The Singular Value Decomposition (SVD) Motivation for SVD [vast]

① Data Compression and Low-rank Approximation: Give a matrix (e.g. an image), its SVD can be used to produce a low-rank approximation that captures the most important features of the data with reduced size.

In image processing, this can be visualized as capturing the main features of an image using fewer singular values and vectors.

② Principal Component Analysis (PCA): By performing SVD on a data matrix, we can obtain the principal components, which are directions of maximal variance. These Principal components are invaluable in dimensionality reduction, visualization, noise reduction.

③ Numerical Stability: In solving systems of linear equations or inverting matrices, direct methods can be numerically unstable or ill-posed. SVD provides a numerically stable way to pseudo-invert a matrix (Moore-Penrose inverse) and solve linear systems, even for ill-conditioned matrices.

④ Determining Rank and Null space: The rank of a matrix can be quickly identified using its SVD by counting the number of non-zero singular value. The singular vectors corresponding to zero singular values form a basis for the null space of the matrix.

⑤ Solving Differential Equations: In computational science, especially when dealing with boundary value problems or systems that can be linearized, SVD can be used to decompose the problem's fundamental modes and make it more tractable.

⑥ Geometry and Computer Graphics: In computer graphics, understanding the orientation and deformation of objects often relies on decomposing matrices describing these objects. SVD is especially valuable for this because it can be decompose a matrix into rotation, scaling, and another rotation, providing insight into the structure and behavior of geometrical transformations.

Lecture 4.

4.1 Determine SVDs of the following matrices

(a) $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$

Let $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ $\quad A^*A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$

$\det(\lambda^2 I - A^*A) = 0 \quad (\lambda-9)(\lambda-4)=0 \quad \lambda_1^2 = 9 \quad \lambda_2^2 = 4$ $\qquad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

① $\lambda_1^2 = 9 \qquad A^*A V_1 = \lambda_1^2 V_1$

$\begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 9V_1 \\ 9V_2 \end{bmatrix}$ $\quad V_2 = 0$ so we can take $V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$AV_1 = \lambda_1 u_1$

$\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3u_1 \\ 3u_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2u_{12} \\ 2u_{22} \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\quad U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Let $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad A^*A = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$ $\det(\lambda^2 I - A^*A) = 0$

$\lambda^2(\lambda^2 - 4) = 0 \quad \begin{cases} \lambda_1^2 = 4 \\ \lambda_2^2 = 0 \end{cases}$

① $\lambda_1^2 = 4$

$\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} 4V_{11} \\ 4V_{21} \end{bmatrix} \Rightarrow \begin{matrix} V_{11} = 0 \\ V_{21} = 1 \end{matrix}$ $\quad V = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad AV_1 = 2u_1$

$\begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}[2]\begin{bmatrix} 0 \\ 1 \end{bmatrix}^*$ $\quad \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2u_{11} \\ 2u_{21} \\ 2u_{31} \end{bmatrix} \Rightarrow \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

4.2. Suppose A is an $m \times n$ matrix and B is the $n \times m$ matrix and ~~B is the~~ obtained by rotating A ninety degree clockwise on paper (not exactly a standard mathematical transformation!) Do A and B have the same singular value? Prove that the answer is yes or give a counterexample. For $m \leq n$

$tr(B^*B) = tr(A^*A)$

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ $\quad B = A^T = \begin{bmatrix} a_{m1} & \cdots & a_{11} \\ a_{m2} & \cdots & a_{12} \\ \vdots & & \vdots \\ a_{mn} & & a_{1n} \end{bmatrix}_{n \times m}$

$A = U_1 \Sigma_1 V_1^*$ $\qquad A^T = U_2 \Sigma_2 V_2^*$

Because $A^*A = \underset{n\times n}{\ }$ $\qquad B^*B = \underset{m\times m}{\ } \begin{bmatrix} a_{11}^* & a_{12}^* & \cdots & a_{1n}^* \\ \vdots & & & \\ a_{m1}^* & \cdots & & a_{mn}^* \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ a_{12} & \cdots & a_{m2} \\ \vdots & & \vdots \\ & & a_{mn} \end{bmatrix}$

Note B is obtained from A by first taking a transpose and then doing some column swaps (essentially a mirror image)

$B = A^T[U] \Rightarrow U$ is a unitary matrix since diagonal remains same.

$\rightarrow$ A and A^T have the same singular values since $\det(A-\lambda I) = \det(A^T - \lambda I)$

Let $B = AU$ $\quad BB^* = AUU^*A^* = AA^* \Rightarrow$ singular values remain the same.

4.4: Two matrices $A, B \in \mathbb{C}^{m \times m}$ are $\boxed{\text{unitarily equivalent}}$ if $A = QBQ^*$ for some unitary $Q \in \mathbb{C}^{m \times m}$. Is it true or false that A and B are unitarily equivalent iff they have the same singular values?

$\Rightarrow$ A and B are unitarily equivalent

$A = QBQ^*$ $\qquad A = U\Sigma_1 V^* = QBQ^*$ $\qquad \Sigma V^* = U^*QBQ^*$

$\Sigma V^*V = U^*QBQ^*V$ $\qquad \Sigma = (Q^*U)^* B (Q^*V)$. $\qquad$ But SVD of a matrix is unique.

$B = (Q^*U)\Sigma_2(Q^*V)^*$ so they have the same values.

$\Sigma_1 = \Sigma_2$

$\Leftarrow$ $\quad A = U_1 \Sigma V_1^* \quad U_2 \Sigma V_2^* = B$

Not true $\boxed{\text{Counter example}}$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$ $\rightarrow$ they share the same singular values $\boxed{\text{but not unitarily equivalent}}$

4.5 Theorem 4.1 asserts that every $A \in \mathbb{C}^{m \times n}$ has a SVD $A = U\Sigma V^*$. Show that if A is real, then it has a real SVD ($U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$)

when $A \in \mathbb{R}^{m \times n}$

$A^*A = A^TA = \mathbb{R}^{n \times n}$ i.e. real and symmetric ($A^* = A^T$ since A is real)

$A^*A = VDV^T \rightarrow$ real diagonal matrix

$V \hookrightarrow$ real orthogonal matrix

If $m > n$, we can add $m-n$ zero rows to V to get another real matrix, V.