Fundamental Algorithms
Lecture #2

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Agenda

- Review – conclusions
- Divide et impera evaluation
- Particular cases
- Master Theorem
- Sorting
  - Heap Sort

Review – conclusions

- **Complexity**
  - Evaluate **time** and **space** requirements
  - **Time** as an estimation of the ***amount of work*** done
    - As an expression of ***# of atomic*** operations
    - Identify the operations done, count their ***number*** and estimate their growths
    - Depends on the ***size of the input data*** (***n***)
    - Depends on ***case*** (best, worst, average to be evaluated)
  - **Space** requirements as an expression of ***supplementary*** memory
    - Need algorithms using ***constant extra space***
    - Some times, algs with ***lgn*** extra space are accepted.

Review – conclusions

- **Complexity**
  - Time = amount of work = as a function of n (size of input data)
  - We need its asymptotic growth
  - Lower bound **Ω** depends on the **problem**
  - Upper bound **O** depends on the **algorithm**
  - **Efficiency** compare algorithms (their corresponding **O** function) among each other – one is more/less efficient
  - **Optimality Ω = O** in the worst case scenario - compare an algorithm with the lower bound

Review – conclusions

- **Correctness**
  - How do we know an algorithm is correct?
  - **Testing** never shows an algorithm is correct. It can only show it is INCORRECT (by finding bugs)
  - **Absence of evidence ≠ Evidence of absence**
  - Dijkstra: "Testing shows the presence, not the absence of bugs. "
  - So, how can we know an algorithm is correct?
  - **Proof**!
  - if the ***pre-conditions*** are satisfied, the ***post-conditions*** will be true when the algorithm *terminates*;
  - ***partial*** correctness = whenever preconditions are satisfied, the post-conditions are true;
  - ***total*** correctness = partial correctness + termination condition

Review – conclusions

- **Stability**
  - The property of an algorithm to preserve the **relative order of equal elements** from the input (initial/original data) in the output (final data/result)
  - Desired property
    - Choose stable algorithms, if possible
    - Sometimes difficult/impossible to estimate

Review – conclusions

- Correctness
- Efficiency
- Stability

Divide et impera evaluation

- Eval alg. Divide et impera

```
divide_et_impera(n, I, O)
    if n<=n0
        then    direct_solution(n, I, O)
        else    divide(n, I1,I2,…,Ia)
                divide_et_impera(n/b,I1,O1)      //a rec. calls
                divide_et_impera(n/b,I2,O2)      //b division factor
                …
                divide_et_impera(n/b,Ia,Oa)
                combine(O1,O2,…,Oa,O)
```

Divide et impera evaluation – contd.

- $f(n)= n^c$
- $$t(n)= \begin{cases} t_0 & \text{if } n<n0 \\ at(n/b)+f(n) & \text{if } n>=n0 \end{cases}$$

a = number of recursive calls

b = de ratio to which the original domain is divided

c = degree of the polynomial expressing the execution time of the divide et impera sequence except for the recursive calls

It is reasonable to assume f(n) is polynomial as we are seeking for overall polynomial running time algorithms

Divide et impera evaluation – contd.

$t(n)=n^c[1+a/b^c+(a/b^c)^2+\ldots(a/b^c)^{\log_b n-1}]$

Cases:
1. $q<1$; $a<b^c$ => $O(n^c)$
2. $q=1$; $a=b^c$ => $O(n^c \cdot \log_b n)$
3. $q>1$; $a>b^c$ => $O(n^{\log_b a})$!!

It's polynomial

Small power

Independent of c

Obs: b should be scalar (**b>1**)

composition should comply the **partition** rule!

In most cases, either divide, or combine is O(1)

Ex: quick sort combine = done by default (sort in situ) – no time at all

merge sort divide O(1): compute the middle index

Particular cases

1. $c=1 \quad \Rightarrow f(n)=n$

$$t(n)=\begin{cases} O(n) & \text{if } a<b \\ O(n\cdot\log_b n) & \text{if } a=b \\ O(n^{\log_b a}) & \text{if } a>b \end{cases}$$

Ex: qsort $a=b=2 \Rightarrow O(n\cdot\log_2 n)=O(n\cdot\log n)$

Is qsort optimal? Justify!

It ($a=b=2$) is NOT the worst case!

Are there means of avoiding worst case?

Se the following courses/seminars.

Particular cases – cont.

2. c=0 => f(n)=ct

Q? Is this possible ? Does such algs exist?

$$t(n)=\begin{cases} N/A & \text{if } a<b^{o} \Leftrightarrow a<1 \text{ not possible!} \\ O(\log_b n) & \text{if } a<b^{o} \Leftrightarrow a=1 \\ O(n^{\log_b a}) & \text{if } a<b^{o} \Leftrightarrow a>1 \end{cases}$$

Ex: a=1, b=2 search in BST => O(logn)

a=2, b=2 tree traversal => O(n)

Master Theorem to remember/to keep close

- $f(n)= n^c$
- $$t(n)= \begin{cases} t_0 & \text{if } n<n_0 \\ a t(n/b)+f(n) & \text{if } n>=n_0 \end{cases}$$

1. $q<1;\ a<b^c => O(n^c)$
2. $q=1;\ a=b^c => O(n^c * \log_b n)$
3. $q>1;\ a>b^c => O(n^{\log_b a})$

Sorting algorithms

- Sorting problem **Ω (nlgn)**
- What is all about?
- Direct strategies – seminary
- Advanced strategies – course

Heap sort

- Sorting with the aid of a heap structure
- Heap = **array** viewed (logical perspective) as a BT
- Representation (logical persp.) based on the index

  i = parent

  2·i 2·i+1 = children
- Property: A[parent(i)]**>=**A[i] Other properties may be defined
- Parent/child property => implies a **partial order** relation
- Q? What is a partial order relation?
- There is NO property between siblings
- Example - blackboard

Heap sort – cont.

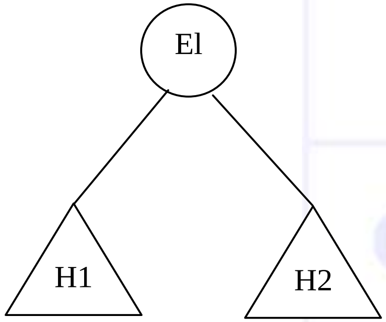
- Q1: identify a **maximal subset** on which the partial order relation becomes a **total order relation**.
  - A branch.
- Q2: based on the heap property, what consequence (**post condition**) follows?
  - The root contains the max value;
  - **Max** value in case the property based on which the heap is built is **>=**.
  - The root would have some other particularity in case another property is the choice.

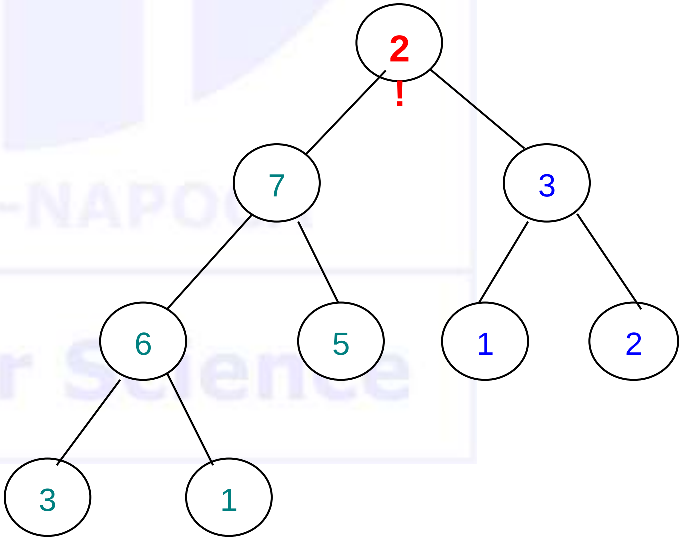
Heap sort (sorting based on heap structure) – cont.

- Procedures (methods)
  - Heapify – Reconstituie heap
    - “Adds ” the root to 2 left and right children rooted heaps
  - Build-Heap
    - Constructs the whole heap structure (on the entire array), by repeatedly applying heapify
  - Heapsort
    - Reorganizes the array by repeatedly extracting the root of the heap and placing it in the “right” position of the sorted array

Heapify (Reconstituie heap)

- **Pre-condition** – 2 heaps (H1, H2)
- Goal: add a single element El s.t. the triple (El and H1, H2) represents a larger heap: H
- **Post-condition** – 1 single heap H (Root+H1+H2)
- The strategy: **top-down** = **sink the root** to its correct place in the heap



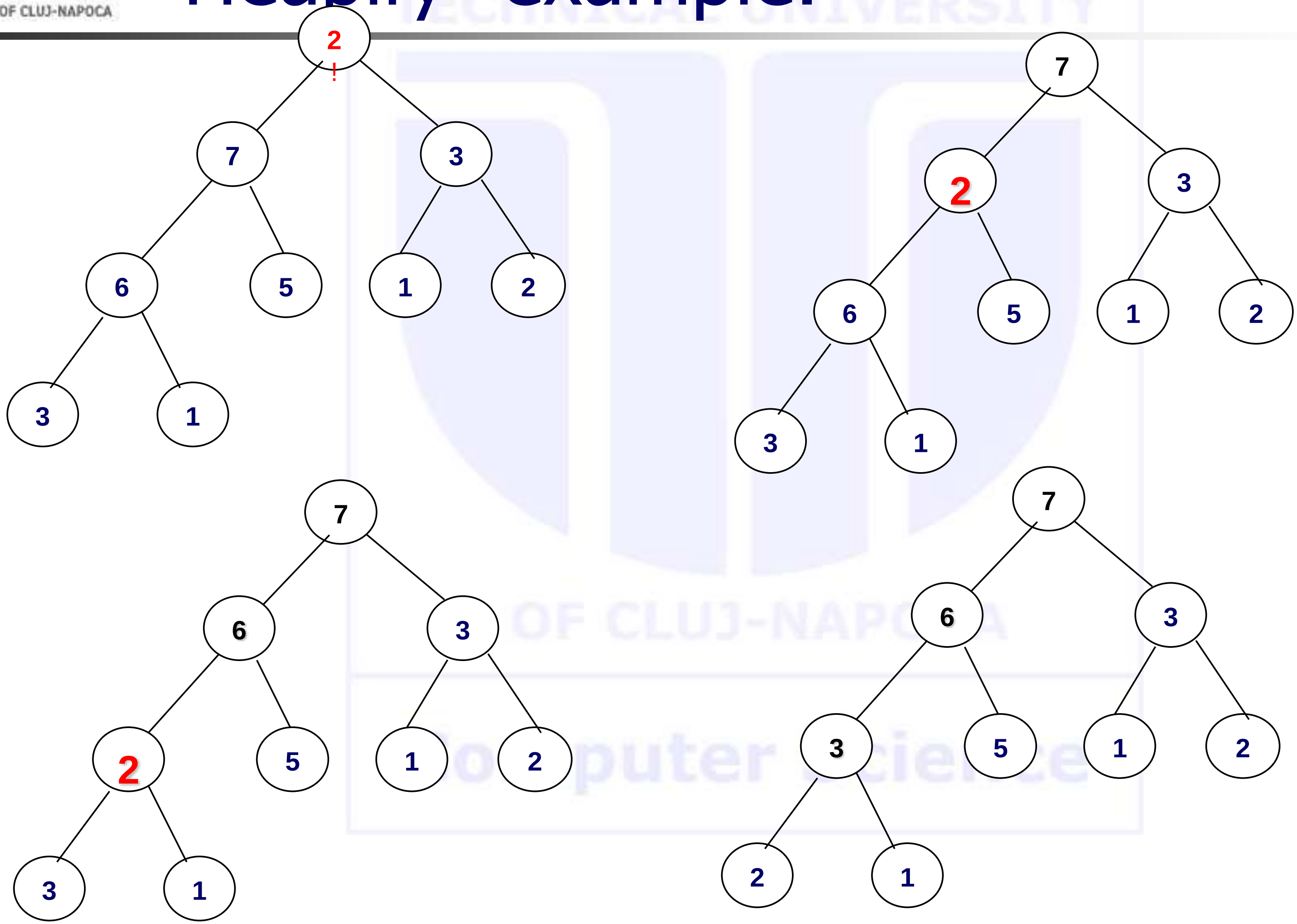


Heapify (Reconstituie heap) – cont.

```
heapify (A,i) //i-index of the root (= El to be added on top of the heap)
largest<-                              //root, left or right child index
   index_of_the_max_bet(A[i],A[left(i)],A[right(i)])
if largest <>i                         //one of the children larger than root
   then    A[i]<->A[largest]           //swap root with largest child
           heapify(A, largest)         //continue the process on the heap
                //branch of the largest child. The other branch (i.e. heap)
                //is not   affected at all
```

- it applies a **top-down** strategy

Heapify -example.



Heapify (Reconstituie heap) – cont.

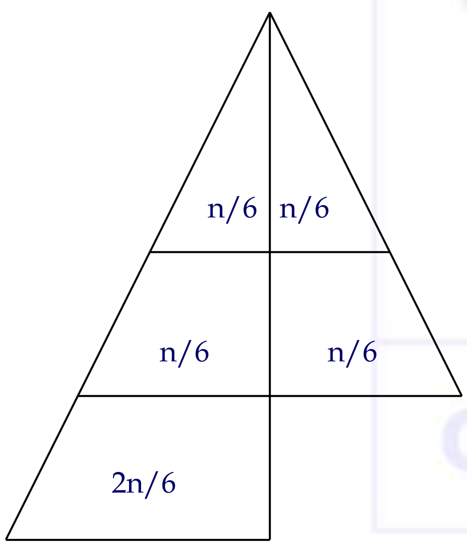
- Running time:
  - O(1) running time at one level
  - Recursive calls (how many times?)
    - Best: none => O(1)
    - **Worst**: every time => repeated down to the level of the leaves
      - ***Intuitive***: height of a full BT=lgn; you have to "sink" the root down to the level of a leaf (O(h), h=lgn for a complete tree)
      - ***Exact*** evaluation:
      - The last row of the tree is exactly half full, and we go on that branch
      - If full BT, half of the nodes are leaves
      - $t(n)=t(2n/3)+O(1)$: $a=1$, $b=3/2$, $c=0$

        Why b=3/2? Explained later (next 2 slides)
      - => Apply Master (case #2) and get $O(\log_b n) = O(\log_{3/2} n) =$

      $= O(\lg n/\lg(3/2)) = O(\lg n/(\lg 3-1)) = O(c \cdot \lg n) =$ **$O(\lg n)$**

Heapify (Reconstituie heap) – cont.

- Why t(n)=t(2n/3)+O(1)?
- Why 2n/3 nodes on the rec call (b=3/2)?
- Picture 3*n/6 (internal) and 3n/6 (leaves)



All other levels – multiple levels

Leaves' parents (on the left) and leaves (on the right) – 1 level

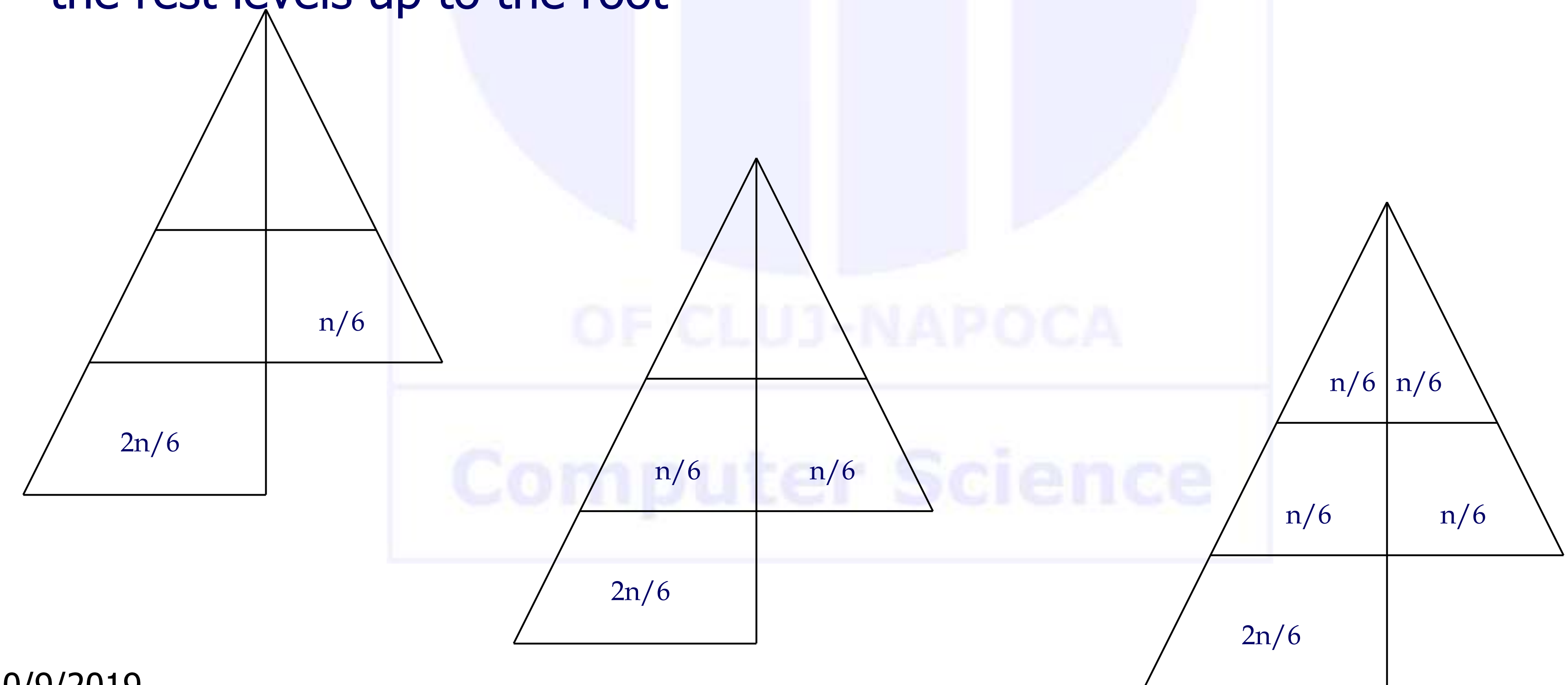
Leaves (on the left half only) – 1 level

Justification of # of nodes

Half of the nb. of nodes are leaves (2/3 on the left, 1/3 on the right)

Nb. of parents of leaves from left= half the number of those leaves (and at the same time = nb. of leaves from right)

The rest of the elements (= n-2n/6-n/6-n/6) are equal split (left/right) on the rest levels up to the root



Evaluation

Nb. Of nodes on the worst case: nodes on the largest branch (left one)

$=n/6+n/6+2n/6=4n/6=2n/3$

So $t(n)=t(2n/3)+O(1)$ (claimed 3 slides before) is justified

Build-Heap

- Heapify starts from the assumption we already have 2 heaps. Where are they from?
- 1 single node **is** a (very basic) **heap**.
- So, half of the # of nodes are already heaps; we get the strategy
  - Start with 2 heaps each of dimension 1
  - Add their common parent node to build a heap of dimension 3
- Adopt a **bottom-up** strategy:
  - ½ out of all nodes are heaps from the very beginning (leaves in a complete binary tree)
  - Apply heapify to the first non-leaf node (the node in the tree with the largest index, having at least one child)
  - Go to the “next” indexed node (sibling to the left of the first processed element)
  - Continue the process until reach the root

Build-Heap– code

```
Build-Heap(A)
for i<-|A|/2 downto 1            //from the non-leave nodes to the root
   do heapify(A,i)               // build the heap out of 2 already built
                                 // heaps and 1 node
```

It applies a **bottom-up** strategy

Running time:

- it **seems** to be n/2 ·lg n
  - We apply n/2 times (on all non-leaf nodes) heapify
  - heapify in worst case is O(lgn)
  - Means n/2 times O(lgn) goes to n/2 ·lg n
  - CL: only building the heap takes n/2 ·lg n
  - So we cannot sort on n/2 ·lg n!!!

Build-Heap– eval.

- Running time – a first evaluation:
  - n/2 times heapify => nlgn. Not good ☹
- Running time – a closer approach:
  - For all leaves, heapify does **not** apply
    - Half of the nodes are leaves – no operation applied
  - For all the parents of all the leaves it only takes O(1)
    - nb. of leaves' parents = half of the nb. of leaves
    - time require to heapify all of them (=nb*time): $½^2 \cdot n \cdot 1$
  - For half of the remaining elements, it takes 2 steps to "heapify" them:
    - half of the rest is $½ \cdot ½ \cdot ½ \cdot n = 1/8 \cdot n$
    - time require to heapify them: $1/8 \cdot n \cdot 2$
  - At each of the next steps, the nb. of elements halves, while the nb. of steps required to heapify each increases by 1

Build-Heap– eval.

t(n) =

// #· individual time

n/2·0+ //(leaves)

$n/2^2$·1+ // (leaves'parents) ….

$n/2^3$ ·2+

$n/2^4$ ·3+…

$$= \sum_{0}^{[\lg n]} [n/2^{h+1}] \cdot O(h)$$

Build-Heap – formal evaluation

To evaluate the sum on the prev slide, start from:

$\sum x^k = (1-x^{n+1})/(1-x)$ (geom prog., first =1, q=x)

$\sum x^k = 1/(1-x)$ For x<1, n->∞ we get:

$(\sum x^k)' = [1/(1-x)]'$ (derive)

$\sum k \cdot x^{k-1} = 1/(1-x)^2$ (multiply by x)

$$\boldsymbol{\sum k \cdot x^k = x/(1-x)^2 \qquad (1)}$$

Use the result (for a particular value of x) to calculate the desired sum from before

Build-Heap – formal evaluation contd.

$$t(n) = \sum_{0}^{[\lg n]} [n/2^{h+1}] \cdot O(h)$$

$$= \sum_{0}^{[\lg n]} n[1/2^{h+1}] \cdot h$$

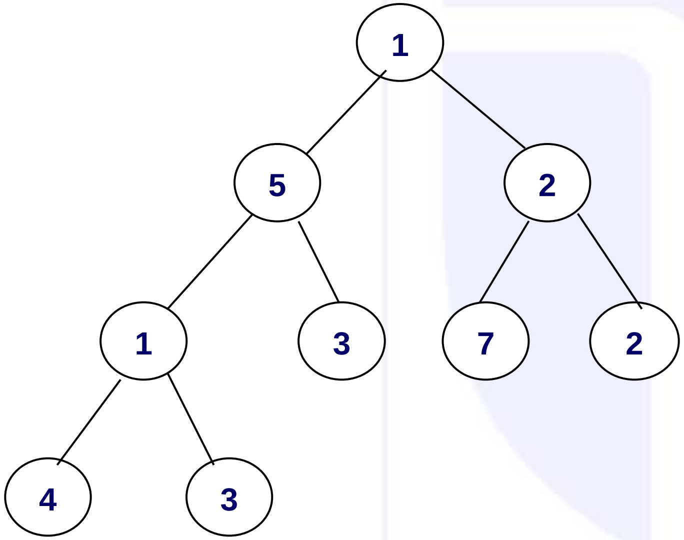
$$= n/2 \cdot \sum_{0}^{[\lg n]} [1/2^{h}] \cdot h = n/2 \cdot \sum h \cdot (1/2)^{h}$$

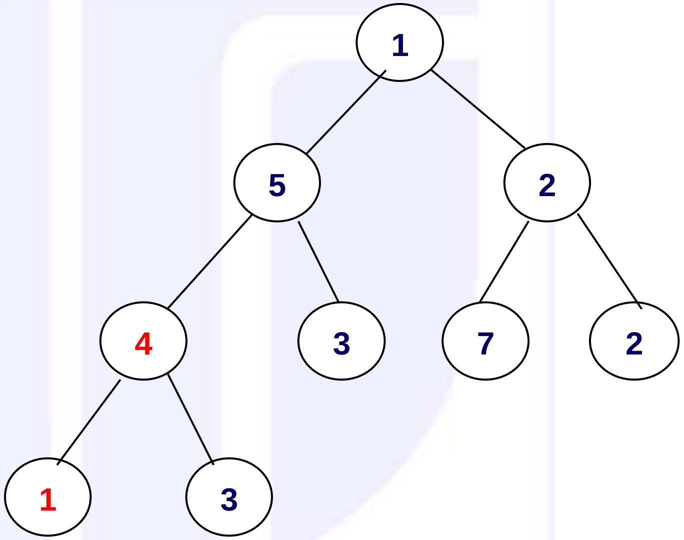
But since $\sum k \cdot x^k = x/(1-x)^2$ (from (1) previous slide), for $\mathbf{x=1/2}$ we get

$\sum k \cdot x^k = (1/2)/(1-1/2)^2 = (1/2)/(1/2)^2 = 2$

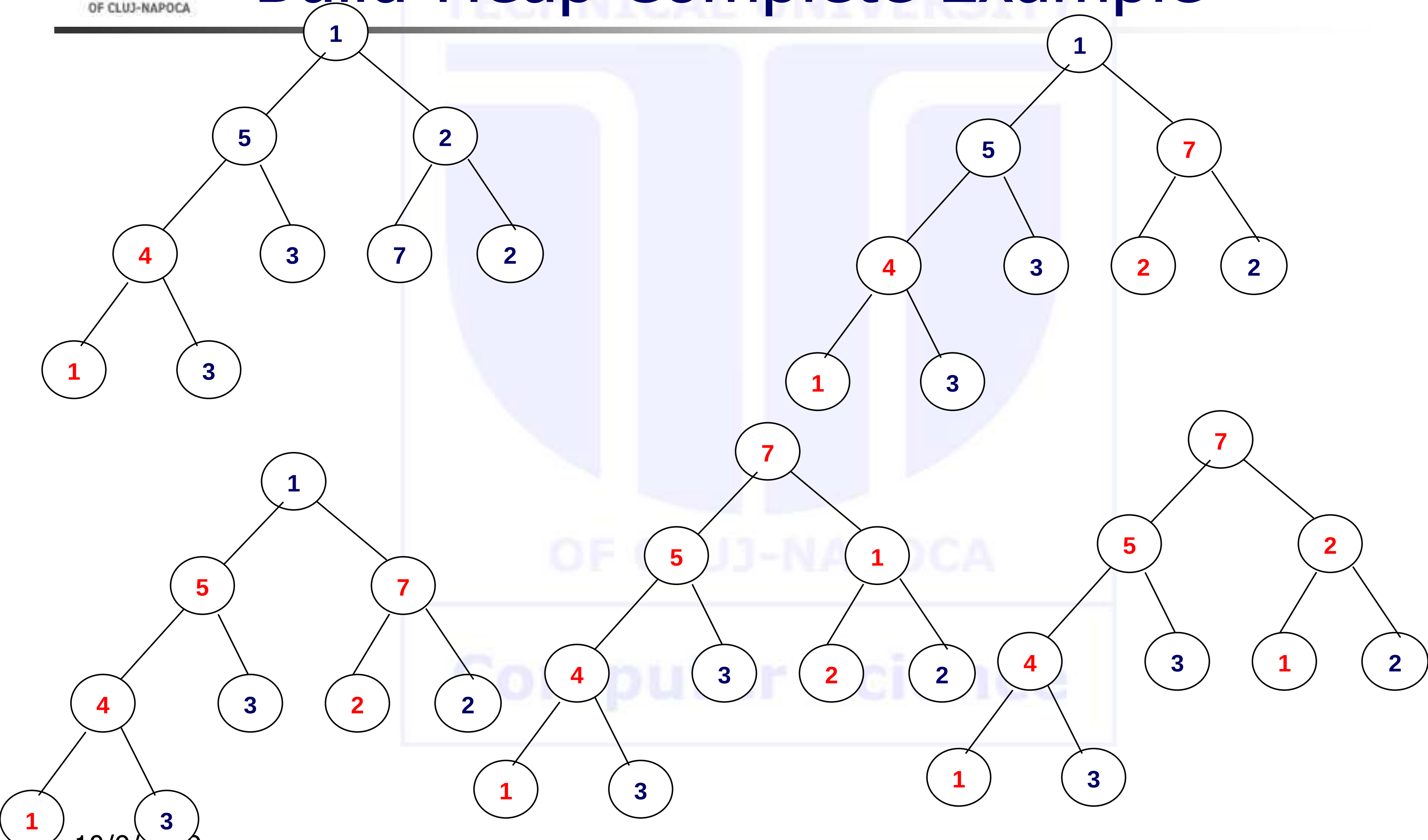
So $\sum h \cdot (1/2)^h = 2$, therefore **$t(n) = O(n)$**

Build-Heap Complete Example





Build-Heap Complete Example



Heapsort

- Heapsort – the complete technique
  - Build Heap which selects the max on the top of the heap
  - swap the top element (root) with the bottom one (last leaf) (i.e. move the max element in the last position of the array, where it belongs in the ordered array)
  - At this point, we destroyed both the heap structure, and we don't have an ordered one!

Heapsort cont.

- Heapsort – the technique –cont.
  - except for the **first** and **last** elements, we have a heap
  - from the second A[2] to the one before the last A[|A|-1] we have a heap
  - BUT the last element is in its right place in the ordered array already; consider it not more in the heap (thus, heap_size should decrement by 1)
  - apply heapify again on the new, smaller heap (without the last), for A[1] to sink that element in the right position
  - repeat the process until the dim of the heap becomes 1
  - while the heap's dimension decreases (by 1 each step, from the right), the already ordered array's dimension increases (with 1 each step, on the left)

Heapsort - code

```
HeapSort(A)
Build-Heap(A)   //generate the initial heap structure
heap_size[A]<-|A|
for i<-|A|downto 2//from the non-leave nodes
  do A[1]<->A[i]   //swap the root of the heap
                   //with the bottom element in the  current heap;
                   //array A[1..i-1] is a heap, array A[1..|A|] is
                   //an ordered structure
  heap_size[A]<-heap_size[A]-1
  heapify(A,1)// rebuild the heap struct. rom 1 to i
```

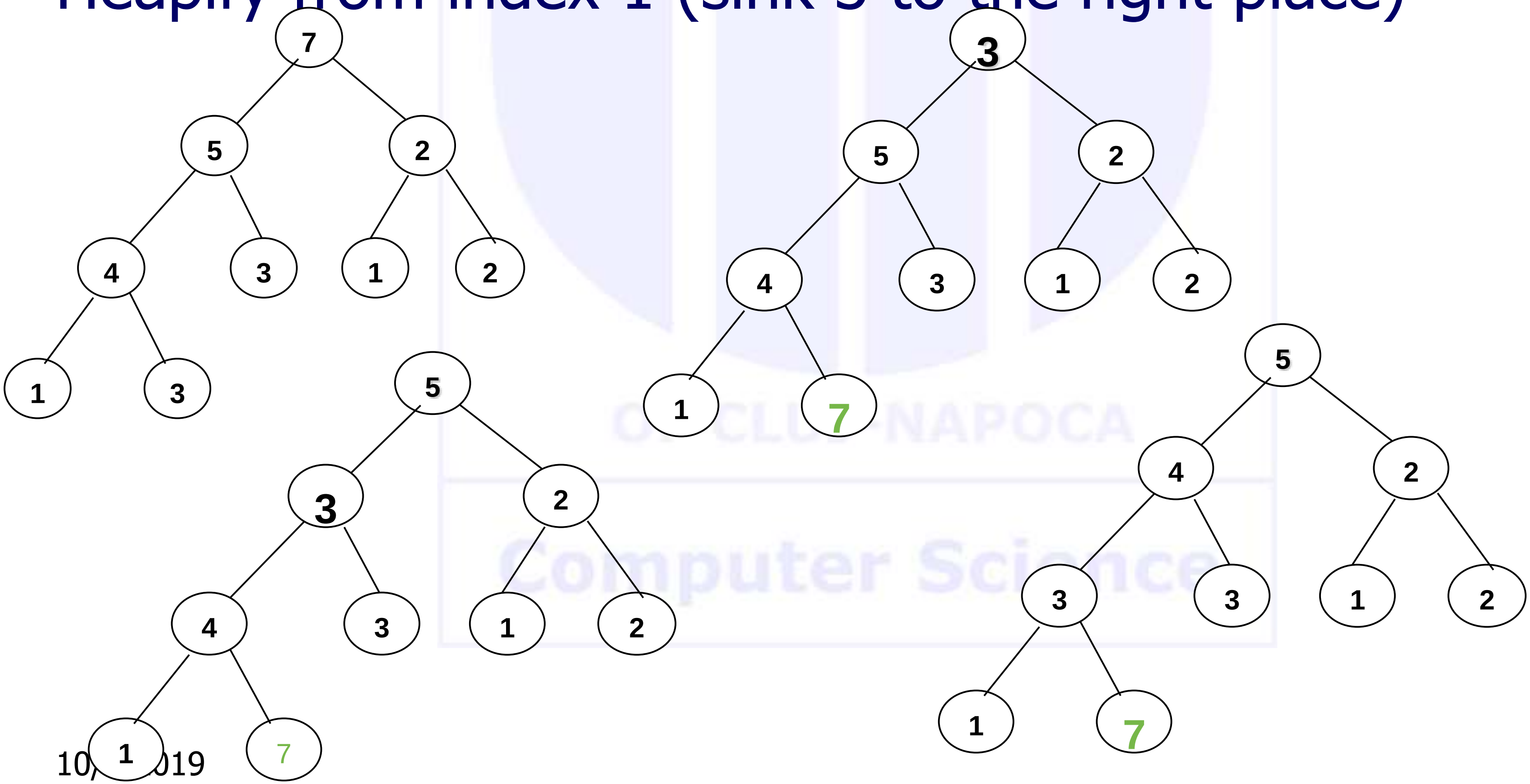
Heapsort - evaluation

- `Build-Heap(A` takes O(n)
- `for i<-|A|downto 2` repeats n times
- `heapify(A,1)` takes O(h) where h goes down from lgn to 1, so loop<=n·lgn
- $O(n)+O(n \cdot \lg n) = O(n \cdot \lg n)$
- $t_{HeapSort}$ = $\mathbf{O(n \cdot \lg n) = \Omega(n \cdot \lg n)}$
- Eval in worst case => **optimal algorithm**

Heapsort – complete example (after the heap was built – the for loop)

Swap 7 (top) with 3 (bottom)

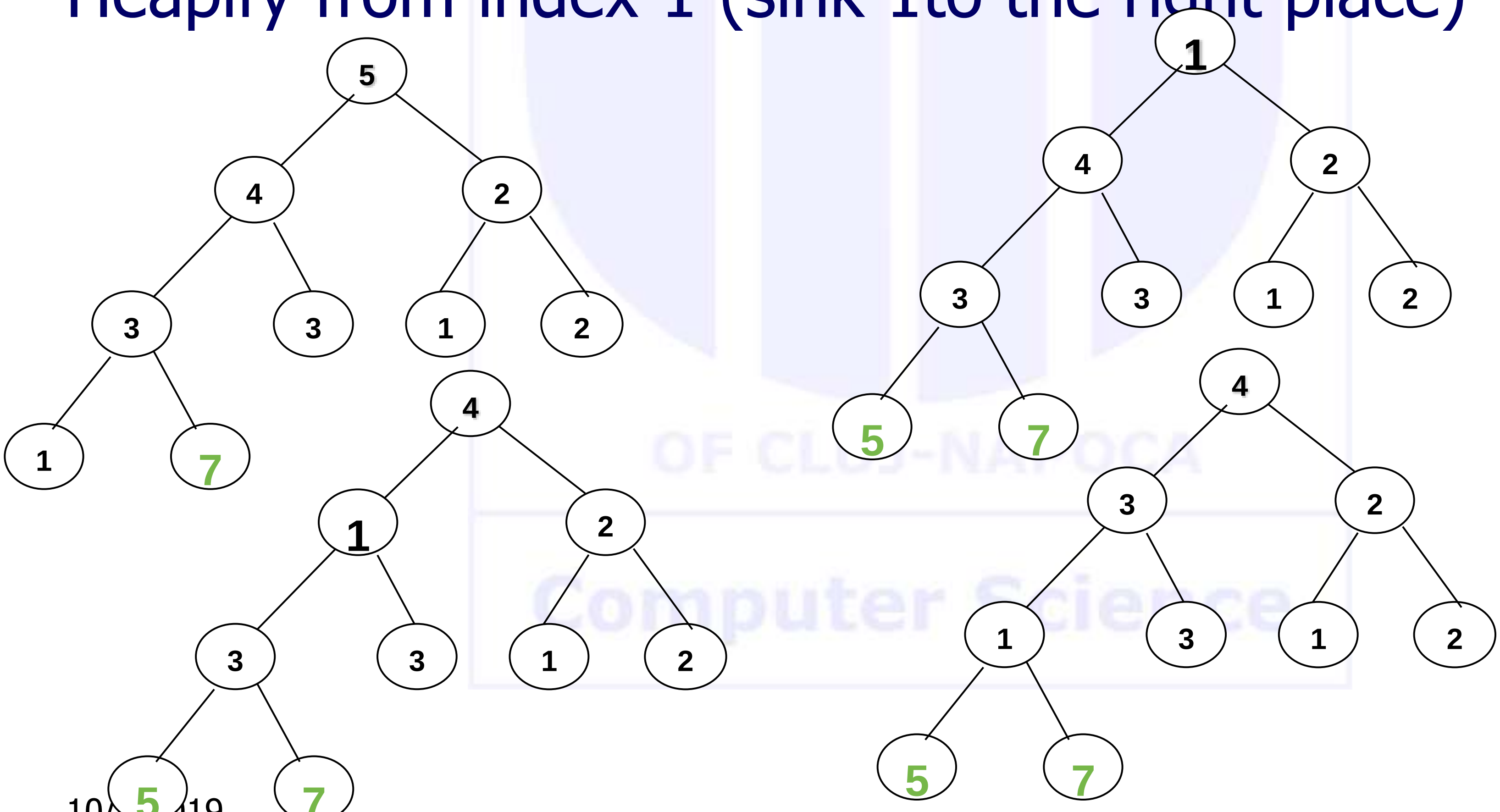
Heapify from index 1 (sink 3 to the right place)



Heapsort – complete example (green=sorted part; blue =heap part)

Swap 5 (top) with 1 (bottom)

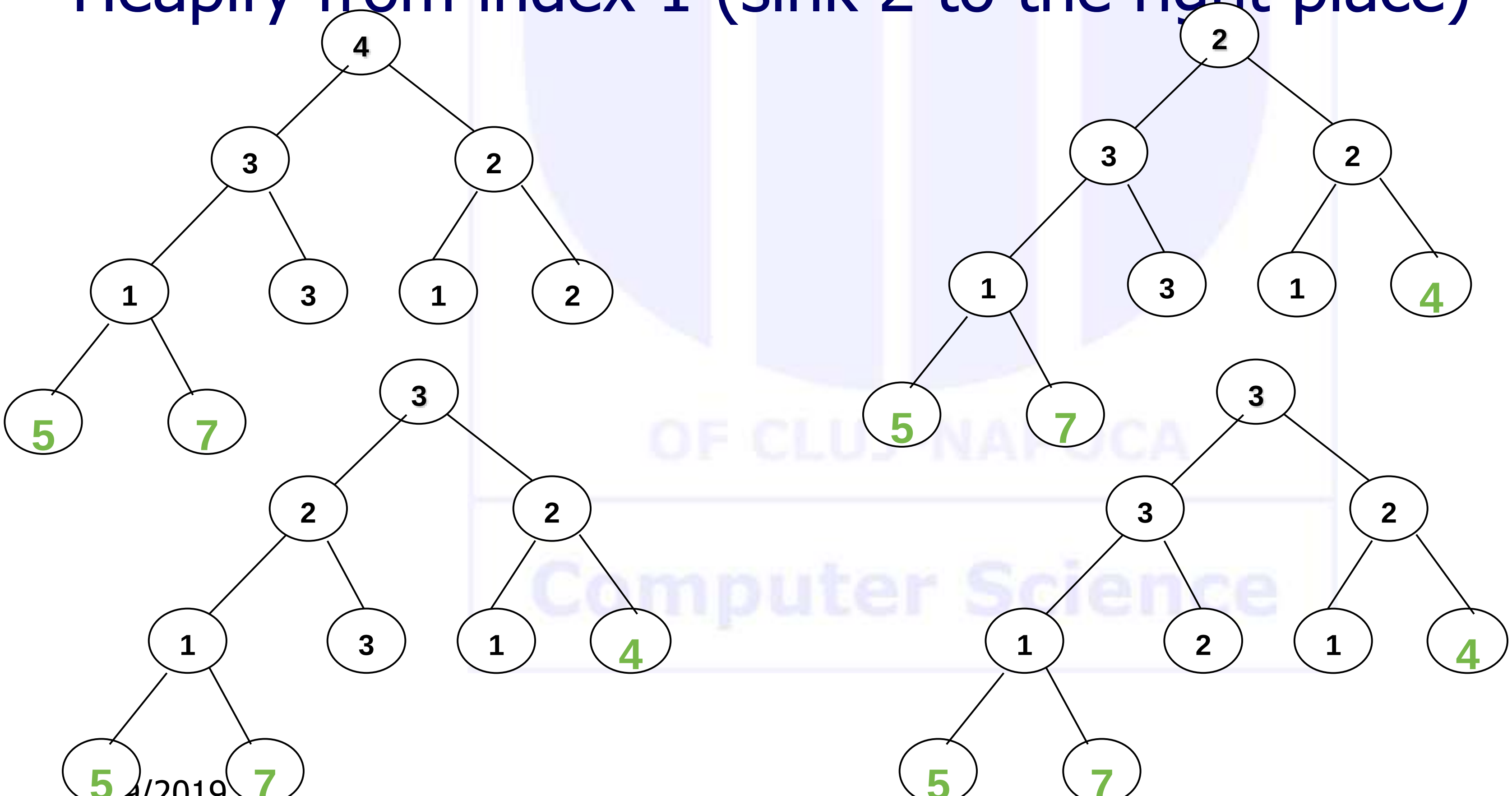
Heapify from index 1 (sink 1to the right place)



Heapsort – complete example (green=sorted part; blue =heap part)

Swap 4 (top) with 2 (bottom)

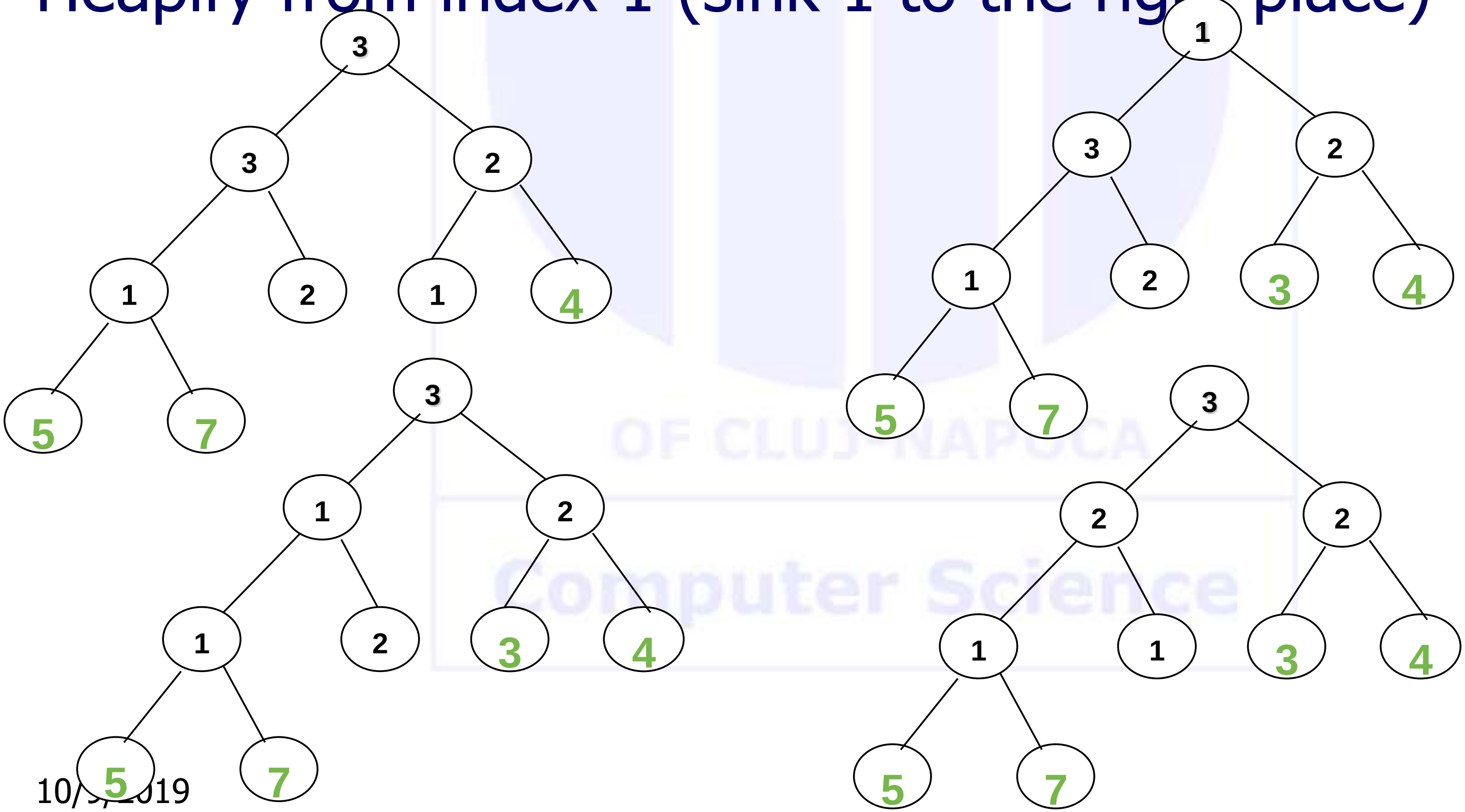
Heapify from index 1 (sink 2 to the right place)



Heapsort – complete example (green=sorted part; blue =heap part)

Swap 3 (top) with 1 (bottom)

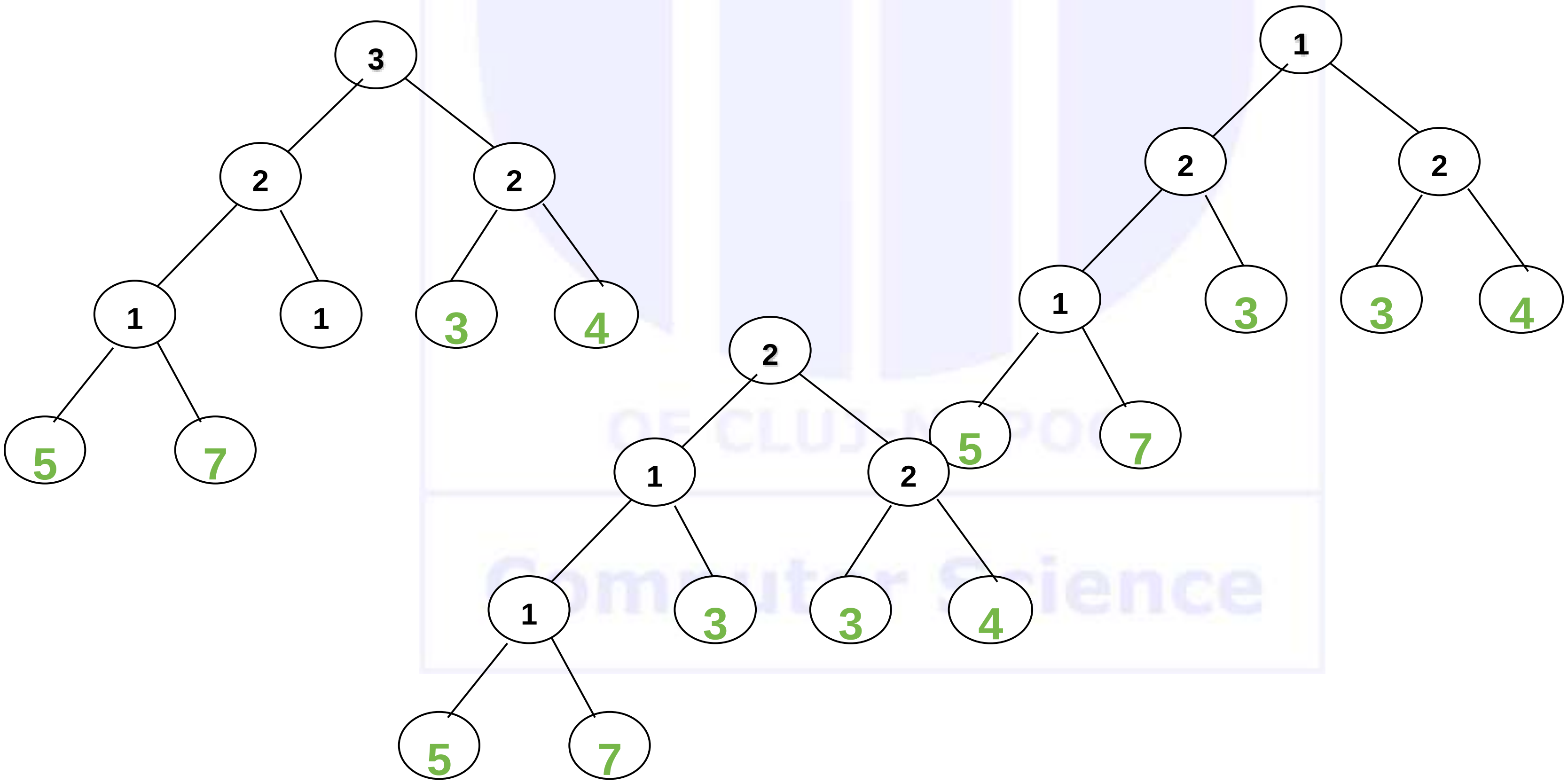
Heapify from index 1 (sink 1 to the right place)



Heapsort – complete example (green=sorted part; blue =heap part)

Swap 3 (top) with 1 (bottom)

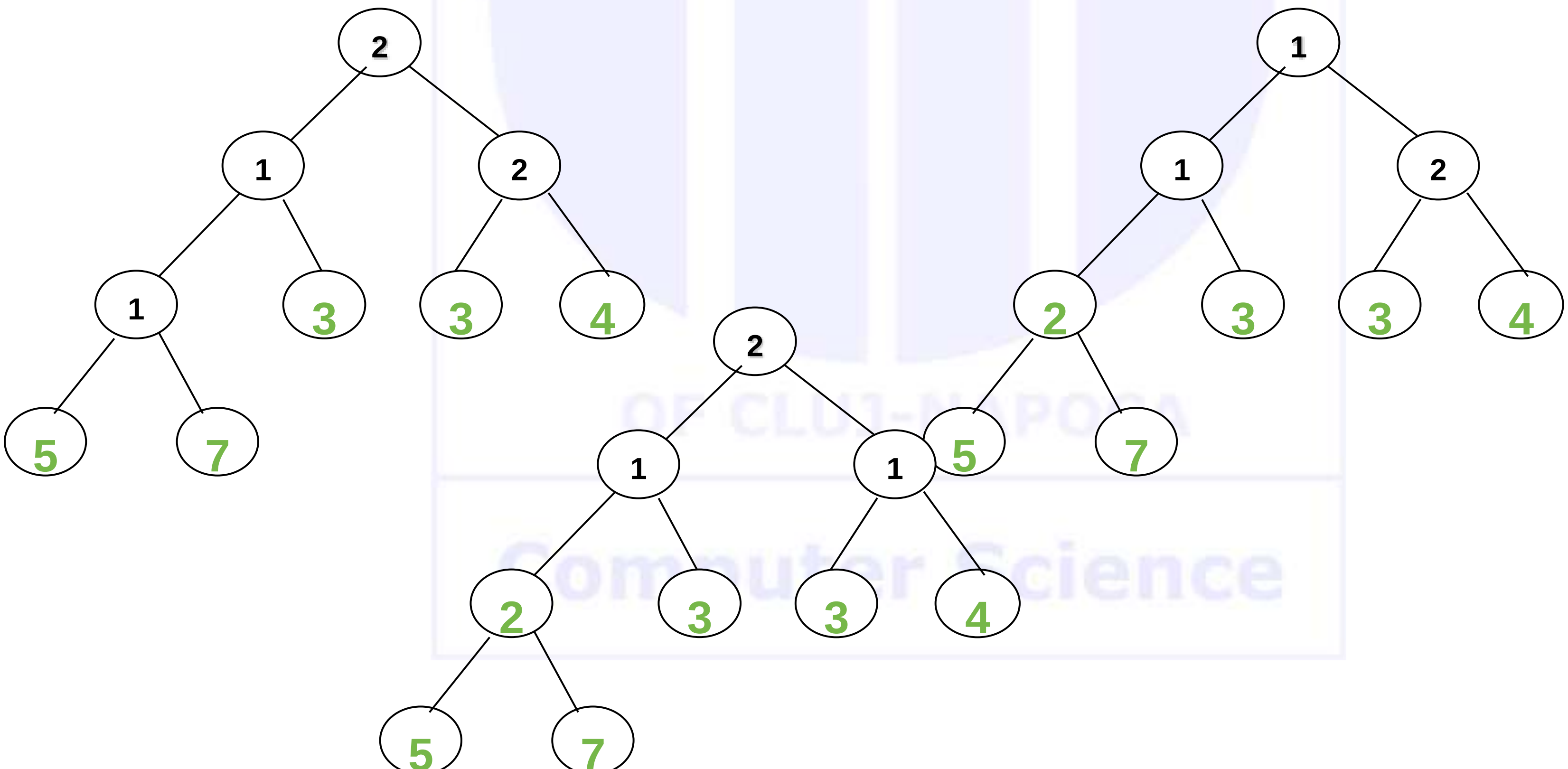
Heapify from index 1 (sink 1 to the right place)



Heapsort – complete example (green=sorted part; blue =heap part)

Swap 2 (top) with 1 (bottom)

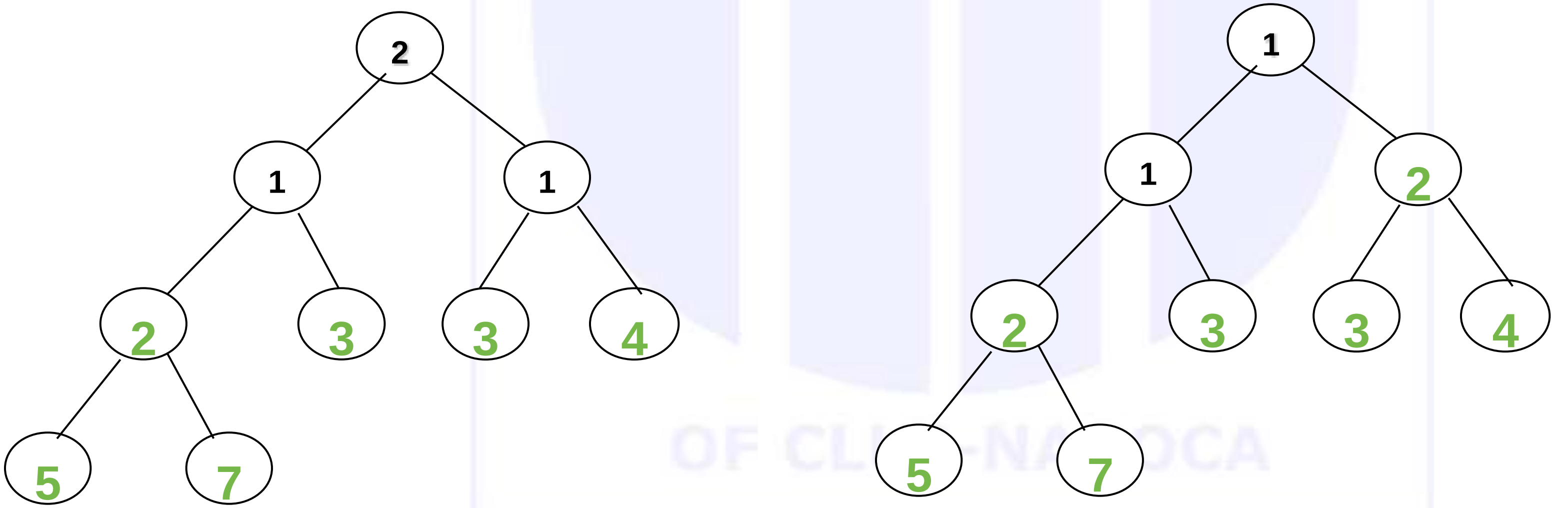
Heapify from index 1 (sink 1 to the right place)



Heapsort – complete example (green=sorted part; blue =heap part)

Swap 2 (top) with 1 (bottom)

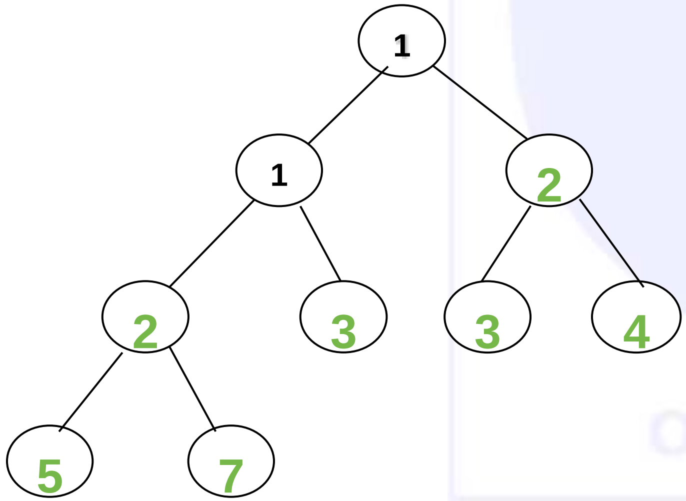
Heapify from index 1 (sink 1 to the right place)

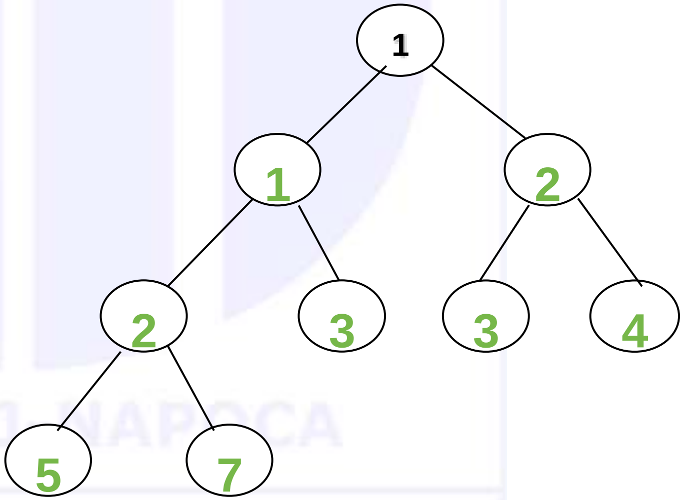


Heapsort – complete example (green=sorted part; blue =heap part)

Swap 1 (top) with 1 (bottom)

Only 1 element in the heap =>smallest=>all is array





Blue = elements in the heap

Green = elements in the ordered array

http://www.eecs.wsu.edu/~cook/aa/lectures/applets/sort1/heapsort.html

Heap – as a data structure

- Build heap strategy applies in case the **dimension** of the array is **known in advance** and has a **constant** value
- If not, define and use a heap as a data-structure => add dimension associated with the structure (size of the heap)
- Operations:
  - pop_heap extract the top from the heap
  - push_heap add one item to the heap

Heap – as a data structure – cont.

- pop_heap Extracts the top element
  - Move bottom element on top (swaps last with top, similar to 1 step of heapsort)
  - Decrements the heap size
  - Heapify the whole (from 1 to the new size), to update the heap structure =>O(lgn)
- push_heap
  - Adds a new element at the bottom
  - Rebuild heap, a bottom-up approach (bubble the bottom element upper in the heap, until it finds a larger-value parent) => O(h)=O(lgn)
- Examples on the blackboard

Heap – as a data structure – cont.

- build_heap
  - Repeats push_heap procedure
  - It takes $1+2\cdot 1+4\cdot 2+\ldots+n/2\cdot \lg n=O(n\lg n)$
- heap_sort
  - Build the heap (build_heap takes $O(n\lg n)$)
  - pop_heap (takes $O(\lg n)$)
  - add the poped element at bottom+1 (i.e. out of the heap, in the array)
  - It takes $O(n\lg n)$ (to build the heap)+ $O(n\lg n)$ (n times a pop operation)

Heap – comparison in building the heap

Approach	**Sol 1** (heapify)	**Sol2**(pop/push)
1 el approach	sinks the top (root) O(h)	bubbles a leaf O(h)
all els(build heap) approach	bottom-up (starts with the last nonleaf el)	top-down (adds a new leaf)
Time to build	O(n)	O(nlgn)
advantage	faster	variable dim
drawback	fixed dim	slower
usage	sorting	priority queues

Heap-Sort - Conclusions

- Optimal sorting algorithm
- In practice, quicksort, even not optimal by initial design (with its default/classic approach) behaves better
- Good quicksort implementations (avoid worst case OR ensure best case always) ARE optimal

- Review
- Divide et impera evaluation
- Particular cases
- Master Theorem
- Sorting
  - Heap Sort