CSE 4190.101: Discrete Mathematics Fall 2021

Problem Set 1 Solution

Instructor: Yongsoo Song **Due on:** Oct 04, 2021

Please submit your answer through eTL. It should be a single PDF file (not jpg), either typed or scanned. **Please include your student ID and name (e.g. 2020-12345 YongsooSong.pdf).** You may discuss with other students the general approach to solve the problems, but the answers should be written in your own words. You should cite any reference that you used, and mention what you used it for. You should follow the academic integrity rules that are described in the course information.

Overall Grading Policy

1. No submissions or late submissions. (0 points received.)
2. Cheating detected. (0 points received.)
3. Not a single file. (including multiple files compressed into a single ZIP file) (10 points deducted.)
4. Not in a PDF format. (10 points deducted.)

Problem 1 (5 points)

Let the following statements be given:

(p) "You can install the program"

(q) "Your computer has **less** than 2GB of RAM"

(r) "Your computer has 20GB of free disk space"

1. Translate the following statement into symbols of formal logic:

   "In order to install the program you need at least 2GB of RAM and 20GB of free disk space"

   $p \to (\neg q \land r)$
   (1 point. No partial credit.)

2. Give the *converse* of this statement in the symbols of formal logic.

   $(\neg q \land r) \to p$
   (1 point. No partial credit.)

3. Give the *converse* in English

   If your computer has at least 2GB of RAM and 20GB of free disk space, then you can install the program.
   (1 point. Answers may vary. No partial credit.)

4. Give the *contrapositive* (of the original statement) in the symbols of formal logic

   $(q \lor \neg r) \to \neg p$
   (1 point. No partial credit.)

5. Give the *contrapositive* in English

   If your computer has less than 2GB of RAM or does not have 20GB of free disk space, then you cannot install the program.
   (1 point. Answers may vary. No partial credit.)

Problem 2 (5 points)

Determine if the expression $(P \to Q) \to R$ is logically equivalent to $P \to (Q \to R)$? Prove or disprove using the truth table method.

P	Q	R	$P \to Q$	$(P \to Q) \to R$	$Q \to R$	$P \to (Q \to R)$
F	F	F	T	F	T	T
F	F	T	T	T	T	T
F	T	F	T	F	F	T
F	T	T	T	T	T	T
T	F	F	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	F	F
T	T	T	T	T	T	T

By the truth table above, two propositions are not logically equivalent.
(5 points. No partial credit.)

Problem 3 (5 points)

Construct a compound proposition that asserts that every cell of a 9×9 Sudoku puzzle contains at least one number (using the propositions $p(i, j, n)$ defined in the lecture note).

$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigvee_{n=1}^{9} p(i, j, n)$
(5 points. No partial credit.)

Problem 4 (10 points)

You are on a treasure island, and find a note with the following hints:

1. If this house is next to a lake, then the treasure is not in the kitchen
2. If the tree in the front yard is an elm, then the treasure is in the kitchen
3. This house is next to a lake
4. Either the tree in the front yard is an elm or the treasure is not buried under the flagpole (or both. This "or" is inclusive.)
5. The treasure is either under the flagpole or in the garage. (but not both. This is an exclusive or, there is only one treasure!)

In order to find where the treasure is, do the following:

(a) Identify 5 atomic statements that can be used to express the above statements in logical form. (Atomic statements should **not** contain any word corresponding to a logical connective.) List the 5 statements in the order in which they appear in the above messages, and call them L, K, E, P, G. Express each of the above compound statements (1-5) using L, K, E, P, G and logical connectives.

L: This house is next to a lake
K: The treasure is in the kitchen
E: The tree in the front yard is an elm
P: The treasure is burried under the flagpole
G: The treasure is in the garage
1. $L \rightarrow \sim K$
2. $E \rightarrow K$
3. L
4. $E \vee \sim P$
5. $P \oplus G$

(b) Determine where the treasure is, and justify your answer.

the treasure is in the garage
1. L
2. $L \rightarrow \sim K$
3. $\sim K$
4. $E \rightarrow K$
5. $\sim E$
6. $E \vee \sim P$
7. $\sim P$
8. $P \oplus G$
9. G

Problem 5 (10 points)

Let $D = \{-48, -14, -8, 0, 1, 3, 16, 23, 26, 32, 36\}$. Determine which of the following statements are true and which are false. Provide a counterexample for the statements that are false. In all statements, the variables x, y range over the set D.

1. $\forall x$, if x is odd then $x > 0$.

   True
   (2 points. No partial credit.)

2. $\forall x$, if $x < 0$ then x is even

   True
   (2 points. No partial credit.)

3. $\forall x$, $\exists y$ such that $y > x$

   False
   If $x = 36$, there is no such y that $y > x$.
   (2 points. 1 point earned for stating **False**, 1 point earned for a correct counterexample.)

4. $\forall x$, (x is even or $\exists y, y > x$)

   True
   (2 points. No partial credit.)

5. $\forall x$, (x is odd or $\exists y, y > x$)

   False

   If $x = 36$, x is not odd and there is no such y that $y > x$.

   (2 points. 1 point earned for stating **False**, 1 point earned for a correct counterexample.)

Problem 6 (5 points)

Prove the following statement by contraposition:

> "For any integer n, if $(n^2 + n + 1)$ is even, then n is odd."

Contrapositive of the statement:

> "For any integer n, if n is even, then $(n^2 + n + 1)$ is odd."

Since either n or $n+1$ is an even number for any integer n, $n^2 + n = n(n+1)$ is even and (n^2+n+1) is odd. Thus, contrapositive of the given statement is true and the given statement is also true.
(5 points. 2 points for stating correct contrapositive, 3 points for being logically correct.)

Problem 7 (10 points)

Suppose that a, b and c are odd integers. Assume that a real number x satisfies the equation $ax^2+bx+c = 0$. Prove by contradiction that x is irrational.

If $ax^2 + bx + c = 0$ has a rational root, the other root has to be rational as well.
Then, it should be possible that $ax^2+bx+c = 0$ is factorized to some form such as $(Ax+B)(Cx+D) = 0$, where $A, B, C, D \in \mathbb{Z}$
Then, $a = AC$, $c = BD$, and since a and c are odd. all A, B, C, D are odd integers.
But, $b = AD + BC$, which makes b even. Thus, a contradiction.

(3 points for using proof by contradiction, 7 points for being logically correct)

Problem 8 (10 points)

Prove or disprove that you can tile a 10×10 checkerboard using straight tetrominoes.

Any numbering or coloring technique is acceptable.
The following three induction can prove the above problem.
1. Number or color tiles in multiples of 4.
2. It specifies that the cover range of straight tetrominoes is 4.
3. Count the numbered or colored tiles and show that there is a number or color that does not come out as 25.(Proof by contradiction law is also possible.)

Problem 9 (10 points)

For each of the following statements, determine if it is true or false. If true, prove it. Otherwise, disprove it by giving a counterexample.

(a) For any sets A, B, C, if $A \cup C \subseteq B \cup C$, then $A \subseteq B$.

False
For any element s in $S = \{x|x \in A, x \notin B, x \in C\}$, since $s \in A$ and $s \notin B$, we can conclude that $A \nsubseteq B$.
(5 points. 3 points earned for stating **False**, 2 points earned for a correct counterexample.)

(b) For any sets A, B, C, if $A \cup C \subseteq B \cup C$ and $A \cap C \subseteq B \cap C$, then $A \subseteq B$.

True
Solution 1
$\forall x$ such that $x \in A$, we want to show that $x \in B$ as well.
1) $x \in C$
We can say that $x \in A \cap C$, and therefore $x \in B \cap C$ by the second premise. Then, $x \in B$ since $B \cap C \subseteq B$.
2) $x \notin C$
As $A \subseteq A \cup C$, we can say that $x \in A \cup C$, and thus $x \in B \cup C$ by the first premise. Since $x \notin C$, then x must be $x \in B$.
1) and 2) cover all the possibilities of x, so we can conclude that $\forall x, x \in A \to x \in B$. Therefore, $A \subseteq B$.
End of proof.

Solution 2
$(A \cup C \subseteq B \cup C) \wedge (A \cap C \subseteq B \cap C)$
$\Leftrightarrow \forall x((\{x|x \in A \vee x \in C\} \subseteq \{x|x \in B \vee x \in C\}) \wedge (\{x|x \in A \wedge x \in C\} \subseteq \{x|x \in B \wedge x \in C\}))$ (Definition of set)
$\Leftrightarrow \forall x((\{x|x \in A \vee x \in C\} \to \{x|x \in B \vee x \in C\}) \wedge (\{x|x \in A \wedge x \in C\} \to \{x|x \in B \wedge x \in C\}))$ (Expression of subset to conditional statement)
$\Leftrightarrow \forall x((\{x|x \notin A \wedge x \notin C\} \cup \{x|x \in B \vee x \in C\}) \wedge (\{x|x \notin A \vee x \notin C\} \cup \{x|x \in B \wedge x \in C\}))$ (Logical equivalence of conditional statement, De Morgan's Law)
$\Leftrightarrow \forall x((\{x|(x \notin A \wedge x \notin C) \vee (x \in B \vee x \in C)\}) \wedge (\{x|(x \notin A \vee x \notin C) \vee (x \in B \wedge x \in C\}))$ (Union definition of set)
$\Leftrightarrow \forall x((\{x|(x \notin A \wedge x \notin C) \vee (x \in C \vee x \in B)\}) \wedge (\{x|(x \notin A \vee x \notin C) \vee (x \in C \wedge x \in B\}))$ (Commutative law)
$\Leftrightarrow \forall x((\{x|((x \notin A \vee (x \in C \vee x \in B)) \wedge (x \notin C \vee (x \in C \vee x \in B))\}) \wedge (\{x|((x \notin A \vee x \notin C) \vee x \in C) \wedge ((x \notin A \vee x \notin C) \vee x \in B)\}))$ (Distributive law)
$\Leftrightarrow \forall x((\{x|((x \notin A \vee (x \in C \vee x \in B)) \wedge ((x \notin C \vee x \in C) \vee x \in B))\}) \wedge (\{x|(x \notin A \vee (x \notin C \vee x \in C)) \wedge ((x \notin A \vee x \notin C) \vee x \in B)\}))$ (Associative law)
$\Leftrightarrow \forall x((\{x|x \notin A \vee (x \in C \vee x \in B)\}) \wedge (\{x|(x \notin A \vee x \notin C) \vee x \in B\}))$ (Negation law, Identity law, Domination law)
$\Leftrightarrow \forall x((\{x|(x \notin A \vee x \in B) \vee x \in C\}) \wedge (\{x|(x \notin A \vee x \in B) \vee x \notin C\}))$ (Commutative law, Associative law)
$\Leftrightarrow \forall x((\{x|(x \notin A \vee x \in B) \vee x \in C\}) \cap (\{x|(x \notin A \vee x \in B) \vee x \notin C\}))$ (Exchange with set operation)
$\Leftrightarrow \forall x(\{x|x \notin A \vee x \in B\})$ (Set intersection)
$\Leftrightarrow \forall x(\{x|x \notin A\} \cup \{x|x \in B\})$ (Union definition of set)
$\Leftrightarrow \forall x(\{x|x \in A\} \to \{x|x \in B\})$ (Logical equivalence of conditional statement, De Morgan's law)
$\Leftrightarrow A \to B$ (Definition of set)
$\Leftrightarrow A \subseteq B$ (Expression of conditional statement to subset)
End of proof.
(5 points. 3 points earned for stating **True**, 2 points earned for a correct proof. Proofs may vary.)

Problem 10 (10 points)

Let F be the set of all nonempty finite sets of integers, and let $f : F \to \mathbb{Z}$ be the function $f(S) = \sum_{x \in S} x$ mapping each set to the sum of its elements. So, for example, $f(\{1, -7, 9\}) = 3$.

(a) Determine if f is surjective, and prove your answer.

$$\forall y, f(\{0, y\}) = 0 + y = y \quad (1)$$
$$\Rightarrow \forall y \exists x, f(x) = y \quad (2)$$

Therefore, f is surjective.

(b) Determine if f is injective, and prove your answer.

$$f(\{0\}) = 0, f(\{-1, 1\}) = -1 + 1 = 0 \quad (3)$$
$$\Rightarrow \exists a, b, f(a) = f(b) \quad (4)$$

Therefore, f is not injective.

Problem 11 (10 points)

Let $f : \mathbb{N}^2 \to \mathbb{Z}$ be the function $f(x, y) = x - y$, where $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ is the set of positive integers.

(a) Show that $f : \mathbb{N}^2 \to \mathbb{Z}$ is not a bijection.

Counterexample: $f(2, 1) = 1$ and $f(3, 2) = 1$
Thus, f is not injective, and therfore not bijective.
(2 points)

(b) Give a subset $S \subseteq \mathbb{N}^2$ such that $f : S \to \mathbb{Z}$ is a bijection, and prove the correctness of your answer.

$S := \{(1, x) | x \in \mathbb{N}\} \cup \{(x, 1) | x \in \mathbb{N}\} = \{(1, 1), (1, 2), (1, 3)..., (2, 1), (3, 1)...\}$

WTS S is injective
for $x_1, x_2 \in \mathbb{N}$
1) let $(1, x_1), (1, x_2) \in S$, then $f(1, x_1) = 1 - x_1$, and $f(1, x_2) = 1 - x_2$
$\to$ if $f(1, x_1) = f(1, x_2)$
$\to 1 - x_1 = 1 - x_2$
$\to x_1 = x_2$
2) let $(x_1, 1), (x_2, 1) \in S$, then $f(x_1, 1) = x_1 - 1$, and $f(x_2, 1) = x_2, 1$
$\to$ if $f(x_1, 1) = f(x_2, 1)$
$\to x_1 - 1 = x_2 - 1$
$\to x_1 = .x_2$

WTS S is surjective.
for any $z \in \mathbb{Z}$
1) if $z = 0, \exists (1, 1) \in S$ s.t. $f(1, 1) = z$
2) if $z > 0, \exists x_1 \in \mathbb{N}$ s.t. $x_1 = z + 1$ i.e. $z = x_1 - 1$
$\to \exists (x_1, 1) \in S$ s.t. $f(x_1, 1) = z$
3) if $z < 0, \exists x_2 \in \mathbb{N}$ s.t. $x_2 = 1 - z$ i.e. $z = 1 - x_2$
$\to \exists (1, x_2) \in S$ s.t. $f(1, x_2) = z$

Thus, f is bijection

(2 points for constructing S, 3 points for injection, 3 points for surjection. 0 points if consturction of S is

wrong.)
(If proofs of surjection/injection are incomplete - such as just writing down the definition, then deducted 2points each)

Problem 12 (10 points)

Show that if S is a set, then there does not exist an onto function f from S to $\mathcal{P}(S)$, the power set of S. Conclude that $|S| < |\mathcal{P}(S)|$. [Hint: Suppose such a function f existed and consider the set $T = \{s \in S | s \notin f(s)\}$.]

Proof.
Suppose that there is a onto function f from $\mathcal{S}$ to $\mathcal{P}(S)$.
Define a set $\mathcal{T} = \{x \in S | x \notin f(x)\}$
$T \in P(S)$
$\therefore there\ is\ a\ x \in S, f(x) = T$
Contradiction case 1. $x \in T$
By the definition of the set T, $x \notin f(x)$, but $f(x) = T$ and $x \in T$, So $f(x) \neq T$
Contradiction case 2. $x \notin T$
By the definition of the set T, $x \in f(x)$, So $f(x) \neq T$
$\therefore |S| < |P(S)|$