Discrete Mathematics Problem Set 1

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Problem 1

1. $\neg q \land r \to p$
2. $p \to \neg q \land r$
3. If you can install the program your computer has at least 2GB of RAM and 20GB of free disk space.
4. $\neg p \to q \lor \neg r$
5. If you can't install the program your computer has less than 2GB of RAM or 20GB of free disk space.

Problem 2

We can construct a truth table.

P	Q	R	$P \to Q$	$(P \to Q) \to R$	$Q \to R$	$P \to (Q \to R)$
True	True	True	True	True	True	True
True	True	False	True	False	False	False
True	False	True	False	True	True	True
True	False	False	False	True	True	True
False	True	True	True	True	True	True
False	**True**	**False**	**True**	**False**	**False**	**True**
False	False	True	True	True	True	True
False	**False**	**False**	**True**	**False**	**True**	**True**

As it shows, when $(P, Q, R) =$ (False, True, False) or (False, False, False), $(P \to Q) \to R$ and $P \to (Q \to R)$ represent different values. Therefore the two expressions are logically inequivalent.

Problem 3

For every i, j $(i, j = 1, 2, \cdots, 9)$, at least one of the propositions "The number at cell (i, j) is n" $(= p(i, j, n))$ when $n = 1, 2, \cdots, 9$ should be true. Therefore the compound proposition is:

$$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigvee_{n=1}^{9} p(i, j, n)$$

Problem 4

(a) The 5 atomic statements are:

L : This house is next to a lake.
K : The treasure is in the kitchen.
E : The tree in the front yard is an elm.
P : The treasure is buried under the flagpole.
G : The treasure is in the garage.

Then the above compound statements can be expressed as:

1. $L \to \neg K$
2. $E \to K$
3. L
4. $E \vee \neg P$
5. $P \oplus G \Leftrightarrow P \wedge \neg G \vee \neg P \wedge G$

(b) The treasure is in the garage. The proposition G is true.
The proof can be written in two ways, either using the $\oplus$ operator or not. Using it gives:

(1)	$L \to \neg K$	Premise
(2)	L	Premise
(3)	$\neg K$	Modus Ponens from (1) and (2)
(4)	$E \to K$	Premise
(5)	$\neg E$	Modus Tollens from (3) and (4)
(6)	$E \vee \neg P$	Premise
(7)	$\neg P$	Disjunctive Syllogism from (5) and (6)
(8)	$P \oplus G$	Premise
(9)	G	Definition of $\oplus$(exclusive or) using (8)

Thus the treasure is in the garage. If we do not use the operator, lines (8) and (9) should be changed as:

(8)	$P \wedge \neg G \vee \neg P \wedge G$	Premise
(9)	$\neg P \vee G$	Addition from (7)
(10)	$\neg(P \wedge \neg G)$	De Morgan's Law from (9)
(11)	$\neg P \wedge G$	Disjunctive Syllogism from (8) and (9)
(12)	G	Simplification from (11)

Thus the treasure is in the garage.

Problem 5

1. True
2. True
3. False / Counterexample : $x = 36$
4. True
5. False / Counterexample : $x = 36$

Problem 6

Let's suppose n is even when $(n^2 + n + 1)$ is even, for any integer n.

We can set $n = 2k$ where $k \in \mathbb{Z}$. Then it follows that

$$\begin{aligned} n^2 + n + 1 &= (2k)^2 + 2k + 1 \\ &= 2 \times (2k^2 + k) + 1 \end{aligned}$$

This shows that $(n^2 + n + 1)$ is odd, but according to the assumption $(n^2 + n + 1)$ should be even.

Therefore, using 'proofs by contradiction', we can say that n should be odd.

Problem 7

Let's suppose that a real number x which satisfies the equation is a rational number.

Solving the equation gives:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For x to be a rational number, $\sqrt{b^2 - 4ac}$ should be rational. Especially, since a, b, c are integers, $(b^2 - 4ac)$ should be a perfect square. Thus we can set $b^2 - 4ac = k^2$, where $k \in \mathbb{N}$. This gives:

$$b^2 - k^2 = 4ac \tag{1}$$
$$(b + k)(b - k) = 4ac \tag{2}$$

From (1), since b^2 is odd and $4ac$ is even, we can know that k^2 and k should be odd. Then we can set $b = 2b^* + 1$, $k = 2k^* + 1$ and write (2) as:

$$4(b^* + k^* + 1)(b^* - k^*) = 4ac \tag{3}$$

Now since $(b^* + k^* + 1) + (b^* - k^*) = 2b^* + 1$ and is odd, either $(b^* + k^* + 1)$ or $(b^* - k^*)$ is even. Then from (3), we can know that $4(b^* + k^* + 1)(b^* - k^*)$ is a multiple of 8 and ac should be even.

However, since this question is supposing a and c are odd, this is a contradiction. Therefore, x is irrational.

Problem 8

Let's name each tile with an alphabet like the following. There are four types of tiles in total.

a	b	c	d	a	b	c	d	a	b
b	c	d	a	b	c	d	a	b	c
c	d	a	b	c	d	a	b	c	d
d	a	b	c	d	a	b	c	d	a
a	b	c	d	a	b	c	d	a	b
b	c	d	a	b	c	d	a	b	c
c	d	a	b	c	d	a	b	c	d
d	a	b	c	d	a	b	c	d	a
a	b	c	d	a	b	c	d	a	b
b	c	d	a	b	c	d	a	b	c

Let's suppose we can tile the checkerboard with straight tetrominoes.

A straight tetromino is a shape that has four tiles in a row. Therefore whererver we place a straight tetromino in the checkerboard, it will cover one of each tile. ('a', 'b', 'c', and 'd')

If we count the number of each tile, there are 25 'a' tiles, 26 'b' tiles, 25 'c' tiles, 24 'd' tiles.

However, since we should fill the checkerboard only with straight tetrominoes, there should be an equal number of each tile. This is a contradiction, and therefore we cannot tile the checkerboard.

Problem 9

(a) False / Counterexample : $A = \{1\}, B = \{2\}, C = \{1, 2\}$

(b) True

$A \cup C \subseteq B \cup C$ and $A \cap C \subseteq B \cap C$ are equivalent to the following.

$$A \cup C \subseteq B \cup C \Leftrightarrow (A \cup C) \cap (B \cup C) = A \cup C \tag{1}$$
$$A \cap C \subseteq B \cap C \Leftrightarrow (A \cap C) \cap (B \cap C) = A \cap C \tag{2}$$

From (1), we can know

$$\begin{aligned}(A \cap B) \cup (A \cap C) &= ((A \cap B) \cup A) \cap ((A \cap B) \cup C) && \text{(Distributive Law)}\\ &= A \cap ((A \cup C) \cap (B \cup C)) && (A \cap B \subseteq A \text{ / Distributive Law})\\ &= A \cap (A \cup C) && \text{(From (1))}\\ &= A && (A \subseteq A \cup C) \qquad (3)\end{aligned}$$

Also (2) gives $(A \cap C) \cap (B \cap C) = B \cap (A \cap C) = A \cap C$, therefore

$$A \cap C \subseteq B \tag{4}$$

From (3) and (4), we can know

$$\begin{aligned}
A \cap B &= ((A \cap B) \cup (A \cap C)) \cap B && \text{(From (3))} \\
&= ((A \cap B) \cap B) \cup ((A \cap C) \cap B) && \text{(Distributive Law)} \\
&= B \cup (A \cap C) && (B \subseteq A \cap B \text{ / From (4))} \\
&= B && \text{(From (4))}
\end{aligned}$$

Therefore $A \subseteq B$.

Problem 10

(a) f is surjective.

The statement "f is surjective" can be written as

$$\forall n \in \mathbb{Z},\ \exists S \in F \text{ such that } f(S) = n$$

Choosing $S = \{n\}$ gives $f(S) = n$. Therefore the proposition is true.

(b) f is not injective.

The statement "f is injective" can be written as

$$\forall S_1, S_2 \in F,\ \text{if } S_1 \neq S_2 \text{ then } f(S_1) \neq f(S_2)$$

We can find a counterexample: $f(\{0\}) = f(\{1, -1\}) = 0$

Problem 11

(a) Since f is not injective, it is not a bijection.

We can find a counterexample: $f(1, 1) = f(2, 2) = 0$

(b) $S = \{(n, 1) \mid n \in \mathbb{N}\} \cup \{(1, n) \mid n \in \mathbb{N}\} = \{(1, 1), (2, 1), (1, 2), (3, 1), (1, 3), \cdots\}$

f is a bijection over the given set S. We can prove this by showing f is both surjective and injective.

First, to show f is surjective, we should prove "$\forall k \in \mathbb{Z},\ \exists T \in S$ such that $f(T) = k$"

Choosing $T = \begin{cases} (1 + k, 1) & (k \geq 0) \\ (1, 1 - k) & (k < 0) \end{cases}$ from S gives $f(T) = k$. Therefore f is surjective.

Second, to show f is injective, we should prove "$\forall T_1, T_2 \in S,$ if $f(T_1) = f(T_2)$ then $T_1 = T_2$".

Let's say $S_1 = \{(n,1) \,|\, n \in \mathbb{N}\}$ and $S_2 = \{(1,n) \,|\, n \in \mathbb{N},\ n > 1\}$. Then $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$.

We can choose T_1 and T_2 from S in four ways.

(1) $T_1, T_2 \in S_1$
When $T_1 = (n_1, 1)$ and $T_2 = (n_2, 1)$, if $f(T_1) = f(T_2)$ then $n_1 - 1 = n_2 - 1$ and $n_1 = n_2$. $\therefore T_1 = T_2$.

(2) $T_1, T_2 \in S_2$
When $T_1 = (1, n_1)$ and $T_2 = (1, n_2)$, if $f(T_1) = f(T_2)$ then $1 - n_1 = 1 - n_2$ and $n_1 = n_2$. $\therefore T_1 = T_2$.

(3) $T_1 \in S_1$, $T_2 \in S_2$
When $T_1 = (n_1, 1)$ and $T_2 = (1, n_2)$, $f(T_1) = n_1 - 1 \geq 0$ and $f(T_2) = 1 - n_2 < 0$. Therefore always $f(T_1) \neq f(T_2)$, and the statement is always true.

(4) $T_1 \in S_2$, $T_2 \in S_1$
Same as case (3), statement is true.

Therefore f is injective. Since f is both surjective and injective, f is a bijection.

Problem 12

If S is a finite set of n elements $(n = 1, 2, 3, \cdots)$, $\mathcal{P}(S)$ has 2^n elements.

When S is an infinite set, let's suppose such a function f exists. We can also define $T = \{s \in S | s \notin f(s)\}$.

Since T consists of elements of S, it is a subset of S and is an element of $\mathcal{P}(S)$. Let's think of an arbitary element $s \in S$. We can consider s in two ways.

(1) $s \in T$
By the definition of T, $s \notin f(s)$ and therefore $T \neq f(s)$.

(2) $s \notin T$
By the definition of T, $s \in f(s)$ and therefore $T \neq f(s)$.

Since f is an onto function, for all s, $f(s)$ makes up $\mathcal{P}(S)$. Also $T \neq f(s)$ for any s, so $T \notin \mathcal{P}(S)$. However, $T \in \mathcal{P}(S)$ according to its definition, so this is a contradiction. Therefore an onto function f does not exist.

$\therefore$ For both finite and infinite sets, an onto function f from S to $\mathcal{P}(S)$ does not exist.