$\boxed{\text{Theorem 5.8}}$ For any v with $0 \le v \le r$, define $A_v = \sum_{j=1}^{v} \sigma_j u_j v_j^*$
if $v = p = \min\{m, n\}$, define $\sigma_{v+1} = 0$. Then.

$$\|A - A_v\|_2 = \inf_{\substack{B \in \mathbb{C}^{m\times n} \\ \text{rank}(B) \le v}} \|A - B\|_2 = \sigma_{v+1}$$

proof: $\boxed{\text{Step 1}}$ Using the properties of matrix norms and the SVD:

$\|A - A_v\|_2 = \|\sum_{j=v+1}^{p} \sigma_j u_j v_j^*\| = \sigma_{v+1}$ (since σ_{v+1} is the largest singular value ~~the maximum above~~.

That establish one parts of the equality.

$\boxed{\text{Step 2}}$ Let's prove that for any matrix B with rank $B \le v$. $\|A-B\|_2 \ge \sigma_{v+1}$

$\boxed{\text{Proof by contradiction}}$ $\Rightarrow$ Assume there exists a matrix B with rank $B \le v$ s.t. $\|A - B\|_2 < \sigma_{v+1}$ $\boxed{\|(A-B)x\|_2 < \sigma_{v+1}}$

$w \in W \Rightarrow n-v$ dimensional

$\|Aw\|_2 = \|(A-B)w\|_2 \le \|A-B\|_2 \|w\|_2 < \sigma_{v+1} \|w\|_2$ $\rightarrow$ $\|AX = \sum_{j=1}^{p} \sigma_j (u_j^* x) v_j$.

But using the SVD, $Ax = \sum_{j=1}^{p} \sigma_j (u_j^* x) u_{j+1}$

Since B has rank at most v, it can be expressed in terms of at most v of the singular vectors of A. Thus the vector $(A-B)x$ has a component in the direction of V_{v+1}, which has magnitude at least σ_{v+1} $\Rightarrow$ $\|A-B\|_2 \ge \sigma_{v+1}$

$\boxed{\text{step 3}}$ $\sigma_{v+1} \le \|A - B\|_2 \xrightarrow{\text{with the equality } B = A_v} \|A - A_v\|_2 = \inf_{\substack{B \in \mathbb{C}^{m\times n} \\ \text{rank}(B) \le v}} \|A-B\|_2 = \sigma_{v+1}$

$\boxed{\text{Theorem 5.9}}$ For any v with $0 \le v \le r$, the matrix A_v of $A_v = \sum_{j=1}^{v} \sigma_j u_j v_j^*$ also satisfies

$$\|A - A_v\|_F = \inf_{\substack{B \in \mathbb{C}^{m\times n} \\ \text{rank}(B) \le v}} \|A - B\|_F = \sqrt{\sigma_{v+1}^2 + \cdots + \sigma_r^2}$$

proof: $\boxed{\text{Step 1}}$ Using the properties of matrix norms and the SVD

$\|A - A_v\|_F = \|\sum_{j=v+1}^{p} \sigma_j u_j v_j^*\|_F \Rightarrow u_j v_j^*$ for ~~$(i \ne j)$~~ for $(i$ in vector$)$ $u_i v_j^* = 0$

$= \sqrt{\sigma_{v+1}^2 + \cdots + \sigma_r^2}$

$\boxed{\text{step 2}}$ $\|A - B\|_F \ge \sqrt{\sigma_{v+1}^2 + \cdots + \sigma_r^2}$ ※

Suppose $\|A - B\|_F < \sqrt{\sigma_{v+1}^2 + \cdots + \sigma_r^2}$ $\sqrt{\sigma_?^2 \cdots +}\|A-B\|_F$

$\|(A-B)X\|_F \le \|A-B\|_F \|X\|_F < \sqrt{\sigma_{v+1}^2 + \cdots + \sigma_r^2}\, \|X\|_F.$

$\|A\|_F$ $BX = 0$ for $(n-v)$-dimensional subspace

$B = A_v$

$\|A - A_v\|_F = \inf \|A - B\|_F = \sqrt{\sigma_{v+1}^2 + \cdots + \sigma_r^2}$

But $\|A - B\|_F \ge \sqrt{\sigma_{v+1}^2 + \cdots + \sigma_r^2}\, \|X\|_F$ $\Leftarrow$ when rank $B \le v$ the inequality holds

Exercise 5.

5.1 In example 3.1 we considered the matrix (3.7) $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $\left\{ \begin{array}{ll} \infty \text{ Norm} \Rightarrow & 4 \\ 1 \text{ Norm} & 3 \\ 2 \text{ Norm} & 2.9208 \end{array} \right.$
and asserted, among other things, that its 2-norm is approximately 2.9208. Using the SVD, work out (on paper) the exact values of $\sigma_{min}(A)$ and $\sigma_{max}(A)$ for this matrix.

Solution: $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ $A^* = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ $A^*A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$

$\det|\lambda^2 I - A^*A| = \begin{vmatrix} \lambda^2 - 1 & -2 \\ -2 & \lambda^2 - 8 \end{vmatrix} = (\lambda^2-1)(\lambda^2-8) - 4 = 0$ $\quad \lambda^4 - 9\lambda^2 + 4 = 0$

$\lambda^2 = \frac{9 \pm \sqrt{65}}{2}$

Thus $\sigma_{min}(A) = \left(\frac{9-\sqrt{65}}{2}\right)^{\frac{1}{2}}$ $\quad \sigma_{max}(A) = \left(\frac{9+\sqrt{65}}{2}\right)^{\frac{1}{2}}$.

5.2 Using the SVD, prove that any matrix in $\mathbb{C}^{m\times n}$ is the limit of a sequence of matrices of full rank. In other words, prove that the set of full-rank matrices is a dense subset of $\mathbb{C}^{m\times n}$. Use the 2-norm for your proof. (The norm doesn't matter, since all norms on a finite-dimensional space are equivalent)

To show that the set of full-rank matrices is dense in $\mathbb{C}^{m\times n}$, we need to prove that for any matrix A in $\mathbb{C}^{m\times n}$ and for any $\varepsilon > 0$, there exists a full-rank matrix B. s.t. $\|A-B\|_2 < \varepsilon$.

Without loss of generality, Let's consider the case where $m \le n$.

$\boxed{\text{1. Case when } A \text{ is full rank}}$ If A already has full rank, then we can set $B = A$ $\quad \|A-B\|_2 = 0 < \varepsilon \ \ \forall \varepsilon > 0$

$\boxed{\text{2. Case when } A \text{ is rank-deficient}}$ Let r be the rank of A, where $r < m$.

$A = U\Sigma V^*$ and Σ has singular values like $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > \sigma_{r+1} = \cdots = \sigma_m = 0$

Define a new diagonal matrix $\tilde{\Sigma}$ s.t. it retains the non-zero singular values of Σ and replace σ_{r+1} with a small value $\delta > 0$ and $\tilde{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r, \delta, 0, \cdots 0)$

Then we will use SVD to construct B by using the same unitary matrix

$B = U\tilde{\Sigma}V^*$ $\qquad$ To ensure $\|A-B\|_2 < \varepsilon$, we need to choose δ s.t.

$\|A-B\|_2 = \|U(\Sigma - \tilde{\Sigma})V^*\|_2 = \|\Sigma - \tilde{\Sigma}\|_2 = \delta < \varepsilon$

This shows for any full rank ~~or~~ rank-deficient matrix A and any $\varepsilon > 0$, there exists B

$\|A-B\|_2 < \varepsilon$ ∎

5.3 Consider the matrix $A=\begin{bmatrix}-2 & 11\\ -10 & 5\end{bmatrix}$

(a) Determine, on paper, a real SVD of A in the ~~form~~ form $A=U\Sigma V^T$. The SVD is not uniqu

So find the one that has the minimal number of minus signs in U and V.

$$A^TA=\begin{bmatrix}-2 & -10\\ 11 & 5\end{bmatrix}\begin{bmatrix}-2 & 11\\ -10 & 5\end{bmatrix}=\begin{bmatrix}104 & -72\\ -72 & 146\end{bmatrix}$$

$\det(bI-A^TA)=\begin{vmatrix}b-104 & 72\\ 72 & b-146\end{vmatrix}$ $\quad (b-104)(b-146)-72^2=0$ $\quad b^2-250b+10000=0$ $\quad b=\begin{cases}50\\200\end{cases}$ $\quad \sigma_1=5\sqrt{2}$ $\quad \sigma_2=10\sqrt{2}$

① Let $\sigma_1=10\sqrt{2}$ $\quad V_1=\begin{bmatrix}V_{11}\\ V_{21}\end{bmatrix}$

$A^TAV_1=200V_1$ $\quad \begin{bmatrix}104 & -72\\ -72 & 146\end{bmatrix}\begin{bmatrix}V_{11}\\ V_{21}\end{bmatrix}=\begin{bmatrix}200V_1\\ 200V_2\end{bmatrix}$ $\quad \begin{cases}104V_{11}-72V_{21}=200V_1\\ -72V_{11}+146V_{21}=200V_2\end{cases}$

$\begin{cases}-96V_{11}-72V_{21}=0\\ -72V_{11}-54V_{21}=0\end{cases}$ $\quad \begin{cases}\cancel{48}^{4}V_{11}+\cancel{36}^{3}V_{21}=0\\ 4V_{11}+3V_{21}=0\end{cases}$ $\quad 2\cancel{V_{11}+V_{21}}=0$ $\quad +4V_{11}+3V_{21}=0$ $\quad 3V_{21}=-4V_{11}$ $\quad V_1=\begin{bmatrix}-3\\ 4\end{bmatrix}$

② Let $\sigma_2=5\sqrt{2}$ $\quad V_2=\begin{bmatrix}V_{12}\\ V_{22}\end{bmatrix}$

$A^TAV_2=50V_2$ $\quad \begin{bmatrix}104 & -72\\ -72 & 146\end{bmatrix}\begin{bmatrix}V_{12}\\ V_{22}\end{bmatrix}=\begin{bmatrix}50V_{12}\\ 50V_{22}\end{bmatrix}$ $\quad \begin{cases}104V_{12}-72V_{22}=50V_{12}\\ -72V_{12}+146V_{22}=50V_{22}\end{cases}$

$\begin{cases}54V_{12}-72V_{22}=0\\ -72V_{12}+96V_{22}=0\end{cases}$ $\quad 54V_{12}=72V_{22}$ $\quad 3V_{12}=4V_{22}$ $\quad 72V_{12}=96V_{22}$ $\quad 36V_{12}=48V_{22}$ $\quad 3V_{12}=4V_{22}$. $\quad V_2=\begin{bmatrix}4\\ 3\end{bmatrix}$

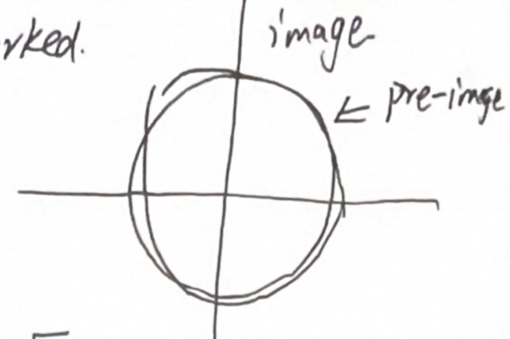
$V=\begin{bmatrix}-3 & 4\\ 4 & 3\end{bmatrix}$ Then we normalize it $\begin{bmatrix}-3/5 & 4/5\\ 4/5 & 3/5\end{bmatrix}$

According to u $\quad u=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$

$A\begin{bmatrix}-3/5\\ 4/5\end{bmatrix}=\begin{bmatrix}10\sqrt{2}\,u_{11}\\ 10\sqrt{2}\,u_{21}\end{bmatrix}$ $\quad \begin{bmatrix}-2 & 11\\ -10 & 5\end{bmatrix}\begin{bmatrix}-3/5\\ 4/5\end{bmatrix}=\begin{bmatrix}6/5+44/5=10\sqrt{2}\,u_{11}\\ 30/5+20/5=10\sqrt{2}\,u_{21}\end{bmatrix}$ $\quad u_{11}=\frac{1}{\sqrt{2}}$ $\quad u_{21}=\frac{1}{\sqrt{2}}$

$A\begin{bmatrix}4/5\\ 3/5\end{bmatrix}=\begin{bmatrix}5\sqrt{2}\,u_{12}\\ 5\sqrt{2}\,u_{22}\end{bmatrix}$ $\quad u_2=\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}$ $\quad U=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$

(b) List the singular values, left singular vectors, and right singular vectors of A. Draw a careful, labeled picture of the unit ball in $\mathbb{R}^2$ and its image under A, together with the singular vectors, with the coordinates of their vertices marked.



$$\begin{cases} \sigma_1 = 10\sqrt{2} \\ \sigma_2 = 5\sqrt{2} \end{cases}$$

left singular vectors: $\pm\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \pm\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

right singular vectors: $\pm\begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix} \quad \pm\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$

(c) $\|A\|_2 = \sigma_{max} = 10\sqrt{2} \qquad \|A\|_F = \sqrt{\sigma_1^2+\sigma_2^2} = \sqrt{200+50} = 5\sqrt{10}$

$\|A\|_1 = \max^m$ column sum of $|A| = \max(12, 16) = 16$

$\|A\|_\infty = \max^m$ row sum of $|A| = \max(13, 15) = 15$

(d). Find A^{-1} not directly, but via the SVD.

$$A^{-1} = (U\Sigma V^*)^{-1} = V\Sigma^{-1}U^* = \frac{1}{5\sqrt{2}}\begin{bmatrix} -3 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} \frac{1}{10\sqrt{2}} & 0 \\ 0 & \frac{1}{5\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 5 & -11 \\ 10 & -2 \end{bmatrix}$$

(e) $\det(\lambda I - A) = 0 \Rightarrow (\lambda+2)(\lambda-5)+110 = 0 \Rightarrow \lambda^2 - 3\lambda + 100 = 0 \quad \lambda = \frac{3 \pm \sqrt{9-400}}{2} = \frac{3 \pm i\sqrt{391}}{2}$

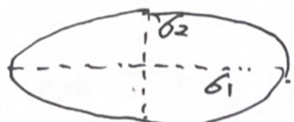
(f) $\det A = \begin{vmatrix} -2 & 11 \\ -10 & 5 \end{vmatrix} = \begin{vmatrix} -2 & 11 \\ 0 & -50 \end{vmatrix} = 100 = \lambda_1\lambda_2$ $\qquad$ $2 \ \ -11 \qquad 10 \ \ -55$ / $-10 \ \ 5$

$$|\det A|^2 = \begin{vmatrix} 104 & -72 \\ -72 & 146 \end{vmatrix} = \begin{vmatrix} 32 & 74 \\ -72 & 146 \end{vmatrix} = \begin{vmatrix} 32 & 106 \\ -72 & 74 \end{vmatrix} = \begin{vmatrix} 32 & 138 \\ -72 & 2 \end{vmatrix} = \begin{vmatrix} 32 & 138 \\ -40 & 140 \end{vmatrix}$$

$$= \begin{vmatrix} -8 & 2 \\ -40 & 140 \end{vmatrix} = 8\times\begin{vmatrix} 1 & 1 \\ 5 & 70 \end{vmatrix} = 10000 = (\sigma_1\sigma_2)^2$$

(g) What is the area of the ellipsoid onto which A maps the unit ball of $\mathbb{R}^2$?

Area $= \pi\cdot\sigma_1\sigma_2 = 100\pi$



5.4. Suppose $A \in \mathbb{C}^{m\times m}$ has an SVD $A = U\Sigma V^*$. Find an eigenvalue decomposition of the $2m\times 2m$ hermitian matrix SVD

$$B = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = \begin{bmatrix} 0 & V\Sigma^*U^* \\ U\Sigma V^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}\begin{bmatrix} U\Sigma V^* & 0 \\ 0 & V\Sigma^*U^* \end{bmatrix}$$

$$X = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}$$

$$\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma^* \end{bmatrix}\begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix}$$

$$X^{-1} = \begin{bmatrix} U^* & 0 \\ 0 & V^* \end{bmatrix}\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix} = \begin{bmatrix} 0 & U^* \\ V^* & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} V^* & 0 \\ 0 & U^* \end{bmatrix} = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}X^{-1}$$

$$B = X\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}X^{-1}$$

$$B = X\begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma^* \end{bmatrix}\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}X^{-1} = X\begin{bmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{bmatrix}X^{-1}$$

$$\begin{bmatrix} 0 & \Sigma \\ \Sigma & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}\begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \Sigma & -\Sigma \\ \Sigma & +\Sigma \end{bmatrix}\begin{bmatrix} I_m & I_m \\ I_m & -I_m \end{bmatrix}$$