★★ For $B = AC$, each column of B is a linear combination of the columns of A.

$$[b_j] = C_{1j}[a_1] + C_{2j}[a_2] + \cdots + C_{mj}[a_m]$$ we will use this

① $e_1 = r_{11} z_1 \rightarrow z_1 = r_{11}^{-1} e_1 \Rightarrow z_1$ has only one non-zero entry

② Let $C^m(k)$ denote the column space of vectors of dimensionality m which have at most non-zero entries

$C^m(m) = C^m$ and each $C^m(k)$ is a linear subspace of C^m

$z_1 \in C^m(1)$ from ①

$$e_{i+1} = \sum_{k=1}^{m} z_k r_{k(i+1)} = \sum_{k=1}^{i} z_k r_{k(i+1)} + z_{i+1} r_{(i+1)(i+1)}$$

$$z_{i+1} = \underbrace{r_{(i+1)(i+1)}^{-1}}_{\text{scalar}} \underbrace{(e_{i+1} - \boxed{\textstyle\sum_{k=1}^{i} z_k r_{k(i+1)}})}_{\in C^m(i+1)} \rightarrow C^m(i)$$

1-4. $\sum_{j=1}^{8} C_j f_j(i) = d_i, \ i = 1, \cdots, 8$

(a) we let $FC = D$ for any $D \in C^8$. Range $(F) \in C^8$

$$\begin{bmatrix} f_1(1) & f_2(1) & \cdots & f_8(1) \\ f_1(2) & \cdots & & \vdots \\ \vdots & & & \vdots \\ f_1(8) & \cdots & & f_8(8) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_8 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_8 \end{bmatrix}$$

F is a full-rank matrix

So $C \rightarrow d$ is a one-to-one maping

(b) $Ad = C \Rightarrow A^{-1} = F$ the element of A^{-1} is $f_j(i)$