Engineering Mathematics 2
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Homework 6-1
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Exercise 6.2

(a) We can choose a total of $\binom{n}{4}$ different copies of K_4 in K_n. Let X the number of monochromatic copies, then X is the sum of X_is, where X_i the random variable having 1 if the ith copy is monochromatic and 0 otherwise. Each copy has 6 edges, so with probability 2^{-5}, the copy is monochromatic. Therefore $\mathbf{E}[X_i] = 2^{-5}$, so $\mathbf{E}[X] = \binom{n}{4}2^{-5}$. From the expectation argument, we can say there exists a coloring of K_n that makes the number of monochromatic copies at most $\binom{n}{4}2^{-5}$.

(b) The proof of (a) guarantees that randomly coloring each edges will give at least one case where there are at most $\binom{n}{4}2^{-5}$ monochromatic copies of K_4. Since there are $\binom{n}{2}$ edges and $\binom{n}{4}$ copies in total, coloring all edges and checking all copies will take time of $O(n^4)$. Now let p the probability that the algorithm succeeds, i.e. $p = \Pr\left(X \le \binom{n}{4}2^{-5}\right)$. Then

$$\begin{aligned}\binom{n}{4}2^{-5} &= \mathbf{E}[X] \\ &= \sum_{x \le \binom{n}{4}2^{-5}} x\Pr(X = x) + \sum_{x \ge \binom{n}{4}2^{-5}+1} x\Pr(X = x) \\ &\ge \sum_{x \le \binom{n}{4}2^{-5}} 1 \cdot \Pr(X = x) + \sum_{x \ge \binom{n}{4}2^{-5}+1} \left(\binom{n}{4}2^{-5} + 1\right)\Pr(X = x) \\ &= p + (1-p)\left(\binom{n}{4}2^{-5} + 1\right),\end{aligned}$$

so we obtain

$$p \ge \frac{1}{\binom{n}{4}2^{-5}}, \quad \frac{1}{p} \le \binom{n}{4}2^{-5}.$$

Therefore the expected number of trials is at most $\binom{n}{4}2^{-5} = O(n^4)$, so the total running time is $O(n^8)$.

(c) Label the color of edges as $c_i \in \{A, B\}$ where $i = 1, \cdots, \binom{n}{2}$. Then start coloring the edges in order of ascending i, by choosing c_i as the color that gives the smaller expected number of monochromatic copies of K_4, given edges up to i are colored. We know that $\mathbf{E}[X] = \binom{n}{4}2^{-5}$, and by symmetry, also $\mathbf{E}[X \mid c_1] = \binom{n}{4}2^{-5}$. Now for all i, since

$$\mathbf{E}[X \mid c_1, \cdots, c_i] = \frac{1}{2}\mathbf{E}[X \mid c_1, \cdots, c_i, c_{i+1} = A] + \frac{1}{2}\mathbf{E}[X \mid c_1, \cdots, c_i, c_{i+1} = B],$$

choosing appropriate c_{i+1} will give $\mathbf{E}[X \mid c_1, \cdots, c_i, c_{i+1}] \le \mathbf{E}[X \mid c_1, \cdots, c_i] \le \cdots \le \mathbf{E}[X \mid c_1] = \binom{n}{4}2^{-5}$. Therefore we can say $\mathbf{E}\left[X \mid c_1, \cdots, c_{\binom{n}{2}}\right] \le \binom{n}{4}2^{-5}$, so this algorithm works properly. In terms of running time, we should check every copy of K_4 for each step of coloring a new edge. There are $\binom{n}{2}$ edges in total and $\binom{n}{4}$ copies in total, so the algorithm takes at most $O(n^6)$ time to find such a coloring.

Exercise 6.6

First we should prove that such k-cut exists for any graph with m edges. Construct disjoint sets $V_1, \cdots, V_k$ such that they are partitions of V. Let C_i the random variable having 1 if the ith edge connects vertices from two different sets in $V_1, \cdots, V_k$ and 0 otherwise. Then

$$\mathbf{E}[C_i] = \Pr(C_i = 1) = 1 - \frac{k}{k^2} = \frac{k-1}{k}$$

since the probability that X_i connects two vertices from a specific partition is $1/k^2$. Let C the value of the k-cut from partitions $V_1, \cdot, V_k$. Then $C = \sum_{i=1}^{m} C_i$, so

$$\mathbf{E}[C] = \sum_{i=1}^{m} \mathbf{E}(C_i) = \frac{(k-1)m}{k}.$$

Therefore using the expectation argument, we can say such k-cut exists for any graph with m edges.

Now we should suggest a deterministic algorithm that gives such cut for any graph. Let n the number of vertices in the graph. Then start placing vertices $v_1, \cdots, v_n$ of the graph in $V_1, \cdots, V_k$ orderly, by choosing the set that will maximize the expected value of the cut. Since $\mathbf{E}[C] = (k-1)m/k$, we can say $\mathbf{E}[C \mid v_1] = (k-1)m/k$ by symmetry. Now for $j = 1, \cdots, n-1$, since

$$\mathbf{E}[C \mid v_1, \cdots, v_j] = \frac{1}{k}\mathbf{E}[C \mid v_1, \cdots, v_j, v_{j+1} \in V_1] + \cdots + \frac{1}{k}\mathbf{E}[C \mid v_1, \cdots, v_j, v_{j+1} \in V_k],$$

choosing appropriate v_{i+1} will give $\mathbf{E}[C \mid v_1, \cdots, v_j, v_{j+1}] \geq \mathbf{E}[C \mid v_1, \cdots, v_j]$. Therefore $\mathbf{E}[C \mid v_1, \cdots, v_n] \geq \cdots \geq \mathbf{E}[C \mid v_1] = (k-1)m/k$, so we end with a proper partition of vertices. The algorithm gives such a cut.

Exercise 6.10

(a) When $\mathcal{F}$ is the family of all subsets with size $\lfloor n/2 \rfloor$, the subsets can not be subsets of each other unless they are the same ones.

(b) By the definition of *antichain*, only one of X_i can be 1, since if more than 1 X_i being 1 indicates the existance of a subset between sets in $\mathcal{F}$. Therefore $X \leq 1$, so we obtain

$$1 \geq \mathbf{E}[X] = \sum_{k=0}^{n} \mathbf{E}[X_k] = \sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}}$$

since the set of the first k numbers of the permutation should be one of the f_k sets in $\mathcal{F}$.

(c) From (b), we can say

$$1 \geq \sum_{k=0}^{n} \frac{f_k}{\binom{n}{k}} \geq \sum_{k=0}^{n} \frac{f_k}{\binom{n}{\lfloor n/2 \rfloor}} = \frac{|\mathcal{F}|}{\binom{n}{\lfloor n/2 \rfloor}}.$$

Therefore $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ for any antichain $\mathcal{F}$.