Lecture 2. Exercise

2.1. Suppose $A = (a_{ij})_{n\times n}$ is the square matrix with a_{ij} entries $(1\le i\le n,\ 1\le j\le n)$

But $i>j$ $a_{ij}=0$ (upper-triangular) $\overline{a_{ji}}=0$ $j>i$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ 0 & \cdots & & & a_{nn} \end{bmatrix}$$

A is unitary
$A^* = A^{-1}$

$$A^* = \begin{bmatrix} \overline{a_{11}} & 0 & 0 & \cdots & 0 \\ \overline{a_{12}} & \overline{a_{22}} & 0 & \cdots & \vdots \\ \overline{a_{13}} & \overline{a_{23}} & \overline{a_{33}} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \overline{a_{1n}} & \overline{a_{2n}} & \overline{a_{3n}} & \cdots & \overline{a_{nn}} \end{bmatrix}$$

$$A^*A = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

So $\overline{a_{ii}}\,a_{ii} = 1$ other equals $0 \Rightarrow$ Can do that.

Since A is upper-triangular, the inverse A^* is also upper-triangular (from the result in Lecture 1)

A and A^* are both upper-triangular

$$\begin{cases} a_{ij} = 0 \text{ for } i>j \\ \overline{a_{ji}} = 0 \text{ for } j>i \end{cases}$$
$\Rightarrow$ This means that $a_{ij} \neq 0$ only if $i=j$ A is diognal.

2.2

(a) $\|x_1+x_2\|^2 = [(x_1+x_2)^*(x_1+x_2)]^2 = [x_1^*x_1 + x_2^*x_1 + x_1^*x_2 + x_2^*x_2]^2$

Because $\{x_i\}$ are orthogonal vectors

$\|x_1+x_2\|^2 = [\|x_1\|^2 + 0 + 0 + \|x_2\|^2]^2$

(b) (general case)

$$\left\|\sum_{i=1}^n x_i\right\|^2 = \left(\sum_{i=1}^n x_i\right)^* \sum_{i=1}^n x_i = \sum_{i=1}^n x_i^* x_i + \overbrace{\cdots\cdots}^{\sum_{i\neq j} x_i^* x_j} = \sum_{i=1}^n \|x_i\|^2$$

For another method suppose $n \rightarrow$ we have $\left\|\sum_{i=1}^n x_i\right\|^2 = \sum_{i=1}^n \|x_i\|^2$

$$\left\|\sum_{i=1}^{n+1} x_i\right\|^2 = \left\|\sum_{i=1}^n x_i + x_{n+1}\right\|^2 = \sum_{i=1}^n \|x_i\|^2 + \sum_{i=1}^n x_i^* x_{n+1} + x_{n+1}^* \sum_{i=1}^n x_i + x_{n+1}^* x_{n+1}$$

$$= \sum_{i=1}^n \|x_i\|^2 + \|x_{n+1}\|^2 = \sum_{i=1}^{n+1} \|x_i\|^2 \quad \blacksquare$$

2.3. $\boxed{\text{There exists a trick in this problem}}$

A is hermitian A^*A, to prove all eigenvalues of A are real.

we need to know $\lambda = \lambda^*$. $\cancel{x^*(Ax) = \lambda^* x^* x}$

$\lambda \|x\|^2 = \lambda(x^* x) = x^*(\lambda x) = x^*(Ax) = x^*(A^* x) = (Ax)^* x = (\lambda x)^* x$

$= \bar{\lambda} x^* x = \bar{\lambda} \|x\|^2$ Thus $\lambda = \bar{\lambda} = \lambda^*$.

(b) If x and y are eigenvectors corresponding to distinct eigenvalues, then. x and y are orthogonal

$Ax = \lambda_1 x$ $\quad$ $\cancel{A}x^*\cancel{A}y = \cancel{A}x^* A^* y = (Ax)^* y = \lambda_1 x^* y = x^*(\lambda_2 y) = \lambda_2 x^* y$

$Ay = \lambda_2 y$

$\lambda_1 \neq \lambda_2$ $\quad$ $(\lambda_1 - \lambda_2) x^* y = 0$ $\quad$ $\lambda_1 - \lambda_2 \neq 0$ $\quad$ $x^* y = 0$

x and y are orthogonal

2.4. Let unitary matrix be Q $\quad$ $Q^* = Q^{-1}$

$\|\lambda\|^2 = 1$

$\Uparrow$

$Qx = \lambda x$

$\cancel{\lambda}\|x\|^2 = \cancel{\lambda}(x^* x) = x^*\cancel{QQ}x = x^* Q^* Q x = (Qx)^*(Qx) = x^* \lambda^* \lambda x = \|\lambda\|^2 x^* x$

2.5. let $S \in \mathbb{C}^{m\times m}$ be skew-hermitian $S^* = -S$

(a)

① $\lambda \|x\|^2 = \lambda(x^* x) = x^*(\lambda x) = x^* S x = x^*(-S^*)x = -(Sx)^* x$

$= -\bar{\lambda} x^* x = -\bar{\lambda}\|x\|^2$ $\quad$ $\bar{\lambda} = \lambda^* = -\lambda \Rightarrow \lambda$ is a purely imaginary eigenvalues

② iS is Hermitian since $(iS)^* = i^* S^* = -i(-S) = iS$.

☆☆☆ From Example 2.3 iS has only real eigenvalues $\Rightarrow S$ must have only imaginary ✓

☆☆☆ ☆☆ Any matrix A can be written as the sum of a Hermitian matrix $\frac{(A+A^T)}{2}$ and a skew-Hermitian matrix $\frac{(A-A^T)}{2}$

(b) $I - S$ is nonsingular $\xrightarrow{\text{means}}$ $\boxed{I - S \text{ has the inverse}}$

Also for a non-singular matrix A, $Ax = 0$ has only the trivial solution $x = 0$

$(I - S)x = 0$ $\quad$ $x = Sx$

$\Rightarrow x = 0$

$x^* x = (Sx)^* x = x^* S^* x = x^*(-S)x = -x^* S x = -x^* x \Rightarrow x^* x = 0$

Hence, $I - S$ is nonsingular

(c) $\underset{\text{unitary}}{Q} = (I-S)^{-1}(I+S) \Rightarrow$ Cayley transform of S

这是线性分数变换 $\frac{1+s}{1-s}$ 的矩阵模拟，将复数 s 平面的左半部分共形映射到单位圆盘上.

$$Q^*Q = [(I-S)^{-1}(I+S)]^*[(I-S)^{-1}(I+S)]$$

$$= (I+S)^{-1}(I-S)(I-S)^{-1}(I+S) = I \Rightarrow Q^* = Q^{-1} \Rightarrow Q \text{ is unitary}$$

2-6. $u, v \Rightarrow$ m-vectors

* 单位矩阵的一级扰动

$A = I + uv^*$ rank-one perturbation of the identity

① Its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and give an expression for α if A is nonsingular

null$(A) = \{\alpha u : \alpha \in \mathbb{C}\}$

$$A^{-1} A = (I + \alpha uv^*)^{-1}(I + uv^*)$$

①Suppose A is singular, i.e.

$x = -u(v^*x)$

$Ax = 0$ for some $x \in \mathbb{C}\setminus\{0\}$ $\quad (I+uv^*)x = x + uv^*x = 0$

neccessary and sufficient

x is a scalar multiple of $u \Rightarrow x = \alpha u$ for some $\alpha \in \mathbb{C}$

$\Rightarrow \alpha u + u(v^*\alpha u) = 0 \Rightarrow \alpha u(1 + v^*u) = 0 \Rightarrow \underline{v^*u = -1}$

② Now suppose A is nonsingular

Let $A^{-1} = [a_1, \cdots, a_m]$ $\quad AA^{-1} = (I+uv^*)[a_1, \cdots, a_m]$

$= [a_1 + uv^*a_1, \ldots, a_m + uv^*a_m] = I$ (by definition)

$a_i + u(v^*a_i) = e_i \quad 1 \le i \le m \Rightarrow a_i + u\alpha_i = e_i \Rightarrow a_i = e_i - u\alpha_i$

$AA^{-1} = (I + uv^*)(I - u\alpha^*)$ $\qquad \alpha^* = \frac{v^*}{1+uv^*}$

$I = I - u\alpha^* + uv^* - uv^*u\alpha^* \Rightarrow u\alpha^*(I + v^*u) = uv^*$

$$A^{-1} = I - \frac{uv^*}{1+uv^*}$$

constant factor

2-7. *A Hadamard matrix $\Rightarrow$ entries ± 1 transpose $\Rightarrow$ equal to inverse

If $m > 2$ then $m = 4p$ for som p ① $k=0$ $\quad H^* = kH^{-1} \quad k \in \mathbb{C}$ ②

$m = 2^k \quad k = 0, 1, 2, \ldots$ $\qquad H_0 = [1] = H_0^* = H_0^{-1} \qquad H_k H_k^* = 2^k I_{2^k}$