Question (from HW 29). Evaluate $\int_{-4}^{4} |x| + \sqrt{16 - x^2} dx$.

Xingjian's solution. The integrand is a sum of two even functions $|x|$ and $\sqrt{16-x^2}$. We can use symmetry and interpret the integral as areas of simple geometric shapes.

First, use that both $|x|$ and $\sqrt{16-x^2}$ are even:

$$\int_{-4}^{4} \left(|x| + \sqrt{16-x^2}\right) dx = 2\int_{0}^{4} \left(x + \sqrt{16-x^2}\right) dx.$$

Compute the area under $y = x$ from 0 to 4:

$$\int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = \frac{16}{2} = 8.$$

By symmetry, the total contribution from $|x|$ over $[-4, 4]$ is

$$2 \cdot 8 = 16.$$

Next, compute the area under $y = \sqrt{16-x^2}$ from 0 to 4. This graph is the upper half of a circle of radius 4 centered at the origin. The area under this curve from -4 to 4 is the area of a semicircle:

$$\text{Area of semicircle} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(4^2) = 8\pi.$$

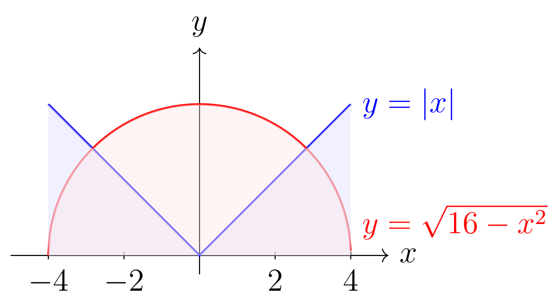
So

$$\int_{-4}^{4} \sqrt{16-x^2}\,dx = 8\pi.$$

Therefore,

$$\int_{-4}^{4} \left(|x| + \sqrt{16-x^2}\right) dx = \underbrace{\int_{-4}^{4} |x|\,dx}_{16} + \underbrace{\int_{-4}^{4} \sqrt{16-x^2}\,dx}_{8\pi} = 16 + 8\pi.$$

$$\boxed{\int_{-4}^{4} \left(|x| + \sqrt{16-x^2}\right) dx = 16 + 8\pi.}$$



Question (from HW29). Find a lower bound for the integral

$$\int_{\pi/6}^{\pi/3} \tan(x)dx.$$

Use our rectangular estimate from class where is the minimum of $y = \tan(x)$on the given interval.

Xingjian's solution. On the interval $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$, the function $\tan(x)$ is increasing. So the minimum value of $\tan(x)$ on this interval is at the left endpoint $x = \frac{\pi}{6}$. A rectangle with height equal to this minimum gives a *lower bound* for the area under the curve.

Since $\tan(x)$ is increasing on $\left(0, \frac{\pi}{2}\right)$, on

$$\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$$

we have

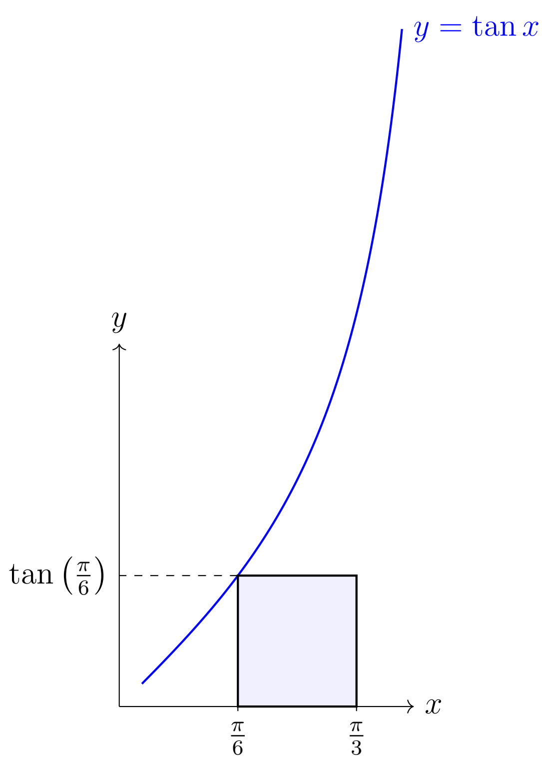
$$\min_{x\in[\pi/6,\pi/3]} \tan(x) = \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}.$$

Our rectangle estimate:

$$\int_{\pi/6}^{\pi/3} \tan(x)\,dx \;\geq\; \left(\frac{\pi}{3} - \frac{\pi}{6}\right) \cdot \left(\tan\left(\frac{\pi}{6}\right)\right).$$

So a rectangular estimate gives

$$\boxed{\int_{\pi/6}^{\pi/3} \tan(x)\,dx \;\geq\; \frac{\pi}{6\sqrt{3}} = \frac{\pi\sqrt{3}}{18}.}$$



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