Engineering Mathematics 2
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Homework 7
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Exercise 7.2

The answer is $P_{0,0}^t = \dfrac{1+(2p-1)^t}{2}$. We can prove this by mathematical induction. When $t=1$, we have $P_{0,0}^1 = p$, so the claim holds. Now suppose the claim holds at t. Then $P_{0,1}^t = 1 - P_{0,0}^t = \dfrac{1-(2p-1)^t}{2}$, so

$$\begin{aligned}
P_{0,0}^{t+1} &= P_{0,0}P_{0,0}^t + P_{0,1}P_{1,0}^t \\
&= \frac{p(1+(2p-1)^t)}{2} + \frac{(1-p)(1-(2p-1)^t)}{2} \\
&= \frac{1+(2p-1)^{t+1}}{2}.
\end{aligned}$$

Therefore the claim holds for all t.

Exercise 7.6

Let h_i the expected number of moves to reach n starting from i. Then we obtain

$$\begin{gathered}
h_0 = \frac{1}{2}h_0 + \frac{1}{2}h_1 + 1 \\
h_1 = \frac{1}{2}c_0 + \frac{1}{2}h_2 + 1 \\
\vdots \\
h_{n-1} = \frac{1}{2}h_{n-2} + \frac{1}{2}h_n + 1 \\
h_n = 0
\end{gathered}$$

For $i = 0, \cdots, n-1$, we can say $h_i = h_{i+1} + 2(i+1)$. We can prove this by mathematical induction. When $i=0$, we have $h_0 = h_1 + 2$, so the claim holds. Now suppose the claim holds at $i < n-1$. Then

$$h_{i+1} = \frac{1}{2}h_i + \frac{1}{2}h_{i+2} + 1 = \frac{1}{2}h_{i+1} + (i+1) + \frac{1}{2}h_{i+2} + 1$$

$$h_{i+1} = h_{i+2} + 2(i+2)$$

Therefore the claim holds for all i. Since $h_n = 0$, we obtain

$$h_i = h_{i+1} + 2(i+1) = \cdots = h_n + 2(i+1) + 2(i+2) + \cdots + 2n = (n+i+1)(n-i)$$

which is the expected number of moves to reach n from i.

Exercise 7.12

Define $Z_i = X_i \pmod{k}$, then the sequence of Z_i becomes a Markov chain since probabilities on Z_i can be obtained just from the previous state. Let P be the corresponding probability matrix. Consider the uniform distribution $\vec{\pi} = \left(\frac{1}{k}, \frac{1}{k}, \cdots, \frac{1}{k}\right)$, then for all j we obtain

$$(\vec{\pi}P)_j = \sum_{i=0}^{k-1} \pi_i P_{i,j} = \frac{1}{k}\sum_{i=0}^{k-1} P_{i,j} = \frac{1}{k} = \pi_j.$$

Therefore $\vec{\pi}$ is a stationary distribution of the chain. Finally we can say

$$\lim_{n\to\infty} \Pr(X_n \text{ is divisible by } k) = \lim_{n\to\infty} \Pr(Z_n = 0) = \lim_{t\to\infty} P_{0,0}^t = \frac{1}{h_{i,i}} = \pi_i = \frac{1}{k}.$$

Exercise 7.13

(a) Using properties of Markov chains, we obtain the following.

$$\begin{aligned}
&\Pr(X_k = a_k \mid X_{k+1} = a_{k+1}, X_{k+2} = a_{k+2}, \cdots, X_m = a_m) \\
&\quad = \frac{\Pr(X_k = a_k, \cdots, X_m = a_m)}{\Pr(X_{k+1} = a_{k+1}, \cdots, X_m = a_m)} \\
&\quad = \frac{\Pr(X_k = a_k, X_{k+1} = a_{k+1})\Pr(X_{k+2} = a_{k+2}, \cdots, X_m = a_m \mid X_k = a_k, X_{k+1} = a_{k+1})}{\Pr(X_{k+1} = a_{k+1})\Pr(X_{k+2} = a_{k+2}, \cdots, X_m = a_m \mid X_{k+1} = a_{k+1})} \\
&\quad = \frac{\Pr(X_k = a_k, X_{k+1} = a_{k+1})}{\Pr(X_{k+1} = a_{k+1})} = \Pr(X_k = a_k \mid X_{k+1} = a_{k+1})
\end{aligned}$$

Therefore the reverse sequence is Markovian.

(b) The stationary distribution gives $\Pr(X_k = j) = \pi_j$ for all j and k. Therefore

$$\begin{aligned}
Q_{i,j} &= \Pr(X_k = j \mid X_{k+1} = i) \\
&= \frac{\Pr(X_k = j)\Pr(X_{k+1} = i \mid X_k = j)}{\Pr(X_{k+1} = i)} \\
&= \frac{\pi_j P_{j,i}}{\pi_i}.
\end{aligned}$$

(c) If $\pi_i P_{i,j} = \pi_j P_{j,i}$, simply substituting this equation to the result of (b) gives the following.

$$Q_{i,j} = \frac{\pi_j P_{j,i}}{\pi_i} = \frac{\pi_i P_{i,j}}{\pi_i} = P_{i,j}$$

Exercise 7.17

When starting from an arbitrary position, moving backward ensures that it will come back to that position, since we cannot move lower than 0. Therefore we can simplify the problem by thinking all situations as starting from 0. When we start fom 0, the probability that we come back to 0 at the first time in $2n+2$ moves is given by

$$\frac{1}{n+1}\binom{2n}{n}p^n(1-p)^{n+1}.$$

This comes from that once we hit 0, we must move to 1, and the number of paths that move from 1 to 1 without hitting 0 is $\frac{1}{n+1}\binom{2n}{n}$, and we should move forward n times and backward $n+1$ times. Then the probability that we first return to 0 after t moves is

$$r_{0,0}^t = \sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}p^n(1-p)^{n+1} = (1-p)\sum_{n=0}^{\infty}C_n(p(1-p))^n = (1-p)\frac{1-\sqrt{1-4p(1-p)}}{2p(1-p)} = \frac{1-|1-2p|}{2p}.$$

(a) If $p < \frac{1}{2}$, $r_{0,0}^t = 1$, so the chain is recurrent. We can determine whether this is positive or negative recurrent using the following equation.

$$h_{0,0}^t = \sum_{n=0}^{\infty}(2n+2)\frac{1}{n+1}\binom{2n}{n}p^n(1-p)^{n+1} = 2(1-p)\sum_{n=0}^{\infty}\binom{2n}{n}(p(1-p))^n = \frac{2(1-p)}{\sqrt{1-4p(1-p)}}$$

Since $h_{0,0}^t$ is finite when $p < \frac{1}{2}$, the chain is positive recurrent.

(b) If $p = \frac{1}{2}$, $r_{0,0}^t = 1$, so the chain is recurrent. Since $h_{0,0}^t$ is infinite when $p = \frac{1}{2}$, the chain is negative recurrent.

(c) If $p > \frac{1}{2}$, $r_{0,0}^t = 1 = \frac{1-p}{p} < 1$, so the chain is transient.

Exercise 7.21

Let $\vec{\pi}$ a stationary distribution of the given Markov chain. Since $P_{i,0} = 1/2$ for $i \le n$, we have

$$\pi_0 = (\vec{\pi}P)_0 = \sum_{i=0}^{n}\pi_i P_{i,0} = \frac{1}{2}\sum_{i=0}^{n}\pi_i = \frac{1}{2}.$$

Now from $P_{i,i+1} = 1/2$, we obtain

$$\pi_1 = (\vec{\pi}P)_1 = \sum_{i=0}^{n}\pi_i P_{i,1} = \pi_0 P_{0,1} = \frac{1}{2^2}$$

$$\pi_2 = (\vec{\pi}P)_2 = \sum_{i=0}^{n}\pi_i P_{i,2} = \pi_1 P_{1,2} = \frac{1}{2^3}$$

$$\vdots$$

$$\pi_{n-1} = (\vec{\pi}P)_{n-1} = \sum_{i=0}^{n}\pi_i P_{i,n-1} = \pi_{n-2}P_{n-2,n-1} = \frac{1}{2^n}.$$

From $P_{n,n} = 1/2$, we obtain

$$\pi_n = (\vec{\pi}P)_n = \sum_{i=0}^{n} \pi_i P_{i,n} = \pi_{n-1}P_{n-1,n} + \pi_n P_{n,n} = \frac{1}{2^{n+1}} + \frac{1}{2}\pi_n, \quad \therefore \pi_n = \frac{1}{2^n}$$

Therefore the stationary distribution is $\vec{\pi} = \left(\frac{1}{2}, \frac{1}{2^2}, \cdots, \frac{1}{2^n}, \frac{1}{2^n}\right)$.