CSC236 Fall 2018 PS2 Sample Solutions

Due: Oct 12, 23:59

General Instructions

Please read the following instructions carefully before starting the exercise. They contain important information about general exercise expectations, exercise submission instructions, and reminders of course policies.

- Your problem set is graded on both correctness and clarity of communication. Solutions which are technically correct but poorly written will not receive full marks. Please read over your solutions carefully before submitting them.
- Each problem set may be completed in groups of up to three. If you are working in a group for this problem set, please consult `https://github.com/MarkUsProject/Markus/wiki/Student_Groups` for a brief explanation of how to create a group on MarkUs.
- Solutions must be typeset electronically, using LaTeX and submitted as a PDF with the correct filename. **Other forms of submission will receive a grade of ZERO**.
  The required files for this problem set are **ps2.pdf** and **ps2.tex**.
- Problem sets must be submitted online through MarkUs. If you havent used MarkUs before, give yourself plenty of time to figure it out, and ask for help if you need it!
- Submissions must be made *before* the due date and time on MarkUs. Late submissions are not accepted.

Problem 1.

(Warmup - this problem will NOT be marked).

Show that $\log n! \in \mathcal{O}(n \log n)$.
(Here $m!$ stands for m factorial, the product of first m non-negative integers. By convention, $0! = 1$.)

Let $P(n) : \log_2(n!) \leq n \log_2 n$. We will prove using induction that $\forall n \in \mathbb{N}^+, P(n)$ where we denote $\mathbb{N}^+$ the set of natural numbers ≥ 1.
Prove $P(1)$.

Let $n_0 = 1$.
Then $\log_2(n_0!) = 0$.
Then $\log_2(n_0!) \leq n_0 \log_2(n_0)$..
Then $P(1)$..

Prove $\forall n \in \mathbb{N}^+, P(n) \implies P(n+1)$.

Let $n \in \mathbb{N}^+$.
Assume $P(n)$.
Then $\log_2(n!) \leq n \log_2(n)$.
Then
$$\begin{aligned}\log_2((n+1)!) &= \log_2((n+1) \cdot n!)\\ &= \log_2(n+1) + \log_2(n!)\\ &\leq \log_2(n+1) + n \log_2 n \text{ \# inductive hypothesis}\\ &\leq \log_2(n+1) + n \log_2(n+1) \text{ \# log is non decreasing}\\ &= (n+1)\log_2(n+1)\end{aligned}$$
Then $\log_2((n+1)!) \leq (n+1)\log_2(n+1)$.
Then $P(n+1)$.
Then $P(n) \implies P(n+1)$.
Then $\forall n \in \mathbb{N}^+, P(n) \implies P(n+1)$.
Then $P(1) \wedge \forall n \in \mathbb{N}^+, P(n) \implies P(n+1)$.
Therefore, $\forall n \in \mathbb{N}^+, P(n)$
Then $\forall n \in \mathbb{N}, n \geq 1 \implies \log_2(n!) \leq n \log_2 n$.
To complete the proof, we just let $c_0 = 1, n_0 = 1$ and check that the definition of big-Oh is satisfied (details omitted here).
Then $\log_2(n!) \in \mathcal{O}(n \log_2 n)$.

Problem 2.

(4 Marks) Suppose you are coding an algorithm for finding the maximum sum of two elements in a list of positive integers. Suppose you have access to a helper function sort(L) that takes in a list of positive integers and returns a list of the same elements but sorted in non-decreasing order. Moreover, suppose `sort(L)` runs in time $\Theta(n \log n)$ (e.g., `mergesort`). Write a Python program `fastMaxSum` calling `sort(L)` as a helper function that runs in time $\Theta(n \log n)$. Justify why it has this running time.

```
1 def fastMaxSum(L):
2     tempList = sort(L)
3     return (tempList[len(L) - 2] + tempList[len(L) - 1])
```

Justification: Line 2 runs in $\Theta(n \log n)$, line 3 can be counted as one elementary operation.

Problem 3.

(6 Marks) **Practice Θ.**

$$\forall k \in \mathbb{N}, 1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1}).$$

Solution 1:

First fix an arbitrary k. Now, we shall show that $1^k + 2^k + \cdots + n^k \in O(n^{k+1})$. Let $c = 1, n_0 = 0$, then

$$\begin{aligned} 1^k + 2^k + \cdots + n^k &\leq n^k + n^k + \cdots + n^k \\ &= n \cdot n^k \\ &= n^{k+1}, \end{aligned}$$

and so $1^k+2^k+\cdots+n^k \in O(n^{k+1})$. Now we shall show that $1^k+2^k+\cdots+n^k \in \Omega(n^{k+1})$.

Claim: $1^k + 2^k + \cdots + n^k \geq \frac{1}{2^{k+1}} n^{k+1}$ for all $n \in \mathbb{N}$.

Proof: We shall handle this proof in odd and even cases. Suppose that n is even. Then we have that

$$\begin{aligned} 1^k + 2^k + \cdots + n^k &= 1^k + 2^k + \cdots + \left(\frac{n}{2}\right)^k + \cdots + \left(\frac{n}{2} + \frac{n}{2}\right)^k \\ &\geq \left(\frac{n}{2}\right)^k + \cdots + \left(\frac{n}{2} + \frac{n}{2}\right)^k \\ &\geq \left(\frac{n}{2} + 1\right)\left(\frac{n}{2}\right)^k \\ &\geq \frac{n^{k+1}}{2^{k+1}}. \end{aligned}$$

Now suppose that n is odd. Then

$$\begin{aligned} 1^k + 2^k + \cdots + n^k &= 1^k + 2^k + \cdots + \left(\frac{n+1}{2}\right)^k + \cdots + \left(\frac{n+1}{2} + \frac{n-1}{2}\right)^k \\ &\geq \left(\frac{n+1}{2}\right)^k + \left(\frac{n+1}{2} + 1\right)^k + \cdots + \left(\frac{n+1}{2} + \frac{n-1}{2}\right)^k \\ &\geq \left(\frac{n-1}{2} + 1\right)\left(\frac{n+1}{2}\right)^k \\ &= \left(\frac{n+1}{2}\right)^{k+1} \\ &\geq \frac{n^{k+1}}{2^{k+1}}. \end{aligned}$$

Hence the claim holds true. Then for $c = \frac{1}{2^{k+1}}$ and $n_0 = 0$, we have that $1^k + 2^k + \cdots + n^k \in \Omega(n^{k+1})$ and thus $1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1})$, which completes the proof.

Solution 2:

Although the statement does not specify the domain of n, the notation does suggest $n \in \mathbb{N}$. Further, $n \geq 1$. Therefore we may write the fully quantified statement as

$$\forall n \in \mathbb{N}^+, \forall k \in \mathbb{N}, 1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1}).$$

Thus, we let n be a generic element of $\mathbb{N} \setminus \{0\}$, obtaining the predicate:

$$P(k) : 1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1}).$$

which we will prove by induction.

The proof invloves estimating the sum $1^k + 2^k + \cdots + n^k$ from above and from below. The difficult part is the estimation from below. To this end, we will prove first a helper statement:

Lemma. $\forall n \in \mathbb{N}^+, \forall k \in \mathbb{N} \setminus \{0\}, \frac{n}{2}(1^k + 2^k + \cdots + n^k) \leq 1^{k+1} + 2^{k+1} + \cdots + n^{k+1}$.

Proof (by cases). Let $F = (1^{k+1} + 2^{k+1} + \cdots + n^{k+1}) - \frac{n}{2}(1^k + 2^k + \cdots + n^k)$.

Case 1: $n = 1$**.** For any k, $F = 1^{k+1} - \frac{1}{2}1^k = \frac{1}{2} \geq 0$.
Case 2: $n > 1$**.**

Sub-case 2.0: $k = 0$**.**
Compute

$$\begin{aligned} F &= 1^{0+1} + 2^{0+1} + \cdots + n^{0+1} - \frac{n}{2}(1^0 + 2^0 + \cdots + n^0) \\ &= \frac{n(n+1)}{2} - \frac{n^2}{2} \geq 0. \end{aligned}$$

Sub-case 2.1: $k = 1$**.**
Compute

$$\begin{aligned} F &= 1^2 + 2^2 + \cdots + n^2 - \frac{n}{2}(1 + 2 \cdots + n) \\ &= \frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)}{4} \end{aligned}$$

$$= \frac{n(n+1)(n+2)}{12} \geq 0.$$

Sub-case 2.2: $k = 2$.

$$F = 1^3 + 2^3 + \cdots + n^3 - \frac{n}{2}(1^2 + 2^2 \cdots + n^2)$$

$$= \frac{n^2(n+1)^2}{4} - \frac{n^2(n+1)(2n+1)}{12}$$

$$= \frac{n^2(n+1)^2(n+2)}{12} \geq 0.$$

Sub-case 2.3: $k > 2$ and n even.
Then exists $m \in \mathbb{Z}$, such that $n = 2m$ and $m > 0$ since $n > 1$.

Compute

$$F = \sum_{j=1}^{m}(j^{k+1} - mj^k) + \sum_{j=m+1}^{2m}(j^{k+1} - mj^k)$$

In the second sum, we do an index change as follows: let $j' = j - m$. Then the second sum looks like:

$$\sum_{j'=1}^{m}((j'+m)^{k+1} - m(j'+m)^k)$$

Rename the index j' to j and substitute in the original sum:

$$F = \sum_{j=1}^{m}(j^{k+1} - mj^k) + \sum_{j=1}^{m}((j+m)^{k+1} - m(j+m)^k)$$

$$= \sum_{j=1}^{m}(j^k(j-m) + j(j+m)^k)$$

$$= \sum_{j=1}^{m} j((j+m)^k - j^{k-1}(m-j))$$

$$= \sum_{j=1}^{m} j((j+m)^k - j^{k-2}(\frac{m^2}{4} - (\frac{m}{2} - j)^2)), \text{ but, since } (\frac{m}{2} - j)^2 \geq 0,$$

$$\geq \sum_{j=1}^{m} j((j+m)^k - j^{k-2}\frac{m^2}{4}))$$

$$\geq \sum_{j=1}^{m} j(m^k - \frac{m^k}{4})$$

$$\geq 0.$$

Sub-case 2.4: $k > 2$ and n odd.
Then exists $m \in \mathbb{Z}$, such that $n = 2m+1$ and $m > 0$ since $n > 1$.

Compute

$$F = \sum_{j=1}^{m}(j^{k+1} - mj^k) + \sum_{j=m+1}^{2m}(j^{k+1} - mj^k) + (2m+1)^{k+1} - \frac{2m+1}{2}(2m+1)^k.$$

The first and second sum are no diferent from the previous case, so their combined sum is nonegative. So is the last term, so here we may also conclude $F \geq 0$.

Combinging all cases altogether, we conclude the lemma is proved.

Proof of the main statement, by induction on k.
Base case $P(0)$: $1^0 + 2^0 + \cdots + n^0 \in \Theta(n^1)$

A direct computation shows $1^0 + 2^0 + \cdots + n^0 = n \in \Theta(n)$ because he definition of Θ implies immediately every function is in Θ of itself.

Inductive step:

$$\forall k \in \mathbb{N}, 1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1}) \implies 1^{k+1} + 2^{k+1} + \cdots + n^{k+1} \in \Theta(n^{k+2}).$$

Let $k \in \mathbb{N}$.
Assume $1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1})$.
The exists $C_1', C_2', n_0' \in \mathbb{R}^+$ such that

$$\forall n \in \mathbb{N}, n \geq n_0 \implies C_1' n^{k+1} \leq 1^k + 2^k + \cdots + n^k \leq C_2' n^{k+1}.$$

The lemma proved above, implies

$$1^{k+1} + 2^{k+1} + \cdots + n^{k+1} \geq \frac{n}{2}(1^k + 2^k + \ldots n^k)$$

Let $C_1 = C_1'/2, C_2 = C_2', n_0 = n_0'$.
Then $C_1, C_2, n_0 \in \mathbb{R}^+$.
Let $n \geq n_0$.
Then $n \geq n_0'$.
Compute:

$$1^{k+1} + 2^{k+1} + \cdots + n^{k+1} \geq \frac{n}{2}(1^k + 2^k + \cdots + n^k) \text{ (by Lemma)}$$
$$\geq \frac{n}{2} C_1' n^{k+1} \text{ (by inductive hypothesis)}$$
$$= C_1 n^{k+2}.$$

Also

$$1^{k+1} + 2^{k+1} + \cdots + n^{k+1}$$
$$\leq n1^k + n2^k + \cdots + nn^k$$
$$= n(1^k + 2^k + \cdots + n^k)$$
$$\leq nC_2 n^{k+1}$$
$$= C_2 n^{k+2}.$$

Combining everyhting altogether, we conclude the given statement is true.

Problem 4.

(10 MARKS) **Recursive functions**.

Consider the following recursively defined function:

$$T(n) = \begin{cases} c_0 & n = 0 \\ c_1 & n = 1 \\ aT(n-1) + bT(n-2) & n \geq 2 \end{cases}$$

where a, b are real numbers.

Denote (*) the following relation:

$$T(n) = aT(n-1) + bT(n-2) \quad n \geq 2 \tag{$*$}$$

We say a function $f(n)$ satisfies (*) iff $f(n) = af(n-1) + bf(n-2)$ is a true statement for $n \geq 2$.

Prove the following:

(i) For all functions $f, g : \mathbb{N} \to \mathbb{R}$, for any two real numbers α, β, if $f(n)$ and $g(n)$ satisfy (*) for $n \geq 2$ then also $h(n) = \alpha f(n) + \beta g(n)$ satisfies it for $n \geq 2$.

(ii) Let $q \neq 0$ be a real number. Show that if $f(n) = q^n$ satisfies (*) for $n \geq 2$ then q is a root of quadratic equation $x^2 - ax - b = 0$.

(iii) State and prove the converse of (ii). Use this statement and part (i) to show that if q_1, q_2 are the roots of $x^2 - ax - b = 0$ then $h(n) = Aq_1^n + Bq_2^n$ satisfies (*) for any two numbers A, B.

(iv) Consider $h(n)$ from part (iii). What additional condition should we impose on the roots q_1, q_2 so $h(n)$ serves as a closed-form solution for $T(n)$ with A, B uniquely determined?

(v) Use the previous parts of this exercise to solve the following recurrence in closed form:

$$T(n) = \begin{cases} 5 & n = 0 \\ 17 & n = 1 \\ 5T(n-1) - 6T(n-2) & n \geq 2 \end{cases}$$

Solution

(i)

$$h(n) = \alpha f(n) + \beta g(n)$$
$$= \alpha(af(n-1) + bf(n-2)) + \beta(ag(n-1) + bg(n-2)) \text{ (since both } f, g \text{ satisfy *)}$$
$$= a(\alpha f(n-1) + \beta g(n-1)) + b(\alpha f(n-2) + \beta g(n-2)) \text{ (rearranging)}$$
$$= ah(n-1) + bh(n-2).$$

(ii) Substituting and canceling q^{n-2} from both sides we get $q^2 - aq - b = 0$, therefore q is a root of $x^2 - ax - b$.

(iii) The converse: If q is a root of $x^2 - ax - b = 0$ then $f(n) = q^n$ satisfies (*). The proof can be obtained by mutiplying the relation $q^2 - aq + b = 0$ by q^{n-2} and rearranging. The last part of the question follows from (i) setting $f(n) = Aq_1^n$ and $g(n) = Bq_2^n$ where q_1, q_2 are the roots of $x^2 - ax - b = 0$.

(iv) What remains to be checked, is the satisfaction of bases cases, namely:

$$c_0 = A + B$$

$$c_1 = Aq_1 + Bq_2$$

For A, B be uniquely determined, we need the system have a unique solution, therefore its determinat, equal to $q_2 - q_1$ must be nonzero, leading to $q_1 \neq q_2$.

(v) For the equation $x^2 - 5x + 6 = 0$; this equation has two solutions $q_1 = 2, q_2 = 3$; also $2 = q_1 \neq q_2 = 3$. Form the system:

$$A + B = 5$$

$$2A + 3B = 17$$

Solving it we find $A = -2, B = 7$, so finally the solution to our recurrence is $T(n) = 7 \cdot 3^n - 2 \cdot 2^n$.