Engineering Mathematics 2
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Homework 3
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Exercise 3.3

Let X_i random variables of the number that appears at the ith roll. Then all X_is are independent and identically distributed, and $\mathbf{E}[X_i] = \dfrac{7}{2}$, $\mathbf{Var}[X_i] = \dfrac{35}{12}$. Since $X = \sum_{i=1}^{100} X_i$, $\mathbf{E}[X] = 350$ and $\mathbf{Var}[X] = \dfrac{875}{3}$. Therefore, using Chebyshev's inequality, we obtain the following.

$$\Pr\big(|X - 350| \geq 50\big) = \Pr\big(|X - \mathbf{E}[X]| \geq 50\big) \leq \frac{\mathbf{Var}[X]}{50^2} = \frac{7}{60}$$

Exercise 3.6

Let X_i random variables of the number of flips made from the appearance of the $(i-1)$th head to the appearance of the ith head. Then all X_is are independent and identically distributed over a geometric distribution of probability p. Therefore $\mathbf{Var}[X_i] = \dfrac{1-p}{p^2}$. Now let X the random variable of the number of flips until the kth head appears. Then $X = \sum_{i=1}^{k} X_i$, and since all X_is are independent,

$$\mathbf{Var}[X] = \mathbf{Var}\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} \mathbf{Var}[X_i] = \frac{k(1-p)}{p^2}.$$

Exercise 3.7

Let X_i random variables of the rate of increase between the price at the $(i+1)$th day and the ith day. Then $\Pr(X_i = r) = p$, $\Pr\left(X_i = \dfrac{1}{r}\right) = 1 - p$. Therefore

$$\mathbf{E}[X_i] = pr + \frac{1-p}{r}, \quad \mathbf{E}\big[X_i^2\big] = pr^2 + \frac{1-p}{r^2}.$$

Now let X the random variable of the price after d days. Then $X = 1 \times \prod_{i=1}^{d} X_i$, and since all X_is are independent,

$$\begin{aligned}
\mathbf{E}[X] &= \mathbf{E}\left[\prod_{i=1}^{d} X_i\right] = \prod_{i=1}^{d} \mathbf{E}[X_i] = \left(pr + \frac{1-p}{r}\right)^d \\
\mathbf{Var}[X] &= \mathbf{E}\big[X^2\big] - (\mathbf{E}[X])^2 \\
&= \mathbf{E}\left[\left(\prod_{i=1}^{d} X_i\right)^2\right] - \left(pr + \frac{1-p}{r}\right)^{2d} = \mathbf{E}\left[\prod_{i=1}^{d} X_i^2\right] - \left(pr + \frac{1-p}{r}\right)^{2d} \\
&= \prod_{i=1}^{d} \mathbf{E}\big[X_i^2\big] - \left(pr + \frac{1-p}{r}\right)^{2d} \\
&= \left(pr^2 + \frac{1-p}{r^2}\right)^d - \left(pr + \frac{1-p}{r}\right)^{2d}
\end{aligned}$$

Exercise 3.15

$$
\begin{aligned}
\mathbf{Var}[X] &= \mathbf{E}\left[X^2\right] - (\mathbf{E}[X])^2 \\
&= \mathbf{E}\left[\left(\sum_{i=1}^n X_i\right)^2\right] - \left(\mathbf{E}\left[\sum_{i=1}^n X_i\right]\right)^2 = \mathbf{E}\left[\left(\sum_{i=1}^n X_i\right)^2\right] - \left(\sum_{i=1}^n \mathbf{E}[X_i]\right)^2 \\
&= \mathbf{E}\left[\sum_{i=1}^n X_i^2 + \sum_{i \neq j} X_i X_j\right] - \sum_{i=1}^n (\mathbf{E}[X_i])^2 - \sum_{i \neq j} \mathbf{E}[X_i]\mathbf{E}[X_j] \\
&= \sum_{i=1}^n \mathbf{E}\left[X_i^2\right] + \sum_{i \neq j} \mathbf{E}[X_i X_j] - \sum_{i=1}^n (\mathbf{E}[X_i])^2 - \sum_{i \neq j} \mathbf{E}[X_i]\mathbf{E}[X_j] \\
&= \sum_{i=1}^n \left(\mathbf{E}\left[X_i^2\right] - (\mathbf{E}[X_i])^2\right) + \sum_{i \neq j} (\mathbf{E}[X_i X_j] - \mathbf{E}[X_i]\mathbf{E}[X_j]) \\
&= \sum_{i=1}^n \mathbf{Var}[X_i] \quad (\because \forall i \neq j, \quad \mathbf{E}[X_i X_j] = \mathbf{E}[X_i]\mathbf{E}[X_j])
\end{aligned}
$$

Exercise 3.20

Since Y is a nonnegative random variable, we can apply Marcov's inequality, which gives

$$
\Pr(Y \geq 1) \leq \frac{\mathbf{E}[Y]}{1}.
$$

Since $\Pr(Y \neq 0) = \Pr(Y \geq 1)$, we obtain $\Pr(Y \neq 0) \leq \mathbf{E}[Y]$. Now let $X = Y \mid Y \neq 0$, then Jensen's inequality gives

$$
(\mathbf{E}[X])^2 \leq \mathbf{E}\left[X^2\right], \quad (\mathbf{E}[Y \mid Y \neq 0])^2 \leq \mathbf{E}\left[Y^2 \mid Y \neq 0\right].
$$

Each can be simplified as

$$
\begin{aligned}
\mathbf{E}[Y \mid Y \neq 0] &= \sum_{y=0}^\infty y \Pr(Y = y \mid Y \neq 0) \\
&= \sum_{y=0}^\infty y \cdot \frac{\Pr(Y = y \cap Y \neq 0)}{\Pr(Y \neq 0)} = \sum_{y=1}^\infty y \cdot \frac{\Pr(Y = y)}{\Pr(Y \neq 0)} \\
&= \frac{1}{\Pr(Y \neq 0)} \sum_{y=1}^\infty y \Pr(Y = y) = \frac{\mathbf{E}[Y]}{\Pr(Y \neq 0)} \\
\mathbf{E}\left[Y^2 \mid Y \neq 0\right] &= \sum_{y=0}^\infty y^2 \Pr(Y = y \mid Y \neq 0) \\
&= \sum_{y=0}^\infty y^2 \cdot \frac{\Pr(Y = y \cap Y \neq 0)}{\Pr(Y \neq 0)} = \sum_{y=1}^\infty y^2 \cdot \frac{\Pr(Y = y)}{\Pr(Y \neq 0)} \\
&= \frac{1}{\Pr(Y \neq 0)} \sum_{y=1}^\infty y^2 \Pr(Y = y) = \frac{\mathbf{E}\left[Y^2\right]}{\Pr(Y \neq 0)}.
\end{aligned}
$$

Therefore

$$
\left(\frac{\mathbf{E}[Y]}{\Pr(Y \neq 0)}\right)^2 \leq \frac{\mathbf{E}\left[Y^2\right]}{\Pr(Y \neq 0)}, \quad \frac{(\mathbf{E}[Y])^2}{\mathbf{E}[Y^2]} \leq \Pr(Y \neq 0).
$$

$$
\therefore \frac{(\mathbf{E}[Y])^2}{\mathbf{E}[Y^2]} \leq \Pr(Y \neq 0) \leq \mathbf{E}[Y]
$$

Exercise 3.26

Let $X = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$, then since all X_is are independent,

$$\mathbf{E}[X] = \mathbf{E}\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right] = \frac{\mu + \mu + \cdots + \mu}{n} = \mu$$
$$\mathbf{Var}[x] = \mathbf{Var}\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right] = \frac{\sigma^2 + \sigma^2 + \cdots + \sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

Now Chebyshev's inequality gives

$$\Pr(|X - \mathbf{E}[X]| > \epsilon) \leq \frac{\mathbf{Var}[X]}{\epsilon^2}, \ \ \Pr(|X - \mu| > \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}.$$

Therefore

$$0 \leq \lim_{n\to\infty} \Pr(|X - \mu| > \epsilon) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2} = 0, \ \ \lim_{n\to\infty} \Pr(|X - \mu| > \epsilon) = 0.$$