Engineering Mathematics Spring 2022

Problem Set 1

Instructor: Yongsoo Song **Due on:** April 11, 2022

Please submit your answer through eTL. It should be a single PDF file (not jpg), either typed or scanned. **Please include your student ID and name (e.g. 2020-12345 YongsooSong.pdf).** You may discuss with other students the general approach to solve the problems, but the answers should be written in your own words. You should cite any reference that you used, and mention what you used it for.

Problem 1 (20 points)

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be $n \times n$ matrices. Prove or disprove the following statements:

(a) If $\mathbf{A}$ and $\mathbf{B}$ are symmetric, then $\mathbf{AB} = \mathbf{BA}$.

(b) If $\mathbf{A}$ and $\mathbf{B}$ are symmetric, then so is $\mathbf{AB}$.

(a) If $\mathbf{AB} = \mathbf{BA}$ and $\mathbf{BC} = \mathbf{CB}$, then $\mathbf{AC} = \mathbf{CA}$.

(a) If $\mathbf{A}$ and $\mathbf{B}$ are upper triangular, then so is $\mathbf{AB}$.

Problem 2 (15 points)

Show that for any square matrix $\mathbf{A}$, it can be expressed uniquely as $\mathbf{A} = \mathbf{S} + \mathbf{T}$ where $\mathbf{S}$ is symmetric and $\mathbf{T}$ is skew-symmetric.

Problem 3 (15 points)

Let $\mathbf{A}$ be an $n \times n$ strictly upper triangular matrix (entries on the main diagonal are 0). Then show that $\mathbf{A}^n = \mathbf{0}$.

Problem 4 (25 points)

Let $L_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation of the Cartesian coordinate system in the plane about the origin, where θ is the angle of rotation.

(a) Show (geometrically) that L_θ is a linear transformation.

(b) Calculate the matrix $\mathbf{A}_\theta \in \mathbb{R}^{2\times 2}$ corresponding to L_θ (such that $L_\theta(\mathbf{x}) = \mathbf{A}_\theta \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$).

(c) Use (a) and (b) to show the addition formulas for cosine and sine:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Problem 5 (20 points)

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix such that $a_{ij} = i + j + c$ for a fixed constant c. Find the rank of $\mathbf{A}$.

Problem 6 (20 points)

Let $\mathbf{A}$ be an $\ell \times m$ matrix and $\mathbf{B}$ be an $m \times n$ matrix. Show that $\mathrm{rank}(\mathbf{AB}) \leq \min\{\mathbf{A}, \mathbf{B}\}$. Provide an example where the equality does not hold.

Problem 7 (25 points)

Let $\mathbf{A} = [a_{ij}]$ be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$ be a vector.

(a) Prove that $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}$ have the same rank. (Hint: kernel)

(b) Suppose that $m \geq n$ and and $\mathbf{A}$ is full rank, i.e., $\mathrm{rank}\,\mathbf{A} = n$. Solve the least squares problem: find $\mathbf{x} \in \mathbb{R}^n$ which minimizes $\|\mathbf{Ax} - \mathbf{b}\|$.

Problem 8 (20 points)

An $m \times n$ matrix $\mathbf{X}$ is called a **Vandermonde matrix** if its (i, j)-th entry is x_i^{j-1} for some $x_1, x_2, \ldots, x_n \in \mathbb{R}$, i.e.,

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 & \ldots & x_1^{n-1} \\ 1 & x_2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & \ldots & x_m^{n-1} \end{bmatrix}.$$

(a) Let X be an square Vandemonde matrix of dimension $n \times n$. Show that $\det X = \prod_{1 \leq i < j \leq n}(x_j - x_i)$.

(b) Suppose that we are given n pairs $(x_i, y_i) \in \mathbb{R}^2$ where x_i's are distinct. Show that there exists a unique polynomial $p(x)$ of degree $< n$ such that $p(x_i) = p(y_i)$.

Problem 9 (20 points)

Describe the Gauss–Jordan Method in pseudocode and analyze its complexity.

Problem 10 (20 points)

Suppose that V is a real inner product space of dimension n and $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of V. Show that the Gram–Schmidt process outputs an orthonormal basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. (Visit `https://en.wikipedia.org/wiki/Gram-Schmidt_process` to study about the Gram–Schmidt process. The same algorithm can be applied to an arbitrary finite-dimensional inner product space over the real numbers). Analyze its complexity in terms of the number of inner products, scalar multiplications, and additions.