Engineering Mathematics 1 Problem Set 2

Department of Computer Science and Engineering
2021-16988 Jaewan Park

Problem 1

(a) The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned}
D(\lambda) &= \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} \\
&= (a_{11}-\lambda) \begin{vmatrix} a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1n} \\ a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} \\
&\quad + \cdots + (-1)^{n+1} a_{n1} \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{22}-\lambda & a_{23} & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{(n-1)2} & a_{(n-1)3} & \cdots & a_{(n-1)(n-1)}-\lambda & a_{(n-1)n} \end{vmatrix}
\end{aligned}$$

The coefficient of the λ^{n-1} term in $D(\lambda)$ only comes from $(a_{11}-\lambda)\cdots(a_{nn}-\lambda)$ from $(a_{11}-\lambda)C_{11}$, considering the above calculation. The coefficient is $(-1)^{n+1}(a_{11}+\cdots+a_{nn})$.

Since $\mathbf{A}$ has eigenvalues $\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_n$, $D(\lambda)$ can also be written as

$$\begin{aligned}
D(\lambda) &= (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n) \\
&= (-1)^n\left\{\lambda^n - (\lambda_1+\cdots+\lambda_n)\lambda^{n-1} + \cdots\right\}
\end{aligned}$$

Therefore comparing the coefficients gives

$$(-1)^{n+1}(a_{11}+\cdots+a_{nn}) = (-1)^{n+1}(\lambda_1+\cdots+\lambda_n)$$

$$\therefore \operatorname{tr}\mathbf{A} = a_{11}+\cdots+a_{nn} = \lambda_1+\cdots+\lambda_n$$

(b) For arbitrary $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\operatorname{tr}(\mathbf{AB}) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij}b_{ji}\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} b_{ji}a_{ij}\right) = \operatorname{tr}(\mathbf{BA})$$

If $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are similar, there exists some $\mathbf{P}$ where $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. Therefore

$$\begin{aligned} \mathrm{tr}\mathbf{B} &= \mathrm{tr}\big(\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\big) = \mathrm{tr}\big(\mathbf{A}\mathbf{P}^{-1}\mathbf{P}\big) = \mathrm{tr}(\mathbf{A}\mathbf{I}) \\ &= \mathrm{tr}\mathbf{A} \end{aligned}$$

Problem 2

(a) Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$. Then

$$\begin{aligned} \langle \mathbf{A}\mathbf{u}, \mathbf{v} \rangle &= \sum_{i=1}^{m} (\mathbf{A}\mathbf{u})_i v_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} u_j \right) v_i \\ &= \sum_{j=1}^{n} u_j \left(\sum_{i=1}^{m} a_{ij} v_i \right) = \sum_{j=1}^{n} u_j \left(\mathbf{A}^T \mathbf{v}\right)_j \\ &= \left\langle \mathbf{u}, \mathbf{A}^T \mathbf{v} \right\rangle \end{aligned}$$

(b) Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$ where all are complex numbers. Then

$$\begin{aligned} \langle \mathbf{A}\mathbf{u}, \mathbf{v} \rangle &= \sum_{i=1}^{m} (\mathbf{A}\mathbf{u})_i \overline{v_i} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} u_j \right) \overline{v_i} \\ &= \sum_{j=1}^{n} u_j \left(\sum_{i=1}^{m} a_{ij} \overline{v_i} \right) = \sum_{j=1}^{n} u_j \overline{\left(\sum_{i=1}^{m} \overline{a_{ij}} v_i \right)} = \sum_{j=1}^{n} u_j \overline{(\mathbf{A}^* \mathbf{v})_j} \\ &= \langle \mathbf{u}, \mathbf{A}^* \mathbf{v} \rangle \end{aligned}$$

Problem 3

Let the eigenvalue corresponding to $\mathbf{v}_i$ as λ_i. Then if $i \neq j$, since $\mathbf{A}$ is symmetric,

$$\lambda_i \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \langle \lambda_i \mathbf{v}_i, \mathbf{v}_j \rangle = \langle \mathbf{A}\mathbf{v}_i, \mathbf{v}_j \rangle = \left\langle \mathbf{v}_i, \mathbf{A}^T \mathbf{v}_j \right\rangle = \langle \mathbf{v}_i, \mathbf{A}\mathbf{v}_j \rangle = \langle \mathbf{v}_i, \lambda_j \mathbf{v}_j \rangle = \lambda_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle$$

Therefore, since $\lambda_i \neq \lambda_j$, $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$. All eigenvectors are pairwise orthogonal.

Problem 4

(a) Let $\mathbf{A} = \begin{bmatrix} 3 & 11 \\ 11 & 3 \end{bmatrix}$, then $Q = \mathbf{x}^T \mathbf{A} \mathbf{x}$.

Since $D(\lambda) = \begin{vmatrix} 3-\lambda & 11 \\ 11 & 3-\lambda \end{vmatrix} = (\lambda - 14)(\lambda + 8)$, the two eigenvalues are $\lambda_1 = 14$, $\lambda_2 = -8$.

Therefore the canonical form can be written as

$$Q = \mathbf{y}^T \mathbf{D} \mathbf{y} = 14y_1^2 - 8y_2^2$$

We can draw the graph of $14y_1^2 - 8y_2^2 = 0 \left(\Leftrightarrow \frac{y_1^2}{4} - \frac{y_2^2}{7} = 0 \right)$. (In **Figure 1-(a)**)

Let the two eigenvectors are $\mathbf{v}_1$ and $\mathbf{v}_2$.

When $\lambda_1 = 14$, $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_1 = \begin{bmatrix} -11 & 11 \\ 11 & -11 \end{bmatrix} \mathbf{v}_1 = \mathbf{0}$, so $\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

When $\lambda_2 = -8$, $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2 = \begin{bmatrix} 11 & 11 \\ 11 & 11 \end{bmatrix} \mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Therefore

$$\mathbf{x} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \mathbf{y} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{y}$$

(b) Let $\mathbf{A} = \begin{bmatrix} -11 & 42 \\ 42 & 24 \end{bmatrix}$, then $Q = \mathbf{x}^T \mathbf{A} \mathbf{x}$.

Since $D(\lambda) = \begin{vmatrix} -11-\lambda & 42 \\ 42 & 24-\lambda \end{vmatrix} = (\lambda - 52)(\lambda + 39)$, the two eigenvalues are $\lambda_1 = 52$, $\lambda_2 = -39$.

Therefore the canonical form can be written as

$$Q = \mathbf{y}^T \mathbf{D} \mathbf{y} = 52y_1^2 - 39y_2^2$$

We can draw the graph of $52y_1^2 - 39y_2^2 = 156 \left(\Leftrightarrow \frac{y_1^2}{3} - \frac{y_2^2}{4} = 1 \right)$. (In **Figure 1-(b)**)

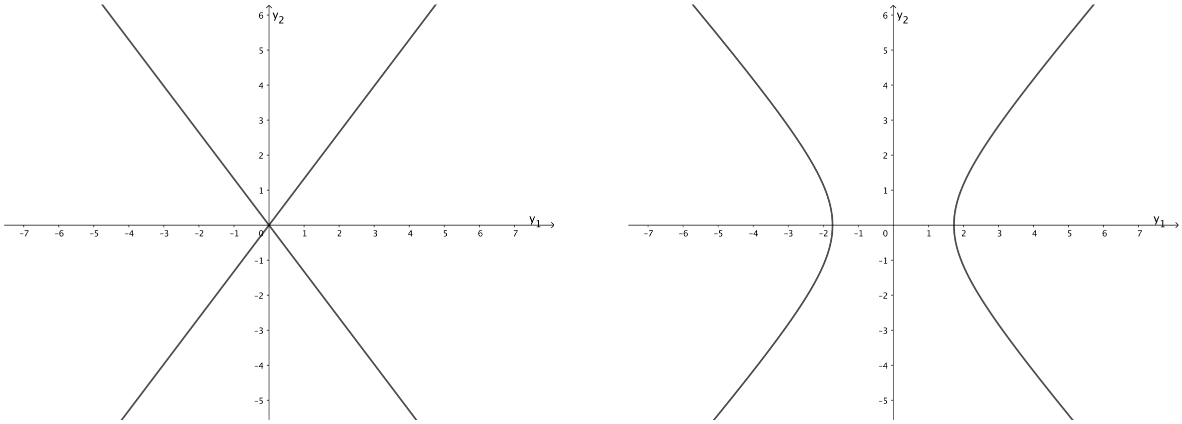
Let the two eigenvectors are $\mathbf{v}_1$ and $\mathbf{v}_2$.

When $\lambda_1 = 52$, $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_1 = \begin{bmatrix} -63 & 42 \\ 42 & -28 \end{bmatrix} \mathbf{v}_1 = \mathbf{0}$, so $\mathbf{v}_1 = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

When $\lambda_2 = -39$, $(\mathbf{A} - \lambda \mathbf{I})\mathbf{v}_2 = \begin{bmatrix} 28 & 42 \\ 42 & 63 \end{bmatrix} \mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_2 = \frac{1}{\sqrt{13}} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

Therefore

$$\mathbf{x} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \mathbf{y} = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \mathbf{y}$$



(a) $\dfrac{y_1^2}{4} - \dfrac{y_2^2}{7} = 0$ (b) $\dfrac{y_1^2}{3} - \dfrac{y_2^2}{4} = 1$

Figure 1: Graphs for **Problem 4**

Problem 5

(a) To Prove : eigenvalues of $\mathbf{A}$ are all positive $\rightarrow Q$ is positive definite

Q can be written in a canonical form of $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, where $\mathbf{x} = \mathbf{X}\mathbf{y}$ and $\mathbf{X}$ is the orthogonal matrix with eigenvectors corresponding to eigenvalues $(\lambda_1, \cdots, \lambda_n)$ as columns. Since $\mathbf{x} \neq \mathbf{0}$, $\mathbf{y} \neq \mathbf{0}$ and at least one $y_i \neq 0$.

Then $Q = \mathbf{y}^T\mathbf{D}\mathbf{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$. Since all $\lambda_i > 0$ and $\mathbf{y} \neq 0$, $Q > 0$.

(b) To Prove : Q is positive definite $\rightarrow$ eigenvalues of $\mathbf{A}$ are all positive

If some eigenvalue $\lambda = 0$, there is a corresponding eigenvector $\mathbf{v}$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x} = \mathbf{0}$. Then $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{0}$, therefore Q is not positive definite.

If some eigenvalue $\lambda < 0$, there is a corresponding eigenvector $\mathbf{v}$ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Then $\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{x}^T\lambda\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x} = \lambda\|\mathbf{x}\|^2 < 0$, since $\lambda < 0$ and $\mathbf{x} \neq 0$. Therefore Q is not positive definite.

Therefore, if Q is positive definite, all eigenvalues of $\mathbf{A}$ are positive.

Problem 6

(a)

$$\frac{1}{y^2}y' = e^{2x-1}, \quad \int \frac{1}{y^2}dy = e^{2x-1}dx$$

$$-\frac{1}{y} = \frac{1}{2}e^{2x-1} + C', \quad y = -\frac{2}{e^{2x-1} + C}$$

(b)

$$y' = 1 + \frac{y}{x}$$

Let $u = \frac{y}{x}$, then $y' = u'x + u$.

$$u'x + u = 1 + u, \quad u' = \frac{1}{x}$$

$$\int 1 \cdot du = \int \frac{1}{x} dx$$

$$u = \ln x + C$$

$$\therefore y = x \ln x + Cx$$

(c)

$$y' = \frac{y}{x} + 3x^3 \cos^2 \left(\frac{y}{x}\right)$$

Let $u = \frac{y}{x}$, then $y' = u'x + u$.

$$u'x + u = u + 3x^3 \cos^2 u, \quad \frac{1}{\cos^2 u} u' = 3x^2$$

$$\int_0^u \frac{1}{\cos^2 u} du = \int_1^x 3x^2 dx$$

$$\tan u = x^3 - 1, \quad u = \arctan\left(x^3 - 1\right)$$

$$\therefore y = x \arctan\left(x^3 - 1\right)$$

(d)

$$\left(1 + \frac{y^2}{x^2}\right) dx - \frac{2y}{x} dy = 0, \quad d\left(x - \frac{y^2}{x}\right) = 0$$

$$\therefore x - \frac{y^2}{x} = C$$

(e)

$$y \cos(x+y) dx + \cos(x+y)(y + \tan(x+y)) dy = 0, \quad d(y \sin(x+y)) = 0$$

$$\therefore y \sin(x+y) = C$$

(f)

$$y' + \frac{4}{x} y = 8x^3, \quad x^4 y' + 4x^3 y = 8x^7$$

$$\int_1^x \left(x^4 y' + 4x^3 y\right) dx = \int_1^x 8x^7 dx$$

$$\left[x^4 y\right]_1^x = \left[x^8\right]_1^x, \quad x^4 y - 2 = x^8 - 1$$

$$\therefore y = x^4 + \frac{1}{x^4}$$

(g)

$$\frac{y}{1-y^2}y' = x$$

$$-\int_3^y \frac{y}{y^2-1}dy = \int_0^x xdx$$

$$\left[-\frac{1}{2}\ln\left(y^2-1\right)\right]_3^y = \left[\frac{1}{2}x^2\right]_0^x, \quad \frac{1}{2}\ln\left(\frac{8}{y^2-1}\right) = \frac{1}{2}x^2$$

$$\therefore e^{x^2}\left(y^2-1\right) = 8$$

Problem 7

We can first solve the problem in the standard way. (Nonhomogenous Linear ODE)

$$y' - y = x, \quad e^{-x}(y'-y) = e^{-x}x$$

$$\int_0^x e^{-x}(y'-y)dx = \int_0^x e^{-x}xdx$$

$$\left[e^{-x}y\right]_0^x = \left[-(1+x)e^{-x}\right]_0^x, \quad e^{-x}y = 1-(1+x)e^{-x}$$

$$\therefore y = e^x - x - 1$$

Now we can solve it using the Picard iteration method.

$$\begin{aligned} y_n(x) &= y_0 + \int_{x_0}^x f(t,\ y_{n-1}(t))dt \quad \Big(f(x,\ y) = x+y\Big) \\ &= \int_0^x (t + y_{n-1}(t))dt \end{aligned}$$

We can assert $y_n(x) = \displaystyle\sum_{k=2}^{n+1}\frac{1}{k!}x^k$, and prove it using mathematical induction.

When $n=0$, $y_0(x) = 0$. The statement is true.

Suppose the statement is true at $n=m$ $(m \geq 0)$. Then $y_m(x) = \displaystyle\sum_{k=2}^{m+1}\frac{1}{k!}x^k$, and this gives

$$\begin{aligned} y_{m+1}(x) &= \int_0^x (t + y_m(t))dt = \int_0^x \left(t + \sum_{k=2}^{m+1}\frac{1}{k!}t^k\right)dt \\ &= \frac{1}{2}t^2 + \sum_{k=2}^{m+1}\frac{1}{(k+1)!}x^{k+1} \\ &= \sum_{k=2}^{m+2}\frac{1}{k!}x^k \end{aligned}$$

Therefore the statement is true at $n = m + 1$, and is true at all $n \geq 0$.

$$\therefore y = \lim_{n\to\infty} y_n(x) = \sum_{n=2}^{\infty} \frac{1}{n!} x^n = e^x - x - 1$$

We can see that the two methods give the same result for solving the ODE.