(\bar{\Omega})$ with $|\underline{u}| \leqslant Ct^{-1}$ and the property: if $x \in M$, $r \in (0, r_{\mathcal{D}}(x))$ and $v \in C(\bar{B}(x, r)) \cap C^{2,\alpha}(B(x, r))$ is the unique solution of the Dirichlet problem

$$\begin{aligned} L_t v &= 0 \text{ in } B(x, r), \\ v &= \underline{u} \text{ on } \partial B(x, r) \end{aligned}$$

whose existence of is guaranteed by [12, Theorem 2.2], then $v \geqslant \underline{u}$ on $\bar{B}(x, r)$; if $x \in \mathcal{N}$, there exists a relatively open domain $W := W_x$ such that $W_x \cap \bar{\mathcal{D}} = \emptyset$ and $\partial' W$ intersects $\mathcal{N}$ at acute angles, and a function $v \in C(\bar{W}) \cap C^{2,\alpha}(W) \cap C^{1,\alpha}(\partial W \cap \mathcal{N})$ satisfying

$$L_t v = 0 \text{ in } W \cap M, \partial_\nu v = 0 \text{ on } W \cap \mathcal{N}, v = \underline{u} \text{ in } \partial' W, \tag{3.1}$$

then $v \geqslant \underline{u}$. The existence of such a function v is guaranteed by Theorem **??**.

For the definition of Perron super solution, one reverse the inequalities of the above definition.

Definition 3.4. (Lift of a Perron sub solution) Let $\underline{u}$ be a Perron sub solution, the modified version $\widehat{\underline{u}}$

$$\widehat{\underline{u}} = \begin{cases} \underline{u} \text{ in } M \backslash W; \\ v \text{ in } W, \end{cases}$$

is called a Perron lift of the Perron sub solution $\underline{u}$ with v from (3.1).

Definition 3.5. Let $\bar{u}_t$ be a Perron super solution of L_t on Ω. We denote by

$$\mathcal{S}_{\bar{u}_t} = \{\underline{u} \in C(\Omega) : \underline{u} \text{ is a sub solution of} L_t \text{ on } \Omega \text{ with } \underline{u} \leqslant \bar{u}_t\}$$

the class of all Perron sub solutions lying below $\bar{u}_t$.

Lemma 3.6. *(Basic properties of Perron sub solutions, see* [12], [25]*) Let $\bar{u}_t$ be a Perron super solution of L_t on Ω. We have the following two basic properties of the class $\mathcal{S}_{\bar{u}_t}$:*

(1) *If $\underline{u}, \underline{v} \in \mathcal{S}_{\bar{u}_t}$, then* $\max\{\underline{u}, \underline{v}\} \in \mathcal{S}_{\bar{u}_t}$.
(2) *If $\underline{u} \in \mathcal{S}_{\bar{u}_t}$ and $\widehat{\underline{u}}$ is the Perron lift of $\underline{u}$ with respect to an admissible domain W, then $\widehat{\underline{u}} \in \mathcal{S}_{\bar{u}_t}$.*

Proof. The lemma is a simple consequence of the maximum principle. See for instance [25]. □

Definition 3.7. (Perron solution) Let $\bar{u}_t$ be a Perron super solution such that $\mathcal{S}_{\bar{u}_t} \neq \emptyset$. Define the Perron solution u_t^P on M with respect to $\bar{u}_t$ pointwise for $x \in M$ by

$$u_t^P(x) = \sup\{\underline{u}(x) : \underline{u} \in \mathcal{S}_{\bar{u}_t}\}. \tag{3.2}$$

The following is the solvability of Jang equation with mixed boundary condition using Perron method, and it is motivated by [12, Lemma 3.2].

Lemma 3.8. *Let u_t^P be the Perron solution constructed in Definition 3.7, then $u_t^P \in C^{2,\alpha}_{\text{loc}}(M) \cap C^{1,\alpha}_{\text{loc}}(N)$, also that $L_t u_t^P = 0$ in M and $\partial_\nu u_t^P = 0$ along N holds classically.*

Proof. The Perron method of linear elliptic equations with oblique boundary condition was first proposed by Lieberman [25]. Since the Jang equation is elliptic, so Theorem 1 of [25] applies here. See also [27].

We include the proof. From the a-priori bound (**??**), we know that $-\frac{C}{t}$ is a sub solution and $\frac{C}{t}$ is a super solution. By (3.2), $u := u_t^P$ is everywhere finite.

Now fix some point $y \in \Omega \cup \mathcal{N}$, and an open set W if $y \in N$ or a ball $B(y, r)$ (still denoted by W) if $y \in \Omega$. Let $\{v_i\}$ be a sequence in $\mathcal{S}_{\bar{u}_t}$ with $v_i(y) \to u(y)$. We can further assume that the sequence $\{v_i\}$ is non-decreasing by Lemma 3.6.

Now we take the lift $\bar{v}_i$ of v in W, i.e. $\bar{v}_i$ solve $L\bar{v}_i = 0$ in W and $\partial_\nu \bar{v}_i = 0$ $N \cap \partial W$ classically. By Schauder estimates [28], we have that there exists a solution v which is C^2 in W and C^1 on $\partial W \cap N$ such that there exists a convergent subsequence $\{\bar{v}_{i_k}\}$ with $\lim_k \bar{v}_{i_k}(x) = v(x)$ for all $x \in W \cup (\partial W \cap \mathcal{N})$ and that

$$Lv = 0 \text{ in } W,$$
$$\partial_\nu v = 0 \text{ on } \partial W \cap \mathcal{N}.$$

Clearly, $v \leqslant u$ in $W \cup (\partial W \cap \mathcal{N})$. Also, by maximum principle and Definition 3.4, $v_i(y) \leqslant \bar{v}_i(y)$ for all i, therefore $u(y) \leqslant v(y)$ as well. We claim now that $v \equiv u$ in $W \cup (\partial W \cap \mathcal{N})$.

If $v(z) < u(z)$ for some $z \in W \cup (\partial W \cap \mathcal{N})$, then we can pick a function $u_0 \in \mathcal{S}_{\bar{u}_t}$ with $u_0(z) > v(z)$ and take $w_i = \max\{u_0, v_i\}$ which is also a sub solution by Lemma 3.6. We take the lifts $\{\bar{w}_i\}$ again and a convergent subsequence $\{\bar{w}_{i_k}\}$ of $\{\bar{w}_i\}$ with $\lim_k \bar{w}_{i_k} = w$ in $W \cup (\partial W \cap \mathcal{N})$. By construction $v \leqslant w$ with a strict inequality at z and w solves the following

$$\begin{aligned} Lw &= 0 \text{ in } W, \\ \partial_\nu w &= 0 \text{ on } \partial W \cap \mathcal{N}. \end{aligned}$$

By maximum principle, $v \equiv w$. However, this is a contradiction. So $v \equiv u$ in $W \cup (\partial W \cap \mathcal{N})$. Since the point y is arbitrarily chosen, we have that $L_t u = 0$ in M and $\partial_\nu u = 0$ along $\mathcal{N}$ holds classically. □

4. Existence of MOTS with free boundary

We use the theory established in previous sections to find a MOTS with a free boundary homologous to some hypersurfaces with certain geometric conditions. This section is a generalization of a theorem proposed by Schoen which was proven in both [12] and [7] via different methods. Eichmair's Perron method carries over to this settings with modifications.

Proof of Theorem **??**. We refer the reader to [12, Lemma 3.3] and paragraphs after. The Perron sub and super solutions constructed there are still Perron sub and super solutions in our settings. We only have to note that the angle conditions along I just as we did in the proof of Lemma **??**. We write these solutions here:

$$\underline{u}_t(x) := \begin{cases} \frac{\chi}{t} - \frac{\chi + C}{\delta t} \operatorname{dist}_{\Sigma_1}(x) & \text{if } \operatorname{dist}_{\Sigma_1}(x) \leqslant \delta \\ -\frac{C}{t} & \text{if } \operatorname{dist}_{\Sigma_1}(x) > \delta \end{cases}$$

and

$$\bar{u}_t(x) := \begin{cases} -\frac{\chi}{t} + \frac{\chi + C}{\delta t} \operatorname{dist}_{\Sigma_2}(x) & \text{if } \operatorname{dist}_{\Sigma_2}(x) \leqslant \delta \\ +\frac{C}{t} & \text{if } \operatorname{dist}_{\Sigma_2}(x) > \delta. \end{cases}$$

The fact that Σ is Λ-minimizing with a free boundary is easily checked following Example A.1 and Lemma A.1 of [12]. □

Remark 4.1. Andersson-Metzger's approach [7] of bending the data along the light cone only affects the ν direction. So the normal to the boundary bends into the vector

$$\tilde{\gamma} = [\gamma - \langle \nu, \gamma \rangle \nu] + \tilde{\nu} \langle \nu, \gamma \rangle.$$

We see that $\langle \tilde{\gamma}, \tilde{\nu} \rangle = \langle \nu, \gamma \rangle$ and $|\tilde{\gamma}| = 1$. So the angle condition still holds for the new data. Deforming the mean curvature and P is the same.

5. Λ-minimizing currents with a free boundary

To study the regularity of the free boundary MOTS, we develop a regularity theory of the Λ-minimizing currents with a free boundary in this section. Without loss of generality, we study Λ-minimizing currents with a free boundary in $\mathbb{R}^{n+1}$. Let $S \subset \mathbb{R}^{n+1}$ be a compact C^2 hypersurface without boundary and B be a compact C^2 oriented $(n-1)$-dimensional submanifold (with or without boundary) such that B and S intersect transversally and $\partial B \subset S$. We call S the *supporting hypersurface* and B the *fixed boundary*.

For any point $x \in S$, define $\tau(x)$ to be the projection of $\mathbb{R}^{n+1}$ to the tangent space $T_x S$ and $\nu(x) = I - \tau(x)$. We assume a curvature condition on S: let κ be the smallest number such that

$$|\nu(x)(x - x')| \leqslant \tfrac{\kappa}{2}|x - x'|^2$$

holds for all $x, x' \in S$.

Let $\rho(x) = \mathrm{dist}(x, S)$, for any y with $\rho(y) < \kappa^{-1}$ define $\xi(y)$ to be the unique point in S such that $\rho(y) = |\xi(y) - y|$. ξ is also called nearest point projection. We refer readers to [2] for properties for these quantities.

Define the class $\mathcal{C}$ of admissible currents to be the set of all n-dimensional integer multiplicity rectifiable currents with $\mathrm{spt}\, T$ being compact and $\mathrm{spt}(\llbracket B \rrbracket - \partial T) \subset S$.

Definition 5.1. Given T with $\mathrm{spt}\, \partial T \subset S \cup B$, we say that T is Λ-minimizing in an open set U if for all Q with $B \cap \mathrm{spt}\, \partial Q \neq \emptyset$, $\mathrm{spt}\, Q$, being a compact set in U, and that

$$\mathbf{M}_U(T) \leqslant \mathbf{M}_U(T + X) + \Lambda \mathbf{M}_U(Q) \tag{5.1}$$

for some $\Lambda \geqslant 0$ where X is an integer multiplicity rectifiable current with $\mathbf{M}(\partial X) < \infty$, $\mathrm{spt}\, \partial X \subset S$ and $\mathrm{spt}(\partial Q - X) \subset S$.

First note that $\mu_T(S) = 0$ since otherwise we can take $X = -T \llcorner S$ and $Q = 0$.

The Λ-minimizing condition reduces to the notion of area-minimizing current with free boundary if $\Lambda = 0$. Also, it reduces to the Λ-minimizing condition in the interior if $\partial X = 0$. Similar to [11, (2.3)], we have that

$$\int (P_T \cdot Dg + \vec{H} \cdot g) \mathrm{d}\|T\| = 0 \tag{5.2}$$

if g is a compactly supported vector field tangent to S and $|\vec{H}_T| \leqslant \Lambda$. Here P_T is the projection to the approximate tangent space of T. We see from (5.2) that $\mathrm{spt}\, T$ meets S at a regular point of $\mathrm{spt}\, \partial T$ forming an angle $\pi/2$. We can also derive this fact later by observing that the boundary tangent cone is area minimizing with a free boundary. Because of this, we also refer T as a Λ-minimizing current with a free boundary. We denote by $F_\Lambda(S)$ the class of Λ-minimizing current with a free boundary on S and we use the shorthand F_Λ if the dependence on S is clear.

We follow Grüter [16] to develop a theory of Λ-minimizing currents with a free boundary. Hypersurfaces with a prescribed mean curvature and a free boundary is a special case of Λ-minimizing currents. In the work of Duzaar [10], a theory of currents of prescribed mean curvature and a free boundary was developed and we note some results extend easily to the case of Λ-minimizing currents. The difference is that Λ-minimizing currents is not necessarily variational. Another approach due to [16] using reflection across the free boundary works as well. The interior and (fixed) boundary regularity theory was developed in [11], we focus here on regularity near the free boundary. We prove first a monotonicity formula for the mass of T in a tubular neighborhood of S.

Theorem 5.2. *Let $m(h)$ be defined by*

$$\mathbf{M}(T \llcorner \{x : \rho(x) < h\})$$

for $0 < h < \frac{1}{2\kappa}$. There is a constant $C = C(n, \kappa, S, B, \Lambda)$ such that

$$\mathrm{e}^{\tilde{C}h} \frac{m(h)}{h} + C\tilde{C}^{-1} \mathrm{e}^{\tilde{C}h} + \kappa \mathrm{e}^{\tilde{C}h} m(h)$$

is increasing where $0 < h < \frac{1}{2\kappa}$ *and* $\tilde{C} = \frac{n+2}{n+1}\Lambda$.

Proof. We follow [17, Theorem 3.1] and outline arguments. See also [10]. We use the homotopy

$$f(t, x) = x + t(\xi(x) - x)$$

for $\rho(x) < \kappa^{-1}$. Define

$$Q(h) = f_{\#}([\![(0,1)]\!] \times (T \llcorner \{0 < \rho < h\})).$$

By the homotopy formula, we have that

$$\partial Q(h) = -T\llcorner\{\rho < h\} - T_1(h) - T_2(h) + \xi_{\#}(T\llcorner\{\rho < h\}).$$

From the λ-minimizing condition (5.1), we have

$$\mathbf{M}(T\llcorner\{\rho < h\}) \leqslant \mathbf{M}(T_1(h)) + \mathbf{M}(T_2(h)) + \Lambda\mathbf{M}(Q(h)).$$

We estimate the mass of $Q(h)$ via

$$\begin{aligned} &Q(h)(\omega) \\ =&\int_0^1 \int_{\mathbb{R}^{n+1}} \langle \omega(f(t,x)), (\xi(x) - x) \wedge (tD\xi(x) + (1-t)I)_{\#}\vec{T}(x)\rangle \mathrm{d}\|T\|\mathrm{d}t \end{aligned}$$

where $\vec{T}(x)$ orients the current T. Similar to the estimate of the mass of $T_1(h)$, we have that

$$\begin{aligned} &\mathbf{M}(Q(h)) \\ =&\sup_{|\omega|=1} T_1(h)(\omega) \\ \leqslant&\mathbf{M}(T\llcorner\{\rho < h\}) \int_0^1 [t^n h(2\kappa h)^n + h]\mathrm{d}t \\ =&(h + (2\kappa)^n \tfrac{h^{n+1}}{n+1})m(h) \\ \leqslant&\tfrac{n+2}{n+1}\Lambda h m(h). \end{aligned}$$

The last line is from $0 < h < \frac{1}{2\kappa}$. We have that

$$m(h) \leqslant hm'(h) + \kappa h^2 m'(h) + Ch^2 + \tfrac{(n+2)}{n+1}\lambda h m(h).$$

We have that

$$(\mathrm{e}^{\tilde{C}h}\frac{m}{h} + C\tilde{C}^{-1}\mathrm{e}^{\tilde{C}h} + \kappa \mathrm{e}^{\tilde{C}h} m)' \geqslant \tilde{C}\mathrm{e}^{\tilde{C}h} m$$

and since $m \geqslant 0$ we finish the proof. □

Since $\partial T - [\![B]\!] = \langle T, \rho, 0_+\rangle$, directly from the above theorem, we have for $0 < h < \frac{1}{2\kappa}$ that

$$\mathbf{M}(\partial T - [\![B]\!]) \leqslant \liminf_{h\searrow 0} \tfrac{m(h)}{h} \leqslant \mathrm{e}^{\tilde{C}h}(\tfrac{m(h)}{h} + C\tilde{C}^{-1} + \kappa m(h)).$$

By boundary rectifiable theorem, we see that ∂T is an integer multiplicity rectifiable current. Define the $(n-1)$-dimensional upper density of $\mu_{\partial T}$ by

$$\theta^{n-1*}(\mu_{\partial T}, x) = \limsup_{r\searrow 0} \frac{\mathbf{M}(\partial T\llcorner B_r(x))}{\omega_{n-1} r^{n-1}}$$

where ω_m is the volume of the m-dimensional unit ball. We have the following upper bound of the density $\theta^{n-1*}(\partial T, x)$ by using a different comparison current.

Theorem 5.3. *There is a constant* $C \leqslant n 2^n \frac{\omega_n}{\omega_{n-1}}$ *such that*

$$\theta^{n-1*}(\partial T, x) \leqslant C\theta^n(T, x) \tag{5.3}$$

for any $x \in \operatorname{spt} \partial T \backslash \bar{B}$.

Proof. Suppose that $x_0 \in \operatorname{spt} \partial T \backslash \bar{B}$ and choose $r > 0$ such that $B_{3r}(x_0) \cap \bar{B} = \emptyset$ and such that $18r\kappa \leqslant 1$. Assume $h > 0$ is sufficiently smaller than r, we define

$$Q_{h,r} = \{x \in \mathbb{R}^{n+1} : \rho(x) < h, |x - x_0| < r\} = B_r(x_0) \cap \{\rho < h\}.$$

There exists a Lipschitz map $F_{h,r}$ (See [10]) with

$$\begin{aligned} F_{h,r}(x) &= x \text{ if } x \in S \cup (\mathbb{R}^{n+1} \backslash B_{2r}(x_0)), \\ F_{h,r}(x) &= \xi(x) \text{ if } x \in Q_{h,r}, \\ \operatorname{Lip} F_{h,r} &\leqslant 1 + \tfrac{5h}{r}, \\ |F_{h,r}(x) - x| &\leqslant h \text{ in } B_{2r}(x_0). \end{aligned}$$

We use the homotopy

$$f(t, x) = x + t(F_{h,r}(x) - x).$$

Let $Q = f_\#([\![(0,1) \times T]\!])$, we have from the Λ-minimizing condition that

$$\mathbf{M}(T \llcorner B_{2r}) \leqslant \mathbf{M}(T_{h,r} \llcorner (B_{2r} \backslash S)) + \Lambda \mathbf{M}(Q),$$

where $T_{h,r} = F_{h,r\#}T$. We have that

$$\mathbf{M}(T_{h,r} \llcorner (B_{2r} \backslash S)) \leqslant (1 + \tfrac{5h}{r})^n M(T \llcorner (B_{2r} \backslash Q_{h,r}))$$

from similar arguments as in [17] and we have that

$$\begin{aligned} &\mathbf{M}(Q) \\ =&\sup_{|\omega|=1} \int_0^1 \int_{B_{2r}} \langle \omega, (F_{h,r} - x) \wedge (tDF_{h,r} + (1-t)I)_\# \vec{T}(x) \rangle \mathrm{d}\|T\| \mathrm{d}t \\ \leqslant&(\int_0^1 (1 + \tfrac{5ht}{r})^n h) \mathrm{d}t \mathbf{M}(T \llcorner B_{2r}) \\ \leqslant&(h + o(h)) \mathbf{M}(T \llcorner B_{2r}). \end{aligned}$$

So we have now that

$$\begin{aligned} \mathbf{M}(T \llcorner Q_{h,r}) &\leqslant (\tfrac{nh}{r} + o(h)) \mathbf{M}(T \llcorner B_{2r}) + \Lambda(h + o(h)) \mathbf{M}(T \llcorner B_{2r}) \\ &\leqslant [(\tfrac{n}{r} + o(\tfrac{1}{r}))h + o(h)] \mathbf{M}(T \llcorner B_{2r}). \end{aligned}$$

We infer that

$$\begin{aligned} &\mathbf{M}(\partial T \llcorner B_r(x_0)) \\ \leqslant&\liminf_{h \searrow 0} \tfrac{\mathbf{M}(T \llcorner Q_{h,r})}{h} \\ \leqslant&(\tfrac{n}{r} + o(\tfrac{1}{r})) \mathbf{M}(T \llcorner B_{2r}(x_0)). \end{aligned}$$

Divide the above by $\omega_{n-1} r^{n-1}$ on both sides and let $r \to 0$ and noting that $\mathbf{M}(T \llcorner B_{2r})$ is of order r^n (See [18] for finiteness of $\theta^n(T, x)$) we have the desired bound for the upper density $\theta^{n-1*}(\partial T, x)$. □

Now we show a compactness property of F_Λ and we show the existence of oriented tangent cones at the free boundary. The statement is valid for any hypersurfaces.

Lemma 5.4. *Let $T_j \in F_\Lambda$ with a free boundary on a C^2 hypersurface S and $T_j \rightharpoonup T$ as currents. Then $T \in F_\Lambda$ and μ_{T_j} converges to μ_T in the sense of Radon measures.*

Proof. We use a modification on [17]. Let K be any compact set in $\mathbb{R}^{n+1}$, $\phi : \mathbb{R}^{n+1} \to [0,1]$ be a smooth function identically equal to one near K and vanishing outside an ε-neighborhood of K. For $\lambda \in [0,1)$ denote by $W_\lambda := \{x : \phi(x) > \lambda\}$.

Let $R_j = T - T_j$, we have that

$$T - T_j = \partial R_j + R_j', \mathbf{M}_{W_0}(R_j) + \mathbf{M}_{W_0}(R_j') \to 0$$

for integer multiplicity current R_j and R_j'. Slicing these currents with respect to ϕ, we fix $0 < \alpha < 1$ and $\{j'\} \subset \{j\}$ (we assume that $j' = j$) such that

$$\partial(R_j \llcorner W_\alpha) = (\partial R_j) \llcorner W_\alpha + P_j$$

where P_j is integer multiplicity, $\operatorname{spt} P_j \subset \partial W_\alpha$ and $\mathbf{M}(P_j) \to 0$. Note that $P_j = \langle R_j, -\phi, -\alpha \rangle$. We may additionally assume that for this α the conditions

$$\mathbf{M}(T_j \llcorner \partial W_\alpha) = 0, \mathbf{M}(T \llcorner \partial W_\alpha) = 0$$

for all j.

For α, we have that

$$T \llcorner W_\alpha = T_j \llcorner W_\alpha + \partial \tilde{R}_j + R_j'' \tag{5.4}$$

where $\tilde{R}_j = R_j \llcorner W_\alpha$ and $R_j'' = R_j' \llcorner W_\alpha - P_j$ with

$$\mathbf{M}(\tilde{R}_j) + \mathbf{M}(R_j'') \to 0$$

as $j \to \infty$.

Let X be an integer multiplicity rectifiable n-current with $\mathbf{M}(\partial X) < \infty$, Q an integer multiplicity rectifiable $(n+1)$-current with $\operatorname{spt} Q \subset K$ and $\operatorname{spt}(\partial Q - X) \subset S$. We want to show that

$$\mathbf{M}_{W_\alpha}(T) \leqslant \mathbf{M}_{W_\alpha}(T + X) + \Lambda \mathbf{M}(Q).$$

From (5.4), we have that

$$\begin{aligned} &\mathbf{M}_{W_\alpha}(T + X) \\ =&\mathbf{M}_{W_\alpha}(T_j + \partial \tilde{R}_j + R_j'' + X) \\ \geqslant&\mathbf{M}_{W_\alpha}(T_j + \partial \tilde{R}_j + X) - \mathbf{M}_{W_\alpha}(R_j''). \end{aligned} \tag{5.5}$$

We define $\hat{Q}_j = \tilde{R}_j + Q$ and $\hat{X}_j = \partial \tilde{R}_j + X$, obviously $\operatorname{spt} \partial \hat{X}_j \subset S$ and

$$\operatorname{spt}(\partial \hat{Q}_j - \hat{X}) = \operatorname{spt}(\partial Q - X) \subset S.$$

Also, $\operatorname{spt}(\hat{Q}_j) \subset \overline{W_\alpha}$, and from the Λ-minimizing condition on T_j, we have that

$$\begin{aligned} &\mathbf{M}_{W_\lambda}(T_j + \partial \tilde{R}_j + X) \\ \geqslant&\mathbf{M}_{W_\lambda}(T_j) - \Lambda(\mathbf{M}_{W_\lambda}(\hat{Q}_j)) \\ \geqslant&\mathbf{M}_{W_\lambda}(T_j) - \Lambda(\mathbf{M}_{W_\lambda}(Q) + \mathbf{M}_{W_\lambda}(\tilde{R}_j) \end{aligned}$$

for any $\lambda \in [0, \alpha)$. Letting $\lambda \nearrow \alpha$, we see that

$$\begin{aligned} &\mathbf{M}_{W_\alpha}(T_j + \partial \tilde{R}_j + X) \\ \geqslant&\mathbf{M}_{W_\alpha}(T_j) - \mathbf{M}(P_j) - \Lambda(\mathbf{M}_{W_\alpha}(Q) + \mathbf{M}_{W_\alpha}(\tilde{R}_j)). \end{aligned} \tag{5.6}$$

From (5.5), (5.6) and $\tilde{R}_j = R_j \llcorner W_\alpha$, we have that

$$\mathbf{M}_{W_\alpha}(T+X) \geqslant \mathbf{M}_{W_\alpha}(T_j) - \Lambda \mathbf{M}_{W_\alpha}(Q) - \varepsilon_j$$

for some $0 < \varepsilon_j \to 0$ as $j \to \infty$. Using the lower semi-continuity of the mass under weak convergence, we have that

$$\mathbf{M}_{W_\alpha}(T+X) \geqslant \mathbf{M}_{W_\alpha}(T) - \Lambda \mathbf{M}_{W_\alpha}(Q).$$

So we have that $T \in F_\Lambda$.

Next we show the convergence of μ_{T_j} to μ_T in the sense of Radon measures. From $X = 0$ and $Q = 0$ in (5.6), we have

$$\limsup_j \mu_{T_j}(K) \leqslant \limsup_j \mathbf{M}_{W_\lambda}(T_j) \leqslant \mathbf{M}_{\{x : d(K,x) < \varepsilon\}}(T)$$

for any sufficiently small $\varepsilon > 0$, we have

$$\limsup_j \mu_{T_j}(K) \leqslant \mu_T(K).$$

It follows from the lower semi-continuity of mass with respect to the convergence $T_j \rightharpoonup T$ that

$$\liminf_j \mu_{T_j}(W) \geqslant \mu_T(W)$$

for any open set $W \subset U$. Hence $\mu_{T_j} \to \mu_T$ as Radon measures. □

Now we show the existence of an oriented tangent cone at the free boundary. For the the existence of an oriented tangent cone at an interior point, we refer the readers to [12].

Lemma 5.5. *The class F_Λ is closed under dilation and translation by $\lambda \geqslant 1$. The class F_Λ is sequentially compact with respect to current convergence. The tangent cone of any $T \in \mathcal{F}_C$ is area minimizing viewed as currents. In particular at the free boundary, if $x_0 \in \operatorname{spt} \partial T \backslash \bar{B}$, there is an oriented tangent cone C at x_0 satisfying*

$$\theta^n(C, x_0) = \theta^n(T, x_0)$$

and satisfying the minimization property:

$$\mathbf{M}_W(C) \leqslant \mathbf{M}_W(C+X)$$

where X is an integer multiplicity n-current with $\operatorname{spt} X \subset W$, $\mathbf{M}(\partial X) < \infty$ and $\operatorname{spt} \partial X \subset T_{x_0} S$.

Proof. Since $\lambda(T - x) \in F_{\Lambda/\lambda}$ for and $\lambda \geqslant 1$ and any $x \in \mathbb{R}^{n+1}$. Note that the support hypersurface also translates and dilates. The lemma follows from Lemma 5.4 and an application of the monotonicity formula (See [18] or [19]). One could also follow the reflection argument of Grüter [16, Theorem 3.5].

Alternatively, it is not difficult to follow the proofs of [17, Theorem 4.3] by using the density bound (5.3). □

Definition 5.6. Let $T \in \mathcal{F}_C$ with free boundary supported on B, define $\operatorname{reg} T$ to be the set of points $x \in \operatorname{spt} T$ such that $T \llcorner B(x, \rho)$ is a C^1 graph for some $\rho > 0$. Define $\operatorname{sing} T = \operatorname{spt} T - \operatorname{reg} T$.

We have the following regularity theorem concerning the size of the singular set for Λ-minimizing currents with a free boundary.

Theorem 5.7. *Let $T \in F_\Lambda$, then* $\dim(\operatorname{sing} T) \leqslant n - 7$.

Proof. The proof is the same with [12, Theorem A.1] with the monotonicity formula from [18]. See also [16] for the case $\Lambda = 0$. □

Appendix A. Mixed boundary value problems for quasilinear elliptic equations in Riemannian manifolds

Fix $n \geq 2$. Let (M^n, g) be an n-dimensional C^∞ Riemannian manifold with non-empty C^∞ boundary ∂M. For our purpose, we shall always assume that M is contained as a closed subset in another n-dimensional C^∞ Riemannian manifold $\tilde{M}$ *without boundary*, whose metric is also denoted by g by abuse of notation. We will use K, Ric and R to denote respectively the sectional, Ricci and scalar curvature of the metric g.

Remark A.1. While it is always possible (for example, by constructing a collar neighborhood of ∂M using Seeley's extension [35]) to extend (M^n, g) locally across ∂M to a Riemannian manifold $(\tilde{M}^n, g)$ without boundary, it is a highly nontrivial problem if one wants to control the geometry (like completeness or curvature bounds) of the extended manifold $\tilde{M}$ in terms of the geometry of the original manifold M. This is known as the smooth Riemannian extension problem (c.f. [30]). For simplicity, we assume that an extension $\tilde{M}$ of (M, g) has always been fixed and we refer to any geometric quantities depending on (M, g) as such which depends on (M, g) and its extension $(\tilde{M}, g)$.

In order to establish a complete regularity theory for the mixed boundary value problems considered in this paper, we restrict ourselves to certain domains inside M with non-degenerate corners and edges.

Definition A.2 (Regular domains)**.** A subset $\Omega \subset M$ is called a *regular domain of* M if Ω is a bounded, connected, relatively open subset of M such that there exists a bounded connected open subset $\tilde{\Omega} \subset \tilde{M}$ satisfying the following conditions:

(i) $\tilde{\Omega}$ is an extension of Ω, i.e. $\Omega = \tilde{\Omega} \cap M$;
(ii) $\partial\tilde{\Omega}$ is an embedded C^∞ hypersurface in $\tilde{M}$ intersecting ∂M transversely.

For any regular domain $\Omega \subset M$, we will denote $\partial_D \Omega := \partial\tilde{\Omega} \cap M$ to be the *Dirichlet boundary of* Ω and $\partial_N \Omega := \Omega \cap \partial M$ to be the *Neumann boundary of* Ω. Their common intersection $E := \partial_D \Omega \cap \overline{\partial_N \Omega}$ is called the *edge* (or *corner*) of Ω. We shall denote $\theta : E \to (0, \pi)$ to be the angle between $\partial_D \Omega$ and $\partial_N \Omega$ measured inside the domain Ω.

Remark A.3. While the choice of extension $\tilde{\Omega}$ is not unique, (ii) guarantees that $\partial_D \Omega$ and $\partial_N \Omega$ (and hence θ) are uniquely determined by Ω, independent of the extension $\tilde{\Omega}$. As in Remark A.1, we refer to any quantities depending on Ω as such which depends on Ω and its fixed extension $\tilde{\Omega}$.

Remark A.4. From the definition, it is clear that $\partial_D \Omega$ is a properly embedded C^∞ hypersurface in M with (possibly empty) boundary lying in ∂M. On the other hand, $\partial_N \Omega$ is a bounded open subset of ∂M with C^∞ boundary. Note that $\partial_N \Omega$ is possibly disconnected even though Ω is connected. By transversality, E is an embedded C^∞ $(n-2)$-dimensional submanifold of $\tilde{M}$ which is the common boundary of $\partial_D \Omega$ and $\partial_N \Omega$.

Lemma A.5 (Regularized distance function from $\partial_D\Omega$)**.** *For any regular domain Ω of M with the interior ball curvature of $\partial\tilde{\Omega}$ bounded above by $K>0$, there exists a function $d\in C^\infty(\overline{\Omega})$ such that*

$$d=0 \quad \text{and} \quad Dd(\eta_{\partial_D\Omega})=-1 \quad \text{on } \partial_D\Omega,$$

$$0\le d\le (2K)^{-1},\ |Dd|\le 9 \quad \text{and} \quad |D^2 d|\le 200K \quad \text{on } \Omega.$$

Here, D and D^2 are the covariant derivative and Hessian of (M,g) respectively; $\eta_{\partial_D\Omega}$ is the outward unit normal of $\partial_D\Omega$ with respect to Ω. Furthermore, d agrees with the distance function from $\partial\tilde{\Omega}$ inside $\{x\in\Omega: d(x)<(4K)^{-1}\}$ for an extension $\tilde{\Omega}$ (c.f. Remark A.3).

Proof. Let $\tilde{d}$ be the distance function from $\partial\tilde{\Omega}$ with respect to the metric $\tilde{g}$. It is a standard result (c.f. [15, Lemma 14.16]) that $\tilde{d}$ is C^∞ in a small tubular neighborhood of $\partial\tilde{\Omega}$. More precisely, if K is an upper bound of the interior ball curvature of $\partial\tilde{\Omega}$. The hessian of $\tilde{d}$ is bounded by $100K$ in the $(4K)^{-1}$-tubular neighborhood of $\partial\tilde{\Omega}$ (c.f. [8] and [9]). We then define $d=\chi_K\tilde{d}+(1-\chi_K)(2K)^{-1}$ where χ_K is a cutoff function as defined in [20, Section 2] from which the estimates follow. □

The same proof applies to the distance function from ∂M also yields the following:

Lemma A.6 (Regularized distance function from $\partial_N\Omega$)**.** *For any regular domain Ω of M with the interior ball curvature of ∂M bounded above by $K>0$ along $\partial_N\Omega$, there exists a function $d'\in C^\infty(\overline{\Omega})$ such that*

$$d'=0 \quad \text{and} \quad Dd'(\eta_{\partial M})=-1 \quad \text{on } \partial_N\Omega,$$

$$0\le d'\le (2K)^{-1},\ |Dd'|\le 9 \quad \text{and} \quad |D^2 d'|\le 200K \quad \text{on } \Omega.$$

Here, D and D^2 are the covariant derivative and Hessian of (M,g) respectively; $\eta_{\partial M}$ is the outward unit normal of ∂M with respect to M. Furthermore, d' agrees with the distance function from ∂M inside $\{x\in\Omega: d'(x)<(4K)^{-1}\}$.

We have the following useful extension lemma.

Lemma A.7 (Extension lemma)**.** *Let Ω be a regular domain in M. For each $\varphi\in C^2(\partial_D\Omega)$, there exists an extension $\overline{\varphi}\in C^2(\overline{\Omega})$ such that*

$$|\overline{\varphi}|_{C^0(\overline{\Omega})}\le|\varphi|_{C^0(\partial_D\Omega)},$$

$$|D\overline{\varphi}|_{C^0(\overline{\Omega})}\le|D\varphi|_{C^0(\partial_D\Omega)}+8K|\varphi|_{C^0(\partial_D\Omega)},$$

$$|D^2\overline{\varphi}|_{C^0(\overline{\Omega})}\le|D^2\varphi|_{C^0(\partial_D\Omega)}+16K|D\varphi|_{C^0(\partial_D\Omega)}+96K^2|\varphi|_{C^0(\partial_D\Omega)},$$

where K is an upper bound of the interior ball curvature of $\partial\tilde{\Omega}$.

Proof. By Seeley's extension procedure [35], we can extend φ to $\partial\tilde{\Omega}$ with the same C^2 bound. Then we extend constantly along normal geodesics emanating from $\partial\tilde{\Omega}$, multiply by the cutoff function χ_K as defined in [20, Section 2] and restrict it back to $\overline{\Omega}$. □

In the following, we study the a-priori estimates for solutions to the mixed boundary value problems for quasilinear elliptic equations on a regular domain Ω in M. We will denote $\Omega^\circ := \Omega \setminus \partial_N\Omega$ to be the interior of Ω (as a subset in $\tilde{M}$). More precisely, we are interested in the second order quasilinear boundary value problem:

$$\left\{ \begin{array}{rcll} Qu & = & 0 & \text{in } \Omega^\circ \\ u & = & \varphi & \text{along } \partial_D\Omega \\ Nu & = & 0 & \text{along } \partial_N\Omega \end{array} \right. \tag{A.1}$$

Here, Q is a second order quasilinear operator of the form

$$Qu := a^{ij}(x, Du, u)D_iD_ju + a(x, Du, u) \tag{A.2}$$

where $a^{ij}(x,p,z)$ is a function defined on $T\Omega^\circ \times \mathbb{R}$ with values in the symmetric 2-tensor in M, and $a(x,p,z)$ is a real-valued function defined on $T\Omega^\circ \times \mathbb{R}$. Here Du and D_iD_ju is respectively the gradient and Hessian of u with respect to the metric g. We adopt Einstein's summation convention and all the indices are raised and lowered by the metric g. On the other hand, N is a first order non-linear boundary operator of the form

$$Nu := b(x, Du, u) \tag{A.3}$$

where $b(x,p,z)$ is a real-valued function defined on $TM|_{\partial_N\Omega} \times \mathbb{R}$. For our purpose, we will assume that $b(x,p,z)$ is C^1 with respect to p. Moreover, φ is a real-valued function defined on $\partial_D\Omega$. The exact regularity assumptions on the coefficients a^{ij}, a, b and the Dirichlet data φ will be specified under different situations.

The ellipticity of the operator Q are defined as follow. We say that Q is *elliptic* in a subset $U \subset T\Omega^\circ \times \mathbb{R}$ if $a^{ij}(x,p,z)$ is a positive definite symmetric 2-tensor for all $(x,p,z) \in U$. Denoting $\lambda(x,p,z)$ and $\Lambda(x,p,z)$ to be the minimum and maximum eigenvalues of $a^{ij}(x,p,z)$ (viewed as an endomorphism on T_xM), ellipticity of Q in U means that for all $(x,p,z) \in U$ and $\xi \in T_xM \setminus \{0\}$, we have

$$0 < \lambda(x,p,z)|\xi|^2 \leq a^{ij}(x,p,z)\xi_i\xi_j \leq \Lambda(x,p,z)|\xi|^2.$$

If, furthermore, both λ^{-1} and Λ are bounded in U, we say that Q is *uniformly elliptic* in U. When $U = T\Omega^\circ \times \mathbb{R}$, we simply say that Q is elliptic (resp. uniformly elliptic) in Ω°. If $u \in C^1(\Omega^\circ)$ and the matrix $[a^{ij}(x, Du(x), u(x))]$ is positive definite for all $x \in \Omega^\circ$, we say that Q is *elliptic with respect to u*. If λ^{-1} and Λ are bounded on $\{(x, Du(x), u(x) \mid x \in \Omega^\circ\}$, Q is said to be *uniformly elliptic with respect to u*.

The obliqueness of the boundary operator N is defined as follow. We say that N is *oblique* in a subset $V \subset TM|_{\partial_N\Omega} \times \mathbb{R}$ if the vector that is dual to the co-vector $D_pb(x,p,z) = (b_{p_i}(x,p,z))$ points strictly into Ω for all $(x,p,z) \in V$. Denoting $\eta_{\partial M}$ to be the outward unit normal of ∂M with respect to M, obliqueness of N in V means that for all $(x,p,z) \in V$,

$$-b_{p_i}(x,p,z) \cdot \eta^i_{\partial M}(x) > 0.$$

When $V = TM|_{\partial_N\Omega} \times \mathbb{R}$, we simply say that N is oblique in $\partial_N\Omega$. If $u \in C^1(\Omega)$ and $-b_{p_i}(x, Du(x), u(x)) \cdot \eta^i_{\partial M}(x) > 0$ for all $x \in \partial_N\Omega$, we say that N is *oblique with respect to u*.

We now begin with the linear theory that will be fundamental to the study of the quasilinear elliptic mixed boundary value problem (A.1). We consider the following inhomogeneous linear boundary value problem:

$$\begin{cases} Lu &= f \quad \text{in } \Omega^\circ \\ u &= \varphi \quad \text{along } \partial_D\Omega \\ Bu &= f_N \quad \text{along } \partial_N\Omega \end{cases} \tag{A.4}$$

where L is a linear differential operator of the form

$$Lu := a^{ij}(x)D_iD_ju + b^i(x)D_iu + c(x)u \tag{A.5}$$

and B is a linear boundary operator of the form

$$Bu := \beta^i(x)D_iu + \gamma(x)u. \tag{A.6}$$

Here, f, φ and f_N are some given functions. Note that the ellipticity of L means that $[a^{ij}(x)]$ is positive definite and the obliqueness of B means that $\beta_i(x)\cdot\eta^i_{\partial M}(x) < 0$.

We first establish the weak maximum principle under suitable assumptions on the sign of the zeroth order coefficients of L and B. Note that the theorem holds regardless of the regularity of $\partial_D\Omega$.

Theorem A.8 (Weak maximum principle)**.** *Suppose that L and B are operators as in* (A.5) *and* (A.6) *such that:*

(i) *L is elliptic in Ω° with $c \equiv 0$;*
(ii) *B is oblique in $\partial_N\Omega$ with $\gamma \equiv 0$;*
(iii) *$|b^i|/\lambda$ is locally bounded in Ω°;*

Then, for any $u \in C^2(\Omega^\circ)\cap C^1(\Omega)\cap C^0(\overline{\Omega})$ satisfying

$$Lu \geq 0 \text{ in } \Omega^\circ \qquad \text{and} \qquad Bu \geq 0 \text{ along } \partial_N\Omega,$$

we have

$$\sup_{\Omega} u = \sup_{\partial_D\Omega} u. \tag{A.7}$$

Proof. First of all, notice that the proof of [15, Theorem 3.1] is local and thus the maximum of u in $\overline{\Omega}$ is either achieved on $\partial_D\Omega$ or $\partial_N\Omega$. To prove that a maximum must indeed be achieved by some point on $\partial_D\Omega$, we construct a local positive subsolution near $\partial_N\Omega$. Let d' be the regularized distance function from $\partial_N\Omega$ in Lemma A.6. Then, there is a sufficiently large constant $\sigma > 0$ such that $L(e^{\sigma d'}) > 0$ by the same argument as in [15, Theorem 3.1] and

$$B(e^{\sigma d'}) = -\sigma\beta_i \cdot \eta^i_{\partial M} > 0$$

by the obliqueness of B. Therefore, we have $L(u+\epsilon e^{\sigma d'}) > 0$ and $B(u+\epsilon e^{\sigma d'}) > 0$ for any $\epsilon > 0$ locally in a neighborhood of $\partial_N\Omega$ and our assertion follows from the classical maximum principle and letting $\epsilon \to 0$. □

Remark A.9. The same conclusion with u replaced by $u^+ := \max(u, 0)$ on the right hand side of (A.7) provided that $c \leq 0$ in Ω° and $\gamma \leq 0$ along $\partial_N\Omega$.

Using the boundary Hopf lemma, we can establish a strong maximum principle for uniformly elliptic Q and oblique B.

Theorem A.10 (Strong maximum principle)**.** *Suppose that L and B are operators as in* (A.5) *and* (A.6) *such that:*

(i) *L is locally uniformly elliptic in Ω° with $c \equiv 0$;*
(ii) *B is oblique in $\partial_N\Omega$ with $\gamma \equiv 0$;*
(iii) *$|b^i|/\lambda$ is locally bounded in Ω°;*

Then, for any $u \in C^2(\Omega^\circ) \cap C^1(\Omega) \cap C^0(\overline{\Omega})$ *satisfying*

$$Lu \geq 0 \text{ in } \Omega^\circ \qquad \text{and} \qquad Bu \geq 0 \text{ along } \partial_N\Omega,$$

the maximum of u *cannot be achieved in* Ω *unless* u *is a constant function.*

Proof. It follows from [15, Theorem 3.5] that u cannot achieve its maximum in Ω°. Since $\partial_N\Omega$ clearly satisfies an interior sphere condition, the maximum cannot be achieved on $\partial_N\Omega$ by (ii) and the boundary Hopf lemma [15, Lemma 3.4]. □

Remark A.11. If $c \leq 0$, $\gamma \leq 0$ and c/λ is locally bounded, then u cannot achieve a non-negative maximum in Ω unless it is constant.

Next, we look at the Schauder theory for the linear mixed boundary value problem (A.4). Let us recall the definitions of weighted Hölder spaces. For any $\delta > 0$, we define

$$\Omega_\delta := \{x \in \Omega \mid \overline{d}(x) > \delta\}$$

where $\overline{d}$ is the distance function from $\partial_D\Omega \cup \partial_N\Omega$ with respect to the metric g.

Definition A.12 (Weighted Hölder spaces)**.** For k be a nonnegative integer, $\alpha \in (0,1)$ and $b \geq -(k+\alpha)$, we define the weighted Hölder norms $\|\cdot\|^{(b)}_{k,\alpha;\Omega}$ on a regular domain Ω in M by

$$\|u\|^{(b)}_{k,\alpha;\Omega} := \sup_{\delta>0} \delta^{b+k+\alpha} |u|_{C^{k,\alpha}(\overline{\Omega_\delta})}$$

We denote $H^{(b)}_{k,\alpha}(\Omega) := \{u \in C^{k,\alpha}(\Omega^\circ) \mid \|u\|^{(b)}_{k,\alpha;\Omega} < \infty\}$.

We collect some useful facts about these weighted Hölder spaces.

Proposition A.13. Let k, k' be nonnegative integers, $\alpha, \alpha' \in (0,1)$.

(a) $H^{(-k'-\alpha')}_{k,\alpha}(\Omega) \subset C^{k',\alpha'}(\overline{\Omega}) \cap C^{k,\alpha}(\Omega^\circ)$ whenever $k' + \alpha' \leq k + \alpha$.
(b) $H^{(-b)}_{k,\alpha}(\Omega) \hookrightarrow H^{(-b')}_{k',\alpha'}(\Omega)$ is a compact embedding whenever $k' + \alpha' < k + \alpha$ and $0 < b' < b$.

Proof. See [25] and [14]. □

We are now ready to state the global Schauder estimates for the linear mixed boundary value problem (A.4). Note that we only assume the Dirichlet boundary data φ to be in $C^{1,\alpha}(\partial_D\Omega)$ and hence we need to employ the intermediate Schauder estimates of Gilbarg-Hörmander [14] and Lieberman [26, 28]. Note that we require the corner angle θ to be acute in order to have $C^{1,\alpha}$-estimates up to the edge.

Theorem A.14 (Global Schauder estimates)**.** *Let* Ω *be a regular domain in* M*;* L *and* B *be the operators as in* (A.5) *and* (A.6) *such that:*

(i) *there exists some* $\theta_{max} \in (0, \frac{\pi}{2})$ *such that* $0 < \theta \leq \theta_{max}$ *along the edge* E*;*
(ii) L *is uniformly elliptic in* Ω°*;*
(iii) B *is is uniformly oblique in* $\partial_N\Omega$*;*

Then, there exists a constant $\alpha_0 = \alpha_0(\theta_{max}, \frac{\Lambda}{\lambda}) \in (0,1)$ *such that for any* $\alpha \in (0, \alpha_0)$ *where*

(iv) $a^{ij} \in H^{(0)}_{0,\alpha}(\Omega) \cap C^0(\overline{\Omega})$, $b^i, c \in H^{(1-\alpha)}_{0,\alpha}(\Omega)$ *and* $\beta^i, \gamma \in C^{0,\alpha}(\overline{\partial_N\Omega})$*;*

we have the following estimate for any $u \in C^2(\Omega^\circ) \cap C^1(\Omega) \cap C^0(\overline{\Omega})$*:*

$$\|u\|^{(-1-\alpha)}_{2,\alpha;\Omega} \leq C(\|Lu\|^{(1-\alpha)}_{0,\alpha;\Omega} + |Bu|_{C^{0,\alpha}(\overline{\partial_N\Omega})} + |u|_{C^{1,\alpha}(\partial_D\Omega)} + |u|_{C^0(\overline{\Omega})}) \tag{A.8}$$

where $C > 0$ *is a constant depending only on* $n, \alpha, \lambda, \Lambda, \Omega$*, and the norms of the coefficients* $\|a^{ij}\|^{(0)}_{0,\alpha;\Omega}, \|b^i\|^{(1-\alpha)}_{0,\alpha;\Omega}, \|c\|^{(1-\alpha)}_{0,\alpha;\Omega}, |\beta^i|_{C^{0,\alpha}(\overline{\partial_N\Omega})}, |\gamma|_{C^{0,\alpha}(\overline{\partial_N\Omega})}$ *and the modulus of continuity of* a^{ij}*.*

Proof. Using the notations in [28], we have $(\Omega^\circ, \partial_N\Omega) \in H^*_b$ for all $b \geq 1$, $\Lambda^{-1}L \in \mathcal{L}_\mu$ with $\mu = \lambda/\Lambda$. Moreover, (i) implies that Ω satisfies a Σ-wedge condition with angle $\theta_1 = \frac{1}{2}(\theta_{max} + \frac{\pi}{2}) \in (0, \frac{\pi}{2})$. Since $0 < \theta_1 < \frac{\pi}{2}$, we can choose $\eta \in (\theta_1, \frac{\pi}{2})$, depending only on θ_1 and the geometry of Ω, such that the condition $\beta^1 \leq (\cot \eta)\cdot\beta^2$ is satisfied near the edge E. The result then follows from [28, Theorem 4] by taking $\alpha_0 = \lambda_1 - 1 > 0$. □

Remark A.15. By the weak maximum principle (Remark A.9), we can omit the term $|u|_{C^0(\overline{\Omega})}$ on the right hand side of (A.8) provided that $c \leq 0$ and $\gamma \leq 0$.

As in the classical Schauder theory [15, Chapter 6], one can derive from the Schauder estimates in Theorem A.14 the following existence result for the linear mixed boundary value problem (A.4) with coefficients and data in certain Hölder spaces up to the Dirichlet boundary.

Theorem A.16 (Unique solvability for linear mixed boundary value problem)**.** *Suppose all the assumptions (i)-(iii) in Theorem A.14 hold and furthermore that* $c \leq 0$ *in* Ω° *and* $\beta^0 \leq 0$ *along* $\partial_N\Omega$*. Assume, moreover, that for some* $\alpha \in (0, \alpha_0)$ *we have*

(iv)' $a^{ij}, b^i, c \in C^{0,\alpha}(\overline{\Omega})$ *and* $\beta^i, \gamma \in C^{1,\alpha}(\overline{\partial_N\Omega})$.

Then, for any given $\varphi \in C^{1,\alpha}(\partial_D\Omega)$*,* $f \in C^{0,\alpha}(\overline{\Omega})$ *and* $f_N \in C^{0,\alpha}(\overline{\partial_N\Omega})$*, the linear mixed boundary value problem* (A.4) *has a unique solution* $u \in H^{(-1-\alpha)}_{2,\alpha}(\Omega)$ *and*

$$\|u\|^{(-1-\alpha)}_{2,\alpha;\Omega} \leq C(|f|_{C^{0,\alpha}(\overline{\Omega})} + |f_N|_{C^{0,\alpha}(\overline{\partial_N\Omega})} + |\varphi|_{C^{1,\alpha}(\partial_D\Omega)}) \tag{A.9}$$

where $C > 0$ *is a constant depending only on* $n, \alpha, \lambda, \Lambda, \Omega$*, the* $C^{0,\alpha}$*-norms of* a^{ij}, b^i *and* c *in* $\overline{\Omega}$*, and the* $C^{0,\alpha}$*-norms of* β^i *and* γ *in* $\overline{\partial_N\Omega}$*.*

Proof. The uniqueness part follows from (A.8) and Remark A.15 since condition (iv)' implies (iv). For the existence part, we make use of [27, Theorem 2]. Note that Ω satisfies a global Σ-wedge condition, and uniform interior and exterior cone conditions hold on $\partial_D\Omega$. Furthermore, (ii), (iii) and (iv)' imply (1.3), (1.4), (2.3) and (3.3) in [27]. Furthermore, (2.7) in [27] also holds since $c \leq 0$ and $\gamma \leq 0$ (see also Theorem 1 in [27] and the paragraph after). By [27, Theorem 2 (b)], there exists a unique solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to (A.4). The desired estimate (A.9) follows directly from (A.8) and the observation that $\|f\|^{(1-\alpha)}_{0,\alpha;\Omega} \leq C(\Omega)|f|_{C^{0,\alpha}(\overline{\Omega})}$. □

We now turn out attention to the quasilinear mixed boundary value problem (A.1). First, we establish a comparison principle similar to [15, Theorem 10.1]. Note that the theorem holds regardless of the regularity of $\partial_D\Omega$.

Theorem A.17 (Comparison principle)**.** *Let $u, v \in C^2(\Omega^\circ) \cap C^1(\Omega) \cap C^0(\overline{\Omega})$ such that*

$$\begin{cases} Qu & \geq & Qv & \text{in } \Omega^\circ \\ u & \leq & v & \text{along } \partial_D\Omega \\ Nu & \geq & Nv & \text{along } \partial_N\Omega \end{cases} \tag{A.10}$$

where Q and N are as defined in (A.2) *and* (A.3)*. Assume that*

(i) *Q is locally uniformly elliptic with respect to u (or v);*
(ii) *N is oblique with respect to $tu + (1-t)v$ for all $t \in [0, 1]$;*
(iii) *a^{ij} and b are independent of z;*
(iv) *a is non-increasing with respect to z;*
(v) *b is C^1 with respect to p.*
(vi) *a^{ij}, a are C^1 with respect to p;*

Then, we have $u \leq v$ in Ω. If we assume instead

$$\begin{cases} Qu & > & Qv & \text{in } \Omega^\circ \\ u & \leq & v & \text{along } \partial_D\Omega \\ Nu & > & Nv & \text{along } \partial_N\Omega \end{cases} \tag{A.11}$$

and conditions (i) - (v) above hold. Then, we have $u < v$ in Ω.

Proof. Let us assume that Q is locally uniformly elliptic with respect to u. As in [15, Theorem 10.1], by (iii), we write

$$\begin{aligned} Qu - Qv = {} & a^{ij}(x, Du) D_i D_j (u - v) + [a^{ij}(x, Du) - a^{ij}(x, Dv)] D_i D_j v \\ & + [a(x, Du, u) - a(x, Dv, u)] + a(x, Dv, u) - a(x, Dv, v) \end{aligned}$$

and

$$Nu - Nv = b(x, Du) - b(x, Dv).$$

Let $w := u - v$. By (vi), we can write

$$b^i D_i w = [a^{ij}(x, Du) - a^{ij}(x, Dv)] D_i D_j v + [a(x, Du, u) - a(x, Dv, u)]$$

and by (v),

$$\beta^i D_i w = b(x, Du) - b(x, Dv)$$

for some continuous functions $b^i(x)$ and $\beta^i(x)$. We consider the linear operators

$$Lw := a^{ij}(x, Du(x)) D_i D_j w + b^i(x) D_i w$$

and

$$Bw := \beta^i(x) D_i w$$

with vanishing zeroth order coefficients. Suppose on the contrary that $u > v$ at some point in Ω. Then $\Omega_+ := \{x \in \Omega \mid u(x) > v(x)\}$ is a non-empty relatively open subset of Ω. Moreover, by (A.10) and (iii), we have

$$\begin{cases} Lw & \geq & 0 & \text{in } \Omega_+ \cap \Omega^\circ \\ w & \leq & 0 & \text{along } \overline{\Omega_+} \cap \partial_D\Omega \\ Bw & \geq & 0 & \text{along } \Omega_+ \cap \partial_N\Omega. \end{cases}$$

Note that L is locally uniformly elliptic in Ω° by (i), B is oblique in $\partial_N\Omega$ by (ii), and b^i/λ is locally bounded by (i) and the continuity of b^i in Ω°. It then follows from the weak maximum principle (Theorem A.8) that $w \leq 0$ in Ω_+ which is impossible by the definition of Ω_+. This proves the first part of the theorem.

For the second part, we work directly with the expressions of $Qu - Qv$ and $Nu - Nv$. Suppose on the contrary that $u \geq v$ at some point in Ω. Then, the function $w = u - v$ must achieve a non-negative maximum at some point $x_0 \in \Omega$. There are two possible cases: either $x_0 \in \Omega^\circ$ or $x_0 \in \partial_N \Omega$. If $x_0 \in \Omega^\circ$, then $Du = Dv$ and $D_i D_j w \leq 0$ at x_0 and by (A.11), (i) and (iv), we have at x_0 the following

$$0 < Qu - Qv = a^{ij}(x, Du(x)) D_i D_j w + a(x, Dv, u) - a(x, Dv, v) \leq 0,$$

which is a contradiction. If $x_0 \in \partial_N \Omega$, then at the boundary point x_0 we have

$$Du = p^T + r\eta_{\partial M}, \quad Dv = p^T + s\eta_{\partial M}$$

for some $p^T \in T_{x_0}(\partial M)$ and $r \geq s$. It then follows from (ii), (iii), (v) and (A.11) that

$$0 < Nu - Nv = b(x, Du) - b(x, Dv) \leq 0$$

at x_0, which is a contradiction. This completes the proof. □

Remark A.18. If N is an oblique linear boundary operator with $\gamma \equiv 0$ as in (A.6), then the conditions (ii), (iii) and (v) about N are clearly satisfied.

In order to establish certain boundary gradient estimate up to the edge E, we shall need the notion of barriers.

Definition A.19 (Upper and lower barriers)**.** Let Q and N be as defined in (A.2) and (A.3). Suppose $u \in C^2(\Omega^\circ) \cap C^1(\Omega) \cap C^0(\overline{\Omega})$ and $x_0 \in \partial_D \Omega$. An *upper barrier for u at x_0* (with respect to Q and N) is a function $w^+ : W \to \mathbb{R}$ defined on a relatively open neighborhood of x_0 in $\overline{\Omega}$ such that $w^+ \in C^2(W \cap \Omega^\circ) \cap C^1(\overline{W})$ and the following holds:

(i) $Qw^+ < Qu$ in $W \cap \Omega^\circ$,
(ii) $w^+(x_0) = u(x_0)$,
(iii) $w^+ \geq u$ on ∂W (the topological boundary of W in M),
(iv) $Nw^+ < Nu$ on $W \cap \partial_N \Omega$.

A *lower barrier w^- for u at x_0* is similarly defined with the inequalities reversed in (i), (iii) and (iv).

Suppose both an upper barrier w^+ and a lower barrier w^- for u exist at $x_0 \in \partial_D \Omega$ in a neighborhood W. If Q and N satisfies conditions (i)-(v) in Theorem A.17, then $w^- \leq u \leq w^+$ in W with equality at x_0. Therefore, we have for all $x \in \overline{W}$ that

$$\frac{w^-(x) - w^-(x_0)}{\mathrm{dist}_g(x, x_0)} \leq \frac{u(x) - u(x_0)}{\mathrm{dist}_g(x, x_0)} \leq \frac{w^+(x) - w^+(x_0)}{\mathrm{dist}_g(x, x_0)}.$$

If u is differentiable at x_0, then we have

$$\frac{\partial w^-}{\partial \eta_{\partial\Omega}}(x_0) \geq \frac{\partial u}{\partial \eta_{\partial\Omega}}(x_0) \geq \frac{\partial w^+}{\partial \eta_{\partial\Omega}}(x_0)$$

where $\eta_{\partial\Omega}$ is the outward unit normal of $\partial_D \Omega$. Since the tangential derivatives of u along $\partial_D \Omega$ at x_0 is completely determined by the Dirichlet data $u|_{\partial_D \Omega}$. To obtain boundary gradient estimates along $\partial\Omega$, it suffices to show the existence of upper and lower barriers for u at all $x_0 \in \partial_D \Omega$ up to the edge E and these barriers have uniformly bounded gradients.

Finally, we establish the De Giorgi-Nash-Moser theory (with bounded coefficients and data) for certain linear mixed boundary value problems which will be useful in establishing Hölder gradient estimates. We consider a linear operator in divergence form

$$Lu := D_i(a^{ij}D_j u) \tag{A.12}$$

where the coefficients $a^{ij}(x)$ are bounded and measurable. Our first result is an interior estimate.

Theorem A.20. *Let L be the linear operator on Ω° defined as in* (A.12). *Suppose that*

(i) *L is uniformly elliptic in Ω°, i.e. there exists constants $0 < \lambda \le \Lambda$ such that for all $x \in \Omega^\circ$ and $0 \neq \xi \in T_xM$,*
$$0 < \lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2;$$
(ii) *$a^{ij} \in L^\infty(\Omega^\circ)$;*
(iii) *f^i is a bounded measurable vector field in Ω°*

Then, there exists a constant $\alpha = \alpha(n, \frac{\Lambda}{\lambda}, |Ric|_{L^\infty(\Omega^\circ)}) \in (0,1)$ such that for any $u \in W^{1,2}(\Omega^\circ)$ which is a weak solution to $Lu = D_if^i$ in Ω°, i.e.

$$\int_\Omega (a^{ij}D_ju - f^i)D_i\zeta = 0 \qquad \text{for any } \zeta \in C^1_0(\Omega^\circ),$$

we have $u \in C^{0,\alpha}(\Omega^\circ)$. Moreover, we have

$$|u|_{C^{0,\alpha}(\Omega^\circ)} \le C$$

where $C > 0$ is a constant depending only on $n, \lambda, \Lambda, |Ric|_{L^\infty(\Omega^\circ)}$, $|f^i|_{L^\infty(\Omega^\circ)}$ and $\sup_{\Omega^\circ}|u|$.

Proof. It follows from the standard De Giorgi-Nash-Moser theory (e.g. [15, Theorem 8.22] with $q = \infty$ and $\nu = 0$) adapted to the Riemannian setting. ☐

For Hölder gradient estimates, we restrict our case to divergence form operator:

$$Qu = \operatorname{div}\mathbf{A}(x, Du, u) + a(x, Du, u)$$

where $\mathbf{A} \in C^1(T\Omega \times \mathbb{R})$ with values in TM and $a \in C^0(T\Omega \times \mathbb{R})$ is a real-valued function; and a conormal boundary operator:

$$Nu = -\mathbf{A}(x, Du, u) \cdot \eta_{\partial M}(x) + \psi(x, u)$$

where $\psi \in C^0(\partial_N\Omega \times \mathbb{R})$ is a real-valued function. If $u \in C^1(\Omega)$ satisfies $Qu = 0$ in Ω° (hence in Ω) and $Nu = 0$ on $\partial_N\Omega$, we have from integration by part that

$$\int_\Omega \mathbf{A}(x, Du, u) \cdot D\zeta - a(x, Du, u)\zeta - \int_{\partial_N\Omega} \psi(x, u)\zeta = 0$$

for all $\zeta \in C^1_0(\Omega)$. From now on, we assume that $u \in C^1(\overline{\Omega})$ and that $|u|_{C^1(\overline{\Omega})} \le K$ for some $K > 0$. Our goal is to ...

Appendix B. Geometric measure theory

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