Engineering Mathematics 2
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Homework 4
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Exercise 4.2

$X \sim B\left(n, \frac{1}{6}\right)$. Therefore $\mathbf{E}[X] = \frac{n}{6}$, and $\mathbf{Var}[X] = \frac{5n}{36}$.

Markov's inequality gives

$$p = \Pr\left(X \geq \frac{n}{4}\right) \leq \frac{\mathbf{E}[X]}{\frac{n}{4}} = \frac{2}{3}.$$

Since X follows a binomial distribution of expectation $\frac{n}{6}$, $\Pr\left(X \geq \frac{n}{6} + \frac{n}{12}\right) = \Pr\left(X \leq \frac{n}{6} - \frac{n}{12}\right)$. Therefore Chebyshev's inequality gives

$$p = \frac{1}{2}\Pr\left(\left|X - \frac{n}{6}\right| \geq \frac{n}{12}\right) \leq \frac{1}{2} \cdot \frac{\mathbf{Var}[X]}{\left(\frac{n}{12}\right)^2} = \frac{10}{n}.$$

Since X is a sum of independent Poisson trials, Chernoff's bound gives

$$p = \Pr\left(X \leq \left(1 + \frac{1}{2}\right)\frac{n}{6}\right) \leq \exp\left(-\frac{\frac{n}{6} \cdot \left(\frac{1}{2}\right)^2}{3}\right) = \exp\left(-\frac{n}{72}\right).$$

For $n >> 1$, $\exp\left(-\frac{n}{72}\right) \leq \frac{10}{n} \leq \frac{2}{3}$, therefore in the order of Chernoff's bound, Chebyshev's inequality, Markov's inequality, we obtain gives tighter bounds.

Exercise 4.3

(a) Let $X \sim B(n, p)$, then

$$\begin{aligned}
M_X(t) &= \mathbf{E}\left[e^{tX}\right] \\
&= \sum_x \Pr(X = x)e^{tx} \\
&= \sum_x \binom{n}{x} p^x (1-p)^{n-x} e^{tx} = \sum_x \binom{n}{x} \left(pe^t\right)^x (1-p)^{n-x} \\
&= \left(1 - p + pe^t\right)^n.
\end{aligned}$$

(b) $M_X(t) = (1 - p + pe^t)^n$ and $M_Y(t) = (1 - p + pe^t)^m$. Since X and Y are independent,

$$\begin{aligned}
M_{X+Y}(t) &= \mathbf{E}\left[e^{t(X+Y)}\right] = \mathbf{E}\left[e^{tX}e^{tY}\right] \\
&= \mathbf{E}\left[e^{tX}\right]\mathbf{E}\left[e^{tY}\right] = M_X(t)M_Y(t) \\
&= \left(1 - p + pe^t\right)^{m+n}.
\end{aligned}$$

(c) Since $(1 - p + pe^t)^{m+n}$ is a moment generating function for a binomial random variable $B(m + n, p)$, we can say that the sum of two random variables $B(m, p)$ and $B(n, p)$ gives $B(m + n, p)$.

Exercise 4.4

Let X_n the random variable of the number of heads in n flips. Then $X_n \sim B\left(n, \dfrac{1}{2}\right)$. Therefore

$$\Pr(X_{100} \geq 55) = \sum_{x=55}^{100} \Pr(X_{100} = x) \approx 0.1841.$$

Since $\mathbf{E}[X_{100}] = 50$, using Chernoff's bound, we obtain

$$\Pr\left(X_{100} \geq \left(1 + \frac{1}{10}\right) \cdot 50\right) \leq \exp\left(-\frac{50 \cdot \left(\frac{1}{10}\right)^2}{3}\right) \approx 0.8465.$$

Therefore the boundary is too large in this case. If $n = 1000$,

$$\Pr(X_{1000} \geq 550) = \sum_{x=550}^{1000} \Pr(X_{100} = x) \approx 0.0008653.$$

Since $\mathbf{E}[X_{1000}] = 500$, using Chernoff's bound, we obtain

$$\Pr\left(X_{1000} \geq \left(1 + \frac{1}{10}\right) \cdot 500\right) \leq \exp\left(-\frac{500 \cdot \left(\frac{1}{10}\right)^2}{3}\right) \approx 0.1889.$$

Therefore the boundary is still too large, even larger in ratio compared to the explicit value. In the first case it was about 5 times higher, while in the second case it was about 219 times higher.

Exercise 4.9

(a) Let $X' = \dfrac{1}{t}\displaystyle\sum_{i=1}^{t} X_i$, then since all X_is are independent and have an identical distribution with X,

$$\mathbf{E}[X'] = \mathbf{E}\left[\frac{1}{t}\sum_{i=1}^{t} X_i\right] = \frac{1}{t}\sum_{i=1}^{t} \mathbf{E}[X_i] = \frac{1}{t}\sum_{i=1}^{t} \mathbf{E}[X] = \mathbf{E}[X]$$
$$\mathbf{Var}[X'] = \mathbf{Var}\left[\frac{1}{t}\sum_{i=1}^{t} X_i\right] = \frac{1}{t^2}\sum_{i=1}^{t} \mathbf{Var}[X_i] = \frac{1}{t^2}\sum_{i=1}^{t} \mathbf{Var}[X] = \frac{1}{t}\mathbf{Var}[X] = \frac{r^2 \mathbf{E}[X]^2}{t}.$$

Using Chebyshev's inequality, we obtain

$$\Pr\Big(\big|X' - \mathbf{E}[X]\big| \geq \epsilon \mathbf{E}[X]\Big) \leq \frac{\mathbf{Var}[X']}{(\epsilon \mathbf{E}[X])^2} = \frac{r^2}{t\epsilon^2}.$$

In order to conclude in $\Pr\Big(\big|X' - \mathbf{E}[X]\big| \leq \epsilon \mathbf{E}[X]\Big) \geq 1 - \delta$, we need $\dfrac{r^2}{t\epsilon^2} \leq \delta$. Therefore $t \geq \dfrac{r^2}{\epsilon^2 \delta}$, so $t = O\left(\dfrac{r^2}{\epsilon^2\delta}\right)$ samples are sufficient.

(b) This is the case of $\delta = \dfrac{1}{4}$. Therefore $t = O\left(\dfrac{4r^2}{\epsilon^2}\right) = O\left(\dfrac{r^2}{\epsilon^2}\right)$ samples are enough.

(c) Try running the process n times, and let the weak estimates from each process $\mu_1, \mu_2, \cdots, \mu_n$. And let the median of these values M, which we will take as our new estimate of $\mathbf{E}[X]$.

Now let $Y_i = \begin{cases} 1 & \left(\left|\mu_i - \mathbf{E}[X]\right| \geq \epsilon\mathbf{E}[X]\right) \\ 0 & \text{(otherwise)} \end{cases}$ and $Y = \sum_{i=1}^{n} Y_i$. According to problem (b), $\Pr(Y_i = 1) \leq \frac{1}{4}$, so $\mathbf{E}[Y_i] \leq \frac{1}{4}$ and $\mathbf{E}[Y] \leq \frac{n}{4}$. Since M is the median of $\mu_1, \mu_2, \cdots, \mu_n$, we can say

$$\Pr\Big(\left|M - \mathbf{E}[X]\right| \geq \epsilon\mathbf{E}[X]\Big) = \Pr\Big(Y \geq \frac{n}{2}\Big).$$

Also, since Y is a sum of independent Poisson trials, using Chernoff's bound, we obtain

$$\Pr\Big(Y \geq \frac{n}{2}\Big) = \Pr\Big(Y \geq (1+1)\frac{n}{4}\Big) \leq \exp\left(-\frac{\frac{n}{4}\cdot 1^2}{3}\right) = \exp\left(-\frac{n}{12}\right)$$

In order to conclude in $\Pr\Big(\left|M - \mathbf{E}[X]\right| \leq \epsilon\mathbf{E}[X]\Big) \geq 1-\delta$, we need $\exp\left(-\frac{n}{12}\right) \leq \delta$. Therefore $n \geq 12\log\frac{1}{\delta}$, so $n = O\Big(12\log\frac{1}{\delta}\Big) = O\Big(\log\frac{1}{\delta}\Big)$ tries are enough, and $t = n \times O\Big(\frac{r^2}{\epsilon^2}\Big) = O\Big(\frac{r^2}{\epsilon^2}\log\frac{1}{\delta}\Big)$ samples are sufficient.

Exercise 4.10

Let X_i random variables of the result of the ith game. Then $\Pr(X_i = 99) = \frac{1}{200}$, $\Pr(X_i = 2) = \frac{4}{25}$, $\Pr(X_i = -1) = \frac{167}{200}$. Now let $X = \sum_{i=1}^{1000000} X_i$, then for all $t > 0$ Markov's inequality gives

$$\begin{aligned}
\Pr(X \geq 10000) &= \Pr\big(e^{tX} \geq e^{10000t}\big) \\
&\leq \frac{\mathbf{E}\big[e^{tX}\big]}{e^{10000t}} = \frac{\mathbf{E}\Big[\prod_{i=1}^{1000000} e^{tX_i}\Big]}{e^{10000t}} = \frac{\prod_{i=1}^{1000000}\mathbf{E}\big[e^{tX_i}\big]}{e^{10000t}} \\
&= \frac{\left(\frac{1}{200}e^{99t} + \frac{4}{25}e^{2t} + \frac{167}{200}e^{-t}\right)^{1000000}}{e^{10000t}}.
\end{aligned}$$

When t is about 0.0006, the above value is at its minimum. At this point, the resulting bound is approximately 0.00016; the probability is very low.

Exercise 4.13

(a) Markov's inequality gives

$$\begin{aligned}
\Pr(X \geq xn) &= \Pr\big(e^{tx} \geq e^{txn}\big) \\
&\leq \frac{\mathbf{E}[e^{tx}]}{e^{txn}} = \frac{\prod_{i=1}^{n}\mathbf{E}\big[e^{tX_i}\big]}{e^{txn}} = \frac{\big(pe^t + (1-p)\big)^n}{e^{txn}} \\
&= \big(pe^{t-tx} + (1-p)e^{-tx}\big)^n.
\end{aligned}$$

When $t = \log\frac{(1-p)x}{p(1-x)}$, $\frac{d}{dt}(pe^{t-tx} + (1-p)e^{-tx}) = 0$, so $pe^{t-tx} + (1-p)e^{-tx}$ is at its minimum. Therefore

$$\Pr(X \geq xn) \leq \left(\left(\frac{x}{p}\right)^x \left(\frac{1-x}{1-p}\right)^{1-x}\right)^n = e^{-nF(x,p)}$$

(b)

$$\begin{aligned}
\frac{d}{dx}\big(F(x,p) - 2(x-p)^2\big) &= \log\frac{x}{p} - \log\frac{1-x}{1-p} - 4(x-p) \\
\frac{d^2}{dx^2}\big(F(x,p) - 2(x-p)^2\big) &= \frac{1}{x} + \frac{1}{1-x} - 4 \geq 0 \ \ (\because\ 0 < x < 1)
\end{aligned}$$

For p where $0 < p < 1$, when $x = p$ the second derivative is non-negative and the first derivative is 0. Therefore there is a global minimum at $x = p$, so $F(x, p) - 2(x - p)^2 \geq F(p, p) - 2(p - p)^2 = 0$.

(c) From (a) and (b), we obtain the following.

$$\Pr(X \geq (p + \epsilon)n) \leq e^{-nF(p+\epsilon,p)} \leq e^{-n \cdot 2(p+\epsilon-p)^2} = e^{-2n\epsilon^2}$$

(d) Let $Y_i = 1 - X_i$ and $Y = \sum_{i=1}^{n} Y_i$, then $Y = n - \sum_{i=1}^{n} X_i = n - X$. Since Y_is are independent Poisson trials with $\Pr(Y_i = 1) = 1 - p$, we can simmilarly apply (c) to Y, which gives us

$$\Pr(X \leq (p - \epsilon)n) = \Pr(Y \geq (1 - p + \epsilon)n) \leq e^{-2n\epsilon^2}.$$