# Engineering Mathematics 1 Problem Set 3

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## Problem 1

Both problems are in forms of second-order Euler-Cauchy equations.

(a) The characteristic equation is $m^2 - 5m + 6 =$, which gives two real roots $m = 2, 3$. Therefore a general solution would be $y = c_1 x^2 + c_2 x^3$. Substituting the initial conditions gives:

$$y = 1.2x^2 - 0.8x^3$$

(b) The characteristic equation is $m^2 + 2m + 1 =$, which gives a real double root $m = -1$. Therefore a general solution would be $y = (c_1 + c_2 \ln x)x^{-1}$. Substituting the initial conditions gives:

$$y = \frac{3.6 + 4\ln x}{x}$$

## Problem 2

Since $y_1$, $y_2$ are solutions of the ODE, we can say $y_1'' + p(x)y_1' + q(x)y_1 = 0$ and $y_2'' + p(x)y_2' + q(x)y_2 = 0$. Therefore $y_1''y_2 + p(x)y_1'y_2 + q(x)y_1y_2 = 0$ and $y_1y_2'' + p(x)y_1y_2' + q(x)y_1y_2 = 0$, so subtracting the two equations gives:

$$y_1''y_2 - y_1y_2'' + p(x)(y_1'y_2 - y_1y_2') = 0$$

$$\frac{dW}{dx} + p(x)W = 0$$

$$\int_{W_0}^{W} \frac{1}{W} dW = -\int_{x_0}^{x} p(t)dt \ \ \left(W_0 = [W(y_1, y_2)]_{x=x_0} =: c\right)$$

$$\therefore W = c \cdot \exp\left[-\int_{x_0}^{x} p(t)dt\right]$$

## Problem 3

For all questions, we should find (1) a general solution of the corresponding homogenous ODE and (2) a particular solution of the nonhomogenous ODE. The sum of the two solutions will be a general solution of the nonhomogenous ODE. (Used initial conditions $y(0) = 1$, $y'(0) = -1.5$ for problems (a), (b), (c))

(a) (1) A damped system where $m = 1$, $c = 4$, $k = 4$. A general solution is $y_h = (c_1 + c_2x)e^{-2x}$.

(2) Using the method of undetermined coefficients, suppose $y_p = e^{-2x}(K\cos 2x + M\sin 2x)$ is a particular solution. Substituting $y_p$ to the ODE gives:

$$8(K\sin 2x - M\cos 2x) - 8((K+M)\sin 2x + (K-M)\cos 2x) + 4(K\cos 2x + M\sin 2x) = \sin 2x$$
$$-4K\cos 2x - 4M\sin 2x = \sin 2x, \quad K = 0,\ M = -\frac{1}{4}$$
$$y_p = -\frac{1}{4}e^{-2x}\sin 2x$$

Therefore a general solution will be $y = (c_1 + c_2 x)e^{-2x} - \frac{1}{4}e^{-2x}\sin 2x$, and substituting the initial conditions gives:

$$y = (1+x)e^{-2x} - \frac{1}{4}e^{-2x}\sin 2x$$

(b) (1) An undamped system where $m = 1$, $k = 9$. A general solution is $y_h = c_1\cos 3x + c_2\sin 3x$.

(2) $y_1 = \sin 3x$, $y_2 = \cos 3x$ are two solutions of the homogenous ODE and $W = -3$. Using Lagrange's method gives a particular solution:

$$\begin{aligned} y_p &= -\sin 3x\int \frac{\cos 3x \cdot \sec 3x}{-3}dx + \cos 3x\int \frac{\sin 3x \cdot \sec 3x}{-3}dx \\ &= \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\ln(\cos 3x) \end{aligned}$$

Therefore a general solution will be $y = c_1\cos 3x + c_2\sin 3x + \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\ln(\cos 3x)$, and substituting the initial conditions gives:

$$y = \cos 3x - \frac{1}{2}\sin 3x + \frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\ln(\cos 3x)$$

(c) (1) A damped system where $m = 1$, $c = 6$, $k = 9$. A general solution is $y_h = (c_1 + c_2 x)e^{-3x}$.

(2) $y_1 = e^{-3x}$, $y_2 = xe^{-3x}$ are two solutions of the homogenous ODE and $W = e^{-6x}$. Using Lagrange's method gives a particular solution:

$$\begin{aligned} y_p &= -e^{-3x}\int \frac{xe^{-3x}}{e^{-6x}} \cdot \frac{16e^{-3x}}{x^2+1}dx + xe^{-3x}\int \frac{e^{-3x}}{e^{-6x}} \cdot \frac{16e^{-3x}}{x^2+1}dx \\ &= -e^{-3x}\int \frac{16x}{x^2+1}dx + xe^{-3x}\int \frac{16}{x^2+1}dx \\ &= -8e^{-3x}\ln\left(x^2+1\right) + 16xe^{-3x}\arctan x \end{aligned}$$

Therefore a general solution will be $y = (c_1 + c_2 x)e^{-3x} + -8e^{-3x}\ln\left(x^2+1\right) + 16xe^{-3x}\arctan x$, and substituting the initial conditions gives:

$$y = \left(1 + \frac{3}{2}x\right)e^{-3x} + -8e^{-3x}\ln\left(x^2+1\right) + 16xe^{-3x}\arctan x$$

(d) (1) A second order Euler-Cauchy equation where $a = -4$, $b = 6$. A general solution is $y_h = c_1x^2 + c_2x^3$.

(2) $y_1 = x^2$, $y_2 = x^3$ are two solutions of the homogenous ODE and $W = x^4$. Using Lagrange's method gives a particular solution:

$$\begin{aligned} y_p &= -x^2 \int \frac{x^3 \cdot 21x^{-4}}{x^4} dx + x^3 \int \frac{x^2 \cdot 21x^{-4}}{x^4} dx \\ &= -x^2 \int \frac{21}{x^5} dx + x^3 \int \frac{21}{x^6} dx \\ &= \frac{21}{20} x^{-2} \end{aligned}$$

Therefore a general solution will be $y = c_1x^2 + c_2x^3 + \frac{21}{20}x^{-2}$.

## Problem 4

Suppose $y = e^{\lambda x}$ is a solution, and substituting it to the ODE gives the characteristic equation:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

Generally, we can say

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k}$$

where $\lambda_1, \cdots, \lambda_k$ are complex roots and $m_1 + \cdots + m_k = n$. Now the solutions for the ODE are:

$$y_{im}(x) = x^m e^{\lambda_i x} \ (1 \le i \le k,\ 0 \le m < m_i)$$

Define

$$\begin{aligned} y^*(x) &= \sum_{1 \le i \le k,\ 0 \le m < m_i} p_{im} y_{im}(x) \\ &= \left[p_{10}e^{\lambda_1 x} + \cdots + p_{1(m_1-1)}x^{m_1-1}e^{\lambda_1 x}\right] + \cdots + \left[p_{k0}e^{\lambda_k x} + \cdots + p_{k(m_k-1)}x^{m_k-1}e^{\lambda_k x}\right] \end{aligned}$$

then $y^*(x)$ is a linear combination of solutions and is also a solution of the ODE. Considering the equation $y^*(x) = 0$, if all solutions $y_{im}(x)$ are linearly independent, all $a_{im}$ should be $p_{im} = 0$, regardless of $x$.

Also, define

$$\begin{aligned} L &:= (D - \lambda_1)^{m_1} \cdots (D - \lambda_k)^{m_k} \\ L_{im} &:= (D - \lambda_1)^{m_1} \cdots (D - \lambda_{i-1})^{m_{i-1}} (D - \lambda_i)^{m+1} (D - \lambda_{i+1})^{m_{i+1}} \cdots (D - \lambda_k)^{m_k} \end{aligned}$$

where $D$ is the derivative operator. Then for any solution $y$ of the ODE, it is obvious that $L(y) = 0$. Also, since $(D-\lambda_i)^{m+1}\big(x^k e^{\lambda_i x}\big) = 0$ when $0 \le k \le m$,

$$\begin{aligned} L_{im}(y_{ik}) &= (D-\lambda_1)^{m_1}\cdots(D-\lambda_{i-1})^{m_{i-1}}(D-\lambda_i)^{m+1}(D-\lambda_{i+1})^{m_{i+1}}\cdots(D-\lambda_k)^{m_k}\big(x^k e^{\lambda_i x}\big) \\ &= \cdots (D-\lambda_i)^{m+1}\big(x^k e^{\lambda_i x}\big) \\ &= 0 \end{aligned}$$

when $0 \le k \le m$. Also, since $(D-\lambda_j)^{m_j}\big(x^k e^{\lambda_j x}\big) = 0$ for all $j$,

$$\begin{aligned} L_{im}(y_{jk}) &= (D-\lambda_1)^{m_1}\cdots(D-\lambda_{i-1})^{m_{i-1}}(D-\lambda_i)^{m+1}(D-\lambda_{i+1})^{m_{i+1}}\cdots(D-\lambda_k)^{m_k}\big(x^k e^{\lambda_j x}\big) \\ &= \cdots (D-\lambda_j)^{m_j}\big(x^k e^{\lambda_j x}\big) \\ &= 0 \end{aligned}$$

when $i \ne j$. This shows that $L_{im}(y_{jk}) = 0$ if $i \ne j$ or $0 \le k \le m$. Therefore

$$\begin{aligned} L_{im}(y^*) &= L_{im}\Big(\big[p_{10}e^{\lambda_1 x} + \cdots + p_{1(m_1-1)}x^{m_1-1}e^{\lambda_1 x}\big] + \cdots + \big[p_{k0}e^{\lambda_k x} + \cdots + p_{k(m_k-1)}x^{m_k-1}e^{\lambda_k x}\big]\Big) \\ &= L_{im}\big(p_{i(m+1)}x^{m+1}e^{\lambda_i x} + \cdots + p_{i(m_1-1)}x^{m_1-1}e^{\lambda_i x}\big) \end{aligned}$$

Sequentially applying this equation from $L_{i(m_i-1)}(y^*)$ to $L_{i0}(y^*)$ gives $p_{i(m_i-1)} = 0,\ \cdots,\ p_{i0} = 0$, in order. Applying these for all $i$ shows that all $p_{im} = 0$. Therefore solutions $y_{im}(x)$ $(1 \le i \le k,\ 0 \le m < m_i)$ are linearly independent.

# Problem 5

For all questions, we should find (1) a general solution of the corresponding homogenous ODE and (2) a particular solution of the nonhomogenous ODE. The sum of the two solutions will be a general solution of the nonhomogenous ODE.

(a) (1) A homogenous ODE with constant coefficients. The characteristic equation is $\lambda^3 + 2\lambda^2 - \lambda - 2 = 0$, which has three real roots $\lambda = 1, -1, -2$. Therefore a general solution is $y_h = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$.

(2) $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{-2x}$ are three solutions of the homogenous ODE and

$$W = \begin{vmatrix} e^x & e^{-x} & e^{-2x} \\ e^x & -e^{-x} & -2e^{-2x} \\ e^x & e^{-x} & 4e^{-2x} \end{vmatrix} = -6e^{-2x}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} & e^{-2x} \\ 1 & -e^{-x} & -2e^{-2x} \\ 1 & e^{-x} & 4e^{-2x} \end{vmatrix} = -e^{-3x},\ W_2 = \begin{vmatrix} e^x & 0 & e^{-2x} \\ e^x & 0 & -2e^{-2x} \\ e^x & 1 & 4e^{-2x} \end{vmatrix} = 3e^{-x},\ W_3 = \begin{vmatrix} e^x & e^{-x} & 0 \\ e^x & -e^{-x} & 0 \\ e^x & e^{-x} & 1 \end{vmatrix} = -2$$

Using Lagrange's method gives a particular solution:

$$\begin{aligned}
y_p &= e^x \int \frac{-e^{-3x}\big(1-4x^3\big)}{-6e^{-2x}}dx + e^{-x}\int \frac{3e^{-x}\big(1-4x^3\big)}{-6e^{-2x}}dx + e^{-2x}\int \frac{-2\big(1-4x^3\big)}{-6e^{-2x}}dx \\
&= \frac{1}{6}e^x \int e^{-x}\big(1-4x^3\big)dx - \frac{1}{2}e^{-x}\int e^x\big(1-4x^3\big)dx + \frac{1}{3}e^{-2x}\int e^{2x}\big(1-4x^3\big)dx \\
&= 2x^3 - 3x^2 + 15x - 8
\end{aligned}$$

Therefore a general solution will be $y = c_1e^x + c_2e^{-x} + c_3e^{-2x} + 2x^3 - 3x^2 + 15x - 8$.

(b) (1) A third order Euler-Cauchy equation, assume $y = x^\lambda$. Then the characteristic equation is $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda-1)^3 = 0$. Therefore a general soltion is $y_h = \big(c_1 + c_2\ln x + c_3(\ln x)^2\big)x$.

(2) Using the method of undetermined coefficients, suppose $y_p = K_2x^2 + K_1x + K_0$ is a particular solution. Substituting $y_p$ to the ODE gives:

$$\begin{gathered}
x(2K_2x + K_1) - \big(K_2x^2 + K_1x + K_0\big) = x^2 \\
K_2x^2 - K_0 = x^2,\ \ K_2 = 1, K_0 = 0 \\
y_p = x^2 + K_1x
\end{gathered}$$

Therefore a general solution will be $y = \big(c_1 + c_2\ln x + c_3(\ln x)^2\big)x + x^2 + K_1x$, and substituting the initial conditions gives:

$$y = x^2 + x\ln x + \frac{11}{2}x(\ln x)^2$$

(c) (1) A homogenous ODE with constant coefficients. The characteristic equation is $\lambda^3 - 2\lambda^2 - 9\lambda + 18 = 0$, which has three real roots $\lambda = 2, 3, -3$. Therefore a general solution is $y_h = c_1e^{2x} + c_2e^{-2x} + c_3e^{-3x}$.

(2) Using the method of undetermined coefficients, suppose $y_p = Cxe^{2x}$ is a particular solution. Substituting $y_p$ to the ODE gives:

$$\begin{gathered}
C(10+8x)e^{2x} - 2\cdot C(3+4x)e^{2x} - 9\cdot C(1+2x)e^{2x} + 18\cdot Cxe^{2x} = e^{2x} \\
-5Ce^{2x} = e^{2x},\ C = -\frac{1}{5} \\
y_p = -\frac{1}{5}xe^{2x}
\end{gathered}$$

Therefore a general solution will be $y = c_1e^{2x} + c_2e^{-2x} + c_3e^{-3x} - \dfrac{1}{5}xe^{2x}$, and substituting the initial conditions gives:

$$y = \frac{9}{2}e^{2x} - \frac{1}{5}xe^{2x}$$

## Problem 6

(a) Let $\mathbf{y} = (y_1, y_2)$, then the system is:

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} := \begin{pmatrix} 2 & 2 \\ 5 & -1 \end{pmatrix}$$

$\mathbf{A}$ has two eigenvectors $\mathbf{u}_1 = c_1(-2, 5)$, $\mathbf{u}_2 = c_2(1, 1)$ each with corresponding eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 4$. Therefore a general solution is:

$$\mathbf{y} = c_1 e^{-3t}(-2, 5) + c_2 e^{4t}(1, 1) = \left(-2c_1 e^{-3t} + c_2 e^{4t},\ 5c_1 e^{-3t} + c_2 e^{4t}\right)$$

Substituting the initial conditions gives:

$$\begin{aligned} y_1 &= -2e^{-3t} + 2e^{4t} \\ y_2 &= 5e^{-3t} + 2e^{4t} \end{aligned}$$

Since $\lambda_1 \lambda_2 = (-3) \cdot 4 = -12 < 0$, the critical point $(0, 0)$ is a saddle point.

(b) Let $\mathbf{y} = (y_1, y_2)$, then the system is:

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} := \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$$

$\mathbf{A}$ has two eigenvectors $\mathbf{u}_1 = c_1(-1, 1)$, $\mathbf{u}_2 = c_2(1, 1)$ each with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$. Therefore a general solution is:

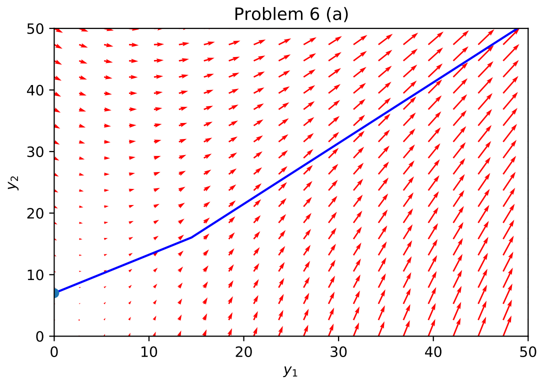
$$\mathbf{y} = c_1 e^{t}(-1, 1) + c_2 e^{5t}(1, 1) = \left(-c_1 e^{t} + c_2 e^{5t},\ c_1 e^{t} + c_2 e^{5t}\right)$$

Substituting the initial conditions gives:

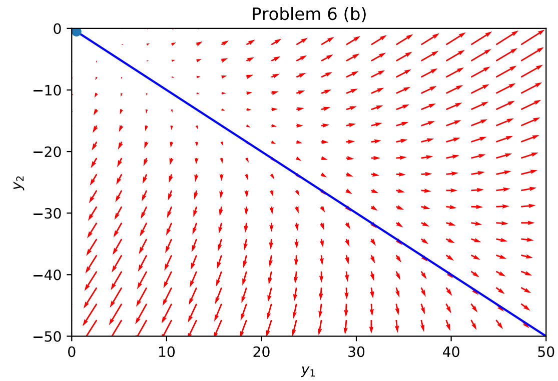
$$\begin{aligned} y_1 &= \frac{1}{2}e^{t} \\ y_2 &= -\frac{1}{2}e^{t} \end{aligned}$$

Since $\lambda_1 \lambda_2 = 1 \cdot 5 = 5 > 0$ and $(\lambda_1 - \lambda_2)^2 = (1-5)^2 = 16 \geq 0$, the critical point $(0, 0)$ is a proper node.

The phase portraits and trajectories for each questions are the following.




(a)




(b)