Engineering Mathematics 2
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Homework 6-2
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Exercise 6.13

Enumerate the $\binom{n}{k}$ possible k-vertex sets in $G_{n,p}$ and define X_i and X such that

$$X_i = \begin{cases} 1 & \text{(The } i\text{th set is a clique.)} \\ 0 & \text{(Otherwise)} \end{cases}, \quad X = \sum_{i=1}^{\binom{n}{k}} X_i$$

Since $\Pr(X_i = 1) = p^{\binom{k}{2}}$, we obtain $\mathbf{E}[X] = \binom{n}{k} p^{\binom{k}{2}} = \Theta\Big(n^k p^{\binom{k}{2}}\Big) = \Theta\Big(\big(pn^{2/(k-1)}\big)^{k(k-1)/2}\Big)$. Therefore in the case of $p = o(n^{-2/(k-1)})$,

$$\lim_{n\to\infty} \Big(pn^{2/(k-1)}\Big)^{k(k-1)/2} = 0,$$

so $\mathbf{E}[X] = o(1)$. Also since X is a nonnegative integer random variable, we have

$$\Pr(X \geq 1) \leq \sum_{i=0}^{\infty} \Pr(X > i) = \mathbf{E}[X] = o(1).$$

Therefore if $p = o(n^{-2/(k-1)})$. the probability that $G_{n,p}$ has a clique of size k is asymptotically $o(1)$.

In the case of $p = \omega(n^{-2/(k-1)})$, we obtain

$$\begin{aligned}
\mathbf{Var}[X] &\leq \mathbf{E}[X] + \sum_{i\neq j} \mathbf{Cov}(X_i, X_j) \leq \mathbf{E}[X] + \sum_{i\neq j} \mathbf{E}[X_i X_j] \\
&= \binom{n}{k} p^{\binom{k}{2}} + \sum_{i=1}^{k-2} \binom{n}{k+i} p^{2\binom{k}{2} - \binom{k-i}{2}} = \sum_{i=0}^{k-2} \binom{n}{k+i} p^{2\binom{k}{2} - \binom{k-i}{2}} \\
&= \sum_{i=0}^{k-2} \Theta\Big(n^{k+i} p^{2\binom{k}{2} - \binom{k-i}{2}}\Big) \\
&= o\Big(n^{2k} p^{2\binom{k}{2}}\Big) = o\Big((\mathbf{E}[X])^2\Big).
\end{aligned}$$

Applying Chebyshev's inequality gives

$$\Pr(X = 0) \leq \Pr(|X - \mathbf{E}[X]| \geq \mathbf{E}[X]) \leq \frac{\mathbf{Var}[X]}{(\mathbf{E}[X])^2} = o(1).$$

Therefore if $p = \omega(n^{-2/(k-1)})$. the probability that $G_{n,p}$ does not have a clique of size k is asymptotically $o(1)$.

Exercise 6.15

We have

$$\mathbf{E}[X] = \binom{n}{3}\left(\frac{1}{n}\right)^3 = \frac{n(n-1)(n-2)}{6n^3} \leq \frac{1}{6}$$

Since X is a nonnegative integer random variable, by Markov's inequality we can conclude

$$\Pr(X \geq 1) \leq \mathbf{E}[X] \leq \frac{1}{6}.$$

Now sample the $\binom{n}{3}$ possible triangles in an arbitrary order, and define X_i such that

$$X_i = \begin{cases} 1 & \text{(The } i\text{th triangle exists in the graph.)} \\ 0 & \text{(Otherwise)} \end{cases}.$$

Then we can consider two cases of calculating $\Pr(X_j = 1 \mid X_i = 1)$ for any i and j. If the jth triangle shares one or no vertices with the ith triangle, $\Pr(X_j = 1 \mid X_i = 1) = 1/n^3$, and there are a total of $\binom{n-3}{3} + \binom{3}{1}\binom{n-3}{2}$ triangles of this case. If the jth triangle shares two vertices with the ith triangle, $\Pr(X_j = 1 \mid X_i = 1) = 1/n^2$, and there are a total of $\binom{3}{2}\binom{n-3}{1}$ triangles of this case. Therefore we obtain

$$\begin{aligned}\mathbf{E}[X \mid X_i = 1] &= \mathbf{E}[X_i \mid X_i = 1] + \sum_{j \neq i} \mathbf{E}[X_j \mid X_i = 1] = 1 + \sum_{j \neq i} \Pr(X_j = 1 \mid X_i = 1) \\ &= 1 + \left(\binom{n-3}{3} + \binom{3}{1}\binom{n-3}{2}\right)\frac{1}{n^3} + \binom{3}{2}\binom{n-3}{1}\frac{1}{n^2}.\end{aligned}$$

The conditional expectation inequality gives

$$\Pr(X \geq 1) = \Pr(X > 0) \geq \sum_{i=1}^{\binom{n}{3}} \frac{\Pr(X_i = 1)}{\mathbf{E}[X \mid X_i = 1]} = \binom{n}{3} \frac{\frac{1}{n^3}}{1 + \left(\binom{n-3}{3} + \binom{3}{1}\binom{n-3}{2}\right)\frac{1}{n^3} + \binom{3}{2}\binom{n-3}{1}\frac{1}{n^2}}.$$

Therefore $\lim_{n\to\infty} \Pr(X \geq 1) \geq \lim_{n\to\infty} \binom{n}{3} \frac{\frac{1}{n^3}}{1 + \left(\binom{n-3}{3} + \binom{3}{1}\binom{n-3}{2}\right)\frac{1}{n^3} + \binom{3}{2}\binom{n-3}{1}\frac{1}{n^2}} = \frac{1}{7}.$

Exercise 6.17

Consider the events where each K_k subgraph of K_n is monochromatic. The probability of each event to happen is $2^{1-\binom{k}{2}}$. Also, for two distinct events to be dependent, the subgraphs should share an edge. One subgraph has less than $\binom{k}{2}\binom{n-2}{k-2}$ dependent graphs, since each edge is in $\binom{n-2}{k-2}$ other graphs. Also $\binom{n-2}{k-2} < \binom{k}{2}\binom{n}{k-2}$. Therefore, by the Lovasz local lemma, when

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} < 1$$

is satisfied, we can say that it is possible to color the edges of K_n so that it has no monochromatic K_k subgraphs.