Homework 1
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Exercise 1.4

The winner and loser should have won n, k games each when the match is over. Since t winner must have won the last game, the loser should have taken k games out of the remaining $n + k - 1$ games. The number of cases for this situation is $\binom{n+k-1}{k}$. All probabilities for a win or lose decision for each game is $\frac{1}{2}$, so the total probability for each case is $\frac{1}{2^{n+k}}$. Therefore the probability for a winner and loser decision is $\frac{\binom{n+k-1}{k}}{2^{n+k}}$, and since the winner and loser can be the opposite of the two, the total probability would be $\frac{\binom{n+k-1}{k}}{2^{n+k-1}}$.

Exercise 1.6

Let E_n^k the event where there are n balls in the bin and k of them are white. We can use mathematical induction on n to obtain the probability of this event. First claim

$$P\big[E_n^k\big] = \begin{cases} \dfrac{1}{n-1} & (1 \le k < n) \\ 0 & (k \ge n) \end{cases}$$

for each n. If $n = 2$, $P\big[E_2^1\big] = 1$ so the hypothesis holds. Now suppose that the the inductive hypothesis holds for $n = m - 1$. For the bin to have k $(1 \le k < n)$ white balls at $n = m$, it should have had $k - 1$ or k white balls in the previous iteration. Therefore

$$\begin{aligned} P\big[E_m^k\big] &= P\big[E_m^k \cap E_{m-1}^k\big] + P\big[E_m^k \cap E_{m-1}^{k-1}\big] \\ &= P\big[E_m^k \mid E_{m-1}^k\big] P\big[E_{m-1}^k\big] + P\big[E_m^k \mid E_{m-1}^{k-1}\big] P\big[E_{m-1}^{k-1}\big] \\ &= \frac{m-1-k}{m-1} \times \frac{1}{m-2} + \frac{k-1}{m-1} \times \frac{1}{m-2} \\ &= \frac{1}{m-1} \end{aligned}$$

and the claim holds also at $n = m$. If $k > m$, $P\big[E_{m-1}^k\big] = P\big[E_{m-1}^k\big] = 0$ so $P\big[E_m^k\big] = 0$, thus the claim holds. Therefore for all n, the probability for the number of white balls to be any number between 1 and $n - 1$ is all equal to $\frac{1}{n-1}$.

Exercise 1.8

Let E_n the event where a number chosen is divisible by $n \in \mathbb{N}$. Then the inclusion-exclusion gives

$$\begin{aligned} P[E_4 \cup E_6 \cup E_9] &= P[E_4] + P[E_6] + P[E_9] - P[E_4 \cap E_6] - P[E_6 \cap E_9] - P[E_9 \cap E_4] + P[E_4 \cap E_6 \cap E_9] \\ &= P[E_4] + P[E_6] + P[E_9] - P[E_{12}] - P[E_{18}] - P[E_{36}] + P[E_{36}] \\ &= \frac{250000}{1000000} + \frac{166666}{1000000} + \frac{111111}{1000000} - \frac{83333}{1000000} - \frac{55555}{1000000} \\ &= \frac{388889}{1000000}. \end{aligned}$$

Exercise 1.15

Let E_n^k the event where the remainder when dividing the sum up to the n-th roll is k. We can use mathematical induction to obtain the probability. First claim that at for all n, $E_n^k = \dfrac{1}{6}$ equally for $k = 0, 1, \cdots, 5$. If $n = 1$, the claim holds. Suppose the hypothesis holds at $n = m - 1$. For any remainder of the sum up to the $(m - 1)$-th roll, the m-th roll can have only one determined value to make the remainder of the sum up to the n-th roll to be k. Therefore for all $i = 0, 1, \cdots, 5$, $P\big[E_m^k \mid E_{m-1}^i\big] = \dfrac{1}{6}$. This gives the following.

$$P\big[E_m^k\big] = P\big[E_m^k \mid E_{m-1}^1\big]P\big[E_{m-1}^1\big] + P\big[E_m^k \mid E_{m-1}^2\big]P\big[E_{m-1}^2\big] + \cdots + P\big[E_m^k \mid E_{m-1}^5\big]P\big[E_{m-1}^5\big]$$
$$= \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} + \cdots + \frac{1}{6} \times \frac{1}{6} = \frac{1}{6}$$

The claim also holds at $n = m$, so it holds for all $n \in \mathbb{N}$. Therefore the probability that the sum up to the tenth roll is divisible by 6 is $\dfrac{1}{6}$.

Exercise 1.18

We can use an algorithm going through the following steps:

1. Choose a random integer $x \in \{0, 1, \cdots, n - 1\}$.

2. Define $y := (z - x) \mod n$.

3. Assume $F(z) = (F(x) + F(y)) \mod m$.

The input is z, and the algorithm uses two lookups $F(x)$ and $F(y)$ to return $(F(x) + F(y)) \mod m$ as the output.

From step 2, we can obtain $z = (x + y) \mod n$, so

$$P[F(z) = (F(x) + F(y)) \mod m] = P[F((x + y) \mod n) = (F(x) + F(y)) \mod m]$$

'$F((x+y) \mod n) = (F(x) + F(y)) \mod m$' is always true when F is not disrupted, but since F is partially uncorrect, the probability of it being true is larger than the probability of $F(x)$ and $F(y)$ not both being wrong. ($F(x) + F(y)$ may have the same $\mod m$ value even if the two are all incorrect, so the probabilty is **larger**, not equal to the probabilty of not both being wrong.) Let E_k the event where $F(k)$ is incorrect, then $\forall k$, $P[E_k] = \dfrac{1}{5}$, so

$$P[F((x + y) \mod n) = (F(x) + F(y)) \mod m] \geq 1 - P[E_x \cap E_y]$$
$$\geq 1 - (P[E_x] + P[E_y]) = \frac{3}{5}$$

Therefore the algorithm has an accuracy larger than $\dfrac{1}{2}$ and works for every possible value of z.

If we run the algorithm three times, choosing the result that appears multiple times (if there exists) would be the best method. The probability of two attempts giving the correct answer is at least $3 \cdot \left(\dfrac{3}{5}\right)^2 \cdot \dfrac{2}{5} = \dfrac{54}{125}$, and the probability of three attempts giving the correct answer is at least $\left(\dfrac{3}{5}\right)^3 = \dfrac{27}{125}$. Therefore the probability of getting a correct answer is at least $\dfrac{54}{125} + \dfrac{27}{125} = \dfrac{81}{125}$, which is enhanced than the non-repeated algorithm.

Exercise 1.23

The randomized min-cut algorithm outputs a specific minimum cut-set with probability at least $\dfrac{2}{n(n-1)}$.
Suppose there are k distinct min-cut sets, then the probability that the algorithm outputs any minimum cut-set is at least $\dfrac{2k}{n(n-1)}$. Since a probability cannot be larger than 1, we obtain the following.

$$\frac{2k}{n(n-1)} \le 1, \;\; k \le \frac{n(n-1)}{2}$$

Therefore there can be at most $\dfrac{n(n-1)}{2}$ distinct min-cut sets.