CSC236 Fall 2018 PS1 Sample Solution

Due: Sep 28, 23:59

General Instructions

Please read the following instructions carefully before starting the exercise. They contain important information about general exercise expectations, exercise submission instructions, and reminders of course policies.

- Your problem set is graded on both correctness and clarity of communication. Solutions which are technically correct but poorly written will not receive full marks. Please read over your solutions carefully before submitting them.

- Each problem set may be completed in groups of up to three. If you are working in a group for this problem set, please consult `https://github.com/MarkUsProject/Markus/wiki/Student_Groups` for a brief explanation of how to create a group on MarkUs.

- Solutions must be typeset electronically, using LaTeX and submitted as a PDF with the correct filename. **Other forms of submission will receive a grade of ZERO**.
  The required files for this problem set are **ps1.pdf** and **ps1.tex**.

- Problem sets must be submitted online through MarkUs. If you havent used MarkUs before, give yourself plenty of time to figure it out, and ask for help if you need it!

- Submissions must be made *before* the due date and time on MarkUs. Late submissions are not accepted.

Problem 1.

(WARMUP - THIS PROBLEM WILL NOT BE MARKED) Let $n \in \mathbb{N}$. Describe the largest set of values n for which you think $2^n < n!$. Use some form of induction to prove that your description is correct.
(Here $m!$ stands for m factorial, the product of first m non-negative integers. By convention, $0! = 1$.)

Solution

Let $P(n) : 2^n > n!$. A direct calculation shows $P(0), P(1), P(2), P(3)$ are false but $P(4)$ is true. Then we state

$$\forall n \in \mathbb{N}, n >= 4 \Rightarrow 2^n < n!$$

One may use simple induction to prove this statement.

Base Case: $n = 4$. $P(4) : 16 = 2^4 < 4! = 24$

Inductive Step: Assume for $n \geq 4, 2^n < n!$. Then:

$$2^{n+1} = 2 \cdot 2^n < (n + 1) \cdot n! = (n + 1)!.$$

(The inequality in the last step is true because $2^n < n!$ by inductive hypothesis and $(n + 1) \geq 4 > 2$).

Problem 2.

(4 Marks) Let $n \in \mathbb{N} \setminus \{0\}$. Using some form of induction, prove that for all such n, there exists and odd natural m and a natural k such that $n = 2^k m$.

Solution

We will use complete induction.

$$P(n) : \exists m, k \in \mathbb{N}, m \text{ is even } \wedge n = 2^k m.$$

Base Case: $n = 1$. Let $m = 1, k = 0$.

Inductive Step: Assume for all $1 \leq s < n, P(s)$.
Case 1: n is prime. If $n = 2$ let $m = 1, k = 1$. Else, n is odd so let $k = 0, m = n$.
Case 2: n is not prime. Let $1 < p, q < n$ be two proper factors of n: $n = p \cdot q$. Then $\exists k_1, k_2, m_1, m_2$ such that m_1, m_2 are odd and $p = 2^{k_1} m_1, q = 2^{k_2} m_2$. Calculate:

$$n = pq = 2^{k_1 + k_2} m_1 m_2$$

Let $m = m_1 m_2$. Then m is odd. Let $k = k_1 + k_2$. Then $n = 2^k m$ and the proof is complete.

Problem 3.

(6 Marks) Denote $\mathbb{Z}[x]$ the set of polynomials on one variable x with integer coefficients. For example, $p(x) = x^2 - 3x + 42$ is such a polynomial, whereas $q(x) = -1.5x^3 + 97x$ is not. Also recall polynomials on one variable with integer coefficients can be added and multiplied with each other using usual rules of high school algebra. (You are allowed to use only the rules of elementary algebra and what is taught is this course in your solution. Any other approaches with receive no credit).

Let's define the set $S \subseteq \mathbb{Z}[x]$ using the following rules:

1. $2 \in S$.
2. $x \in S$.
3. $\forall p(x) \in \mathbb{Z}[x], \forall q(x) \in S,\ p(x)q(x) \in S$.
4. $\forall p(x), q(x) \in S,\ p(x) + q(x) \in S$.

Also define the set $T = \{2p(x) + xq(x) | p(x), q(x) \in \mathbb{Z}[x]\}$.

Using some form of induction, prove $S = T$.

Solution

We are trying to prove the equality of two sets, which is equivalent to proving inclusions $S \subseteq T$ and $S \supseteq T$.

Lets set up a predicate on S:

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in S$.

Predicate: Q(p(x)): Free term of $p(x)$ is an even integer.

We will use structural induction to prove this lemma:

$$\forall p(x) \in S, Q(p(x))$$

Base Case: If $p(x) = 2$ then evidently is free term is 2, hence even. If $p(x) = x$, its free term is 0, hence even.

Inductive Hypothesis: Any existing polynomial is S has an even free term. **Case 1**: $S \ni s(x) = p(x)q(x)$ where $q(x) \in S$. The free term of $s(x)$ is equal to the product of free terms of $p(x)$ and $q(x)$. But the free term of $q(x)$ is even by the inductive hypothesis, and we know the product of an even number with any number is also even.

Case 2: $S \ni s(x) = p(x) + q(x)$ where both $p(x), q(x) \in S$. The free term of $s(x)$ will be the sum of free terms of $p(x)$ and $q(x)$ who by inductive hypothesis have even free terms, hence even.

Lemma is proved.

Using the what we proved, we prove the inclusion $S \subseteq T$. Let $s(x) \in S$. Then $s(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ as a polynomial. Let $p(x) = a_n x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$. Then $p(x) \in \mathbb{Z}[x]$. Since $x \in S$, then $xp(x) \in S$ by (3). The Lemma guarantees that a_0 is even, so $a_0 = 2b_0$ for some integer b_0. But then $b_0 \in \mathbb{Z}[x]$ as a constant polynomial. Since $2 \in S$, $2b_0 \in S$ as well. Finally, $s(x) = xp(x) + 2b_0 \in S$ by (4).

The fact $T \subseteq S$ can be proved directly. Take $f(x) = 2p(x) + xq(x) \in T$. Since $2, x \in S$ (from 1,2) and $p(x), q(x) \in \mathbb{Z}[x]$, 3 and 4 imply directly $f(x) \in S$.

Since both inclusions hold, we conclude $S = T$ as desired.

Problem 4.

(6 MARKS) Let P be a convex polygon with consecutive vertices $v_1, v_2, ..., v_n$. Use some form of induction to show that when P is triangulated into $n-2$ triangles, the $n-2$ triangles can be numbered $1, 2, ..., n-2$ so that v_i is a vertex of triangle i for $i = 1, 2, ..., n-2$.

Note: You may read about polygon triangulation here:
`https://en.wikipedia.org/wiki/Polygon_triangulation`

Solution

We will prove it by using complete induction.

$P(n)$: A triangulation into $n-2$ triangles of a convex polygon with consecutive vertices $\{v_1, \ldots, v_n\}$ can have its triangles labelled $1, \ldots, n-2$ such that vi is a vertex of triangle i.

Base case: Let $n = 3$. The vertices are numbered, $1, 2, 3$, and the one triangle can be numbered 1.

Inductive step: Let $n \in \mathbb{N}, n \geq 3$. Assume $H(n) : \forall j \in \mathbb{N}, 3 \leq j < n, P(j)$.

Let T be a triangulation of a polygon with consecutive vertices $v_1, \ldots, v_n$. Every T contains a diagonal from v_{n-1} or from v_v. (Otherwise, it is not a triangulation.)

Case 1: There is a diagonal from v_n in T. Choose k s.t. there is a diagonal from v_k to v_n in T. This diagonal divides the polygon into two sub-polygons, P_1 and P_2. P_1 has the vertices $v_1, v \ldots, v_k, v_n$ and P2 has the vertices $v_k, v_{k+1}, \ldots, v_n$. Rename the vertex v_n of P_1 as v_{k+1}. Rename each of the vertices of P2, v_i to v_{i-k+1}. By $H(n)$, P_1 triangulates with triangles numbered as claimed, because $k \leq k-2 < n$.. P_2 has $n-k+1$ vertices, so by $H(n)$, P_2 also triangulates as claimed. Add $k-1$ to each triangle number in P_2. The original vertices v_i are now part of the triangle numbered $i, i \in k+1, \ldots, n-2$.

Case 2: There is a diagonal from v_{n-1} in T. Choose k s.t. there is a diagonal from v_k to v_{n-1} in T. This diagonal divides the polygon into two sub-polygons, P_1 and P_2. P_1 has the vertices $v_1, \ldots, v_k, v_{n-1}, v_n$, and P_2 has the vertices $v_k, v_{k+1}, \ldots, v_{n-1}$. Rename the vertices v_{n-1}, v_n of P_1 as v_{k+1}, v_{k+2}. Rename each of the vertices of P_2, v_i to v_{i-k+1}. By $H(n)$, P_1 triangulates with triangles numbered as claimed, because $k \leq n-3 < n$. P_2 has $n-k$ vertices, so by $H(n)$, P_2 also triangulates as claimed. Add $k-1$ to each triangle number in P_2. The original vertices vi are now part of the triangle numbered $i, i \in k+1, \ldots, n-2$.

We conclude $P(n)$ holds for all $n \in \mathbb{N}$, $n \geq 3$.