Find the SVD decomposition for the matrix

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

We let $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

Because $A = U_0 \Sigma_0 V_0^*$ where U_1, V_1 are unitary matrixes, Σ_1 is a diagonal matrix

$$A^*A = (U_0\Sigma_0V_0^*)^*(U_0\Sigma_0V_0) = V_0\Sigma_0^*U_0^*U_0\Sigma_0V_0 = V_0\Sigma_0^*\Sigma_0V_0$$

But for

$$AA^* = (U_0\Sigma_0V_0^*)(U_0\Sigma_0V_0^*)^* = U_0\Sigma_0V_0^*V_0\Sigma_0^*U_0 = U_0\Sigma_0\Sigma_0^*U_0$$

So, there exists 2 different computational methods $\Rightarrow$ We compute it, respectively

① For Computing A^*A

$$A^*A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$b: b_1, b_2, b_3$

We let eigenvalues of A be $\lambda: \lambda_1, \lambda_2, \lambda_3$ and eigenvalues of A^*A be b

then we have the relationship like the following:

$$b_1 = \lambda_1^2 \quad b_2 = \lambda_2^2 \quad b_3 = \lambda_3^2$$

So $bI - A^*A = \begin{bmatrix} b-13 & -12 & -2 \\ -12 & b-13 & 2 \\ -2 & 2 & b-8 \end{bmatrix}$ and $\det(bI - A^*A) = 0$

therefore

$$\begin{vmatrix} b-13 & -12 & -2 \\ -12 & b-13 & 2 \\ -2 & 2 & b-8 \end{vmatrix} = \begin{vmatrix} b-25 & b-25 & 0 \\ -12 & b-13 & 2 \\ -2 & 2 & b-8 \end{vmatrix} = (b-25)\begin{vmatrix} 1 & 1 & 0 \\ -12 & b-13 & 2 \\ -2 & 2 & b-8 \end{vmatrix}$$

$$= (b-25)\begin{vmatrix} 1 & 1 & 0 \\ 0 & b-1 & 2 \\ 0 & 4 & b-8 \end{vmatrix} = (b-25)\begin{vmatrix} 1 & 0 & 0 \\ 0 & b-1 & 2 \\ 0 & 4 & b-8 \end{vmatrix} = (b-25)\begin{vmatrix} b-1 & 2 \\ 4 & b-8 \end{vmatrix}$$

$$= (b-25)[(b-1)(b-8) - 8]$$

$$= (b-25)(b^2 - 9b) = (b-25)(b-9)b$$

So $b_1 = 25$ $b_2 = 9$ $b_3 = 0$, and now we can deduce the eigenvectors of thee

(1) let $v_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ V_{31} \end{bmatrix}$ and

$$b_1 v_1 = A^*A v_1 \Rightarrow \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \\ V_{31} \end{bmatrix} = \begin{bmatrix} 25V_{11} \\ 25V_{21} \\ 25V_{31} \end{bmatrix}$$

$$\begin{aligned} 13V_{11} + 12V_{21} + 2V_{31} &= 25V_{11} \\ 12V_{11} + 13V_{21} - 2V_{31} &= 25V_{21} \\ 2V_{11} - 2V_{21} + 8V_{31} &= 25V_{31} \end{aligned} \Rightarrow \begin{aligned} &[1]\ -12V_{11} + 12V_{21} + 2V_{31} = 0 \\ &[2]\ 12V_{11} - 12V_{21} - 2V_{31} = 0 \\ &[3]\ 2V_{11} - 2V_{21} - 17V_{31} = 0 \end{aligned} \Rightarrow$$

$\uparrow [3] \times 6 + [2]$ $V_{31} = 0$

$V_{11} = V_{21}$

$v_1 = V_{11} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and v_1 can be an arbitrary number in $\mathbb{R}$.

we take $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and normalize it to become $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

(2) let $v_2 = \begin{bmatrix} V_{12} \\ V_{22} \\ V_{32} \end{bmatrix}$ and

$$b_2 v_2 = A^*A v_2 \quad \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \\ V_{32} \end{bmatrix} = \begin{bmatrix} 9V_{12} \\ 9V_{22} \\ 9V_{32} \end{bmatrix}$$

$$\begin{aligned} 13V_{12} + 12V_{22} + 2V_{32} &= 9V_{12} \\ 12V_{12} + 13V_{22} + (-2)V_{32} &= 9V_{22} \\ 2V_{12} - 2V_{22} + 8V_{32} &= 9V_{32} \end{aligned}$$

$[1]\ 4V_{12} + 12V_{22} + 2V_{32} = 0$

$[2]\ 12V_{12} + 4V_{22} - 2V_{32} = 0$

$[3]\ 2V_{12} - 2V_{22} - V_{32} = 0$

$\rightarrow [1] + [2]$ $16(V_{12} + V_{22}) = 0$

$+4V_{12} = V_{32}$

$v_2 = V_{12} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ and V_{12} can be an arbitrary number in $\mathbb{R}$

we take $v_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ and normalize it to become $\begin{bmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ 4/3\sqrt{2} \end{bmatrix}$

(3) let $v_3 = \begin{bmatrix} V_{13} \\ V_{23} \\ V_{33} \end{bmatrix}$ and

$$b_2 v_3 = A^*A v_3 \quad \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} \begin{bmatrix} V_{13} \\ V_{23} \\ V_{33} \end{bmatrix} = \begin{bmatrix} 0V_{13} \\ 0V_{23} \\ 0V_{33} \end{bmatrix}$$

$$\begin{cases} 13V_{13} + 12V_{23} + 2V_{33} = 0 \\ 12V_{13} + 13V_{23} - 2V_{33} = 0 \\ 2V_{13} - 2V_{23} + 8V_{33} = 0 \end{cases}$$

$25(V_{13} + V_{23}) = 0 \Rightarrow V_{13} = -V_{23}$

$V_{33} = \frac{V_{23}}{2}$

$-4V_{23} + 8V_{33} = 0$ $V_{23} = 2V_{33}$

so $v_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ we take $v_3 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$ and normalize it to become $\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

show that if a matrix A is both triangular and unitary, then it's diagonal

Combine (1) (2) (3), we can get

$$\begin{bmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & -2/3 \\ 1/\sqrt{2} & -1/3\sqrt{2} & 2/3 \\ 0 & 4/3\sqrt{2} & 1/3 \end{bmatrix} = V_1$$

For U we can use $A = U\Sigma V^* \Rightarrow AV = U\Sigma$ Thus $\begin{cases} AV_1 = \lambda_1 U_1 \\ AV_2 = \lambda_2 U_2 \\ AU_3 = \lambda_3 U_3 \end{cases}$

① $AV_1 = \lambda_1 U_1$, we take $U_1 = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}$ and $\lambda_1^2 = 25$ ($\lambda = 5 > 0$)

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \\ V_{31} \end{bmatrix} = \begin{bmatrix} 5U_{11} \\ 5U_{21} \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 5U_{11} \\ 5U_{21} \end{bmatrix}$$

$$\begin{bmatrix} 5U_{11} \\ 5U_{21} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \end{bmatrix} \qquad \begin{matrix} U_{11} = 1/\sqrt{2} \\ U_{21} = 1/\sqrt{2} \end{matrix} \qquad \text{Thus } U_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

② $AV_2 = \lambda_2 U_2$ we take $U_2 = \begin{bmatrix} U_{12} \\ U_{22} \end{bmatrix}$ $\lambda_2^2 = 9$ ($\lambda = 3 > 0$)

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \\ V_{32} \end{bmatrix} = \begin{bmatrix} 3U_{12} \\ 3U_{22} \end{bmatrix} \qquad \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ 4/3\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3U_{12} \\ 3U_{22} \end{bmatrix}$$

$$\begin{cases} 3/3\sqrt{2} - 2/3\sqrt{2} + 8/3\sqrt{2} = 3U_{12} \\ 2/3\sqrt{2} + 3\cdot(-1/3\sqrt{2}) - 8/3\sqrt{2} = 3U_{22} \end{cases} \Rightarrow \begin{cases} 3U_{12} = 9/3\sqrt{2} \\ 3U_{22} = -9/3\sqrt{2} \end{cases} \Rightarrow \begin{cases} U_{12} = 1/\sqrt{2} \\ U_{22} = -1/\sqrt{2} \end{cases} \quad U_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Thus $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$

③ $AV_3 = \lambda_3 V_3$ We take $U_3 = \begin{bmatrix} U_{13} \\ U_{23} \end{bmatrix}$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} V_{13} \\ V_{23} \\ V_{33} \end{bmatrix} = \begin{bmatrix} 0\, V_{13} \\ 0\, V_{23} \end{bmatrix}$$ 0 and U_{13} U_{23} can be any value

2 For computing AA^*

$$AA^* = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

we let eigenvalues of A be λ: λ_1, λ_2 and eigenvalues of AA^* be C: $C_1 C_2$ then we have the relationship like the following:

$$C_1 = \lambda_1^2 \qquad C_2 = \lambda_2^2$$

So $CI - AA^* = \begin{bmatrix} C-17 & -8 \\ -8 & C-17 \end{bmatrix} \Rightarrow \det(CI - AA^*) = (C-17)^2 - 64 = 0$ $\quad (C-17)^2 = 64$ $\quad C-17 = \pm 8$ $\quad \begin{cases} C_1 = 25 \\ C_2 = 9 \end{cases}$

and now we can deduce the eigenvectors of these

(1) let $U_1' = \begin{bmatrix} U_{11}' \\ U_{21}' \end{bmatrix}$ $\quad C_1 U_1 = AA^* U_1$ $\quad \begin{bmatrix} 25U_{11}' \\ 25U_{21}' \end{bmatrix} = \begin{bmatrix} 17 & +8 \\ +8 & 17 \end{bmatrix} \begin{bmatrix} U_{11}' \\ U_{21}' \end{bmatrix}$

$$\begin{cases} 8u'_{11} - 8u'_{21} = 0 \\ -8u'_{11} + 8u'_{21} = 0 \end{cases} \quad u_{11} = u'_{21} \quad u'_1 = u'_{11}\begin{bmatrix}1\\1\end{bmatrix}$$ and u'_{11} can be an arbitrary number

u'_1 can be taken as $\begin{bmatrix}1\\1\end{bmatrix}$

Then we normalize it and become $\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$

(2) Let $u'_2 = \begin{bmatrix}u'_{12}\\u'_{22}\end{bmatrix}$ $\quad C_2 u'_2 = AA^* u'_2 \Rightarrow \begin{bmatrix}9u'_{12}\\9u'_{22}\end{bmatrix} = \begin{bmatrix}17 & 8\\8 & 17\end{bmatrix}\begin{bmatrix}u'_{12}\\u'_{22}\end{bmatrix}$

$$\begin{cases} 8u'_{12} + 8u'_{22} = 0 \\ 8u'_{12} + 8u'_{22} = 0 \end{cases} \quad u'_{12} = -u'_{22} \quad u'_2 = u'_{12}\begin{bmatrix}1\\-1\end{bmatrix}$$ and u'_{12} can be an arbitrary number

u'_2 Can be taken as $\begin{bmatrix}1\\-1\end{bmatrix}$

Then we normalize it and become $\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}$

Thus $U = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$

$$A = U\Sigma V^* \qquad A^* = V\Sigma^* U^* \qquad A^* U = V\Sigma^*$$

So $A^* u_1 = \lambda_1 v_1$ and $A^* u_2 = \lambda_2 v_2$ $\quad$ we let $v_1 = \begin{bmatrix}v'_{11}\\v'_{21}\\v'_{31}\end{bmatrix}$

① $\begin{bmatrix}3 & 2\\2 & 3\\2 & -2\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}5v'_{11}\\5v'_{21}\\5v'_{31}\end{bmatrix}$ $\quad \begin{bmatrix}5v'_{11}\\5v'_{21}\\5v'_{31}\end{bmatrix} = \begin{bmatrix}5/\sqrt{2}\\5/\sqrt{2}\\0\end{bmatrix} \Rightarrow v_1 = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}$

② we let $v_2 = \begin{bmatrix}v'_{12}\\v'_{22}\\v'_{32}\end{bmatrix}$ $\quad \lambda_2^2 = 9 = C_2 \quad \lambda_2 = 3 > 0$

$\begin{bmatrix}3 & 2\\2 & 3\\2 & -2\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}3v'_{12}\\3v'_{22}\\3v'_{32}\end{bmatrix}$ $\quad \begin{bmatrix}v'_{12}\\v'_{22}\\v'_{32}\end{bmatrix} = \begin{bmatrix}1/3\sqrt{2}\\-1/3\sqrt{2}\\4/3\sqrt{2}\end{bmatrix}$ $\quad v_2 = \begin{bmatrix}1/3\sqrt{2}\\-1/3\sqrt{2}\\4/3\sqrt{2}\end{bmatrix}$

③ we let $v_3 = \begin{bmatrix}v'_{13}\\v'_{23}\\v'_{33}\end{bmatrix}$ and that doesn't not work

$$A = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}5 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/3\sqrt{2}\\1/\sqrt{2} & -1/3\sqrt{2}\\0 & 4/3\sqrt{2}\end{bmatrix}^*$$

$$= \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}5 & 0\\0 & 3\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\1/3\sqrt{2} & -1/3\sqrt{2} & 4/3\sqrt{2}\end{bmatrix}$$

Show that if a matrix A is both triangular and unitary, then it is diagonal

Solution: we will use the induction to prove it.

① $n=1$, we can give 1×1 matrix about this element $a \in \mathbb{C}^n$,

$[a]$ this is automatically satisfied.

② $n=2$, we can give the matrix $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ (which is upper-triangular)

and entries in matrix are in complex number $\mathbb{C}$.

Therefore $A=\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ $\quad A^*A=\begin{bmatrix} \bar{a} & 0 \\ \bar{b} & \bar{c} \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}=\begin{bmatrix} |a|^2 & \bar{a}b \\ \bar{b}a & |b|^2+|c|^2 \end{bmatrix}$

Because A is unitary so $A^*=A^{-1}$ $\quad \bar{b}a=0 \quad \bar{a}b \quad |\bar{b}|^2+|\bar{c}|^2=1 \quad |\bar{a}|^2=1$

Thus $b=0$ so $A=\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$ This matrix A is diagonal.

③ Suppose $n=k\in\mathbb{N}^+$, this matrix is also diagonal under these conditions.

For $n=k+1$, we can rewrite matrix A into the following:

$$A=\begin{bmatrix} A_k & \alpha \\ \beta & C_2 \end{bmatrix}=\begin{bmatrix} A_k & \alpha \\ 0 & C_2 \end{bmatrix}$$

where β is a raw vector and its value is 0

and α is a column vector with nonzero entries

and $C_2\in\mathbb{C}$ and all nonzero entries in complex number set.

So A^* is upper-triangular matrix $\quad A_k$ is a diagonal as we supposed before.

$$A^*A=\begin{bmatrix} A_k^* & 0 \\ \alpha^* & \bar{C_2} \end{bmatrix}\begin{bmatrix} A_k & \alpha \\ 0 & C_2 \end{bmatrix}=\begin{bmatrix} I_k & A_k^*\alpha \\ \alpha^*A_k & |C_2|^2 \end{bmatrix}$$

where I_k is identity with rank k.

Because A is unitary $\quad A^*A=I$

$\alpha^*A_k=0 \quad A_k^*\alpha=0 \quad |C_2|^2=1 \quad$ A_k is not zero matrix so α is zero vector.

Thus $A=\begin{bmatrix} A_k & 0 \\ 0 & C_2 \end{bmatrix}$ A is diagonal.

$AA^* = A^*A$ A^*A 3×2 AA^* 3×3

$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 4-\lambda \end{bmatrix}$ | III IV

A is square $\Rightarrow$

A^*A AA^*

posit def. sy $\Rightarrow$ >0

$\begin{bmatrix} 1 & \vdots \\ 0 \end{bmatrix}$ $AX = 0$ $\begin{cases} A \text{非} & \det A \neq 0 \quad X = 0 \\ A \text{奇} & \det A = 0 \quad \text{rand}A = \end{cases}$

The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{x_i\}$

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \| x_i \|^2$$

(a) prove this in the case $n=2$ by an explicit computation of $\|x_1 + x_2\|^2$.

(b) show that this computation also establishes the general case, by induction.

(a) Proof: $\left\| \sum_{i=1}^{2} x_i \right\|^2 = \|x_1 + x_2\|^2 = (x_1+x_2)^*(x_1+x_2) = (x_1+x_2)^* x_1 + (x_1+x_2)^* x_2$

$= x_1^* x_1 + x_2^* x_1 + x_1^* x_2 + x_2^* x_2$

Since $\{x_i\}$ is a set of 2 orthogonal vectors. $x_2^* x_1 = 0 = x_1^* x_2$.

$\left\| \sum_{i=1}^{2} x_i \right\|^2 = x_1^* x_1 + x_2^* x_2 = \|x_1\|^2 + \|x_2\|^2 = \sum_{i=1}^{2} \|x_i\|^2$. The equation holds.

(b) For $n=2$, (a) has already proved.

Suppose $n = k \in \mathbb{N}^+$, the equation still holds, i.e.

$$\left\| \sum_{i=1}^{k} x_i \right\|^2 = \sum_{i=1}^{k} \|x_i\|^2$$

Thus, when $n = k+1$, there exists

$$\left\| \sum_{i=1}^{k} x_i + x_{k+1} \right\|^2 = \left\| \sum_{i=1}^{k+1} x_i \right\|^2 = \left\| \sum_{i=1}^{k} x_i \right\|^2 + \left(\sum_{i=1}^{k} x_i\right)^* x_{k+1} + x_{k+1}^* \left(\sum_{i=1}^{k} x_i\right) + x_{k+1}^* x_{k+1}$$

Since $\{x_i\}$ is a set of 2 orthogonal vectors.

$\left(\sum_{i=1}^{k} x_i\right)^* x_{k+1} = \left(\sum_{i=1}^{k} x_i^*\right) x_{k+1} = \sum_{i=1}^{k} x_i^* x_{k+1}$ Then $x_i^* x_{k+1} = 0$ for $1 \le i \le k$

$x_{k+1}^* \left(\sum_{i=1}^{k} x_i\right) = x_{k+1}^* \left(\sum_{i=1}^{k} x_i\right) = \sum_{i=1}^{k} x_{k+1}^* x_i \Rightarrow$ Then $x_{k+1}^* x_i = 0$ for $1 \le i \le k$.

Then $\left\| \sum_{i=1}^{k} x_i + x_{k+1} \right\|^2 = \left\| \sum_{i=1}^{k} x_i \right\|^2 + \|x_{k+1}\|^2 = \sum_{i=1}^{k} \|x_i\|^2 + \|x_{k+1}\|^2 = \sum_{i=1}^{k+1} \|x_i\|^2$.

This computation also holds.

If u and v are m-vectors, the matrix $A = I + uv^*$ is known as a rank-one perturbation of the identity. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and give an expression for α. For what u and v is A singular? If it is singular, what is null(A)?

[1] Consider the condition for A is nonsingular.

Let $A^{-1} = [a_1 \,\vdots\, a_2 \,\vdots\, a_3 \,\vdots\, \cdots \,\vdots\, a_m]$ and a_i be vector in A^{-1} for $1 \le i \le m$.

so $AA^{-1} = (I + uv^*)[a_1 \,\vdots\, a_2 \,\vdots\, a_3 \,\vdots\, \cdots \,\vdots\, a_n]$

$$= [a_1 + uv^*a_1 \,\vdots\, a_2 + uv^*a_2 \,\vdots\, a_3 + uv^*a_3 \,\vdots\, \cdots \,\vdots\, a_n + uv^*a_n] = I$$

we let $I = [e_1 \,\vdots\, e_2 \,\vdots\, e_3 \,\vdots\, \cdots \,\vdots\, e_m]$ where e_i be vector in identity for $1 \le i \le m$

$a_1 + uv^*a_1 = e_1 \qquad a_2 + uv^*a_2 = e_2 \quad \cdots \qquad a_m + uv^*a_m = e_m$

we can summary it into:

$e_i = a_i + uv^*a_i$ for $1 \le i \le m \qquad a_i = e_i - uv^*a_i$

Because v^*a_i is a valuse, when we compute and combine it:

$A^{-1} = I - u\theta^* \qquad \theta^* = (v^*a_1, v^*a_2, \cdots, v^*a_m)$

$AA^{-1} = (I + uv^*)(I - u\theta^*)$

$= I - u\theta^* + uv^* - uv^*u\theta^* = I \Rightarrow uv^* = u\theta^* + uv^*u\theta^*$

Since v^*u is a value

$\cancel{AA} \Rightarrow uv^* = u\theta^* + (v^*u)u\theta^* \qquad uv^* = (1 + v^*u)u\theta^*$

$uv^* = u[1 + v^*u]\theta^* \qquad \theta^* = \dfrac{v^*}{1 + v^*u}$

Thus A has the inverse with the form $A^{-1} = I + \alpha uv^*$, where $\alpha = \dfrac{\text{[illegible]}}{1 + v^*u}$

[2] consider the condition for A is singular.

$\exists X \ne 0 \quad AX = 0$

$(I + uv^*)X = 0 \qquad X = -uv^*X$ Because v^*X is a value, we can denote it as k

$X = ku \qquad$ so $(I + uv^*)ku = ku + uv^*ku = 0$

$ku + kuv^*u = 0$

$ku(1 + v^*u) = 0$

$\begin{cases} ku \ne 0 \Rightarrow \\ 1 + v^*u = 0 \end{cases}$

Combine ~~(1) (2) (3), we can~~ get

$$\begin{bmatrix} \cancel{1/\sqrt{2}} & \cancel{1/3\sqrt{2}} & \cancel{-2/3} \\ 1/\sqrt{2} & \cancel{-1/3\sqrt{2}} & \cancel{2/3} \\ 0 & \cancel{4/3\sqrt{2}} & 1/3 \end{bmatrix} = V_1$$

For U we can use $A = U_\phi \Sigma_\phi V_\phi^*$ $\Rightarrow$ $A V_\phi = U_\phi \Sigma_\phi$

Thus $AV_1 = \lambda_1 U_1$ $\quad AV_2 = \lambda_2 V_2$ $\quad AV_3 = \lambda_3 V_3$.

① $AV_1 = \lambda_1 U_1$ $\quad \lambda_1^2 = 25$ $\quad \cancel{\lambda_1 = 5 > 0}$ ~~we let~~ $U_1 = \begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{bmatrix}$

$$A\begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 5U_{11} \\ 5U_{21} \\ 5U_{31} \end{bmatrix} \Rightarrow \begin{matrix} 25/\sqrt{2} = 5U_{11} \\ 25/\sqrt{2} = 5U_{21} \\ 0 = 5U_{31} \end{matrix} \Longrightarrow \begin{bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{bmatrix} = \begin{bmatrix} 5/\sqrt{2} \\ 5/\sqrt{2} \\ 0 \end{bmatrix}$$

---

②

But $ku \neq 0$ for sure $\quad 1 + V^* u = 0 \quad V^* u = -1. \Rightarrow$ This is condition.

As for $null(A)$ if A is singular

$$null(A) = \{x \mid Ax = 0 \quad x \neq 0\} = span\{x\} = span\{u\}.$$