Question 19:(from Xronos) Calculate the following limit.

$$\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$$

Xingjian's solution

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} &= \lim_{x\to -1} \frac{(\sqrt{x^2+8}-3)(\sqrt{x^2+8}+3)}{(x+1)(\sqrt{x^2+8}+3)} \\
&= \lim_{x\to -1} \frac{x^2+8-9}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x\to -1} \frac{(x-1)(x+1)}{(x+1)(\sqrt{x^2+8}+3)} \\
&= \lim_{x\to -1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-2}{3+3} = -\frac{1}{3}.
\end{aligned}$$

$$\boxed{-\tfrac{1}{3}}$$

Xingjian comment: Strategy for these limits. (1) First check the denominator. If direct substitution makes it undefined (typically 0), you likely have a removable singularity. (2) If there is an irrational term (most commonly a square root), multiply numerator and denominator by the conjugate to use the difference-of-squares identity $(a-b)(a+b) = a^2-b^2$. (3) Simplify and cancel the factor that caused the zero. (4) Evaluate the resulting limit.

Question 16:(from Xronos) If you know that $\lim_{x\to 4} f(x) = 3$ and $\lim_{x\to 0} g(x) = -5$, then evaluate the following limit:

$$\lim_{x\to 0} f(x+4) + g(x)$$

Xingjian's solution Using limit laws and a change of variable $y = x+4 \to 4$,

$$\lim_{x\to 0} \big(f(x+4) + g(x)\big) = \lim_{y\to 4} f(y) + \lim_{x\to 0} g(x) = 3 + (-5) = -2.$$

$$\boxed{-2}$$

Xingjian comment: The calculation is easy, but be sure you can justify that

$$\lim_{x\to 0} f(x+4) \;=\; \lim_{y\to 4} f(y).$$

This change-of-variables step is simple yet fair game on exams. Indeed, let $y = x+4$. Then $x \to 0$ iff $y \to 4$, so

$$\lim_{x\to 0} f(x+4) = \lim_{x\to 0} f(y) = \lim_{y\to 4} f(y).$$

More generally, for any constant c and point a,

$$\lim_{x\to a} f(x+c) \;=\; \lim_{y\to a+c} f(y).$$

Question 7:(from Xronos)

$$\lim_{x\to 2}(x-2)\cos(\ln|x-2|)$$

Xingjian's solution

Let $t = x-2 \to 0$. Then

$$\lim_{x\to 2}(x-2)\cos(\ln|x-2|) = \lim_{t\to 0} t\cos(\ln|t|).$$

Since $|\cos(\ln|t|)| \le 1$, the squeeze theorem gives $t\cos(\ln|t|) \to 0$.

$$\boxed{0}$$

Xingjian comment: Be sure how squeeze theorem works! If you have anything unclear about it, welcome to ask!

Question 22:(from Xronos) Does the equation $\sin(x-4) = -x+13$ have a solution in the interval $(4, 13)$?

Xingjian's solution

Define $F(x) = \sin(x-4) + x - 13$, which is continuous on $[4, 13]$. Compute

$$F(4) = \sin 0 + 4 - 13 = -9 < 0, \qquad F(13) = \sin 9 + 13 - 13 = \sin 9 > 0,$$

since $9-2\pi \in (0, \pi)$, so $\sin 9 > 0$. By the Intermediate Value Theorem, there exists $c \in (4, 13)$ with $F(c) = 0$, i.e. $\sin(c-4) = -c+13$.

$$\boxed{\text{Yes, a solution exists in } (4, 13).}$$

Xingjian comment: For "does an equation have a solution on (a,b)**?"**

(1). Rewrite as a root-finding problem. Move everything to one side:

$$\text{Solve } \ L(x) = R(x) \quad \Longleftrightarrow \quad F(x) := L(x) - R(x) = 0.$$

(2). Continuity check (mandatory). Verify F is continuous on the closed interval $[a, b]$. (Polynomials, trig, exponentials, logs on their domains, and sums/products/compositions of continuous functions are continuous.)

(3). Endpoint test. Compute $F(a)$ and $F(b)$.

$$\text{If } F(a)\cdot F(b) < 0 \quad \Rightarrow \quad \text{opposite signs.}$$

(4). Apply the Intermediate Value Theorem(IVT). Since F is continuous and crosses from negative to positive (or vice versa), there exists $c \in (a, b)$ with $F(c) = 0$.

This will help you to solve most of questions for IVT (I think so). But something you also need to worry: **No continuity $\Rightarrow$ IVT does *not* apply.**

Why the sign change works (intuition): Continuity means the graph of F has no jumps. If $F(a) < 0$ and $F(b) > 0$, the graph must cross the x-axis somewhere between a and b, so $F(c) = 0$ for some $c \in (a, b)$.

Xingjian Xu
xingjianxu@ufl.edu
LIT 453

MAC 2311 Webpage: https://people.clas.ufl.edu/xingjianxu/MAC2311/
Office Hours: Tuesday, Periods 4 and 5 on LIT453;
Monday Period 4 on LIT 215.