CSE 4190.101: Discrete Mathematics Fall 2021

Problem Set 3

Instructor: Yongsoo Song **Due on:** Nov 29, 2021

Please submit your answer through eTL. It should be a single PDF file (not jpg), either typed or scanned. **Please include your student ID and name (e.g. 2020-12345 YongsooSong.pdf).** You may discuss with other students the general approach to solve the problems, but the answers should be written in your own words.

- You should cite any reference that you used, and mention what you used it for.
- The reference information should be specific so that TAs are able to find the exact material you used. For example, it is not allowed to simply mention that "I referred a lecture note of Discrete Mathematics class at * university".
- Similarly, if your reference includes a url, type it or submit a separate text file (instead of handwritten address) so that TAs can easily visit the page.
- All references should be publicly accessible. Otherwise, attach the reference to your submission.

Problem 1 (10 points)

Let b be a positive integer. Use the well-ordering property to show that the following form of mathematical induction is a valid method to prove that $P(n)$ is true for all positive integers n.

- **Basis Step:** $P(1), P(2), \ldots, P(b)$ are true.
- **Inductive Step:** For each positive integer k, if $P(k) \wedge P(k+1) \wedge \cdots \wedge P(k+b-1)$ is true, then $P(k+b)$ is true.

Use the proof by contradiction: assume that the set $S = \{n \in \mathbb{N} \mid P(n) \text{ is false}\}$ is nonempty. Since $S \subseteq \mathbb{N}$ and $S \neq \emptyset$, by well-ordering property, there exists a least element $m \in S$. If $m \leq b$, then given that $P(1)$, $P(2)$, $\cdots$, $P(b)$ are true, $P(m)$ cannot be false. If $m > b$, since m is the least element to make the statement false, $P(m-b)$, $P(m-b+1)$, $\cdots$, $P(m-1)$ are all true. According to the inductive step, $P(m)$ is also true. In both cases, $P(m)$ is true and thus $m \notin S$. This contradicts the assumption: $m \in S$. So the first assumption is false and therefore, this procedure is a valid mathematical induction.

Problem 2 (10 points)

Show that it is possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the average of any two of these numbers never appears between them. [Hint: Show that it suffices to prove this fact when n is a power of 2. Then use mathematical induction to prove the result when n is a power of 2.]

Let m be a non-negative integer such that $2^{m-1} < n \leq 2^m$. Assume that we can arrange the numbers $1, \cdots, 2^m$ to satisfy given condition. In such sequence, eliminate all numbers larger than n. Then the sequence is an arrangement of $1, \cdots, n$ and satisfies given condition. This is because if we pick two numbers a_i and a_j from the sequence, then any number between them is not the average since the sequence is a subset of the original sequence containing $1, \cdots, 2^m$, which their average do not appear between them. We only eliminated $n+1, \cdots, 2^m$, hence the condition is satisfied. Therefore, it suffices to prove the statement only for $n = 2^k$.

If $n = 1 = 2^0$, the statement is obviously true. Suppose that we can arrange the numbers $1, \cdots, 2^k$ in that way. Let us split the sequence $1, \cdots, 2^{k+1}$ into odd and even numbers, i.e. $2\times 1-1, 2\times 2-1, \cdots, 2\times 2^k - 1$ and $2\times 1, 2\times 2, \cdots, 2\times 2^k$.

Since the average of two numbers has linearity, each subsequence can be arranged by induction hypothesis. If we concatenate two sequences, then the whole sequence satisfies given property, because if we pick two numbers from the same subsequence, (both odd numbers or even numbers) then the property holds by induction hypothesis, and if we pick two numbers from different subsequence, then the sum of two numbers is odd and the average is not an integer, then it cannot be in the whole sequence.

By the mathematical induction, the statement is true for all $k \geq 0$.

Problem 3 (10 points)

Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, or any smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

Let $P(n)$ be the number of breaks to break a bar consisting n squares into n separate squares. The number of breaks for 1-square bar is 0 because it is already a single square (i.e. $P(1) = 0$). Suppose that $P(k) = k-1$ is true for $k = 1, 2, \cdots, n-1$, and we want to prove that $P(n) = n-1$ is true. If once a bar which consists of n squares is broken, it is split into two chocolate bars which each split bar consists of less than n squares. Then, the number of breaks to break one bar is the sum of 1 and the number of breaks to break two split bar. Since $P(n) = P(a)+P(b)+1$ where $a+b = n$, $P(n) = P(a)+P(b)+1 = (a-1)+(b-1)+1 = n-1$ is true. Therefore, $P(n) = n-1$ is true, and the number of breaks is $n-1$.

Problem 4 (10 points)

How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

The pattern of bit strings containing five consecutive 0s or five consecutive 1s are:

00000*****, 100000****, *100000***, **100000**, ***100000*, ****100000
11111*****, 011111****, *011111***, **011111**, ***011111*, ****011111

where * can be 0 or 1.

Then, the number of bit strings containing five consecutive 0s (N_0) is $2^5 + 5\times 2^4 = 112$ and the number of bit strings containing five consecutive 1s (N_1) is also $2^5 + 5\times 2^4 = 112$. However, both cases contain bit strings 1111100000, 0000011111. ($N_{0,1} = 2$) Therefore, the number of bit strings containing either five consecutive 0s or five consecutive 1s is $N_0 + N_1 - N_{0,1} = 222$.

Problem 5 (10 points)

Suppose that p, q and r are prime numbers and that $n = pqr$. Use the principle of inclusion–exclusion to find the number of positive integers not exceeding n that are relatively prime to n.

Define the set $P_k = \{x \in \mathbb{N} \mid x \leq n \text{ and } k \text{ divides } x\}$ be a set of natural numbers which are less than or equal to n and are multiples of k. If k divides n, then $|P_k| = \frac{n}{k}$ holds. If p and q are relatively prime, then $P_p \cap P_q = P_{pq}$.

1. If p, q and r are all distinct numbers:

$$\begin{aligned} n - |P_p \cup P_q \cup P_r| &= n - |P_p| - |P_q| - |P_r| + |P_p \cap P_q| + |P_q \cap P_r| + |P_r \cap P_p| - |P_p \cap P_q \cap P_r| \\ &= n - |P_p| - |P_q| - |P_r| + |P_{pq}| + |P_{qr}| + |P_{rp}| - |P_{pqr}| = n - \frac{n}{p} - \frac{n}{q} - \frac{n}{r} + \frac{n}{pq} + \frac{n}{qr} + \frac{n}{rp} - \frac{n}{pqr} \\ &= n - pq - qr - rp + p + q + r - 1 \end{aligned}$$

2. If two of them are same (WLOG, $p = q \neq r$):

$$n - |P_p \cup P_r| = n - |P_p| - |P_r| + |P_p \cap P_r| = n - |P_p| - |P_r| + |P_{pr}| = n - \frac{n}{p} - \frac{n}{r} + \frac{n}{pr} = n - pr - p^2 + p$$

3. If $p = q = r$:

$$n - |P_p| = n - \frac{n}{p} = n - p^2$$

Problem 6 (10 points)

Give a combinatorial proof that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

[Hint: Count in two ways the number of ways to select a committee, with n members from a group of n mathematics professors and n computer science professors, such that the chairperson of the committee is a mathematics professor.]

First, we select k mathematics professors and $(n-k)$ computer science professors and then choose one chairperson among k mathematics professors. This way, the number of ways to choose is $\sum_{k=1}^{n}\binom{n}{k}\binom{n}{n-k}k = \sum_{k=1}^{n} k\binom{n}{k}^2$. Second, we choose the chairperson among n mathematics professors and randomly select other $(n-1)$ committee among $(2n-1)$ remaining professors. This way, the number of ways to choose is $n\binom{2n-1}{n-1}$. $\therefore \sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

Problem 7 (10 points)

Suppose that balls are tossed into b bins one by one so that each ball is equally likely to fall into any of the bins and that the tosses are independent. What is the expected number of balls tossed until every bin contains a ball?

Let the random variable X denote the number of balls tossed until every bin contains a ball and X_i denote the number of balls tossed while exactly $(i-1)$ bins contain a ball. While $(i-1)$ bins contain a ball, the probability that a new ball falls into the empty bin is $(b-i+1)/b$. Since all trials are independent, X_i has the geometric distribution with parameter $(b-i+1)/b$ and $E(X_i) = b/(b-i+1)$. Therefore, $E(X) = E\left(\sum_{i=1}^{b} X_i\right) = \sum_{i=1}^{b} E(X_i) = \sum_{i=1}^{b} \frac{b}{b-i+1} = b\sum_{i=1}^{b}\frac{1}{i}$.

Problem 8 (10 points)

Alice and Bob communicate using bit strings. Suppose that Alice sends a 1 one-third of the time and a 0 two-thirds of the time. When a 0 is sent, the probability that it is received correctly is 0.9 (so that the probability that it is received incorrectly as a 1 is 0.1). When a 1 is sent, the probability that it is received correctly is 0.8.

1. Find the probability that a 0 is received.

2. Use Bayes' theorem to find the probability that a 0 was transmitted, given that a 0 was received.

Let T_* and R_* denote the event of *-bit transmission and *-bit reception.

1. $P(R_0) = P(T_1)P(R_0|T_1) + P(T_0)P(R_0|T_0) = \frac{1}{3} \times 0.2 + \frac{2}{3} \times 0.9 = \frac{2}{3}$

2. $P(T_0|R_0) = \frac{P(R_0|T_0)P(T_0)}{P(R_0|T_0)P(T_0)+P(R_0|T_1)P(T_1)} = \frac{0.9\times 2/3}{2/3} = 0.9$

Problem 9 (10 points)

Use Chebyshev's inequality to find an upper bound on the probability that the number of tails that come up when a biased coin with probability of heads equal to 0.8 is tossed n times deviates from the mean by more than $\sqrt{n}$.

Chebyshev's inequality is $P(|X-\mu| \geq \sqrt{n}) \leq \frac{Var(X)}{n}$. The trials are independent Bernoulli trials, therefore $Var(X) = n \times 0.8 \times 0.2 = 0.16n$. Applying this gives $P(|X-\mu| \geq \sqrt{n}) \leq \frac{Var(X)}{n} = \frac{0.16n}{n} = 0.16$.

Problem 10 (10 points)

Let $X_1, X_2, \ldots, X_n$ be pairwise independent random variables on a sample space S. Show that

$$V(X_1 + X_2 + \cdots + X_n) = V(X_1) + V(X_2) + \cdots + V(X_n).$$

$$V\left(X_1 + X_2 + ... + X_n\right) = E\left[\left(X_1 + X_2 + ... + X_n\right)^2\right] - \left(E\left[X_1 + X_2 + ... + X_n\right]\right)^2$$

$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\right] - \left(\sum_{i=1}^{n} E\left[X_i\right]\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} E\left[X_i X_j\right] - \left(\sum_{i=1}^{n} E\left[X_i\right]\right)^2$$

By using pairwise independent property ($E[X_i]E[X_j] = E[X_i X_j]$),

$$\sum_{i=1}^{n}\sum_{j=1}^{n} E\left[X_i X_j\right] - \left(\sum_{i=1}^{n} E\left[X_i\right]\right)^2 = \sum_{i=1}^{n} E\left[X_i^2\right] - \sum_{i=1}^{n}\left(E\left[X_i\right]\right)^2 = \sum_{i=1}^{n} V\left(X_i\right)$$