CSCI 383 Fall 2011	**Proof by Structural Induction**

Structural induction is used to prove a property P of all the elements of some recursively-defined data type. The following are some examples of recursively defined data types:

Example 1: An *even number* e is either

1. 0,
2. $2 + k$, where k is an even number, or
3. $-k$, where k is an even number.

Example 2: A *non-empty binary tree* T is either

1. a root node r, or
2. a root node r attached by an edge to either one or two of the nodes r_L and r_R, where r_L and r_R are the roots of non-empty binary trees T_L and T_R, respectively.

Example 3: A *fully bracketed arithmetic expression in* x is a string E over the alphabet $\{], [, +, -, *, x\}$ that is either

1. the symbols $0, 1$, or x,
2. $[e + e']$, where e and e' are fully bracketed arithmetic expressions, or
3. $[e * e']$, where e and e' are fully bracketed arithmetic expressions.

A proof by structural induction consists of two main steps:

1. Prove P for the "base cases" of the definition.
2. Prove P for the result of any recursive combination rule, assuming that it is true for all the parts.

For example, structural induction on the non-empty binary tree of example 2 takes the form:

Proof. To prove $P(T)$ holds $\forall$ non-empty binary trees T, show:

Base [T = single root node.] Show P(single node) holds.

Inductive Step [T = root node attached to roots of one or two binary trees T_L and T_R.] Assume $P(T_L)$ and $P(T_R)$ in order to prove $P(T)$.

□

Here's an actual example:

Theorem 1. *For any non-empty binary tree $T = (V, E)$, $|V| = |E| + 1$.*

Proof. We prove this by structural induction on $T = (V, E)$. For any binary tree $T = (V, E)$, let $P(T)$ be the property that $|V| = |E| + 1$.

As a base case, consider when T is a single root node. Then $|V| = 1 = 0 + 1 = |E| + 1$.

For the inductive step, suppose $T = (V, E)$ consists of a root node r that is connected by an edge to the root r_L of a non-empty tree $T_L = (V_L, E_L)$. It may also be connected by an edge to the root r_R of a non-empty tree $T_R = (V_R, E_R)$. As our induction hypothesis, suppose that $P(T_L)$ and $P(T_R)$ hold; that is, suppose that $|V_L| = |E_L| + 1$ and $|V_R| = |E_R| + 1$. Then if T contains only T_L

$$\begin{aligned} |V| &= |V_L| + 1 && V \text{ is } r \text{ plus all nodes of } T_L \text{ (recursive definition of } T\text{)} \\ &= |E_L| + 1 + 1 && \text{by the induction hypothesis applied to } T_L \\ &= |E| + 1\ . && E \text{ is all edges of } T_L \text{ plus the edge } (r, r_L) \text{ (recursive definition of } T\text{)} \end{aligned}$$

If T contains both T_L and T_R then

$$\begin{aligned} |V| &= |V_L| + |V_R| + 1 && V \text{ is } r \text{ plus all nodes of } T_L \text{ and } T_R \text{ (recursive definition of } T\text{)} \\ &= |E_L| + 1 + |E_R| + 1 + 1 && \text{by the induction hypothesis applied to both } T_L \text{ and } T_R \\ &= |E| + 1\ . && E \text{ is all edges of } T_L \text{ and } T_R \text{ plus two (recursive definition of } T\text{)} \end{aligned}$$

In either case, $P(T)$ holds; thus by induction, $P(T)$ is true for every non-empty binary tree T. □

Theorem 2. *The set of even numbers defined in example 1 are elements of* $E = \{2x \mid x \in \mathbb{Z}\}$.

Proof. We prove this by structural induction on even number e. Let $P(e)$ be the property that $e = 2x$ for some $x \in \mathbb{Z}$.

As a base case, consider when $e = 0$. Then $e = 0 = 2 \cdot 0$, and $0 \in \mathbb{Z}$.

For the inductive step, we have two cases, depending on whether $e = e' + 2$ or $e = -e'$, for some even number e'. As our induction hypothesis, suppose that $P(e')$ holds; that is, suppose that $e' = 2x'$ for some $x' \in \mathbb{Z}$. Then if $e = e' + 2$, we have

$$\begin{aligned} e &= e' + 2 \\ &= 2x' + 2, && \text{for some } x' \in \mathbb{Z}, \text{ by the induction hypothesis} \\ &= 2(x' + 1) \\ &= 2x && \text{where } x = x' + 1 \in \mathbb{Z} \text{ since } x' \in \mathbb{Z}\ . \end{aligned}$$

Otherwise, if $e = -e'$ then

$$\begin{aligned} e &= -e' \\ &= -(2x') && \text{for some } x' \in \mathbb{Z}, \text{ by the induction hypothesis} \\ &= 2(-x') \\ &= 2x && \text{where } x = -x' \in \mathbb{Z} \text{ since } x' \in \mathbb{Z}\ . \end{aligned}$$

Therefore, we've shown by induction that for every even number e, the property $P(e)$ holds. □

Comments

- Although many proofs by structural induction have an equivalent proof by mathematical induction (for example, we could prove Theorem 1 by induction on the height of the tree T), there *are* structural induction proofs (namely, those on infinite data objects) that are actually strictly more powerful than ordinary induction.
- Structural induction will make your life easier in this course, so try to pick it up early on!