1. $\vec{y} = A\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \quad \|\vec{y}\| = \sqrt{26}$

$\vec{x} \leftarrow \frac{1}{\sqrt{26}} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

$\vec{y} = A\vec{x} = \frac{1}{\sqrt{26}} \begin{bmatrix} 11 \\ 9 \\ 12 \end{bmatrix}, \quad \|\vec{y}\| = \frac{\sqrt{346}}{\sqrt{26}}$

$\vec{x} \leftarrow \frac{1}{\sqrt{346}} \begin{bmatrix} 11 \\ 9 \\ 12 \end{bmatrix}$

$\vec{y} = A\vec{x} = \frac{1}{\sqrt{346}} \begin{bmatrix} 43 \\ 41 \\ 44 \end{bmatrix}, \quad \|\vec{y}\| = \frac{\sqrt{5466}}{\sqrt{346}}$

$\Rightarrow \vec{x} \leftarrow \frac{1}{\sqrt{5466}} \begin{bmatrix} 42 \\ 40 \\ 42 \end{bmatrix}$

b. $\vec{y} = A\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \|\vec{y}\| = 1$

$\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$y = A\vec{x} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \quad \|\vec{y}\| = \sqrt{2}$

$\vec{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix},$

$\vec{y} = A\vec{x} = \frac{1}{\sqrt{2}} \begin{bmatrix} -2 \\ -2 \\ -1 \end{bmatrix}, \quad \|\vec{y}\| = \frac{3}{\sqrt{2}}$

$\Rightarrow \vec{x} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$

2. choose $i=1$

$$\Rightarrow \vec{x}^T = \frac{[-2 \quad -4 \quad 2]}{6}$$

$$B = A - 6\,\vec{v}_1 \vec{x}^T = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix} - \begin{bmatrix} -2 & -4 & 2 \\ -12 & -24 & 12 \\ -32 & -64 & 32 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 10 & 25 & -10 \\ 36 & 66 & -27 \end{bmatrix}$$

$$C = \begin{bmatrix} 25 & -10 \\ 66 & -27 \end{bmatrix}$$ has the remaining eigenvalues

3. The algorithm should converge

to $\lambda = 5.2361$

with eigenvector $\propto \begin{bmatrix} 0.7795 \\ 0.4817 \\ 0.0920 \\ 0.3897 \end{bmatrix}$

1.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step 1.

$$\rho(\vec{x}) = \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} = [1, 0] \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3$$

$$\vec{y} = (A - \rho I)^{-1} \vec{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\|\vec{y}\| = 1$$

$$\Rightarrow \quad \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Step 2:

$$\rho(\vec{x}) = \vec{x}^T A \vec{x} = 3$$

$$\vec{y} = (A - \rho I)^{-1} \vec{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\|\vec{y}\| = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

2.

$$A^T A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$\rightarrow$ eigenvalues $\quad (\lambda - 3)^2 - 1 = 0$

$\rightarrow \quad \lambda_1 = 4, \quad \lambda_2 = 2$

$$\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$\left(\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \text{ is also ok} \right)$

Singular values: $\sigma_1 = 2, \quad \sigma_2 = \sqrt{2}$

$$\Rightarrow u_1 = \frac{1}{\sigma_1} A v_1 = \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} \frac{1}{2\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2\sqrt{2}} & -\frac{1}{2} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

3. $A = USV^T$

$U \to$ unitary, $S \to$ diagonal, $V^T \to$ unitary

Normal equation :

$$A^T A \vec{x} = A^T b$$

$$\Rightarrow \cancel{V} S \cancel{U^T} \cancel{U} S V^T \vec{x} = \cancel{V} S U^T b$$

$$\Rightarrow S^2 V^T \vec{x} = S U^T b$$

$$\Rightarrow \boxed{\vec{x} = V S^{-1} U^T b}$$

4. $$c_k = \frac{1}{n} \sum_{j=0}^{n-1} y_j e^{-i 2\pi j k/n}$$

$n = 4$

$$c_0 = \frac{1}{4}(y_0 + y_1 + y_2 + y_3) = 0$$

$$c_1 = \frac{1}{4}\left(1 \times e^{-i\frac{\pi}{2}} - 1 \times e^{-i\frac{3\pi}{2}}\right)$$

$$= \frac{1}{4} \times (-1) i - \frac{1}{4} \times i = -\frac{i}{2}$$

$$c_2 = \frac{1}{4}\left(1 \times e^{-i\pi} - 1 \times e^{-i3\pi}\right).$$

$$= \frac{1}{4}(-1 + 1) = 0$$

$$c_3 = \frac{1}{4}\left(1 \times e^{-i\frac{3\pi}{2}} - 1 \times e^{-i\frac{9\pi}{2}}\right)$$

$$= \frac{1}{4}(i + i) = \frac{i}{2}$$

$$\Rightarrow q(x) = -\frac{i}{2} e^{i2\pi x} + \frac{i}{2} e^{i6\pi x}$$

1. By substituting the data points, we find

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \omega^6 & \omega^{12} & \omega^{18} \\ 1 & \omega^7 & \omega^{14} & \omega^{21} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$\omega = e^{i2\pi/8}$
$= e^{i\pi/4}$

$A\vec{x} \approx \vec{b}$ Least square problem

Normal equation

$$A^T A \vec{x} = A^T b$$

$A^T A$: Columns are orthogonal

(A^T: conjugate transpose since some entries are complex)

$$\Rightarrow \quad 8\vec{x} = A^T b$$

$$\Rightarrow \quad c_0 = \frac{1}{8}\left(1+0+(-2)+1+3+0+(-2)+1\right) = \frac{1}{4}$$

$$c_1 = \frac{1}{8}\left(1-2\omega^2+\omega^3+3\omega^4-2\omega^6+\omega^7\right)$$
$$= -\frac{1}{4}$$

$$c_2 = \frac{1}{8}\left(1-2\omega^4+\omega^6+3\omega^8-2\omega^{12}+\omega^{14}\right)$$
$$= 1+\frac{1}{4}i$$

$$c_3 = \frac{1}{8}\left(1-2\omega^3+\omega^9+3\omega^{12}-2\omega^{18}+\omega^{21}\right)$$
$$= -\frac{1}{4}$$

2. $A\vec{v} = \lambda \vec{v}$

$\downarrow$

$\vec{v} = (v_j)$

$\Rightarrow \quad v_{j-2} - 16 v_{j-1} + 30 v_j - 16 v_{j+1} + v_{j+2} = \lambda v_j$

assume $v_j = e^{i 2\pi j k/n} = \omega^{jk}$

$\Rightarrow$

$$\lambda_k = e^{-i4\pi k/n} - 16 e^{-i2\pi k/n} + 30 - 16 e^{i2\pi k/n} + e^{-i4\pi k/n}$$

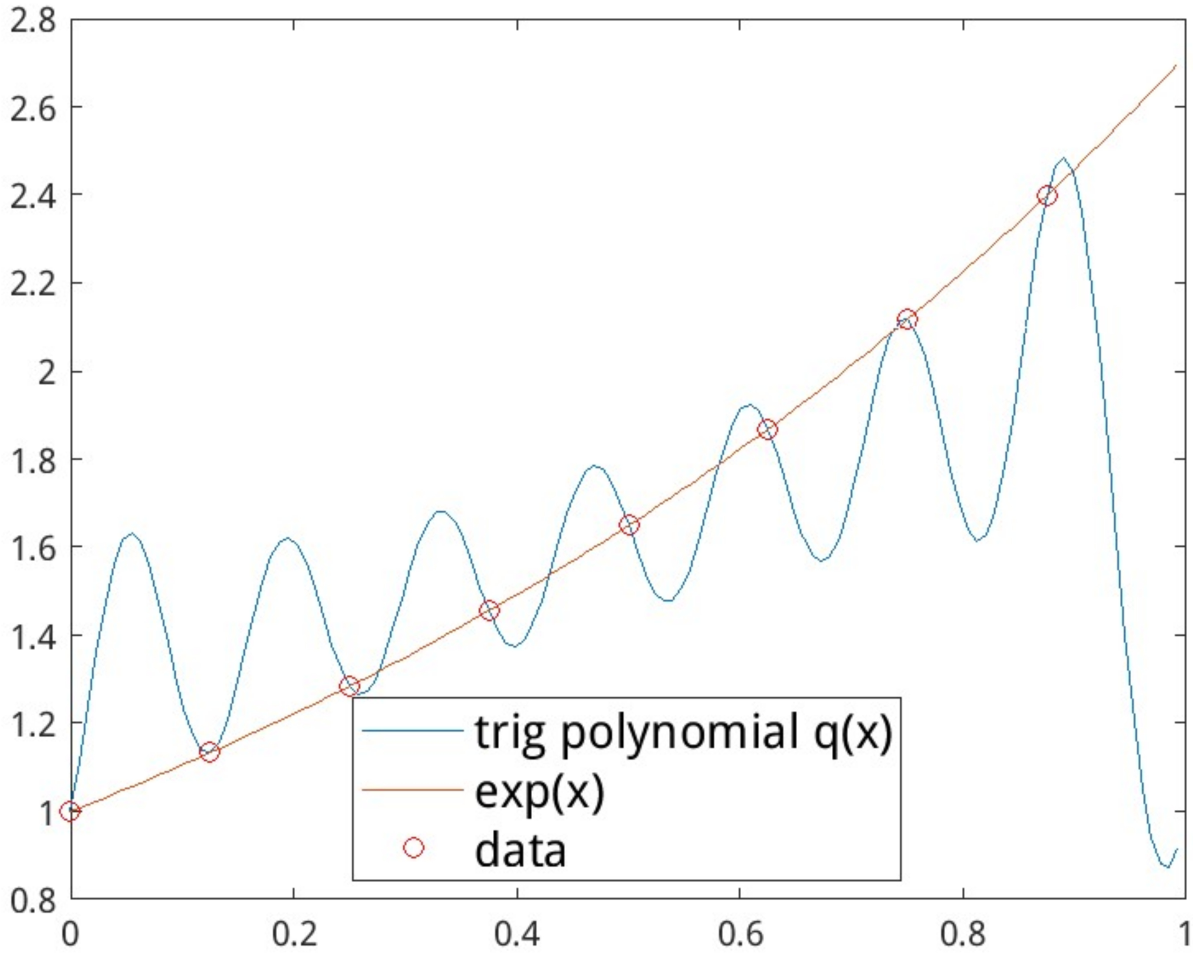
using the identity $e^{-i\theta} + e^{i\theta} = 2\cos\theta$

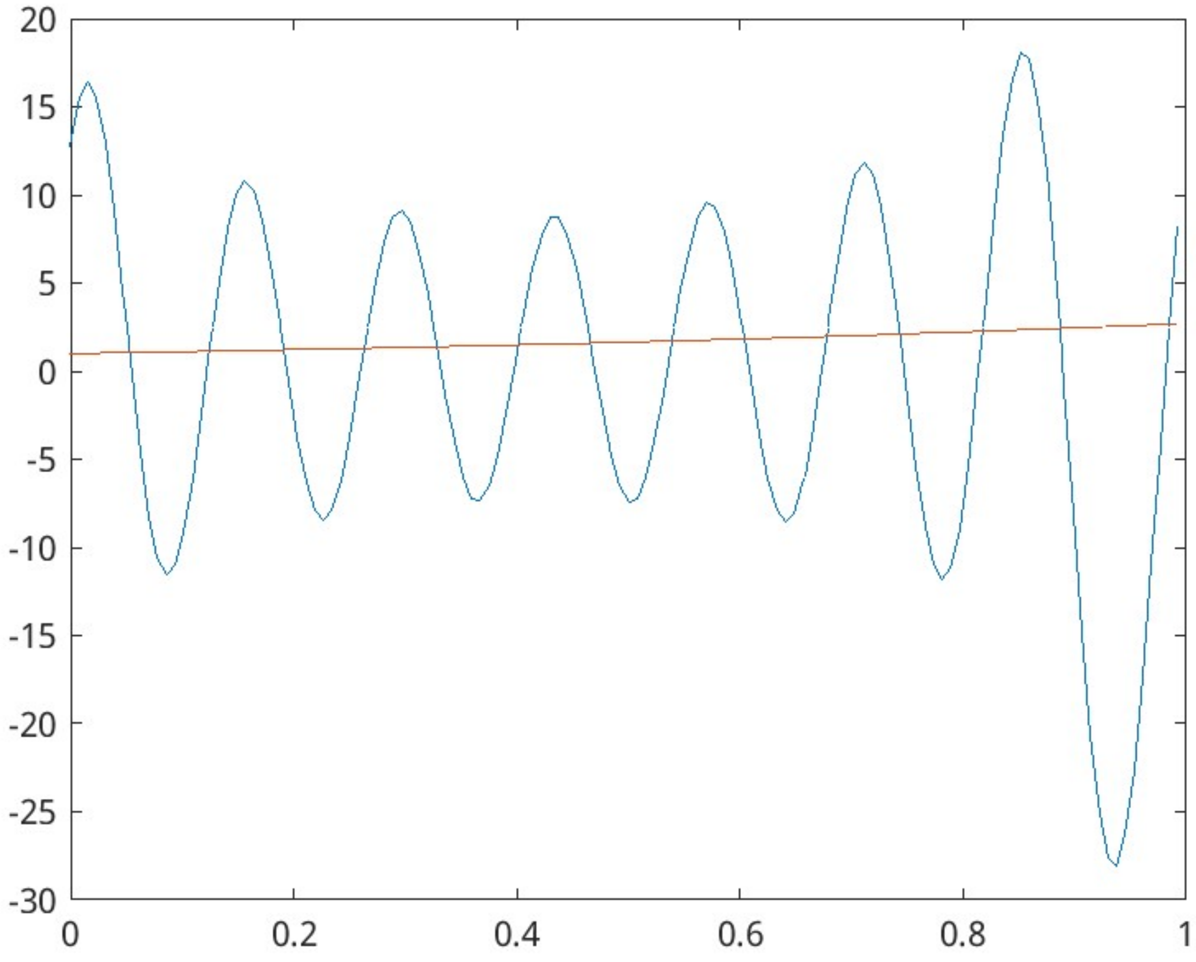
$$\Rightarrow \lambda_k = 2\cos\frac{4\pi k}{n} - 32\cos\frac{2\pi k}{n} + 30$$

3. $\vec{W}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $h = 0.1$

$$f(0, \vec{W}_0) = \begin{bmatrix} 1-2 \\ 3-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\vec{W}_1(0.1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0.1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9 \\ -0.9 \end{bmatrix}$$


2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0
0.2
0.4
0.6
0.8
1
trig polynomial q(x)
exp(x)
data



20
15
10
5
0
-5
-10
-15
-20
-25
-30
0
0.2
0.4
0.6
0.8
1

1. $f(t,y) = -y + t\sqrt{y}$, $t_0 = 0$, $y_0 = 1$, $h = \frac{1}{4}$

$W_1 = W_0 + hf(t_0, W_0)$

$= 1 + \frac{1}{4}(-1) = \frac{3}{4}$

$W_2 = W_1 + hf(t_1, W_1)$

$= \frac{3}{4} + \frac{1}{4}\left(-\frac{3}{4} + \frac{1}{4}\sqrt{\frac{3}{4}}\right) = \frac{9}{16} + \frac{\sqrt{3}}{32} \approx 0.6166$

2 $\vec{f} = \begin{bmatrix} -y + xy \\ x - xy \end{bmatrix}$ $\vec{W} = \begin{bmatrix} y \\ x \end{bmatrix}$

$k_1 = f(t_0, \vec{W}_0) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $\vec{W}_0 + h\,k_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$

$= \begin{bmatrix} \frac{5}{2} \\ \frac{3}{2} \end{bmatrix}$

$k_2 = f(t_1, \vec{W}_0 + hk_1)$

$= \begin{bmatrix} -\frac{5}{2} + \frac{15}{4} \\ \frac{3}{2} - \frac{15}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{4} \\ -\frac{9}{4} \end{bmatrix}$

$k_1 + k_2 = \begin{bmatrix} \frac{13}{4} \\ -\frac{17}{4} \end{bmatrix}$

$\Rightarrow \vec{W}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \frac{h}{2}(k_1 + k_2] = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{13}{32} \\ -\frac{17}{32} \end{bmatrix}$

$= \begin{bmatrix} \frac{77}{32} \\ \frac{47}{32} \end{bmatrix} \approx \begin{bmatrix} 2.40625 \\ 1.46875 \end{bmatrix}$

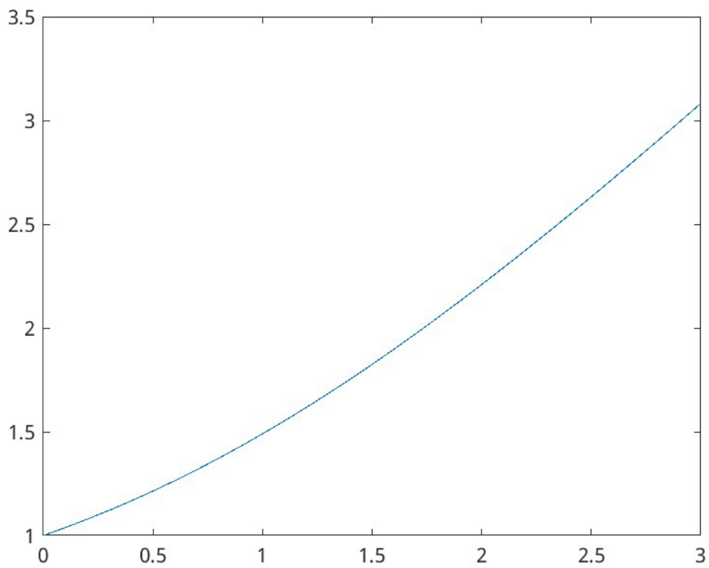
3. Taylor expansion around $h=0$

$$k_2 = f(t_j, w_j) + (f_t + f_y f)\, 2h \quad (f = f(t_j, w_j) = k_1)$$
$$+ O(h^2)$$

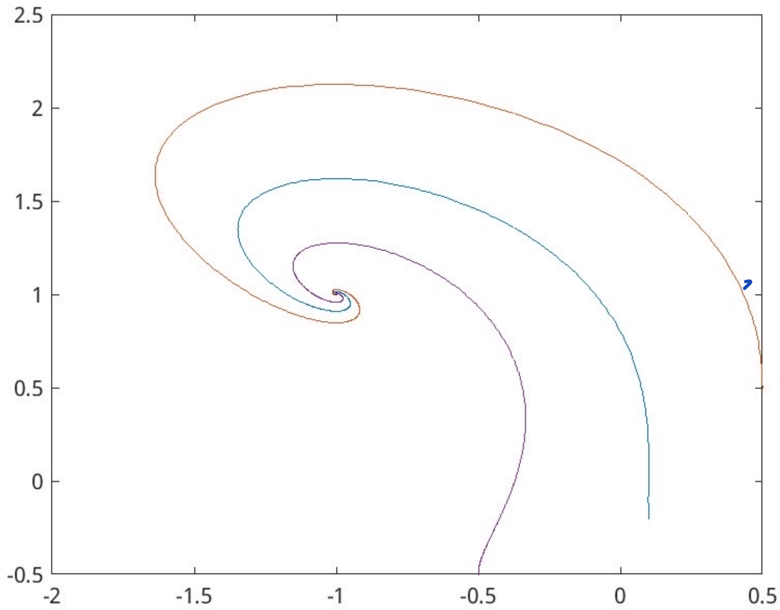
$$\Rightarrow w_{j+1} = w_j + \frac{3}{4} h f + \frac{1}{4} h k_2$$
$$= \underbrace{w_j + hf + \frac{h^2}{2}(f_t + f_y f)}_{\text{same as the expansion of the exact solution}} + O(h^3)$$

$\Rightarrow$ local error $= O(h^3)$

4.



ODE (x, y)



1. Backward Euler

$$y_{n+1} = y_n + h\, y_{n+1}(1 - y_{n+1})$$

$$h = 0.2$$

$$\Rightarrow \quad 0.2\, y_{n+1}^2 + 0.8\, y_{n+1} - y_n = 0$$

$y_0 = 0.1 \Rightarrow y_1 \approx 0.1213$ (discard the root -4.1213)

$$\Rightarrow \quad 0.2\, y_2^2 + 0.8\, y_2 - 0.1213 = 0$$

$\Rightarrow \quad y_2 = 0.1463$ (discarding the negative root)

2. $A = \begin{bmatrix} -2 & -2 \\ 2 & 0 \end{bmatrix}$

$$\lambda^2 + 2\lambda + 4 = 0 \Rightarrow \lambda = -1 \pm i\sqrt{3}$$

Stability condition $|1 + h\lambda| \leq 1$

$$\Rightarrow \quad |1 - h \pm i\sqrt{3}\, h| \leq 1$$

$$\Rightarrow \quad -2h + h^2 + 3h^2 \leq 0 \Rightarrow h \leq \frac{1}{2} = h^*.$$

Stability condition for Heun's

$$|1 + h\lambda + \tfrac{1}{2}h^2\lambda^2| \leq 1$$

$1 + h\lambda + \frac{1}{2}h^2\lambda^2$

$= 1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} + \frac{1}{2}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2$

↙ it is enough to check $\lambda = -1 + \sqrt{3}i$

$= 1 - \frac{1}{2} + \frac{1}{8} - \frac{3}{8} + i\frac{\sqrt{3}}{2} - i\frac{\sqrt{3}}{4}$

$= \frac{1}{4} + i\frac{\sqrt{3}}{4}$

$\Rightarrow \left|1 + h\lambda + \frac{1}{2}h^2\lambda^2\right| = \frac{1}{2} < 1 \Rightarrow$ stable

3,

$k_1 = f(t_n, y_n)$

$k_2 = f(t_n + \frac{h}{2}, y_n + k_1\frac{h}{2})$

$k_3 = f(t_n + \frac{h}{2}, y_n + k_2\frac{h}{2})$

$k_4 = f(t_n + h, y_n + k_3 h)$

$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Consider: $f = \lambda y$

$\Rightarrow k_1 = \lambda y_n$

$k_2 = \lambda y_n + \frac{h\lambda^2}{2} y_n = (1 + \frac{h\lambda}{2})\lambda y_n$

$k_3 = \lambda y_n + \frac{h\lambda}{2}(1 + \frac{h\lambda}{2})\lambda y_n$

$= [1 + \frac{1}{2}h\lambda + \frac{1}{4}(h\lambda)^2]\lambda y_n$

$k_4 = \lambda y_n + h\lambda[1 + \frac{1}{2}h\lambda + \frac{1}{4}(h\lambda)^2]\lambda y_n$

$$= \left[1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{4}(h\lambda)^3\right] \lambda y_n$$

$$\Rightarrow \quad y_{n+1} = y_n + h\lambda y_n + \frac{1}{2}(h\lambda)^2 y_n + \frac{1}{6}(h\lambda)^3 y_n + \frac{1}{24}(h\lambda)^4 y_n$$

$$\Rightarrow \quad \left|1 + h\lambda + (h\lambda)^2/2 + \frac{(h\lambda)^3}{6} + \frac{(h\lambda)^4}{24}\right| < 1$$

4,

Stability condition for Heun's

$$|1 + h\lambda + \tfrac{1}{2}h^2\lambda^2| \leq 1$$

$$A = \begin{bmatrix} 0 & -1 \\ 5 & -6 \end{bmatrix} \Rightarrow \quad \lambda^2 + 6\lambda + 5 = 0$$

$$\Rightarrow \quad \lambda = -1 \quad \text{or} \quad \lambda = -5$$

$\lambda = -1 \Rightarrow$

$$|1 - h + \tfrac{1}{2}h^2| \leq 1$$

$$\Rightarrow -1 \leq 1 - h + \tfrac{1}{2}h^2 \leq 1$$

$$-h + \tfrac{1}{2}h^2 \leq 1 \Rightarrow h \leq 2$$

$$\tfrac{1}{2}h^2 - h + 2 \geq 0$$

$$\Delta = 1 - 4 = -3 < 0$$

$\Rightarrow$ *holds unconditionally*

$\lambda = -5 \Rightarrow \left|1 - 5h + \frac{25}{2}h^2\right| \le 1$

$$-1 \le 1 - 5h + \frac{25}{2}h^2 \le 1$$

$\Downarrow$ $h \le \frac{2}{5}$

$\Rightarrow \frac{25}{2}h^2 - 5h + 2 \ge 0$

$\frac{1}{2}(5h-2)^2 \ge 0$ holds automatically

$\Rightarrow h \le \frac{2}{5}$