MATH456/CMPSC456 Homework 6

Due March 15, 2024

1. (10 points) Consider the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Using $\boldsymbol{x} = (1, 0)^T$ as the initial vector, apply two steps of Rayleigh quotient iterations (12.1.4). Show the approximate eigenvector and eigenvalue at each iteration.

2. (10 points) Determine the SVD of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

3. (15 points) Let $\boldsymbol{x}$ be the solution of a least square problem that is written as an over-determined linear system $A\boldsymbol{x} \approx \boldsymbol{b}$ with $A \in \mathbb{R}^{m \times n}$; $m > n$. Let $A = USV^T$ be the SVD of A. Express the solution $\boldsymbol{x}$ of the least square problem using the matrices U, S, V and the vector $\boldsymbol{b}$.

4. (15 points) Given the following data

x_i	y_i
0	0
1/4	1
1/2	0
3/4	−1

find a trigonometric polynomial $q(x) = c_0 + c_1 e^{i2\pi x} + c_2 e^{i4\pi x} + c_3 e^{i6\pi x}$ that interpolates the data points.

MATH456/CMPSC456 Homework 7

Due March 26, 2024

1. (10 points) Given the following data

x_i	y_i
0	1
1/8	0
1/4	−2
3/8	1
1/2	3
5/8	0
3/4	−2
7/8	1

Find the least square approximation of order $n = 4$: $q(x) = c_0 + c_1 e^{i2\pi x} + c_2 e^{i4\pi x} + c_3 e^{i6\pi x}$.

2. (15 points) Find the eigenvalues of the following matrix. Show that the eigenvalues are non-negative.

$$A = \begin{pmatrix} 30 & -16 & 1 & 0 & \cdots & 0 & 1 & -16 \\ -16 & 30 & -16 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -16 & 30 & -16 & 1 & \ddots & \ddots & 0 \\ 0 & 1 & -16 & 30 & -16 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -16 & 30 & -16 & 1 \\ 1 & 0 & \cdots & 0 & 1 & -16 & 30 & -16 \\ -16 & 1 & 0 & \cdots & 0 & 1 & -16 & 30 \end{pmatrix}$$

3. (10 points) Consider the system of ordinary differential equations (ODEs):

$$\frac{dy}{dt} = y + 2z,$$
$$\frac{dz}{dt} = 3y + 2z,$$

with initial conditions $y(0) = 1$ and $z(0) = -1$. Determine the solution at $t = 0.1$ by applying Euler's method for one step.

4. Computer project. (15 points) Find the order $n = 8$ trigonometric interpolating function for $f(x) = e^x$ in [0,1]. Plot the interpolation function $q(x)$ together with $f(x)$. In a separate figure, plot $q'(x)$ together with $f'(x)$.

MATH456/CMPSC456 Homework 8

Due April 3, 2024

1. (10 points)Apply the Euler's method to the ODE $y' = -y + t\sqrt{y}$ for two steps with $x(0) = 1$ and $h = 1/4$. Show the solution w_2.

2. (10 points)Apply the Heunr's method to the ODEs $y' = -y + xy,\ x' = x - xy$ for one step with $x(0) = 2, y(0) = 2$ and $h = 1/4$. Show the numerical solution $\boldsymbol{w}_1$.

3. (15 points) Show that the following Runge-Kutta method has local error of order h^3

$$
\begin{aligned}
k_1 &= f(t_j, w_j), \\
k_2 &= f(t_j + 2h, w_j + 2hk_1), \\
w_{j+1} &= w_j + \frac{h}{4}(3k_1 + k_2).
\end{aligned}
$$

4. (15 points) Computer project: Write a computer code and implement the 3-stage Runge-Kutta method:

$$
\begin{aligned}
k_1 &= f(t_j, w_j), \\
k_2 &= f(t_j + h/2, w_j + hk_1/2), \\
k_3 &= f(t_j + h, w_j + hk_1), \\
w_{j+1} &= w_j + \frac{h}{6}(k_1 + 4k_2 + k_3).
\end{aligned}
$$

Make sure that the code works for a system of ODEs.

First, test the code on the ODE $x' = e^{t-x}, x(0) = 1$. Use $h = 0.1$ and compute the solution up to $t = 3$. Compare the numerical solution to the exact solution $x(t) = \log(e^t + e - 1)$ with their graphs in the same figure.

Second, use the code and solve the ODE system,

$$
\begin{aligned}
x' &= x^2 - y^2, \\
y' &= xy + y + x + 1,
\end{aligned}
$$

using the $h = 0.01$ and $t = 100$. Plot the trajectories on the $x - y$ plane ($y(t)$ vs $x(t)$).

Math456/CMPSC456 Homework 9

Due April 12, 2024

1. (10 points) For the logistic model,

$$y' = y(1-y), y(0) = 0.1,$$

solve the ODE for two steps using the **backward** Euler's method with $h = 0.2$. You may need a calculator. Recall that this is a population model and the solution is expected to be non-negative.

2. (15 points) For the ODE system,

$$\begin{aligned} x' &= -2x - 2y, \\ y' &= 2x, \end{aligned}$$

find the threshold h^* for which the Euler method, when applied to the linear ODEs, will be stable for all $h \le h^*$, but unstable otherwise.

Check whether $h = h^*$ in Heun method would give stable numerical solutions. Using the stability condition to support your answer.

3. (15 points) Find the stability condition, in terms of λ and h, for the 4-stage Runge Kutta method.

4. (10 points) Consider the Heun's method applied to the differential equations,

$$\begin{aligned} x' &= -\, y, \\ y' &= 5x - 6y. \end{aligned}$$

Determine the stability threshold h^*, such that the numerical solution is stable for all $h \le h^*$, but unstable otherwise.

Math456/CMPSC456 Homework 10

Due April 25, 2024

1. (10 points) Is the multistep method,

$$w_{j+1} - 3w_j + 2w_{j-1} = \frac{h}{12}\Big[13f_{j+1} - 20f_j - 5f_{j-1}\Big],$$

zero stable?

2. (10 points) For the same multistep method,

$$w_{j+1} - 3w_j + 2w_{j-1} = \frac{h}{12}\Big[13f_{j+1} - 20f_j - 5f_{j-1}\Big].$$

Find the order of the local error.

3. (10 points) Find the coefficients in the following multistep method

$$w_{j+1} - w_j = h[b_0 f_{j+1} + b_1 f_j + b_2 f_{j-1}],$$

so that the order of the local error is maximized.

4. (10 points) Consider the two-step Adams-Bashforth method,

$$w_{j+1} = w_j + \frac{3}{2}hf_j - \frac{1}{2}hf_{j-1},$$

applied to the differential equations,

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -1000y_1 - 70y_2. \end{aligned}$$

For the step size $h = 0.05$, would the numerical solution be stable?

5. (10 points) Write a program to implement the 3-step Adams-Bashforth method. Test it on the ODE

$$x' = y, y' = -x - y^2,$$

with initial conditions $x(0) = -25, y(0) = 6$. Find the solution on the time interval $[0, 30]$. Plot the solution $(x(t), y(t))$ on the x-y plane.