MathDNN Homework 12

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Problem 2

Let $\mathcal{L}(\theta, \phi)$ the objective of the minimax problem, then

$$\begin{aligned}\mathcal{L}(\theta, \phi) :&= \mathbb{E}_{X \sim p_{\text{true}}}[\log D_\phi(X)] + \lambda \mathbb{E}_{\tilde{X} \sim p_\theta}\left[\log\left(1 - D_\phi\left(\tilde{X}\right)\right)\right] \\ &= \int \Big(p_{\text{true}}(x) \log D_\phi(x) + \lambda p_\theta(x) \log\left(1 - D_\phi(x)\right)\Big) dx.\end{aligned}$$

Since

$$\frac{d}{dy}(a \log y + b \log(1-y)) = \frac{a}{y} - \frac{b}{1-y} = 0 \Leftrightarrow y = \frac{a}{a+b}$$

we can say $\mathcal{L}(\theta, \phi)$ is maximized by

$$D_\phi(x) = \frac{p_{\text{true}}(x)}{p_{\text{true}}(x) + \lambda p_\theta(x)}.$$

and let the corresponding ϕ as ϕ^*. Then we obtain

$$\begin{aligned}\max_{\phi \in \mathbb{R}^p} \mathcal{L}(\theta, \phi) &= \mathcal{L}(\theta, \phi^*) \\ &= \int \left(p_{\text{true}}(x) \log\left(\frac{p_{\text{true}}(x)}{p_{\text{true}}(x) + \lambda p_\theta(x)}\right) + \lambda p_\theta(x) \log\left(\frac{\lambda p_\theta(x)}{p_{\text{true}}(x) + \lambda p_\theta(x)}\right)\right) dx \\ &= D_f(p_{\text{true}} \,||\, p_\theta) - \Big((1+\lambda)\log(1+\lambda) - \lambda \log \lambda\Big).\end{aligned}$$

Therefore we can say that the following problems are equivalent.

$$\begin{aligned}\underset{\theta \in \mathbb{R}^p}{\text{minimize}}\ \underset{\phi \in \mathbb{R}^p}{\text{maximize}}\ \mathcal{L}(\theta, \phi) &\Leftrightarrow \underset{\theta \in \mathbb{R}^p}{\text{minimize}} \left(D_f(p_{\text{true}} \,||\, p_\theta) - \Big((1+\lambda)\log(1+\lambda) - \lambda \log \lambda\Big)\right) \\ &\Leftrightarrow \underset{\theta \in \mathbb{R}^p}{\text{minimize}}\ D_f(p_{\text{true}} \,||\, p_\theta)\end{aligned}$$