Multivariate calculus

> Reference: https://www.coursera.org/learn/multivariate-calculus-machine-learning

Derivative

Definition

$$f'(x) = \frac{df(x)}{dx} = \lim_{\Delta x \to 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \right)$$

Derivatives of some named functions

$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$\frac{d}{dx}(sin(x)) = cos(x)$

$\frac{d}{dx}(cos(x)) = -sin(x)$

$\frac{d}{dx}(e^x) = e^x$

Derivative rules

These rules help computing the derivation faster.

- *Sum rule*

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

- *Power rule*

$$\frac{d}{dx}(ax^b) = abx^{b-1}$$

- *Product rule*

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

- *Chain rule*

$$\frac{d}{dx}g(h(x)) = g'(h(x))h'(x)$$

in other words

$$\text{Given } g = g(u) \text{ and } u = h(x)$$

$$\text{then } \frac{dg}{dx} = \frac{dg}{du}\frac{du}{dx}$$

- *Total derivative*: for the function $f(x, y, z, ...)$, where each variable is a function of parameter t, the total derivative is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + ...$$

  where

$$\frac{\partial f}{\partial x}$$

  is the *partial derivative* of f with respect to x

Derivative structures

Given $f = f(x, y, z)$,

- **Jacobian**

$$J_f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

  where J is a *row vector* of the partial derivatives of f. This vector points in the *direction of the greatest slope* from the point (x, y, z), and the *bigger the norm* of this vector, the *steeper the slope* is.

- **Hessian**

$$H_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

  or, in a more compact notation

$$H = \begin{bmatrix} \partial_{xx} f & \partial_{xy} f & \partial_{xz} f \\ \partial_{yx} f & \partial_{yy} f & \partial_{yz} f \\ \partial_{zx} f & \partial_{zy} f & \partial_{zz} f \end{bmatrix}$$

When the *determinant of the the Hessian matrix* is positive, we know we are either at a minimum or a maximum (the gradient is zero). If the element e_{11} of the Hessian is positive, we have a minimum; if it is negative, we have a maximum. If the determinant is negative, we have a **saddle point**.

Notes:

- to calculate an Hessian matrix, it is easier to calculate first the Jacobian
- the Hessian matrix is symmetrical

Multi-variable chain rule

Example with $f(\boldsymbol{x}(t))$

If

$$f(x_1, x_2, \ldots, x_n) = f(\boldsymbol{x}) = f(\boldsymbol{x}(t))$$

with

$$x_1 = x_1(t), x_2 = x_2(t), \ldots, x_n = x_n(t)$$

then

$$\frac{df}{dt} = \frac{\partial f}{\partial \boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{dt}$$

where

$$\frac{\partial f}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}, \frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

Note that $\frac{\partial f}{\partial \boldsymbol{x}}$ is the Jacobian as a column-vector = $(J_f)^T$

Example with $f(\boldsymbol{x}(\boldsymbol{u}(t)))$

If

$$f(\boldsymbol{x}(\boldsymbol{u}(t)))$$

with

$$f(\boldsymbol{x}) = f(x_1, x_2), \boldsymbol{x}(\boldsymbol{u}) = \begin{bmatrix} x_1(u_1, u_2) \\ x_2(u_1, u_2) \end{bmatrix}, \boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

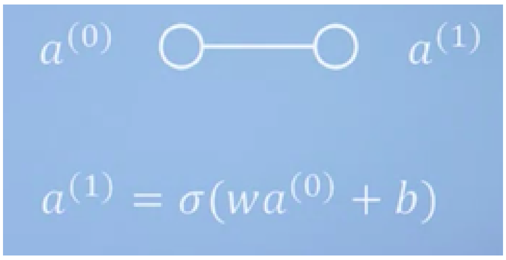
then

$$\frac{df}{dt} = \frac{\partial f}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{u}} \cdot \frac{d\boldsymbol{u}}{dt} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{bmatrix} \cdot \begin{bmatrix} \frac{du_1}{dt} \\ \frac{du_2}{dt} \end{bmatrix}$$

Neural networks

Model of neurons

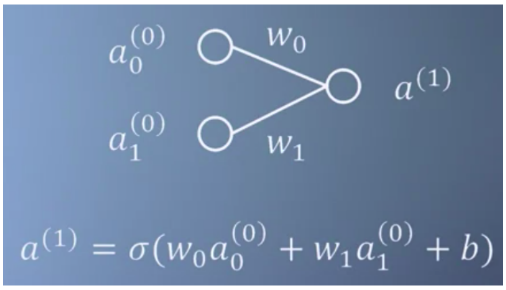
Definitions



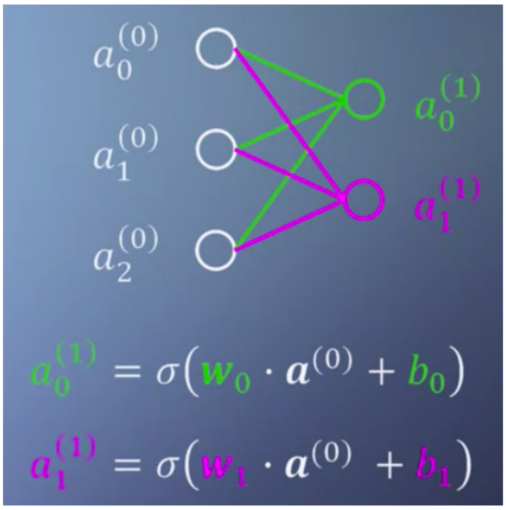
where

- a = activity
- w = weight
- b = bias
- σ = activation function

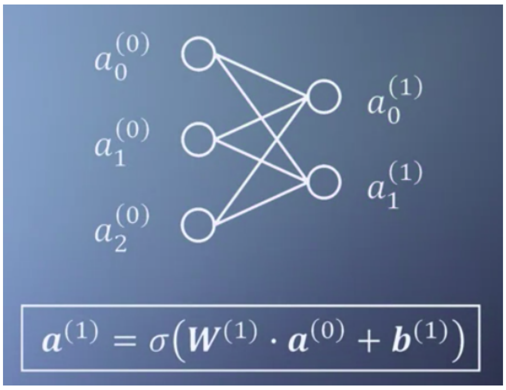
2-1 neuronal network



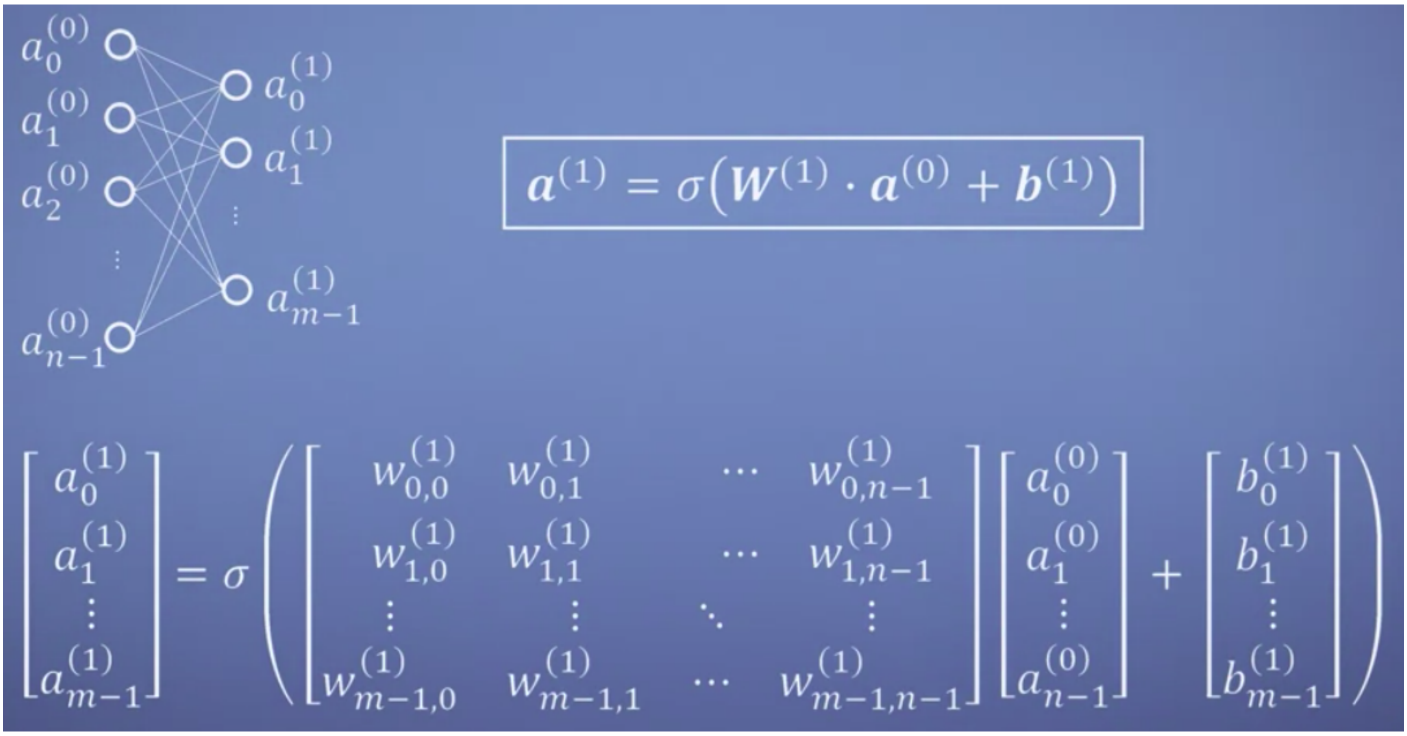
3-2 neuronal network



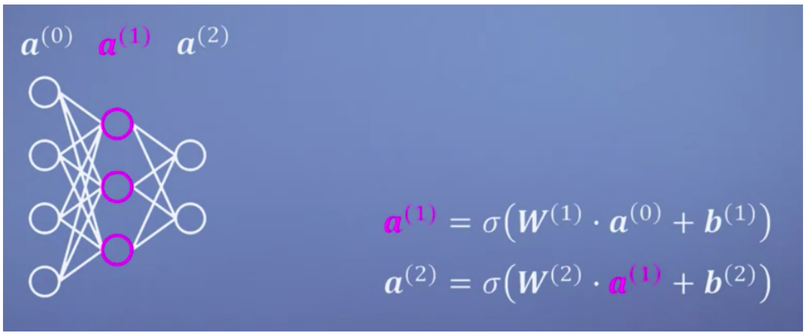
Using a matrix notation:



General case: n-m neuronal network



Hidden layer in a neuronal network



Function linking two layers in a neuronal network