RESEARCH STATEMENT

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1. INTRODUCTION

My primary research focuses on geometric and analytic aspects of some rigidity problems of scalar and mean curvature in Riemannian manifolds or in spacetime. I have established a few theorems in scalar curvature geometry, mathematical relativity and some miscellaneous results. My toolbox includes partial differential equation theory and geometric measure theory of minimal surfaces. The most important ones are summarized as follows.

(1) Gromov [24] proposed the Gromov dihedral rigidity conjecture as an attempt to study the local geometry and convergence of manifolds with lower scalar curvature bounds. The Euclidean dihedral rigidity is a comparison theory of the convex polyhedron in the Euclidean space and is reminiscent of the triangle comparison theory of sectional curvature. Using the stable constant mean curvature surface with a capillary boundary condition, Li confirmed the Euclidean case [30] for 3-prisms and cone type polyhedron under the restrictive conditions and for parabolic cubes in the hyperbolic space [31].

In [19] with Gaoming Wang, I have a satisfying answer for the dihedral rigidity conjecture in the hyperbolic 3-space for the tetrahedron and prisms. In particular, we solved completely the tetrahedra case for both the Euclidean and hyperbolic case, and we removed some restrictive assumptions Li imposed on the cone polyhedron case. Also, my work is a generalization of Gromov's original formulation [24] in hyperbolic 3-space which only involved the cubes.

(2) The conjectured Penrose inequality states that an outermost marginally outer trapped surface (or MOTS) is the lower bound of the ADM mass for an asymptotically flat initial data set satisfying the dominant energy condition. MOTS is an important concept which naturally arises in mathematical relativity. The notion of capillary MOTS was recently proposed by [1].

In [16], with Martin Li, I established the existence theory of free boundary MOTS and some results on the capillary MOTS from the partial differential equation side. In [12], I developed a curvature estimate for stable free boundary MOTS in an initial data set with boundary. This generalizes the Schoen-Simon-Yau curvature estimate [34] to the free boundary settings. The method I used has the merit that the curvature estimate depends on explicit geometry quantities compared to a previous work of [27].

(3) Gromov [25] introduced a new geometric object called bands. A band is just the product of a base manifold and an interval, Gromov related the width of the bands with the lower scalar curvature band. Let (M, g) be an over-torical band with $R_g \geqslant n(n-1)$, then the distance between the boundaries of M is at least $\frac{2\pi}{n}$. With Xueyuan Wan, I [17] extended this band width estimate to the settings

of constant mean curvature initial data sets. We [18] then applied the band width estimate to study an asymptotically hyperbolic manifold with arbitrary ends.

In [10], I also introduced a new asymptotically hyperbolic manifold with a horospherical boundary. The corresponding positive mass theorem was completed by Almaraz-Lima [4].

(4) Positive mass theorem is a scalar curvature geometry for noncompact manifolds as revealed by the works [36], [37]. In [3], Almaraz-Lima-Mari introduced the initial data set with a noncompact boundary. In [13], I further developed the theory using spinors and a new boundary condition. This new positive mass theorem suggested a potential application of the theory of capillary MOTS to re-prove my result in the nonspin settings.

(5) I [14] established the Willmore type inequality for free boundary surfaces in the geodesic balls in hyperbolic space. The method is the inverse mean curvature flow where PDE theory is of fully nonlinear nature.

2. Gromov dihedral rigidity conjecture in hyperbolic 3-space

A central question in differential geometry is to study compactness of manifolds with certain curvature bounds. In the geometry of sectional curvature, a triangle comparison theory was developed in spaces with lower regularity. The Euclidean dihedral rigidity conjecture [24] is Gromov's attempt to characterize the compactness of manifolds with nonnegative scalar curvature.

The dihedral rigidity conjecture goes back to the Schoen and Yau's resolution [35] of the Geroch's conjecture which states that if g is a smooth metric on the torus $\mathbb{T}^n$ where $3 \leqslant n \leqslant 7$, then the scalar curvature R_g of $(\mathbb{T}^n, g)$ cannot be everywhere nonnegative unless is flat.

Conjecture 2.1. *Let M be a convex polyhedron in $\mathbb{R}^n$ and δ be the induced flat metric on M. Suppose that g is another Riemannian metric on M such that the scalar curvature R_g is nonnegative, faces are weakly mean convex and along the intersection of codimension 1 faces the dihedral angles of (M, g) is no larger than the dihedral angles of (M, δ), then (M, g) is isometric to a flat Euclidean polyhedron.*

A simplest example is the Euclidean cube $[0, 1]^n$. Gromov [24] proposed that for a C^0 metric g, $R_g \geqslant 0$ at a point p if and only if there exists no cube M around p with weakly mean convex faces and everywhere acute dihedral angles. Gromov proved Conjecture 2.1 by observing the $\mathbb{Z}^n$-invariance of the cube $[0, 1]^n$ in $\mathbb{R}^n$ and reduced to the Geroch conjecture. Later Li [30], [31] confirmed the Conjecture 2.1 for wider classes of polyhedra and for parabolic cubes in hyperbolic 3-space.

In dimension 3, jointly with Inkang Kim, I [15] gave a harmonic map proof for a special case of the Euclidean cubical dihedral rigidity with dihedral angles identically $\pi/2$. The crux of our proof is a $C^{2,\alpha}$ regularity theory of mixed boundary value problems.

To extend further the Gromov dihedral rigidity to the hyperbolic 3-space, it is crucial to observe that in the upper half space model of the hyperbolic space, the linear planes are totally umbilic and forming a polyhedron whose faces intersect at constant angles.

Theorem 2.2 ([19])**.** *Let M be a tetrahedron or a prism (see Figure 1) in $(\mathbb{R}^3_+, \delta)$ and δ be the induced flat metric on M and $\bar{g}$ be the hyperbolic metric. Suppose that the bottom face of $(M, \bar{g})$ is a horosphere, g is another Riemannian metric on*

M such that the scalar curvature $R_g \geqslant -6$, faces has mean curvatures than those of $(M, \bar{g})$ and along the intersection of codimension 1 faces the dihedral angles of (M, g) is no larger than the dihedral angles of (M, δ), then (M, g) is isometric to $(M, \bar{g})$.

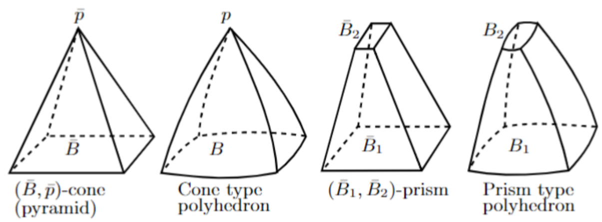

FIGURE 1. Tetrahedrons and prisms.

Other than generalizing to more general polyhedra, our most important contribution to the subject is the limiting behavior of the mean curvatures of the leaves of the constant mean curvature foliations near the vertex point p. Our method actually filled a major gap in Li's proof [30] for tetrahedra. We expect our method would work for other singular manifolds with a conical boundary point.

The Ricci flow [7] was also used to define the C^0 metrics with weak $R_g \geqslant 0$, it is natural to ask whether the two notions using respectively the Ricci flow and polyhedron comparison theorem are equivalent. This is one aspect of the major open problem of characterizing the nonnegative scalar curvature in C^0 sense.

3. Jang equation and free boundary MOTS

An initial data set (M, g, k) is a Riemannian manifold (M, g) with a symmetric 2-tensor k. In [37], Schoen and Yau studied the Jang equation [28] to settle the positive energy conjecture. Let u be a C^2 function on M, extend k trivially to $M \times \mathbb{R}$. Then

$$H(u) := \mathrm{div}_g \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right), \mathrm{tr}_g\, k(u) = \left(g^{ij} - \frac{\nabla^i u \nabla^j u}{1 + |\nabla u|^2} \right)$$

is respectively the mean curvature of the graph of u and the restricted trace of k to the graph of u. The Jang equation is

$$H(u) + \mathrm{tr}_g\, k(u) = 0.$$

It turns out the Jang equation will blow up at regions whose boundary are the so called marginally outer trapped surface (in short MOTS; see [37]).

Definition 3.1 ([6, 1]). *A surface Σ is called a marginally outer trapped hypersurface in a initial data set if it satisfies*

$$H + \mathrm{tr}_\Sigma\, k = 0.$$

It is called stable if there is a deformation $f\nu$, $f > 0$ that does not decrease $H + \mathrm{tr}_g\, k$. It is called free boundary if further the boundary $\partial\Sigma$ intersecting ∂M orthogonally; it is stable if the deformation preserves the free boundary condition.

I established the curvature estimate for a stable free boundary marginally outer trapped hypersurface. This curvature estimate is not only important in the theory of MOTS but also in the minimal hypersurface theory. Before this result, a curvature estimate for free boundary minimal hypersurface was only known via a blowup argument [27].

Theorem 3.2 ([12])**.** *If a marginally outer trapped surface $\Sigma \subset M$ is stable with a free boundary, then*

$$|A|(x) \leqslant C$$

for any $x \in \bar{\Sigma}$ with C depending on explicit geometric quantities of M.

Jointly with Martin Li, I developed a PDE and geometric theory of the capillary regularized Jang equation with a Neumann boundary:

$$\begin{aligned} H(u) + \operatorname{tr}_g k(u) &= \tau u \text{ in } M, \tau > 0, \\ \langle \nabla u, \eta \rangle &= 0 \text{ on } \partial M, \end{aligned} \tag{3.1}$$

where η is the outward normal.

Theorem 3.3. *If M is compact with boundary, the Jang equation* (3.1) *has a $C^{2,\alpha}$ estimate of the form*

$$\|u\|_{C^{2,\alpha}(\bar{M})} \leqslant \tfrac{C}{\tau},$$

where C and α depends on $\sup_M |k|_g$, Riemann curvature operator Rm *of M and the boundary geometry of ∂M.*

Therefore an existence result of (3.1) immediately follows from the method of continuity. In fact, in the PDE aspect, we can have a more general boundary condition in the following

$$\tfrac{\langle \nabla u, \eta \rangle}{\sqrt{1+|\nabla u|^2}} = \cos\alpha$$

where $\cos\alpha \leqslant 1 - \delta < 1$ for some small constant δ.

The free boundary MOTS arises as the boundary of the blow up limit as $\tau \to 0^+$. Using the theory developed above, we are able to show an existence result of free boundary MOTS modulo a technical issue of quasilinear PDE with mixed boundary value problems. This is motivated by Eichmair [22].

Theorem 3.4 ([16])**.** *Given (M^n, g, k) where $\partial M \neq \emptyset$, let $\Omega \subset M$ be an open set with smooth embedded boundary. One part of $\partial\Omega$ lies in ∂M and the other parts can be decomposed into two nonempty hypersurfaces with boundary denoted as Σ_1 and Σ_2. They satisfy*

$$H_{\Sigma_1} + \operatorname{tr}_{\Sigma_1} k > 0, \quad H_{\Sigma_2} - \operatorname{tr}_{\Sigma_2} k > 0,$$

and Σ_i, $i = 1, 2$ intersect ∂M transversely. In fact, we assume that they form with ∂M a corner with angle in the range $(0, \pi/2)$. In terms of normal vectors, the inner product between the outward normal of ∂M and the outward normal of $\Sigma_1 \cup \Sigma_2$ at the intersection place is everywhere less than 0. Then there exists a marginally outer trapped surface Σ with free boundary homologous to Σ_1. Σ is C-almost minimizing with free boundary and with a singular set of Hausdorff dimension $\leqslant n - 8$. When the dimension is greater than 8, the MOTS equation and the free boundary condition are satisfied distributionally.

There is also a work in progress (jointly with Martin Li) on the existence theory of partially free boundary MOTS. It should be expected that these results could be used to give a proof of the positive mass theorem in [5] where the proof works only for spin manifolds.

Also, it should be expected that there is an existence theory of MOTS where Σ intersect with the ambient boundary ∂M with varying prescribed angle. The most important case is of constant contact angle.

4. Band width estimates in CMC initial data sets

Let (M, g) be the product $(\mathbb{T}^{n-1} \times [-1, 1])$ with a Riemannian metric, the length of an object is a fundamental geometric property. Denote $\partial_\pm M = \mathbb{T}^{n-1} \times \{\pm 1\}$ the two boundary components of M. Gromov established the following band width estimate.

Theorem 4.1 ([25])**.** *If $R_g \geqslant n(n-1)$, then*

$$\operatorname{dist}(\partial_- M, \partial_+ M) \leqslant \tfrac{2\pi}{n}.$$

It should be noted that the rigid band is the metric

$$\mathrm{d}t^2 + \cos(\tfrac{nt}{2})^{\frac{4}{n}} g_{\mathbb{T}^{n-1}}.$$

The central technique for establishing the estimate is the so-called stable prescribed mean curvature surfaces (or μ-bubble) introduced by Gromov to the scalar curvature geometry. The μ-bubble helps establish several important results about scalar curvature most notably the Schoen-Yau's $K(\pi, 1)$ conjecture [35]. See Gromov four lectures [26] for a collection of more results.

The quantities

$$\theta^\pm = \pm H + \operatorname{tr}_\Sigma k$$

are called outward (inward) null expansion of the surface. The quantities

$$2\mu = R_g + (\operatorname{tr}_g k)^2 - |k|_g^2$$

and

$$J = \operatorname{div}(k) - \mathrm{d}(\operatorname{tr}_g k) g$$

are the energy density and the current density.

In [17], with Xueyuan Wan, I introduced the surface with prescribed null expansion and the stability condition which generalizes the μ-bubble in which case k vanishes identically.

Definition 4.2 ([17])**.** *Given a Lipschitz function h in M, a surface Σ is said to be a surface of prescribed null expansion h in (M, g, k) if along Σ*

$$\theta^+ = h.$$

The quantity $H + \operatorname{tr}_\Sigma k$ is called the null expansion of Σ. The surface Σ is said to be stable if there exists a positive function $\phi \in W^{2,p}(\Sigma)$ such that

$$\delta_{\phi\nu}(H + \operatorname{tr}_\Sigma k - h) \geqslant 0,$$

where ν is the unit normal to Σ.

Let σ and λ be two real constants, $\eta = \eta(t)$ be the solution to the ordinary differential equation

$$\sigma + \tfrac{n}{n-1}\eta^2 - 2\eta\lambda + 2\eta' = 0, \eta' < 0.$$

On an over-torical band $M = \mathbb{T}^{n-1} \times [-1, 1]$, denote by s the coordinate of the line segment. Given any representative hypersurface Σ in the homology class $H_{n-1}(M; \mathbb{Z})$ we compute the null expansion with respect to the normal pointing the same direction as ∂_s.

We call (M, g, k) a constant mean curvature λ initial data set if $\operatorname{tr}_g k = \lambda$. We find the following with the help of the surface of prescribed null expansion of Definition 4.2.

Theorem 4.3. *Let $M = \mathbb{T}^{n-1} \times [-1, 1]$, assume on (M, g, k) that $\operatorname{tr}_g k = \lambda$,*

$$\mu - |J| \geqslant \tfrac{1}{2}\sigma$$

and the null expansions at the boundaries $\partial_\pm M$ satisfy

$$\theta(\partial_- M) \leqslant \eta(t_-), \theta(\partial_+ M) \geqslant \eta(t_+).$$

The numbers $t_\pm$ are given so that η is a well defined smooth function on $[t_-, t_+]$. Then

$$\operatorname{dist}(\partial_- M, \partial_+ M) \leqslant t_+ - t_-.$$

We have found an application of the width estimate to the positive mass theorem of an asymptotically hyperbolic manifold with arbitrary ends.

Definition 4.4. *A subset $\mathcal{E}$ in a Riemannian manifold is said to be an asymptotically hyperbolic end if $\mathcal{E}$ is diffeomorphic via Ψ to the hyperbolic space minus a compact set and the metric on $\mathcal{E}$ satisfies the decay*

$$|g - \bar{g}|_{\bar{g}} + |\bar{\nabla} g|_{\bar{g}} + |\bar{\nabla}^2 g|_{\bar{g}} = o(\mathrm{e}^{-nr/2})$$

where $\bar{g}$ is the hyperbolic metric and r is the distance function to a fixed point in the hyperbolic space.

Let $V_0 = \cosh r$, $V_i = \theta^i \sinh r$ be the basis of the space $\mathcal{N}$ of the static potentials where θ^i is the spherical coordinates in the hyperbolic model $\mathrm{d}r^2 + \sinh^2 r \mathrm{d}\theta^2$, $e = g - \bar{g}$, and $\mathbb{U}(V, e)$ the 1-form associated with a chosen static potential $V \in \mathcal{N}$ defined via

$$\mathbb{U}(V, e) = V(\operatorname{div}_{\bar{g}} e - \mathrm{d}(\operatorname{tr}_{\bar{g}} e)) - e(\bar{\nabla} V,) + \operatorname{tr}_{\bar{g}} e \mathrm{d}V,$$

the mass functional (see [21]) of a given end $\mathcal{E}$ is

$$m_{\mathcal{E}}(V) = \lim_{r \to \infty} \int_{S_r} \langle \mathbb{U}(V, e), \nu \rangle, \tag{4.1}$$

where S_r is the hyperbolic geodesic sphere of radius r and ν is the g-unit normal to S_r. Our theorem below generalizes the positive mass theorem for an asymptotically flat manifold with arbitrary ends by [29] and [8].

Theorem 4.5 ([18])**.** *Let (M, g) be a complete connected n-dimensional Riemannian spin manifold without boundary such that $R_g + n(n-1) \geqslant 0$ and let $\mathcal{E} \subset M$ be an asymptotically hyperbolic end. Then the mass functional $m_{\mathcal{E}}$ satisfies*

$$-m_{\mathcal{E}}(V_0)^2 + \sum_{i=1}^n m_{\mathcal{E}}(V_i)^2 \leqslant 0$$

with equality occurring if and only if (M,g) is isometric to the hyperbolic space.

A natural follow-up question would be to obtain the corresponding theorem for the asymptotically hyperboloidal initial data sets (M,g,k) with k asymptotic to $-g$. It is also interesting to remove the spin condition in Theorem 4.5.

I made an observation [11] about the evaluation of the mass of asymptotically hyperbolic manifolds on polyhedra using the half space model of hyperbolic space. Recall the upper half-space model

$$\bar{g} = \tfrac{1}{(x^n)^2}((\mathrm{d}x^1)^2 + \cdots + (\mathrm{d}x^n)^2), x^n > 0 \tag{4.2}$$

of hyperbolic n-space $\mathbb{H}^n$. Such polyhedra were used as models in [19] to find the hyperbolic Gromov dihedral rigidity. I [10] defined the asymptotically hyperbolic manifolds with a horospherical boundary which is asymptotic to $\{x \in \mathbb{H}^n : 0 < x^n < 1\}$ and related mass like quantity.

It also turns out that we can modify some of the $x^1, \ldots, x^{n-1}$ in (4.2) to be the parameter of standard circles in the model (4.2). Now the distance r would only be calculated in terms of the non-circular coordinates. This will give a new space overlooked by previous researchers. And the mass is defined in the same way as in (4.1) with the static potential V being $\frac{1}{x^n}$. The asymptotically locally hyperbolic with the torus as the conformal infinity is a special case of this new space. And hopefully, this new space could give an analog of an asymptotically flat manifold with flat fibers of [20] and would generate interesting results. For example, the positive mass theorem with arbitrary ends as in [20].

5. A tilted spacetime positive mass theorem

In [5], Alamraz-de Lima-Mari introduced the asymptotically flat initial data sets with a noncompact boundary.

Definition 5.1 ([5])**.** *We say that (M,g,k) is an n-dimensional asymptotically flat initial data set of order $\tau > \frac{n-2}{2}$ if M outside a compact set is diffeomorphic to Euclidean half space minus the unit ball and using a specific coordinate, g and k satisfies the decay rate*

$$|g-\delta| + |x||\partial g| + |x|^2|\partial\partial g| + |x||k| + |x|^2|\partial k| = O(|x|^{-\tau}).$$

The quantities defined as

$$E = \lim_{r\to\infty}\left[\int_{S_+^{n-1,r}} (g_{ij,j} - g_{jj,i})\nu^i - \int_{S^{n-2,r}} e_{\alpha n}\vartheta^\alpha\right],$$

and

$$P_i = 2\int_{S_+^{n-1,r}} (k - (\mathrm{tr}_g\, k)g)_{ij}\nu^j.$$

are respectively called the ADM energy and ADM linear momentum. Here, ν is unit normal to $S_+^{n-1,r}$, ϑ is normal to $S^{n-2,r}$ in ∂M. Denote $\hat{P} = (P_1, \ldots, P_{n-1})$, $S_+^{n-1,r}$ is the upper half of the coordinate sphere of radius r and $S^{n-2,r} = \partial S_+^{n-1,r}$.

Definition 5.2 ([13])**.** *We say that (M,g,k) satisfies the interior dominant energy condition if*

$$\mu \geqslant |J|. \tag{5.1}$$

If $\partial M \neq \emptyset$, let η be the outward normal of ∂M in M, $H_{\partial M} = \mathrm{div}_{\partial M}\,\eta$. We say that (M, g, k) satisfies the tilted boundary dominant energy condition if

$$H_{\partial M} \pm \cos\theta\,\mathrm{tr}_{\partial M}\,k \geqslant \sin\theta|k(\eta,\cdot)^{\top}| \text{ on } \partial M, \tag{5.2}$$

where $\theta \in [0, \frac{\pi}{2}]$ is a constant angle and $k(\eta,\cdot)^{\top}$ denotes the component of the 1-form $k(\eta,\cdot)$ tangential to ∂M.

I found the following via Witten's spinorial proof [38] and [32] of the spacetime positive mass theorem and a boundary chirality operator [4] for spinors.

Theorem 5.3 ([13])**.** *If (M, g) is spin and (M, g, k) satisfies the interior dominant energy condition* (5.1) *and the tilted boundary dominant energy condition* (5.2) *for some nonzero $\theta \in [0, \frac{\pi}{2}]$, then*

$$E \pm \cos\theta P_n \geqslant \sin\theta|\hat{P}|.$$

The time-symmetric case $k = 0$ of the theorem first appeared in [2] where a minimal surface proof was also given. The rough idea is: assume that the energy (mass) E is negative, then the boundary ∂M and a plane asymptotically parallel to ∂M serve as the barriers and we can find an area-minimizing minimal surface without boundary in between. Then the Gauss-Bonnet theorem applied on the stable minimal plane contradicts the nonnegativity of scalar curvature and mean curvature. An alternative proof was given by the author [9]. Instead, the free boundary minimal surface was used.

Observing the two works, one should be able to conclude that the two proofs using the minimal surface are actually proofs of two special cases when p vanishes: (I) $\theta = \pi/2$ in [9]; (II) or $\theta = 0$ in [2]. This suggests that there is a proof via stable minimal surface with capillary boundary conditions for the case $p \equiv 0$ as well. Eventually, one hopes to establish directly Theorem 5.3 implementing [23] using the capillary marginally outer trapped surface [1]. My results on capillary MOTS mentioned previously, in particular, free boundary MOTS are quite important in this program.

6. Willmore inequality in hyperbolic geodesic balls

I proved the following Willmore type inequality in hyperbolic space.

Theorem 6.1 ([14])**.** *Let*

$$\lambda = \omega_{n-1}\int_0^{\rho_0} \sinh^{n-1} s\mathrm{d}s$$

be the volume of n-dimensional hyperbolic geodesic ball B_0 of radius ρ_0 and $\Lambda = 2\coth\rho_0\lambda^{\frac{2-n}{n}}$. Any weakly convex free boundary hypersurface M in B_0 satisfies the Willmore type inequality

$$|M|^{\frac{2-n}{n}}\int_M (H^2 - n^2) + \Lambda|\partial M| \geqslant -n^2\lambda^{\frac{2}{n}} + \Lambda\omega_{n-1}\sinh^{n-1}\rho_0,$$

where ω_{n-1} is the volume of standard $(n-1)$-sphere. Equality occurs if and only if M is totally geodesic.

The result is proved via starting an inverse mean curvature flow with a free boundary from M. Along the way, I proved the convergence result of hypersurface M evolving under the inverse mean curvature flow with a free boundary into totally geodesic discs. Theorem 6.1 is the first result using free boundary inverse mean curvature flow to establish the geometric inequalities in the hyperbolic space. An important potential research direction is to pursue the Alexandrov-Fenchel inequalities as those in [33].

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