# Modelling the oscillation of walking humans

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## Contents

# 1 Introduction

Walking is one of the things that we do most as humans. It is the most fundamental way of transportation, serving us faithfully for roughly 6 million years. As one of the few bipedal mammals on the Earth, our motion during walking is complex and rather effective for the energy we expend doing it. We have tried to analyze the way we walk, in order to create better tools that, well, enable us to walk better, like shoes or corrective soles. One fundamental side product of our peculiar way of motion is that the entire upper body undergoes vertical oscillation, or put simply, we move and sway up and down whenever we walk.

## 1.1 Aim

The aim of this investigation is, using mathematical functions, to model this vertical oscillation of the human body while walking.

## 1.2 Applications

There are numerous applications of modern technology where being able to model the oscillation from walking is invaluable. For example, optical image stabilization present in nearly all modern video cameras aims to minimize the 'screen shake' by compensating with movement in the lens of the camera. A model of oscillation while walking could aid this endeavor by allowing more predictable corrections to be made by software, which would benefit us in several areas such as film entertainment and physiological analysis. Additionally, motion discomfort has been a key barrier into the widespread adoption of virtual reality into our daily lives. Some people do not feel comfortable in an environment disconnected from their physical presence, and being able to model a person's sway while walking could aid the endeavor to smooth, or even recreate it in a virtual world, therefore alleviating this discomfort and opening the possibilities for a more broad adoption of VR technology.

## 1.3 Personal motivation

Personally, I am interested in the ways technology can help us in our daily lives. Being able to reliably apply a model to aid the aforementioned use cases would be a tremendous achievement that would help integrate technology into modern lives. I also have an interest in physiology & fitness, and so being able to apply mathematics to the area is an intriguing & exciting opportunity.

# 2 Data collection

## 2.1 Background

In order to model the vertical oscillation of human walking, it must first be measured accurately. I considered several possibilities for how I can achieve this, including making a kinematic model of a human leg, or simply filming a person walking and then tracing over the footage to obtain the path of the leg. However, both methods posed significant issues with them:

- The kinematic model may not be perfectly applicable to the shape, proportion & movement range of a real human leg. It would be very complex to make it even approximate the properties of a real human leg, which would not be worthwhile since I would only be trying to extract data from it once or twice.

- Tracing over a film of someone walking poses some accuracy & precision errors. For example, determining the location of the part of the leg that needs to be tracked would rely on a visual estimate, which is subject to the many issues of a video camera, such as parallax error (as an orthographic camera cannot physically exist), motion blur, resolution & lens distortion.

## 2.2 Solution

In order to address the issues with the initially proposed data collection methods, I opted for a more robust & accurate to real life measurement technique of actually motion tracking in 3D while walking. For the purposes of this investigation, I did not require full limb-for-limb tracking - only my head, and the bases of my feet.

For the task, I used an Oculus Rift virtual reality headset and motion controller pair in an improvised setup. Although the controllers are meant to be used for hand tracking, I fixed them securely to my ankles, as close as possible to my heel, and found that they still tracked the motion of walking accurately.

For the software, after some research on what popular dance groups such and other "fully virtual dancers" use, I found that they all used the same set of software, albeit with a more elaborate setup than my improvised one.

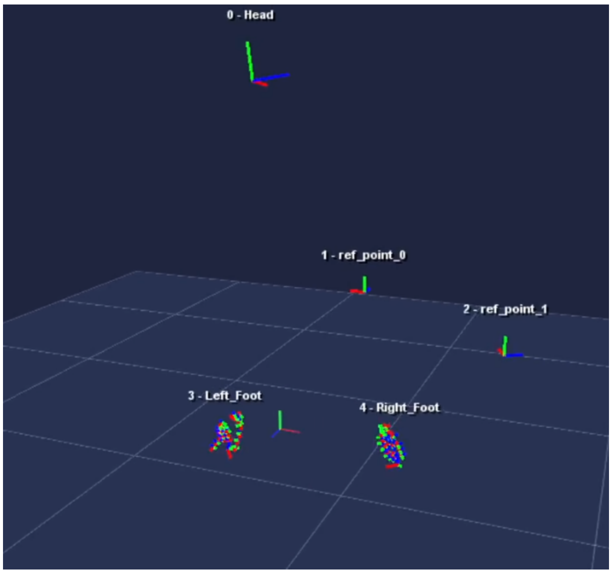



Figure 1: Motion tracking setup with person in the middle of a step

Figure 1 shows the computer side of the tracking setup. I used two IR tracking sensors (`ref_point_0` and `ref_point_1`) pointed at the area I was to walk through, and set up the software so that it tracked my head and feet, to which the tracking points were attached to.

After some testing with moving the tracking point along a ruler, I found that the position data returned was accurate to two millimeters, which is far beyond anything that would have been achieved with the two initially proposed data collection methods.

## 2.3 Results

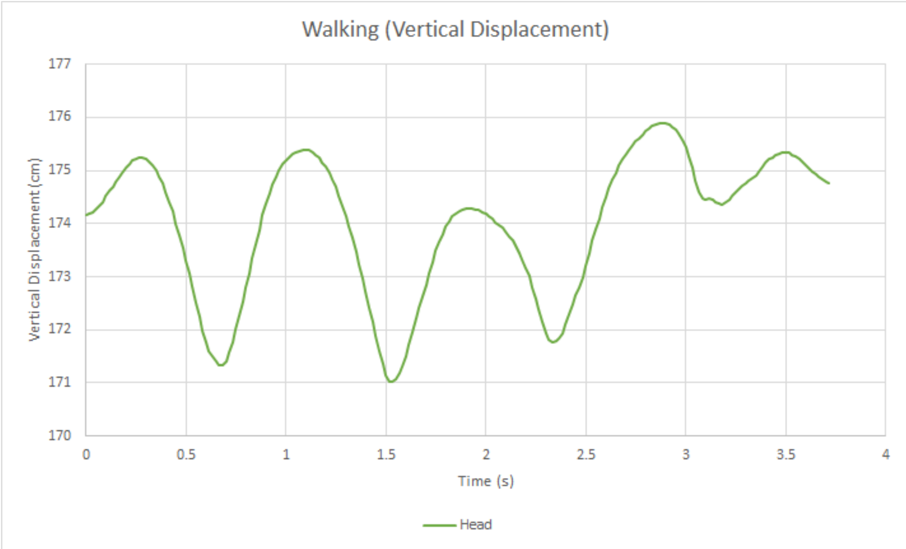



Figure 2: Vertical motion of head

Figure 2 depicts the vertical displacement of my head from the floor. It can be observed that it resembles some sort of wave with crests & troughs, which makes sense - a person walking would go up and down periodically with every step. From the graph, it is evident that the 4 steps were taken, indicated by the 4 crests. It is also noteworthy that the troughs are far sharper than the crests, which may be a product of the foot of the walker hitting the floor (the trough), and the leg moving through the air (the crest). Anyhow, this interesting pattern warrants investigation, as the head is the best indicator of the general vertical motion of the body in this case.

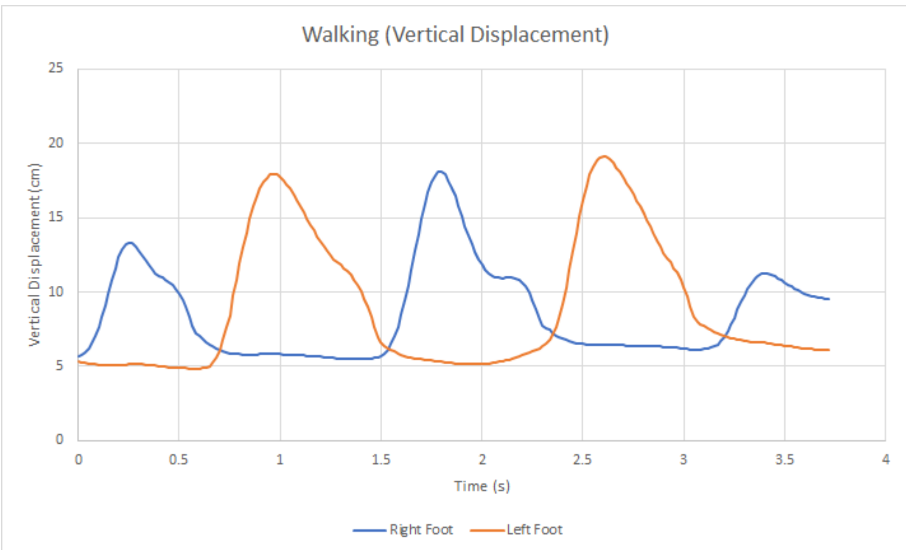



Figure 3: Vertical motion of feet over time

Figure 3 depicts the motion of my feet vertically. It is evident from this that each foot has two crests, for a total of four steps, which matches up nicely with the observations made from Figure 2. The shape of these crests and troughs is peculiar, but anyhow seem to follow an explicable pattern. It seems like the graphs of the right and left foot are completely out of phase, which makes sense given that only one foot can be on the ground at a time, during which the other one must be in the process of taking a step.

The shape of the troughs makes sense as well, as they are almost perfectly flat. The small amount of curvature that precedes the large peak is most likely caused by the heel being lifted in the air as the foot is getting ready for another step. This is further supported by the fact that at the beginning, the left foot's first trough is far sharper than the succeeding ones, because one leg of the person walking would be starting the step from a standstill.

It is also noteworthy to talk about the small secondary bump following the crest, visible in all crests, but most prominent in the second step of the right foot. This could be explained as the foot of the person walking maintaining an approximate level height, then suddenly returning to the ground on the "falling edge" of a step. In reality, the prominence of this secondary bump could be specific to the walking style of the person, however, it is significant enough to the point where it cannot be disregarded as simple error.

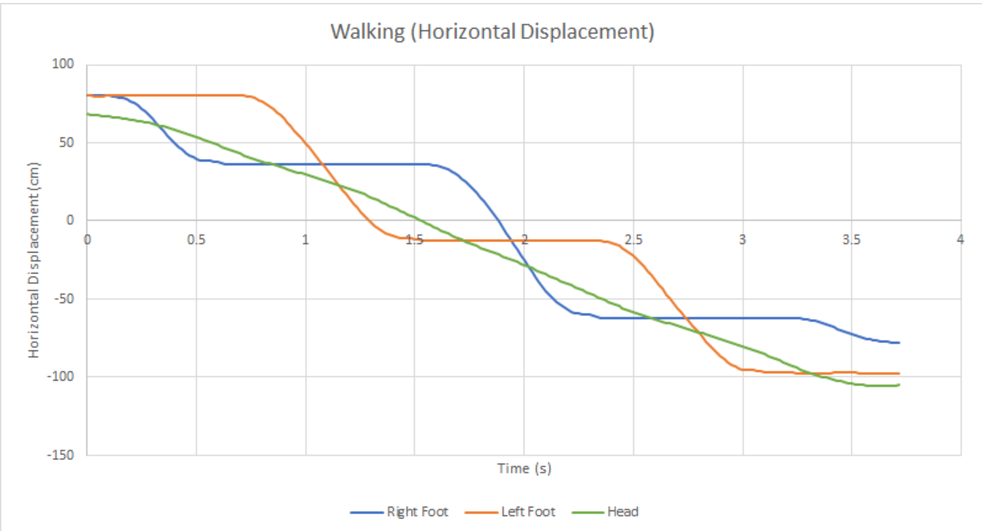



Figure 4: Horizontal motion of ankles and head

Figure 4 shows the horizontal motion of the three tracked points throughout the walk. Its data is not necessarily relevant to modelling the vertical oscillation, however it is able to provide some key insights into the motion of the person during the walk. It is evident that generally, for all three graphs, the motion happens in one direction, which is forward, or, on the graph, towards $-\infty$. The graphs of horizontal motion of the feet alternate in successive "dips", which resemble one step, where one foot is stationary while the other is free to move forward. The graph of the motion of the head is very close to being linear, and it appears to pass through the graphs of both feet. In the regions where the graph of a foot is above the graph of the head, it would be lagging behind in real life, while being below would mean it is ahead. Generally, it can be observed that the regions where it is found above the graph of the head (lagging behind) are greater than the regions where it is behind, which coincides with the data for the vertical motion of the feet, where the falling edge of each crest is generally longer than its rising edge.

# 3 Modelling options

In terms of modelling the curves, a polynomial may prove to provide the most precise model of the dataset, especially for the more peculiar graphs like those of the motion of the feet. There are three main advantage of using polynomials to model these motions. First, polynomials are inexpensive to compute and work with, since they only use multiplication and addition, something both humans and computers are quite adept at. This makes representing something as a polynomial very worthwhile. Based off of this, the second advantage - fitting polynomials to best approximate data is a well-explored topic, with several well-established working methods which can be used to provide shockingly accurate results. The last of these advantages is specific to the shape of our graph - it has smooth curves, which polynomials are excellent at representing.

There are several relatively easy methods to fit polynomials to a function, supposing that you can find its derivative. One possible method is constructing a Taylor polynomial, which would be accurate around the point which is constructed. Unfortunately, for our case, we are trying to fit through a dataset of points, for which finding the derivative (or more importantly, higher-order derivatives) might prove challenging without, paradoxically, knowing how to define the original function.

Ultimately, the method that is most applicable to this type of task is some type of regression. Since we are trying to fit a polynomial to the data, polynomial regression with some cost function will work nicely. For the cost function, the "least squares" has been chosen for its simplicity, effectivity and widespread use.

# 4 Least squares polynomial fitting

## 4.1 Theory

Suppose that there is a polynomial $P$ of degree $k$ so that

$$P(x) = a_0 + a_1 x + \cdots + a_{k-1}x^{k-1} + a_k x^k$$

The coefficients $a_0$ through $a_k$ are the unknowns that must be solved for, defining the shape of the polynomial. Assuming one wants to fit this polynomial to a set of points, it would be useful to know how accurately it can model the dataset, given a guess of its coefficients $a_0 \cdots a_k$.

Figure 5 depicts one such cubic polynomial. It is evident from the graph that $P(x)$ does not exactly pass through every point - rather, there is a slight difference for every point it passes by. Let one of these points be represented as $(x, y)$. The vertical difference $r$ between this point and the graph of $P$ could be simply represented by

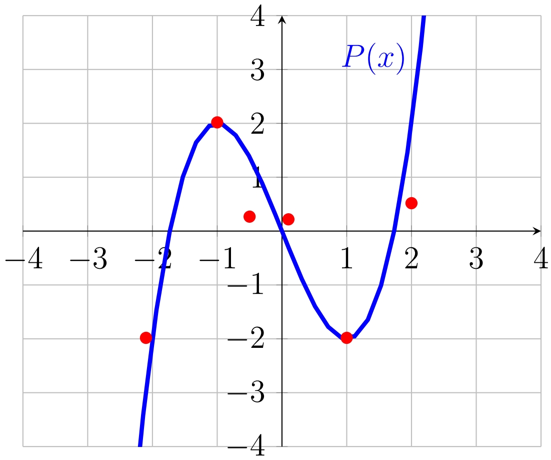



Figure 5: Example cubic ($k = 3$) $P(x)$ fitting 5 points.

$$r = |y - P(x)|$$

The absolute value of the difference is used since $r$ should be indicative of the deviation from the graph, therefore the direction of difference is irrelevant. Furthermore, instead of taking the modulus of the right side, it can simply be squared in order to accentuate any error, but more importantly be derivable, as the absolute value produces discontinuous derivatives. (Weisstein, 2022) This leaves the final equation of the difference $r$ between a point $(x, y)$ as

$$r = (y - P(x))^2$$

The total measure of deviation $R^2$ from the graph $P(x)$, called the "residual", can be obtained by summing up the aforementioned vertical differences for $n$ points with coordinates $(x, y)$: (Weisstein, 2022)

$$R^2 = \sum_{i=1}^{n} [y_i - P(x_i)]^2$$

The partial derivative of the residual with respect to each coefficient can be found, and set to zero in order to ensure that coefficient contributes a minimum of the error in terms of the entire equation - this effectively minimizes $R^2$, which is the goal of this method. This results in the following system of equations with partial derivatives that must be brought to zero: (Weisstein, 2022)

$$\begin{aligned}
\frac{\partial(R^2)}{\partial a_0} &= -2\sum_{i=1}^{n}[y_i - P(x_i)]x^0 && = 0 \\
\frac{\partial(R^2)}{\partial a_1} &= -2\sum_{i=1}^{n}[y_i - P(x_i)]x^1 && = 0 \\
&\vdots \\
\frac{\partial(R^2)}{\partial a_{k-1}} &= -2\sum_{i=1}^{n}[y_i - P(x_i)]x^{k-1} && = 0 \\
\frac{\partial(R^2)}{\partial a_k} &= -2\sum_{i=1}^{n}[y_i - P(x_i)]x^k && = 0
\end{aligned}$$

From this point, the method requires solving the above system for the most closely fitting coefficients, i.e. minimizing $R^2$. There are several methods through which this system of equations can be simplified and solved, the proof of which, however, is beyond the scope of this investigation. After simplification, the matrix system can be represented as (Weisstein, 2022)

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} & x_1^k \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} & x_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{k-1} & x_{n-1}^k \\ 1 & x_n & x_n^2 & \cdots & x_n^{k-1} & x_n^k \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{k-1} \\ a_k \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$$

An interesting point about this method is that if the number of points $n$ that the polynomial must be fit to exceed the degree $k$, the polynomial can never pass through all of them exactly, which makes sense given that the polynomial may only have a maximum of $k-1$ turning points. This means that the above system of matrices is "over-determined", that is, it can never give a perfect solution in terms of coefficients, only the closest approximation.

As for solving the system, multiplying by the transpose $X^T$ of the matrix with x-terms $X$ would yield a square linear system which can then be solved numerically for the coefficients in the vector $\vec{a}$.

$$X^T X \vec{a} = X^T \vec{y}$$

As mentioned above, and in the case that applies to solving this problem, a well-formed solution to this system does not exist most of the time. However, if the degree $k$ is greater than $n$, the system can be solved for a single vector of solutions by simply inverting $X^T X$:

$$\vec{a} = (X^T X)^{-1} X^T \vec{y}$$

## 4.2 Application & Exploration

Now that the underlying method that powers this type of polynomial regression is known, it is possible to apply it to the dataset. While a polynomial might accurately model one crest or trough of a step, in reality it would not be practical to model the entire collected data-set with just one. There are several methods to account for this issue which will be explored below.

As for actually fitting the polynomials using the above method, I wrote a computer program in the Python programming language which solved the system using the NumPy mathematics library and MatPlotLib for visualization. I further used MATLAB to draw some figures and draw insights.

### 4.2.1 Even "naive" subdivision

A naive solution to the problem is to segment the entire domain into even chunks, then fit a polynomial to the portion of the graph (drawn as a punctured line) in those subregions (divided vertically with dotted lines):

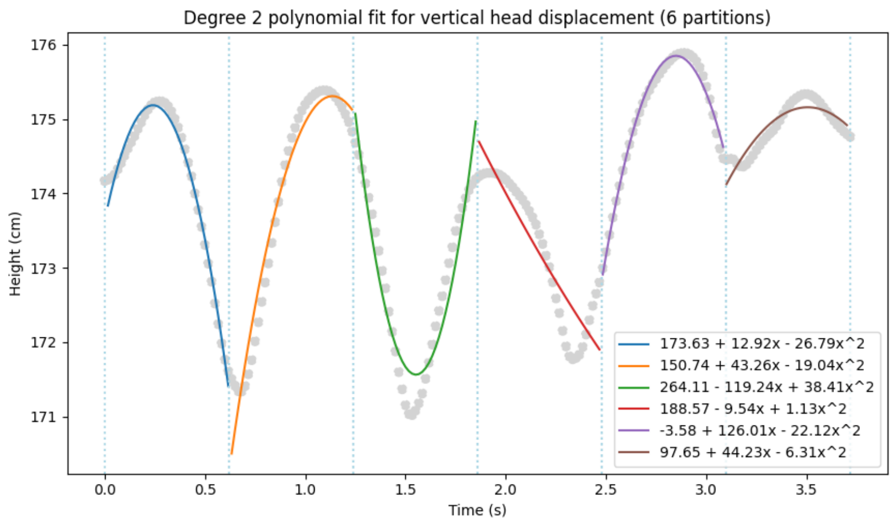



Figure 6: Naive quadratic fitting for vertical head oscillation data

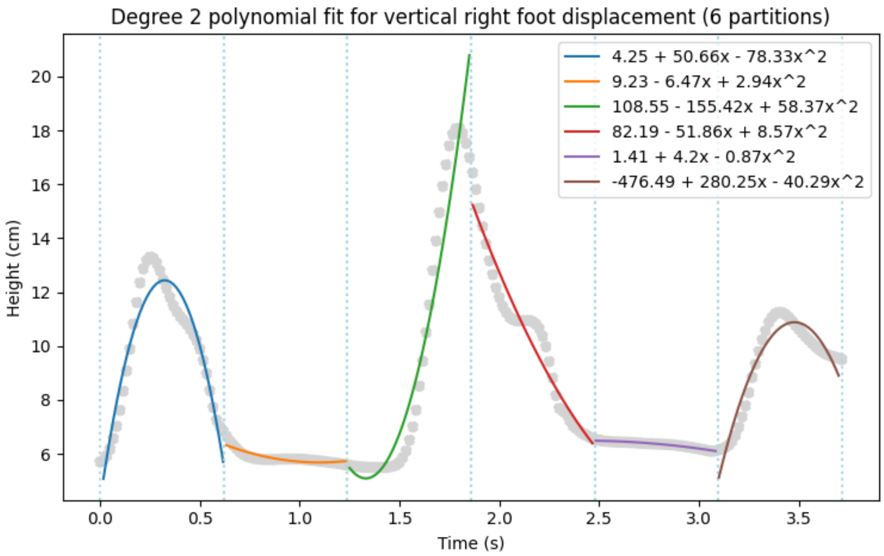



Figure 7: Naive cubic fitting for vertical right foot oscillation data

There are several main observations that can be made from these naive fitting attempts: first, from Figure 6, it is evident that all crests and troughs are smooth and symmetrical enough so that a quadratic can accurately model them. This should be kept in mind as it will become important when trying to refine the fitting method. While the first crest in Figure 6 appears to be well-fitted, the same cannot be said for the succeeding crests or troughs.

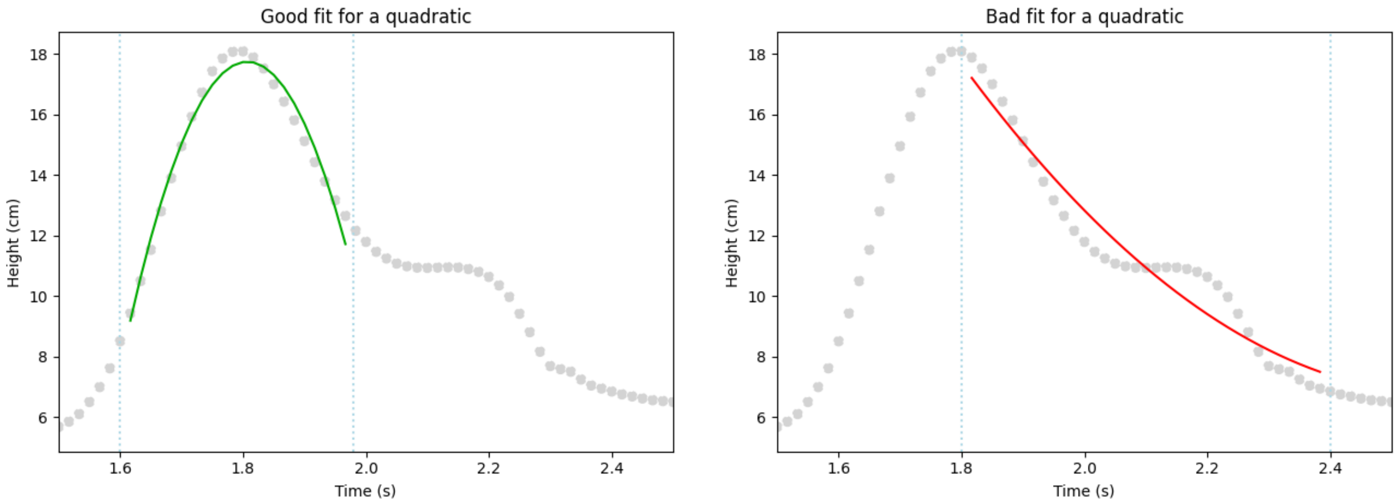
A possible explanation for this is that the selection of subregions does not respect any of the graph features that a quadratic might benefit from, like the aforementioned smooth curvature at the crests and troughs. For example, the third crest's equation (red) is a quadratic that acts very similarly to a linear in the region it is meant to approximate, since there are not enough turning points for it to assume the necessary shape. The result of this, as we can see, is not a very good fit.

In the more complex graph of Figure 7, this is further reciprocated. Even though a higher degree of polynomial is used, some regions of the graph are fitted far better than others, like the first crest compared to the second, which, due to the naive segmentation, is forcefully cut in two.

While this method is a good start, it has a plethora of shortcomings which can be addressed quite easily. In general, the fit of the graph is quite poor.

### 4.2.2 Crest & trough subdivision

In order to address the issues found in 4.2.1, and take advantage of the patterns identified, we are going to have to examine our dataset more closely. Polynomial functions are able to fit a dataset much more closely if the area over which they are fitted is of a similar shape to them. For example, curved crests and troughs are present in the dataset here, and they can almost perfectly be fit by a quadratic.




(a) Example of a quadratic that fits the crest well

(b) Example of a quadratic with a poor fit

Figure 8: Examples of how the region over which a polynomial is fit can drastically affect the accuracy of the fit

In Figure 8a, the crest can be perfectly fit by a concave down quadratic, as the region in which the regression was performed was limited to just around the crest. In contrast, Figure 8b shows a misplaced quadratic that is trying to fit an area not suited for its shape, resulting in heavy deviations from the data.

Knowing this, it is possible to improve the segmentation method to look for areas that might be a good fit for specific polynomial functions. For a quadratic, looking for the crests and troughs to harness the aforementioned benefits from their shape is a good staring point.

Crests and troughs can be considered the local minima and maxima of the data at their peaks. It is a property of functions that these minimum and maximum points are also turning points, meaning the derivative of the function changes sign around them. Applying this to the current data, it is sufficient to find a point around which both neighboring points are either both above or both below in the y-axis, as this effectively compares if the slopes of the two neighboring points relative to the central point have inverted.

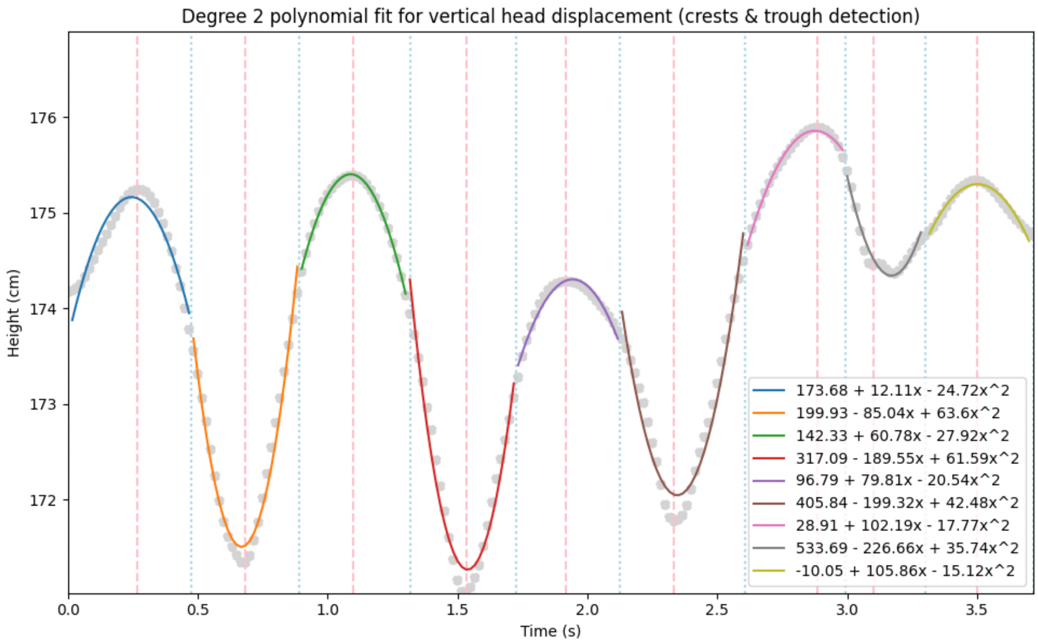



Figure 9: Quadratic fitting for vertical head oscillation data with crest & trough detection

Figure 9 demonstrates this detection method in action - the red dashed vertical lines show the point at which a crest or trough was detected, and the blue dotted lines show the start & end of each fitting region. These region borders are placed between each crest or trough, by averaging their x-coordinates. Compared to the naive fit seen in Figure 6, this is certainly a much better outcome as we can see the quadratics' curves being used more effectively around the peaks.

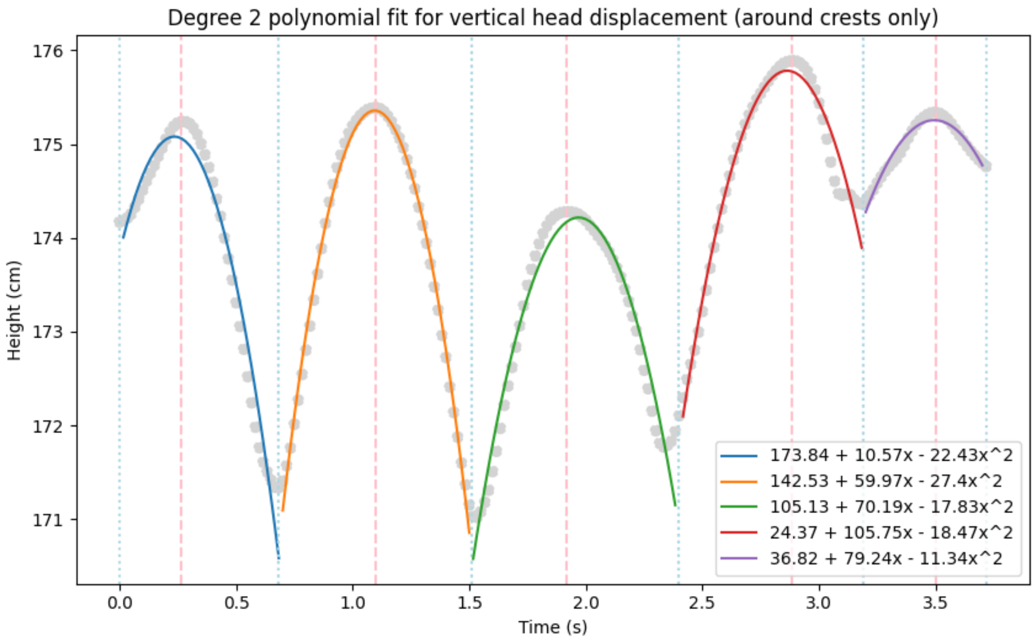



Figure 10: Quadratic fitting for vertical head oscillation data with crest detection only

The sharpness of the troughs seen in the head movement data poses an interesting question - how would the accuracy of the model vary if only the crests were used as the bases around which the quadratic are to be fit? Figure 10 illustrates one such example, and interestingly, it seems that even with this, virtually no important information appears to be lost, yet the number of equations has been reduced to half, all of which are, naturally, concave down.

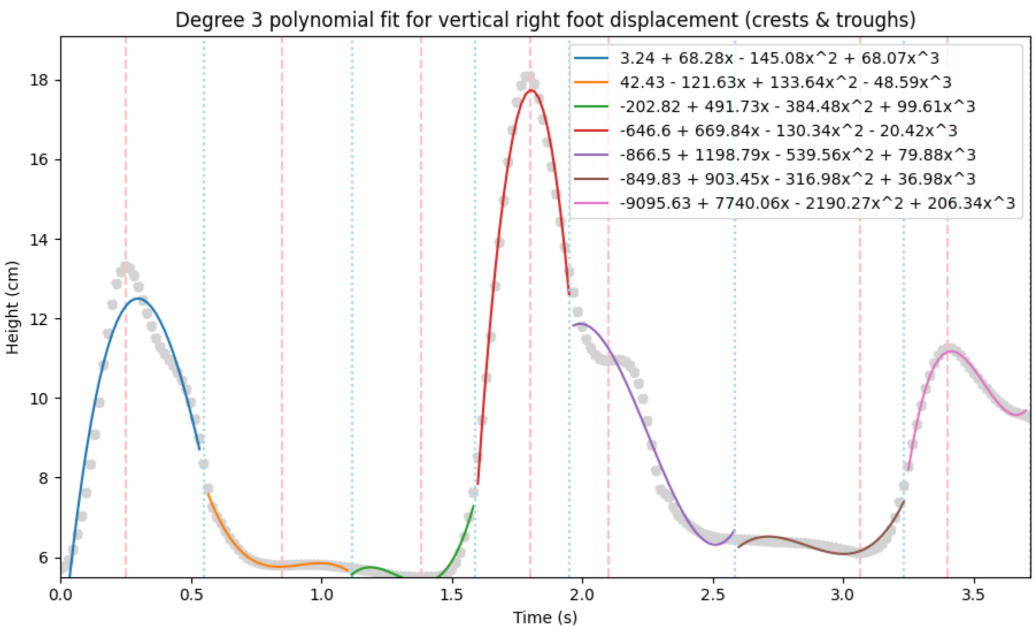



Figure 11: Cubic fitting for vertical right foot oscillation data with crest & trough detection

Figure 11 shows the vertical oscillation of a right foot, with cubics fitted using the new method. It can be seen that with crest & trough detection, the results are still much more accurate than the naive approach. However, it should be noted that the data for the motion of feet is significantly more complex to model than the data for the head, hence the comparative inaccuracy of the fit. In the end, even though there is some evidence of improvement from the previous method for dividing subregions for the vertical oscillation of the feet, it is still too inaccurate to be used for the final model.

### 4.2.3 Manual subdivision

Although a fitting solution has been found for the graph of the vertical oscillation of the head while walking (Figure 10), in order to fit a polynomial to the more complex graphs of the feet, a more precise method is required.

As the crests observed in these graphs appear to repeat over a period, it would potentially be useful to fit a polynomial to one of these by manually specifying the domain. This would then allow us to extrapolate the fitted polynomial to other crests, and apply the insight learned from the previous attempts at fitting.

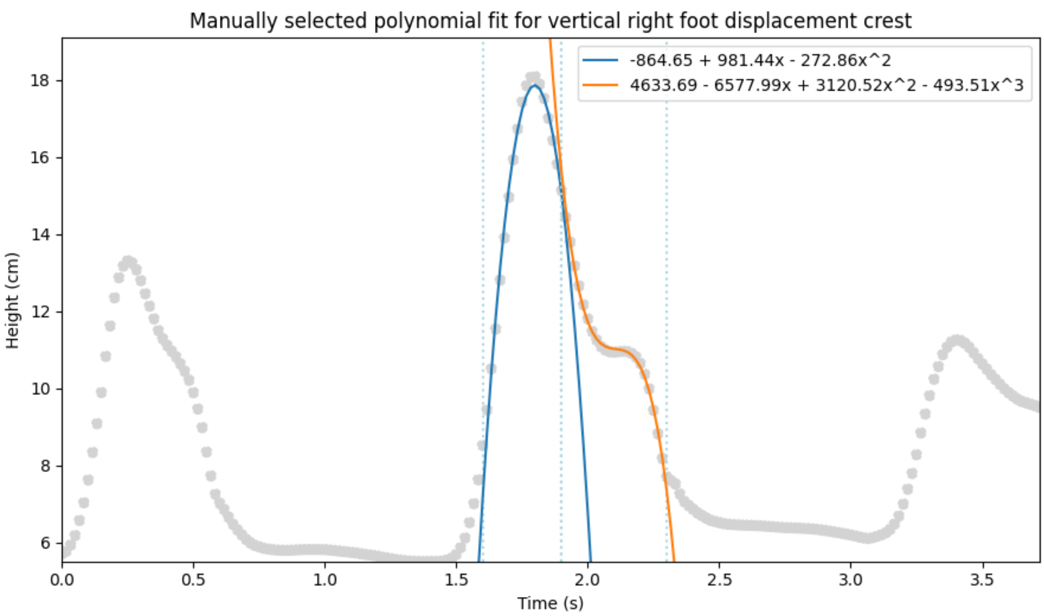



Figure 12: Cubic & quadratic fitting for vertical right foot oscillation data by manually selecting the fitting region

Figure 12 demonstrates a manual fit of a quadratic for $1.6 \leq x \leq 1.9$ and a cubic for $1.9 \leq x \leq 2.3$. Based on the knowledge of the efficacy of concave down quadratics at modeling crests explored in Section 4.2.2, a quadratic has been used in that region, which appears to fit well. Similarly, a flat area can be observed to the right of the peak, which happens to almost perfectly be modelled by a cubic, with a stationary point in the middle of the flat area. From these results, it is safe to say that this combination of a quadratic and cubic is sufficient to model the crests of vertical oscillation in feet.

## 4.3 Assembling

In order to assemble the final model for the vertical oscillation while walking, we can draw on the best fitting methods to model the oscillation of the head and feet. For the head, the "crest-only concave down quadratic" fit depicted in Figure 10 seems like the best choice due to how accurately it represents the original data using very few quadratic equations.

### 4.3.1 Head vertical oscillation

The quadratic equations returned by the regression process are in standard form. In order to make the equations more wieldy to work with, it would be sensible to convert them into vertex form, as this gives us the coordinate of the crest's peak, which can prove useful if we're trying to figure out the periodicity of the dataset.

For this conversion, it is a good start to first find the roots $r_1$ and $r_2$ of the quadratic $Q(x)$ using the quadratic formula:

$$\text{Given } Q(x) = ax^2 + bx + c,$$
$$\Delta = b^2 - 4ac$$
$$r_1 = \frac{-b + \sqrt{\Delta}}{2a}, r_2 = \frac{-b - \sqrt{\Delta}}{2a}$$

As the vertex lies perfectly in the middle of the two roots, finding its x-coordinate $h$ is easy to do by just averaging the roots: Then, evaluating $Q(h)$ would also give us the y-coordinate $k$ of the vertex.

$$h = \frac{r_1 + r_2}{2}$$
$$k = Q(h) = ah^2 + bh + c$$

Using the values for $h$ and $k$, it is now possible to express the original quadratic in the vertex form $Q(x) = a(x-h)^2 + k$. Evaluating this for the first two equations $Q_1(x)$ and $Q_2(x)$ fit to the crests of the vertical head oscillation data:

$$Q_1(x) = -22.43x^2 + 10.57x + 173.84$$
$$\Delta = 10.57^2 - 4(-22.43)(173.84) \approx 15708.650$$
$$r_1 \approx \frac{-10.57 + \sqrt{\Delta}}{2(-22.43)} \qquad r_2 \approx \frac{-10.57 + \sqrt{\Delta}}{2(-22.43)}$$
$$r_1 \approx -2.558 \qquad r_2 \approx 3.030$$
$$h \approx \frac{-2.558 + 3.03}{2} \approx 0.236$$
$$k = Q(h) \approx -22.43(0.236)^2 + (10.57)(0.236) + 173.84 \approx 175.085$$
$$Q_1(x) \approx -22.43(x - 0.236)^2 + 175.085$$

Applying the same process for $Q_2(x)$, $\quad Q_2(x) \approx -27.4(x - 1.094)^2 + 175.344.$

As noted by the discussion in 2.3, the pattern observed in all collected data appears to be cyclical. Here, this is further reciprocated as the values of $a$ for both $Q_1$ and $Q_2$ are very similar, potentially alluding to the fact that all peaks are roughly the same in shape.

This means that it may be a valid approach to represent it as a series of the same equations with a constant offset, called the "period". We can use the already found coordinates of the vertices of $Q_1$ and $Q_2$ to find the value of the period $p$.

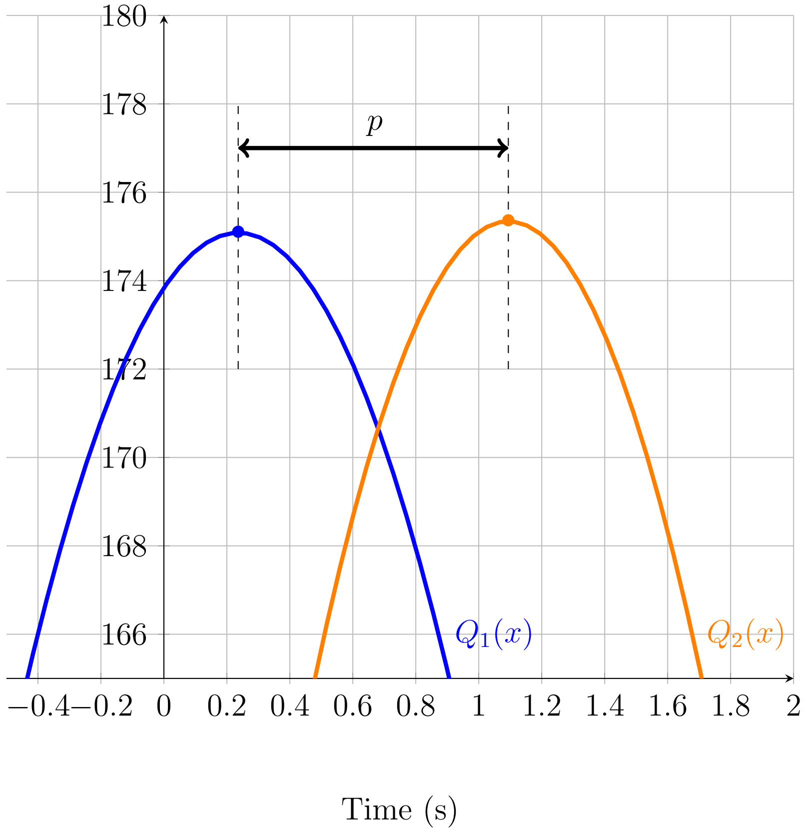



Figure 13: Graphs of $Q_1$ and $Q_2$. $p$ is the horizontal distance between their vertices.

Figure 13 shows how it is possible to figure out the period of vertical head oscillation by finding the horizontal distance between the two vertices. In terms of the original function of the vertical head displacement, the vertices are the peaks of the graph, where the cycle of the function appears to repeat.

Calculating $p$ by subtracting $Q_1$'s vertex x-coordinate $h_1$ from $Q_2$'s vertex x-coordinate $h_2$, it can be found that $p = 1.094 - 0.236 = 0.858$.

As the x-coordinate here represents the time in seconds, the value of $p$ tells us that a full step of the walking cycle takes around 0.858 seconds, or 858 milliseconds, which seems to be inline with what can approximately be observed in real life - it takes about a second to take a full step.

Using the property that $p$ denotes a period, it is then possible to create a series of equations $y_0 \dots y_n$ which describe the step accurately at different points in time $x$.

$$
\begin{aligned}
y_0 &= at^2 + k \\
y_1 &= a(x-p)^2 + k \\
y_2 &= a(x-2p)^2 + k \\
&\vdots \\
y_n &= a(x-np)^2 + k \qquad n \in \mathbb{Z}
\end{aligned}
$$

The value of $a$ defines the shape of the steps. It is reasonable to assume that this value depends on the physical parameters of the person walking. From the data collected, we can calculate the mean of $a$ values for $Q_1$ and $Q_2$ and use it as a good enough estimate. $a = \frac{-22.43-27.4}{2} = -24.915$

Furthermore, by subtracting $p$ along the values of $t$, the graph of any $y_0 \ldots y_n$ is horizontally translated right by $p$ with each succeeding equation. This means that value of $p$ determines the period between each step, which effectively makes it represent how fast the person walking can take steps. This is again dependent on the person, but for our dataset we will use the previously obtained value $p = 0.858$. If a generalization is necessary, one can use $p = h_2 - h_1$.

The value for $k$ denotes the maximum height any equation $y_0 \ldots y_n$ will reach, as all of their vertices will have the y-coordinate $k$. As the height of the head from the floor will be at a maximum whenever the person's leg is fully extended, the value of $k$ can simply be taken as the person's standing height. Similarly to how $a$ was computed here, $k$ can be gotten by averaging the y-coordinates of the vertices of $Q_1$ and $Q_2$: $k = \frac{175.085+175.344}{2} = 175.2145$. This result is further reciprocated by the fact that the person measured walking in this investigation is of approximately the height of 175 cm.

In order to represent the final function as one single definition, the obtained equations can be restricted to a particular domain inside a piece wise function $h$. The domain is simply the meeting point of two equations, which would happen in the middle of their vertices at a distance $\frac{p}{2}$ from either vertex. If any particular equation is horizontally translated by a distance of $np$ where $n \in \mathbb{Z}$, then the x-coordinate of the intersection of that equation with the next would be $(n + \frac{1}{2})p$.

Thus, the function $h(t)$ models the vertical oscillation of a walking human with respect to time $t$.

$$h(x) = \begin{cases} ax^2 + k & 0 \le x < \frac{p}{2} \\ a(x-p)^2 + k & \frac{p}{2} \le x < \frac{3p}{2} \\ a(x-2p)^2 + k & \frac{3p}{2} \le x < \frac{5p}{2} \\ & \vdots \end{cases}$$

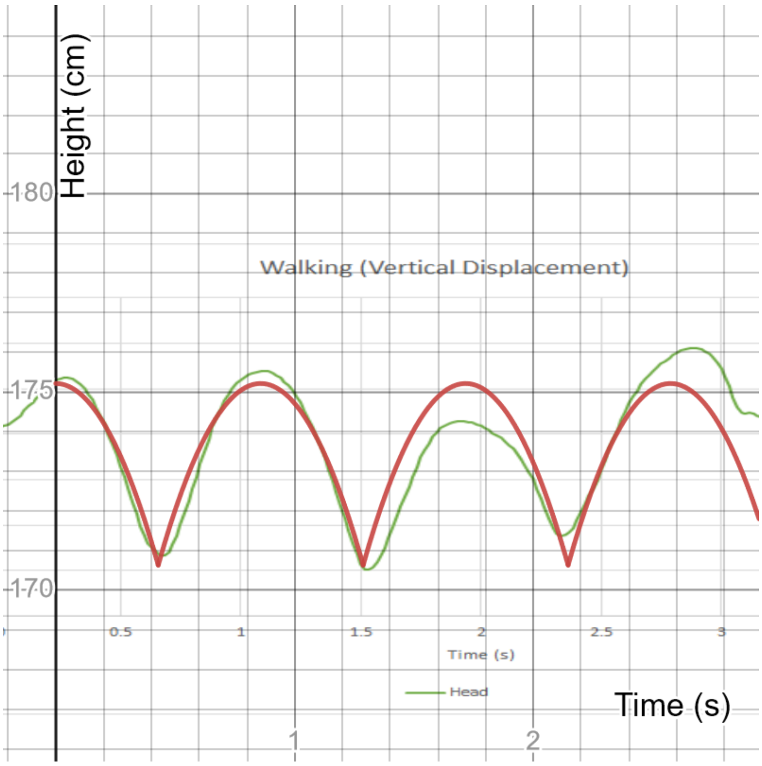


Figure 14: Graph of $h(x)$ overlaid over initial data collected.

It can be observed that even though the later steps do not quite reach the maximum height due to the limited walking space available, the data is quite nicely fit by the graph of $h(x)$, leading to the conclusion that the model is accurate.

### 4.3.2 Feet vertical oscillation

To assemble the final model for the oscillation of the feet, it is a good start to convert the quadratic $Q(x)$ into vertex form, as already shown.

$$\begin{aligned} Q(x) &= -272.86x^2 + 981.44x - 864.65 \\ &\approx -272.86(x - 1.798)^2 + 17.876 \\ &\therefore \text{Vertex at } (1.798,\ 17.876) \end{aligned}$$

It is also possible to find the x-coordinate of the inflection point of the cubic $C(x)$ by finding the root of the second derivative.

$$\begin{aligned} C(x) &= -493.51x^3 + 3120.52x^2 - 6577.99x + 4633 \\ \frac{\mathrm{d}C}{\mathrm{d}x} &= -1480.53x^2 + 6241.04x - 6577.99 \\ \frac{\mathrm{d}^2C}{\mathrm{d}x^2} &= -2961.06x + 6241.04 \end{aligned}$$

$$\begin{aligned} -2961.06x + 6241.04 &= 0 \\ -2961.06x &= -6241.04 \\ x &= \frac{-6241.04}{-2961.06} \\ x &\approx 2.108 \end{aligned}$$

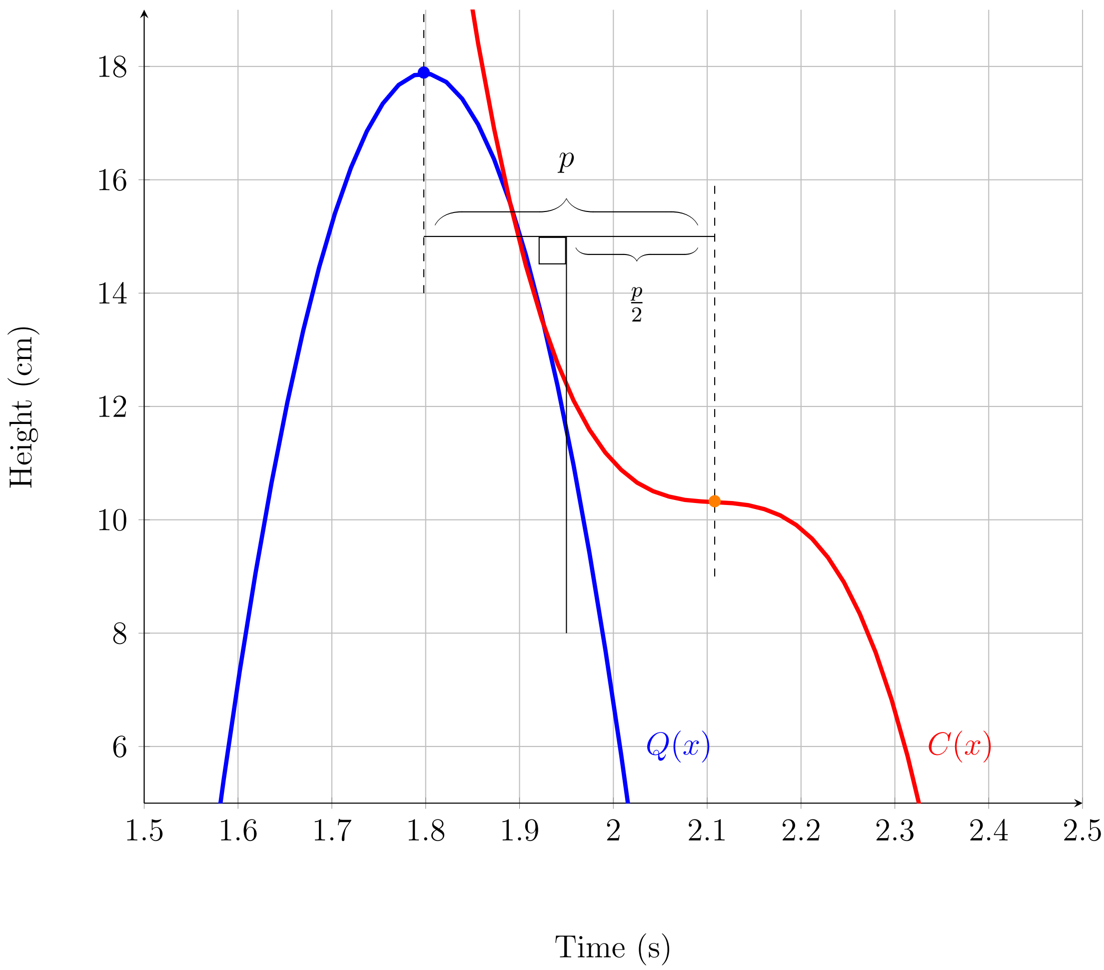



Figure 15: Graphs of $Q$ and $C$. $d$ is the horizontal distance between the vertex of $Q$ and the inflection point of $C$. A perpendicular line bisects the created horizontal line at $d/2$ from either side.

In order to express one peak as a single curve, first it can be expressed as a piece wise $w(x)$ between the cubic and the quadratic, where the functions change over at a horizontal distance $\frac{d}{2}$ away from the quadratic vertex's x-coordinate $h$.

$$w(x) = \begin{cases} -272.86(x - 1.798)^2 + 17.876 & x \leq h + \frac{d}{2} \\ -493.51x^3 + 3120.52x^2 - 6577.99x + 4633 & x > h + \frac{d}{2} \end{cases}$$

However, this produces an undesired effect: the function is now discontinuous at the meeting point $h + \frac{d}{2}$, which is not convenient if one wishes to use this model to smooth some sort of movement as there would be a noticeable "bump" where the functions change over.

In order to find a better place to put the meeting point, we can just equate the cubic and quadratic, then solve for the roots of the resulting cubic.

$$
\begin{aligned}
C(x) &= Q(x) \\
-493.51x^3 + 3120.52x^2 - 6577.99x + 4633 &= -272.86x^2 + 981.44x - 864.65 \\
-493.51x^3 + 3393.38x^2 - 7559.43x + 5497.65 &= 0 \\
\text{Solve using cubic formula, not shown for brevity} & \\
x \approx 1.89006 \qquad x \approx 1.926 \qquad x \approx 3.060 &
\end{aligned}
$$

We can pick any one of these values for the x-coordinate of the transition and get a continuous graph, however the solution for $x \approx 3.060$ is beyond the region of interest where we wish to join the functions together, so it can be disregarded. From the solutions around 2, $x \approx 1.926$ has been chosen as it appears to create a more smooth final function. Using this as a meeting point, the piece wise function $w$ becomes:

$$
w(x) = \begin{cases} -272.86(x - 1.798)^2 + 17.876 & x \leq 1.926 \\ -493.51x^3 + 3120.52x^2 - 6577.99x + 4633 & x > 1.926 \end{cases}
$$

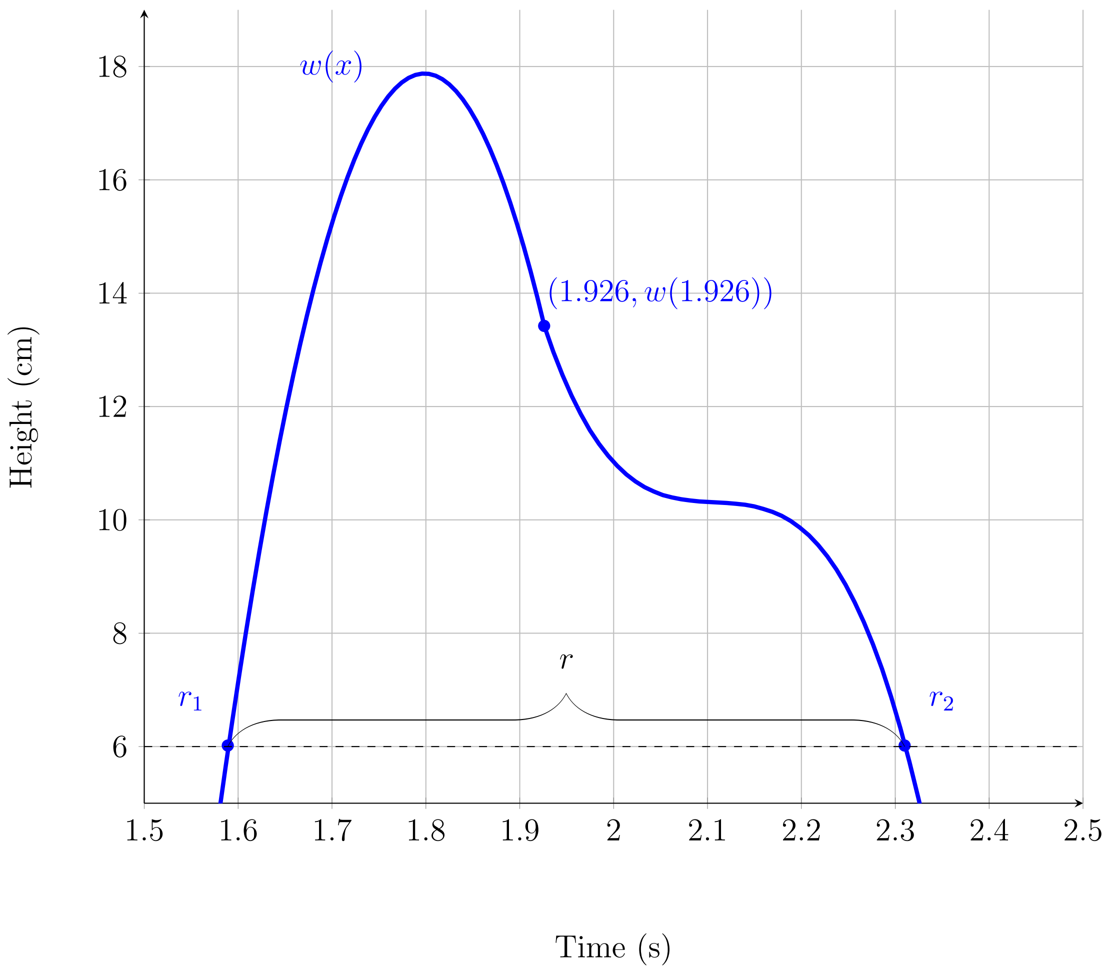



Figure 16: Final graph of continuous $w(x)$, $r$ is the distance between the intersection points $r_1$ and $r_2$ of $w(x)$ with $y = 6$

As suggested by, Figure 3 the true "floor" of the graph of the vertical oscillation in feet is actually above the x-axis. This is due to the measurement being taken from the walker's ankle and not the very base of his foot. Evidently, that point appears to about 6 cm above the true ground. We can find the "roots" $r_1$ and $r_2$ by equating $w(x) = 6$. It would also be useful to simply subtract 6 from the parts of $w$ in order to bring the floor to the true $y = 0$. Solving the equality yields the solutions $x \approx 1.589$ and $x \approx 2.310$, giving us the values for $r_1$ and $r_2$. This, then, allows us to calculate the value of $r$: $r = 2.310 - 1.589 \approx 0.721$.

The significance of $r$ is actually quite big. It was previously observed that the "peaks" modeled by $w(x)$ alternate for each foot, meaning only one foot can be undergoing a peak, while the other must be going through a plateau. This means that the horizontal length of the peak we just found as $r$ must correspond to a plateau on the graph of the other foot, therefore illustrating that plateaus are of the same length as peaks.

Using this insight, it is now possible to model the final piece wise function $h_R(x)$ for the vertical oscillation of the right foot. The function must be a crest, followed by a trough, where both the crest and trough are $r$ units long in the x-axis. The graph of $w$ should be translated horizontally so that $r_1$ is at the origin. This means that there would be a leftward horizontal translation by $r_1$ units. Every pair of crests & troughs, the graph of $w$ should also be translated to the right by $2r$ units.

$$h_R(x) = \begin{cases} w(x + r_1) & 0 \leq x < r \\ 0 & r \leq x < 2r \\ w(x + r_1 - 2r) & 2r \leq x < 3r \\ 0 & 3r \leq x < 4r \\ & \vdots \end{cases}$$

The function for the left foot $h_L(x)$ would simply be the above, but with the order of the crests and plateaus swapped. This would effectively put the two graphs completely out of phase, which is the desired effect.

$$h_L(x) = \begin{cases} 0 & 0 \leq x < r \\ w(x + r_1 - r) & r \leq x < 2r \\ 0 & 2r \leq x < 3r \\ w(x + r_1 - 3r) & 3r \leq x < 4r \\ & \vdots \end{cases}$$

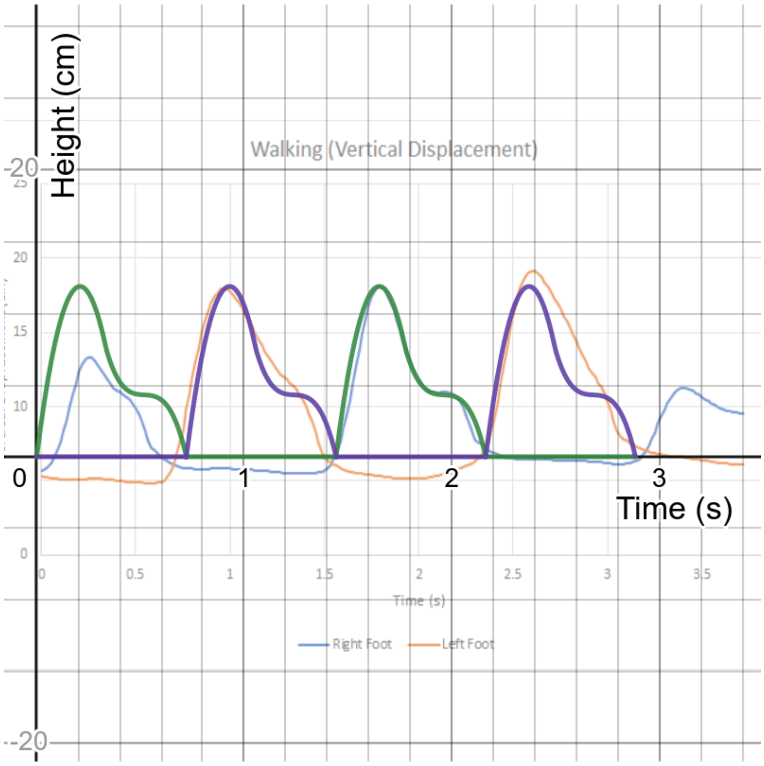


Figure 17: Graph of $h_R(x)$ (green) and $h_L(x)$ (purple) overlaid over initial data collected.

It is evident that the functions fit the three middle peeks well, with their maximums & points of transition lining up almost perfectly, which is indicative of a strong model. Even though the first step did not reach the maximum height, its features still line up horizontally with the fitted function which may indicate that the misalignment is due to an external factor like the first step not being as big as the rest.

# 5 Evaluation & conclusion

From the models obtained for the vertical oscillation of both the head and feet of a walking human, it is evident that polynomials are an effective way to model the obtained data. However, it is important to note that the method through which the polynomial is fitted to the graph plays a highly important role in the accuracy of the final result. Each of the explored fitting methods has its own strengths and limitations, so it is important to choose wisely for the dataset at hand.

The models derived, themselves, may not necessarily have a high ecological validity in that they describe the motion of one person's particular walking style, which in and of itself is not particularly ensured to remain constant. Although the model demonstrates malleability through its parameters, it would potentially be worthwhile, as an extension, to add complexity based on more granular real life physical properties of the person.

Furthermore, while the models fit to the dataset well in some regions, it would potentially be meritous, as an extension to the work done here, to attempt to model the dataset using a wave function like $y = \sin x$. While this would reduce the accuracy of the fit in some places, it would produce a simpler definition of the model which may be of benefit to some users seeking this simplicity over overall accuracy.

In general, the resulting functions obtained throughout this investigation are sufficient at modelling the vertical oscillation of the human body while walking, and are therefore a good solution to the problem of this investigation.

# 6 References

Weisstein, E., 2022. Least Squares Fitting–Polynomial – from Wolfram MathWorld. [online] MathWorld. Available at: https://mathworld.wolfram.com/LeastSquaresFittingPolynomial.html [Accessed August 2022].

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