HW1 for Modern Analysis

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1 Problem1

Let $A \in \mathbb{R}$ and bounded below. Prove that the set $-A = \{-a : a \in A\}$ is bounded above. Prove further that $-\sup(-A)$ is a lower bound for A and indeed is the greatest such.

We divide the problem into three parts to prove:

1. Prove that the set $-A = \{-a : a \in A\}$ is bounded above.
2. Prove that $-\sup(-A)$ is a lower bound for A.
3. Prove that $-\sup(-A)$ is the greatest lower bound.

Proof for the first part. Since $A \in \mathbb{R}$ is nonempty, then $-A$ is nonempty for elements $a \in A$.
Since A is bounded below, then $\forall a \in A, \quad \exists m \in \mathbb{R}, a \geq m$. Thus $-a \leq -m$ for all $a \in A$.
Thus $-A = \{-a : a \in A\}$ is bounded above. $\square$

Proof for the second part. Since $-A = \{-a : a \in A\}$ is bounded above, $\sup(-A)$ exists. Thus $\forall a \in A, -a \leq \sup(-A)$. Thus $a \geq -\sup(-A)$ for all $a \in A$. Thus $-\sup(-A)$ is a lower bound for A. $\square$

Proof for the third part. Suppose it is not the greatest lower bound,i.e. $\exists m$ s.t. $a \geq m > -\sup(-A)$. Therefore $-a \leq -m < \sup(-A)$. Since $\sup(-A)$ is the least upper bound for $-A$. It is contradition. $\square$

2 Problem2

Let $A \in \mathbb{R}$ and $B \in \mathbb{R}$ be nonempty, such that $A \cup B = \mathbb{R}$ and such that $a < b$ whenever $a \in A$ and $b \in B$. Consider those $p \in \mathbb{R}$ that satisfy $(-\infty, p) \subseteq A$ and $(p, \infty) \subseteq B$.Prove(in this order):
(1). there is at most one such p;

(2). there is at least one such p.

Before proving; we give a claim and a proof as following:

Claim 1. *Let $A \in \mathbb{R}$ and $B \in \mathbb{R}$ be nonempty, such that $A \sup B = \mathbb{R}$ and such that $a < b$ whenever $a \in A$ and $b \in B$, then $A \cap B = \emptyset$*

Proof for Claim1. Suppose $A \cap B \neq \emptyset = \{x\}$.
That means $x \in A$ and $x \in B$.
But if $x \in A$ and $x \in B$, then $x < x$. It is contradiction. $\square$

Proof for Problem 2(1). Suppose there exists $p_1 \neq p_2$(we give $p_1 < p_2$) Then we have:

$$(-\infty, p_1) \subseteq A \qquad (-\infty, p_2) \subseteq A \qquad (-\infty, p_1) \subseteq (-\infty, p_2),$$
$$(p_1, \infty) \subseteq B \qquad (p_2, \infty) \subseteq B \qquad (p_2, \infty) \subseteq (p_1, \infty).$$

Thus,

$$(-\infty, p_2) \cap (p_1, \infty) = (p_1, p_2).$$

But $A \cap B = \emptyset$, then the subsets of A has no intersection with the subsets of B. It is contradiction. $\square$

Proof for Problem 2(2). Since $a < b$ whenever $a \in A$ and b, that means A is bounded above.
There exists $p = \sup(A)$ s.t. $a \leq p$ for all $a \in A$.
For any number $x < p$, x can not be an upper bound for A. Thus there exists an $a \in A$. $x < a \leq p$. But since $a \in A, x < a, x \in A$. Hence $(-\infty, p) \subseteq A$.

And if there exists any number $q < p$, then q is not an upper bound for A. This means there must be some elements in A that are greater than q and less than p. So, p cannot belong to A. Since $A \cup B = \mathbb{R}$ and $p \notin A$, we must have $p \in B$.

Furthermore, since for every $a \in A$ and $b \in B$, $a < b$, every number greater than p must also be in B. So, (p, ∞) is contained in B.

Thus, we've shown that the number p, which is the supremum of A, satisfies the conditions that $(-\infty, p) \subseteq A$ and $(p, \infty) \subseteq B$.

$\square$