Problem Set #2 CSC236 Fall 2018

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We declare that this assignment is solely our own work, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters.

This submission has been prepared using LaTeX.

Problem 1.

(Warmup - this problem will NOT be marked).

Show that $\log n! \in \mathcal{O}(n \log n)$.
(Here $m!$ stands for m factorial, the product of first m non-negative integers. By convention, $0! = 1$.)

Problem 2.

(4 Marks) Suppose you are coding an algorithm for finding the maximum sum of two elements in a list of positive integers. Suppose you have access to a helper function sort(L) that takes in a list of positive integers and returns a list of the same elements but sorted in non-decreasing order. Moreover, suppose `sort(L)` runs in time $\Theta(n \log n)$ (e.g., `mergesort`). Write a Python program `fastMaxSum` calling `sort(L)` as a helper function that runs in time $\Theta(n \log n)$. Justify why it has this running time.

```
def fastMaxSum(L:list of int): -> int
    if len(L) < 2:
        return L
    else:
        NewLst = sort(L)
        GetResult = NewLst[-1] + NewLst[-2]
        return GetResult\\
```

Justify:

The run time of the function `fastMaxSum` is T(`fastMaxSum(L)`) = T(`sort(L)`)+T(`GetResult`)

$\because$ The run time of adding two elements of the list is a constant time,
T(`fastMaxSum(L)`) = T(`sort(L)`)+ a constant
So it will not affect the run time of the whole function.

$\therefore$ T(`fastMaxSum(L)`) = T(`sort(L)`)=$\theta(n \log n)$

Problem 3.

(6 MARKS) **Practice** Θ.

$$\forall k \in \mathbb{N}, 1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1}).$$

To show $\forall k \in \mathbb{N}, 1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1})$, we need to show
$\forall k \in \mathbb{N}, 1^k + 2^k + \cdots + n^k \in \bigcirc(n^{k+1})$, and
$\forall k \in \mathbb{N}, 1^k + 2^k + \cdots + n^k \in \Omega(n^{k+1})$

To show $\bigcirc(n^{k+1})$

$$\begin{aligned}
\text{let } S &= 1^k + 2^k + \cdots + (n-1)^k + n^k, \text{for } n_0 = 1 \\
S &\leq n^k + n^k + \cdots + n^k \\
S &\leq n * n^k \\
S &\leq n^{k+1}
\end{aligned}$$

Therefore $S \in \bigcirc(n^{k+1})$

To show $S \in \Omega(n^{k+1})$
Let $S = 1^k + 2^k + \cdots + (n-1)^k + n^k$, for $n_0 = 2$
we have two cases here, when n is odd and when n is even

case 1 : n is even, so $\frac{n}{2}$ is an Integer, so we have

$$S = 1^k + 2^k + \cdots + (n-1)^k + n^k, \text{for } n_0 = 2$$

$$\because 1^k + 2^k + \cdots + (\frac{n}{2} - 1)^k \geq 0$$

$$\begin{aligned}
S &\geq (\frac{n}{2})^k + (\frac{n}{2} + 1)^k \cdots + (n-1)^k + n^k \\
&\geq (\frac{n}{2})^k + (\frac{n}{2})^k \cdots + (\frac{n}{2})^k \\
&\geq n/2 * (n/2)^k \\
&\geq n^{k+1}/2^{k+1} \\
&\geq c * n^{k+1} \\
&\geq n^{k+1}
\end{aligned}$$

case 2, when n is odd

$$\begin{aligned}
S &= 1^k + 2^k + \cdots + (n-1)^k + n^k, \text{for } n_0 = 2 \\
&\geq (\frac{n+1}{2})^k + (\frac{n+1}{2} + 1)^k \cdots + n^k \\
&\geq (\frac{n+1}{2})^k + (\frac{n+1}{2})^k \cdots + (\frac{n+1}{2})^k \\
&\geq (\frac{n+1}{2}) * (\frac{n+1}{2})^k \\
&= (\frac{n+1}{2})^{k+1} \\
&\geq (\frac{n}{2})^{k+1} \\
&\geq n^{k+1}/2^{k+1} \\
&\geq c * n^{k+1} \\
&\geq n^{k+1}
\end{aligned}$$

So we have $S \in \Omega(n^{k+1})$
Therefore $s \in \Theta(n^{k+1})$

Problem 4.

(10 MARKS) **Recursive functions**.

Consider the following recursively defined function:

$$T(n) = \begin{cases} c_0 & n = 0 \\ c_1 & n = 1 \\ aT(n-1) + bT(n-2) & n \geq 2 \end{cases}$$

where a, b are real numbers.

Denote (*) the following relation:

$$T(n) = aT(n-1) + bT(n-2) \quad n \geq 2 \tag{$*$}$$

We say a function $f(n)$ satisfies (*) iff $f(n) = af(n-1) + bf(n-2)$ is a true statement for $n \geq 2$.

Prove the following:

(i) For all functions $f, g : \mathbb{N} \to \mathbb{R}$, for any two real numbers α, β, if $f(n)$ and $g(n)$ satisfy (*) for $n \geq 2$ then also $h(n) = \alpha f(n) + \beta g(n)$ satisfies it for $n \geq 2$.

(ii) Let $q \neq 0$ be a real number. Show that if $f(n) = q^n$ satisfies (*) for $n \geq 2$ then q is a root of quadratic equation $x^2 - ax - b = 0$.

(iii) State and prove the converse of (ii). Use this statement and part (i) to show that if q_1, q_2 are the roots of $x^2 - ax - b = 0$ then $h(n) = Aq_1^n + Bq_2^n$ satisfies (*) for any two numbers A, B.

(iv) Consider $h(n)$ from part (iii). What additional condition should we impose on the roots q_1, q_2 so $h(n)$ serves as a closed-form solution for $T(n)$ with A, B uniquely determined?

(v) Use the previous parts of this exercise to solve the following recurrence in closed form:

$$T(n) = \begin{cases} 5 & n = 0 \\ 17 & n = 1 \\ 5T(n-1) - 6T(n-2) & n \geq 2 \end{cases}$$

(i)
We know that $f(n)$ and $g(n)$ satisfy (*) for $n \geq 2$

$$\begin{cases} \text{f(n)=af(n-1)+bf(n-2)} \\ \text{f(n)=ag(n-1)+bg(n-2)} \end{cases}$$

So we can substitute f(n) and f(n) into h(n):

$$\begin{aligned} h(n) &= \alpha f(n) + \beta g(n) \\ &= \alpha[af(n-1) + bf(n-2)] + \beta[ag(n-1) + bg(n-2)] \\ &= \alpha a f(n-1) + \alpha b f(n-2) + \beta a g(n-1) + \beta b g(n-2) \\ &= a[\alpha f(n-1) + \beta f(n-2)] + b[\alpha g(n-1) + \beta g(n-2)] \\ &= a h(n-1) + b h(n-2) \end{aligned}$$

$\therefore h(n) = \alpha f(n) + \beta g(n)$ satisfies (*) for $n \geq 2$.

(ii)when n = 2

$$\begin{aligned} f(2) &= af(2-1) + bf(2-2) \\ &= af(1) + bf(0) \\ \text{since } f(n) &= q^n \\ f(2) &= q^2 = aq^1 + bq^0 \\ q^2 &= aq + b \\ q^2 - aq - b &= 0 \end{aligned}$$

When x = q, the quadratic formula holds true, therefore q is a root of $x^2 - ax - b = 0$

(iii)
part 1: To state and prove the converse of (ii): If q is a root of the equation $x^2 - ax - b = 0$, then $f(n) = q^n$ satisfies $(*)$ for $n \geq 2$ where $q \neq 0 \in \mathbb{R}$
note: since q is a root of the equation $x^2 - ax - b = 0$, we have

$$\begin{aligned} x^2 - ax - b &= 0 \\ q^2 - aq - b &= 0 \\ q^2 &= aq + b \\ q^2 * q^{n-2} &= aq * q^{n-2} + bq^{n-2} \\ q^n &= aq^{n-1} + bq^{n-2} \end{aligned}$$

Therefore $q^n = f(n)$ satisfies $(*)$ for $n \geq 2$ where $q \neq 0 \in \mathbb{R}$

part 2:
we have showed that q_1 and q_2 are roots of $x^2 - ax - b = 0$ and $q^n = f(n)$ satisfies $(*)$ for $n \geq 2$
So we have $q_1^n = f(n)$, $q_2^n = g(n)$ also satisfy$(*)$ for $n \geq 2$
By (i), we have $h(n) = Af(n) + Bg(n) = Aq_1^n + Bq_2^n$ which also satisfies $(*)$ for $n \geq 2$.

(iv) Since q_1, q_2 are the roots of $x^2 - ax - b = 0$ then

$$\begin{aligned} q_1 &= \frac{-(-a) + \sqrt{(-b)^2 - (4)(1)(-b)}}{(2)(1)} \\ &= \frac{a + \sqrt{b^2 + 4b}}{2} \\ q_2 &= \frac{-(-a) - \sqrt{(-b)^2 - (4)(1)(-b)}}{(2)(1)} \\ &= \frac{a - \sqrt{b^2 + 4b}}{2} \end{aligned}$$

Also, since $h(n) = Aq_1^n + Bq_2^n$ is a closed form of $T(n)$
as $n = 1$, $T(1) = Aq_1^1 + Bq_2^1 = Aq_1 + Bq_2 = C_1$
as $n = 0$, $T(0) = Aq_1^0 + Bq_2^0 = A + B = C_0$

(v)We know that $h(n) = Aq_1^n + Bq_2^n$ is a closed form of $T(n)$, in order to find the closed form of the recurrence we have to find the values of q_1, q_2, A and B.
from (i), we can get $h(n) = ah(n-1) + bh(n-2)$

$$\begin{aligned} \because T(n) &= h(n) \\ \because 5T(n-1) - 6T(n-2) & \quad n \geq 2 \\ \therefore a = 5, b &= -6 \\ \text{Then, we can get} & \\ q_1 = \frac{5 + \sqrt{25-24}}{2} &= 3 \\ q_2 = \frac{5 - \sqrt{25-24}}{2} &= 2 \\ \text{Since } T(1) = Aq_1 + Bq_2 = C_1 &= 17 \\ T(0) = A + B = C_0 &= 5 \\ \text{We can get:} \begin{cases} 3A + 2B = 17 \\ A + B = 5 \end{cases} &\Rightarrow \begin{cases} A = 7 \\ B = -2 \end{cases} \\ \therefore T(n) = h(n) = (7)3^n - (2)2^n &= (7)3^n - 2^{n+1} \end{aligned}$$