Question:(from HW9) Find the equation of the tangent line to $f(x) = \sqrt{3-x}$ at $x = 2$ using the definition of the derivative.

Xingjian's solution Using the definition of the derivative

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Here $a = 2$, then compute $f(2)$, you can get

$$f(2) = \sqrt{3-2} = 1.$$

So the point on the curve is $(2, 1)$.

Then apply the definition:

$$f'(2) = \lim_{h\to 0} \frac{\sqrt{3-(2+h)} - 1}{h} = \lim_{h\to 0} \frac{\sqrt{1-h} - 1}{h}.$$

Multiply numerator and denominator by the conjugate:

$$f'(2) = \lim_{h\to 0} \frac{(\sqrt{1-h} - 1)(\sqrt{1-h} + 1)}{h(\sqrt{1-h} + 1)} = \lim_{h\to 0} \frac{-h}{h(\sqrt{1-h} + 1)}.$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{1-h} + 1} = \frac{-1}{2}.$$

So the slope is $-\frac{1}{2}$ and the tangent line at $(2, 1)$ is:

$$y - 1 = -\tfrac{1}{2}(x-2) \quad \Rightarrow \quad y = -\tfrac{1}{2}x + 2.$$

Question from previous Exam: Let $h(x) = \frac{-6e^3}{x} - 2e^{3x}$. For which value of x will the graph of the function have a horizontal tangent line?

Xingjian's solution A horizontal tangent occurs where the derivative of the function is zero, that is,$h'(x) = 0$. **Xingjian comment:**This is not like horizontal asympotine in previous question. so we can use the definition to calculate the derivatives of this $h(x)$

$$\begin{aligned} h(x)' = \lim_{t\to 0} \frac{h(x+t) - h(x)}{t} &= \lim_{t\to 0} \frac{-6e^3/(x+t) - 2e^{3(x+t)} + 6e^3/x + 2e^{3x}}{t} \\ &= \lim_{t\to 0} \frac{\frac{6e^3}{x} - \frac{6e^3}{x+t} + 2e^{3x} - 2e^{3(x+t)}}{t} \text{then factorize the numerator part} \\ &= \lim_{t\to 0} \frac{6e^3(\frac{1}{x} - \frac{1}{x+t}) + 2e^{3x}(1 - e^{3t})}{t} \\ &= \lim_{t\to 0} \left(\frac{6e^3 \frac{x+t-x}{x(x+t)}}{t} + 2e^{3x}\frac{1-e^{3t}}{t}\right) \\ &= \lim_{t\to 0} 6e^3 \frac{t}{x(x+t)t} + 2e^{3x}\frac{1-e^{3t}}{t}. \end{aligned}$$

Then look at the first term, t can be canceled by denominator and numerator. Then we can do

$$\lim_{t\to 0} 6e^3 \frac{t}{x(x+t)t} = \lim_{t\to 0} 6e^3 \frac{1}{(x+t)x} = \frac{6e^3}{x^2}.$$

For the second term, it is really tricky in my opinion. Let $g(x) = e^{3t}$, then we know it is continuous and it also has derivative. So based on definition, we can write:

$$g'(x) = \lim_{t\to 0} \frac{g(x+t) - g(x)}{t} = \lim_{t\to 0} \frac{e^{3(x+t)} - e^{3x}}{t} = e^{3x} \lim_{t\to 0} \frac{e^{3t} - 1}{t}$$

So the second term is totally the negative part of $g'(x)$ times 2. Thus

$$\lim_{t\to 0} 2e^{3x} \frac{1-e^{3t}}{t} = \lim_{t\to 0} -2g'(x) = \lim_{t\to 0} -2 \times 3e^{3x} = \lim_{t\to 0} -6e^{3x}.$$

Also a face you should remember for e^x, its derivative always is itself. But by chain rule, here $g'(x) = 3e^{3x}$. Thus we let $h'(x) = \frac{6e^3}{x^2} - 6e^{3x} = 0$. Right now choice provides us $x = 1$, then plug in this equation. Then we can get our answer

$$\boxed{x = 1.}$$

Xingjian comment: I didn't check your answer for this question. But in my opinion, it is tricky and hard. And I am very sure that difficulty should not be like this way.

Question from previous exam Let $g(x) = \frac{32}{\sqrt{x}} - x^2 + 4\sqrt{x}$. What is the value of $g'(4)$.

Xingjian comment: I think this might be a doable question in the exam. But you shoud work hard for that. We compute $g'(x)$ using the definition of the derivative:

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}.$$

Plug in $g(x)$:

$$g'(x) = \lim_{h\to 0} \frac{\frac{32}{\sqrt{x+h}} - (x+h)^2 + 4\sqrt{x+h} - \frac{32}{\sqrt{x}} + x^2 - 4\sqrt{x}}{h}.$$

We can split this limit into three parts:

$$g'(x) = \underbrace{\lim_{h\to 0} \frac{\frac{32}{\sqrt{x+h}} - \frac{32}{\sqrt{x}}}{h}}_{\text{Term 1}} + \underbrace{\lim_{h\to 0} \frac{x^2 - (x+h)^2}{h}}_{\text{Term 2}} + \underbrace{\lim_{h\to 0} \frac{4(\sqrt{x+h} - \sqrt{x})}{h}}_{\text{Term 3}}.$$

For Term 1:

$$\frac{\frac{32}{\sqrt{x+h}} - \frac{32}{\sqrt{x}}}{h} = \frac{32}{h} \left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} \right).$$

Rationalize the difference:

$$\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} = \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} = -\frac{\sqrt{x+h} - \sqrt{x}}{\sqrt{x}\sqrt{x+h}}.$$

Next, multiply numerator and denominator by $\sqrt{x+h}+\sqrt{x}$:

$$\sqrt{x+h}-\sqrt{x}=\frac{(x+h)-x}{\sqrt{x+h}+\sqrt{x}}=\frac{h}{\sqrt{x+h}+\sqrt{x}}.$$

Thus

$$\frac{1}{h}\left(\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\right)=-\frac{1}{\sqrt{x}\sqrt{x+h}(\sqrt{x+h}+\sqrt{x})}.$$

Take the limit $h\to 0$:

$$\lim_{h\to 0}\frac{\frac{32}{\sqrt{x+h}}-\frac{32}{\sqrt{x}}}{h}=32\cdot\left(-\frac{1}{2x\sqrt{x}}\right)=-\frac{16}{x^{3/2}}.$$

For Term 2:

$$\frac{x^2-(x+h)^2}{h}=\frac{x^2-x^2-2xh-h^2}{h}=\frac{-2xh-h^2}{h}=-2x-h\to -2x\quad (h\to 0).$$

For Term 3:

$$\frac{4(\sqrt{x+h}-\sqrt{x})}{h}=4\cdot\frac{\sqrt{x+h}-\sqrt{x}}{h}.$$

Rationalize the numerator:

$$\sqrt{x+h}-\sqrt{x}=\frac{h}{\sqrt{x+h}+\sqrt{x}}\implies\frac{\sqrt{x+h}-\sqrt{x}}{h}=\frac{1}{\sqrt{x+h}+\sqrt{x}}\to\frac{1}{2\sqrt{x}}.$$

Multiply by 4:

$$\lim_{h\to 0}\frac{4(\sqrt{x+h}-\sqrt{x})}{h}=\frac{2}{\sqrt{x}}.$$

Then we can combine all three terms to get

$$g'(x)=-\frac{16}{x^{3/2}}-2x+\frac{2}{\sqrt{x}}.$$

Finally we can evaluate at $x=4$:

$$g'(4)=-\frac{16}{4^{3/2}}-2(4)+\frac{2}{\sqrt{4}}=-\frac{16}{8}-8+1=-2-8+1=-9.$$

Question from previous exam: Consider the function

$$f(x)=\frac{x^2+2x-15}{|x-3|}.$$

Which of the following statements is/are true?

P. There is a removable discontinuity at $x = 3$.

Q. $\lim_{x\to 3^+} f(x) = -8$.

R. $\lim_{x\to -\infty} f(x) = +\infty$.

Xingjian comment: This questions look fair to show up in the exam.

Xingjian's solution: First, simplify the numerator: $x^2 + 2x - 15 = (x+5)(x-3)$. So the function becomes

$$f(x) = \frac{(x+5)(x-3)}{|x-3|}.$$

Then we split into two cases:

- When $x > 3$, $|x-3| = x-3$, so

$$f(x) = \frac{(x+5)(x-3)}{x-3} = x+5 \quad (x > 3).$$

- When $x < 3$, $|x-3| = -(x-3)$, so

$$f(x) = \frac{(x+5)(x-3)}{-(x-3)} = -(x+5) \quad (x < 3).$$

So as $x \to 3^+$, $f(x) \to 8$; as $x \to 3^-$, $f(x) \to -8$. Therefore, the two one-sided limits are not equal.

- Statement P: **False**, because the discontinuity at $x = 3$ is *not removable* (jump discontinuity). - Statement Q: **False**, since $\lim_{x\to 3^+} f(x) = 8$, not -8. - Statement R: As $x \to -\infty$,

$$f(x) = \frac{x^2 + 2x - 15}{|x-3|} \sim \frac{x^2}{|x|} \to +\infty,$$

so **True**.

$$\boxed{\text{Answer: (D) R only}}$$

Question from previous exam Let $g(x) = \frac{x^2+2-xe^x}{x}$. **Find** $g'(x)$. (Hint: Simplify $g(x)$ first. You may use power and exponential rules instead of the definition of the derivative.) Then find the equation of the tangent line to $g(x)$ at $x = -1$ using point-slope form.

Xingjian's solution: For first part, simplify the function, we can get

$$g(x) = \frac{x^2 + 2 - xe^x}{x} = x + \frac{2}{x} - e^x.$$

Then take derivative term by term:

$$g'(x) = \frac{d}{dx}\left(x + \frac{2}{x} - e^x\right) = 1 - \frac{2}{x^2} - e^x.$$

$$\boxed{g'(x) = 1 - \frac{2}{x^2} - e^x}$$

For second part, we first compute the point:

$$g(-1) = (-1) + \frac{2}{-1} - e^{-1} = -1 - 2 - \frac{1}{e} = -3 - \frac{1}{e}.$$

Next, compute the slope:

$$g'(-1) = 1 - \frac{2}{1} - e^{-1} = 1 - 2 - \frac{1}{e} = -1 - \frac{1}{e}.$$

Using point-slope form:

$$y - y_0 = m(x - x_0) \Rightarrow y + 3 + \frac{1}{e} = \left(-1 - \frac{1}{e}\right)(x+1).$$

Final answer for this is:

$$\boxed{y + 3 + \frac{1}{e} = \left(-1 - \frac{1}{e}\right)(x+1)}$$

Xingjian comment: This is also fair question. Also similar questions show up in your homework.

Question from previous exam What is the interval of differentiability for the function

$$f(x) = \begin{cases} -2 & x \leq 1 \\ x^2 - 4x + 8 & 1 < x \leq 3 \\ 2x - 1 & 3 < x \end{cases}$$

Xingjian's solution:

To find the interval of differentiability, we first note that each piece of the function is differentiable within its interval. We only need to check whether the function is differentiable at the points where the pieces change: at $x = 1$ and $x = 3$.

At $x = 1$: - For $x \leq 1$, the function is constant (-2), so $f'(x) = 0$. - For $x > 1$, $f(x) = x^2 - 4x + 8$[do it by using definition!], so $f'(x) = 2x - 4$, and at $x = 1$, this gives $f'(1^+) = -2$.

Since the left and right derivatives are not equal, the function is not differentiable at $x = 1$.

At $x = 3$: - Left of 3: $f'(x) = 2x - 4 \Rightarrow f'(3^-) = 2$. - Right of 3: $f(x) = 2x - 1 \Rightarrow f'(3^+) = 2$.

The left and right derivatives are equal, so the function *is differentiable* at $x = 3$.

$$\boxed{\text{Answer: (A) } (-\infty, 1) \cup (1, \infty)}$$

Xingjian comment: For calculate this, you can use directly definition to compute here. Also if you know a computation rule of derivative $(x^a)' = ax^{a-1}$ for $a > 1$and use it, that will be much quicker.

Question from previous exam: Let $f(x) = 2e^{ax} - 5x^2 + 2x - 4$, where a is a constant. For which value of a will $f'(0) = 1$?

Xingjian's solution: Take derivative of $f(x)$, you can use the definition or using $(x^a)' = ax^{a-1}$ and $(e^x)' = e^x$ either way is okay, but $(x^a)' = ax^{a-1}$ and $(e^x)' = e^x$ will accelerate your speed for calculating:

$$f'(x) = \frac{d}{dx}(2e^{ax} - 5x^2 + 2x - 4) = 2ae^{ax} - 10x + 2.$$

Now evaluate at $x = 0$:

$$f'(0) = 2a \cdot e^0 - 0 + 2 = 2a + 2.$$

Set equal to 1:

$$2a + 2 = 1 \Rightarrow 2a = -1 \Rightarrow a = -\frac{1}{2}.$$

$$\boxed{\text{Answer: (D)} \ -\frac{1}{2}}$$

Xingjian's solution for only using definition. We use the definition of the derivative:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h}.$$

I decide to split into several steps like the following: First, I need to compute $f(0)$

$$f(0) = 2e^{a\cdot 0} - 5(0)^2 + 2(0) - 4 = 2e^0 - 0 + 0 - 4 = 2 - 4 = -2.$$

Then I need to plug into the limit

$$f'(0) = \lim_{h\to 0} \frac{f(h) - (-2)}{h} = \lim_{h\to 0} \frac{f(h) + 2}{h}.$$

$$f(h) = 2e^{ah} - 5h^2 + 2h - 4.$$

$$f(h) + 2 = 2e^{ah} - 5h^2 + 2h - 2.$$

Then I write the full expression for the limit

$$f'(0) = \lim_{h\to 0} \frac{2e^{ah} - 5h^2 + 2h - 2}{h}.$$

Break into parts:

$$= \lim_{h\to 0} \left(\frac{2(e^{ah} - 1)}{h} + \frac{2h}{h} - \frac{5h^2}{h} \right) = \lim_{h\to 0} \left(\frac{2(e^{ah} - 1)}{h} + 2 - 5h \right).$$

Take the limits

We use some known limit(I don't know you learned it or not, but similar to the first question):

$$\lim_{h\to 0}\frac{e^{ah}-1}{h}=a.$$

So:

$$\lim_{h\to 0}\frac{2(e^{ah}-1)}{h}=2a,\quad \lim_{h\to 0}-5h=0.$$

Therefore:

$$f'(0)=2a+2.$$

We are given that:

$$f'(0)=1\Rightarrow 2a+2=1\Rightarrow 2a=-1\Rightarrow a=\boxed{-\frac{1}{2}}.$$

Question from the previous exam: Let

$$f(x)=\begin{cases} x^2-1 & x<3\\ 8 & x=3\\ \frac{5}{x^2-9} & x>3\end{cases}$$

Which of the following statements are true?

P. $f(x)$ has a removable discontinuity at $x=3$.

Q. $f(x)$ is continuous from the right at $x=3$.

R. $x=3$ is a vertical asymptote on the graph of $f(x)$.

Xingjian's solution:

Check left-hand limit, you will get $\lim_{x\to 3^-}f(x)=\lim_{x\to 3^-}(x^2-1)=9-1=8$. Check right-hand limit: $f(x)=\frac{5}{x^2-9}=\frac{5}{(x-3)(x+3)}$.

As $x\to 3^+$, denominator will approach 0 as $f(x)$ turning to ∞.

- **P is False:** Left limit $=8$, right limit $=\infty$. Not removable. - **Q is False:** $\lim_{x\to 3^+}f(x)=\infty$, not equal to $f(3)=8$, so not continuous from the right. - **R is True:** As $x\to 3^+$, $f(x)\to\infty$ turns to vertical asymptote.

$$\boxed{\text{Answer: (C) R only}}$$

Xingjian Xu
xingjianxu@ufl.edu
LIT 453
MAC 2311 Webpage: https://people.clas.ufl.edu/xingjianxu/MAC2311/
Office Hours: Tuesday, Periods 4 and 5 on LIT453;
Monday Period 4 on LIT 215.