Engineering Mathematics 2
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Homework 5
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Exercise 5.4

Let $P(n,a,b)$ the probability that out of n people, there are a days that are birthdays of two people and b days that are birthdays of one person. When $n=1$, only $b=1$ is available, so $P(1,0,1)=1$. Considering leap years, assume there are 366 days in a year. Then we obtain the following recurrence relation.

$$P(n,a,b) = P(n-1,a-1,b+1)\frac{b+1}{366} + P(n-1,a,b-1)\frac{366-a-(b-1)}{366}$$

We desire days with three people sharing birthdays to not exist, so the total probability can be calculated as the sum of $P(n,a,b)$ where $2a+b=n$. Therefore, let this event E, then we can exactly calculate the probability using the following conditions.

$$\Pr(E) = \sum_{2a+b=100} P(100,a,b)$$

$$P(n,a,b) = P(n-1,a-1,b+1)\frac{b+1}{366} + P(n-1,a,b-1)\frac{366-a-(b-1)}{366}$$
$$P(1,0,1)=1,\quad P(1,1,0)=0,\quad P(1,0,0)=0$$

Exercise 5.9

Placing the elements into their buckets each takes $O(1)$ time each, and $O(n)$ time in total. Now let X_j the number of elements in the jth bucket. Then in the second step, the time to sort the jth bucket is at most $c(X_j)^2$ for some constant c. Then the expected time for the second step is

$$\mathbf{E}\left[\sum_{j=1}^{n} c(X_j)^2\right] = c\sum_{j=1}^{n}\mathbf{E}\Big[(X_j)^2\Big] = cn\mathbf{E}\Big[(X_1)^2\Big]$$

Since all elements are chosen from $[0,2k)$ with probability at most $\dfrac{a}{2k}$, the expectation of $(X_1)^2$ is at most the expectation of X^2, where X is a random variable distributed over $B\Big(n,\dfrac{a}{n}\Big)$. Therefore the expected time for the second step is bounded by the following.

$$\begin{aligned} cn\mathbf{E}\Big[(X_1)^2\Big] &\le cn\mathbf{E}\big[X^2\big] \\ &= cn\left(n(n-1)\Big(\frac{a}{n}\Big)^2 + n\Big(\frac{a}{n}\Big)\right) \\ &= cn\big(a^2+a\big) - ca^2 \end{aligned}$$

Therefore the total required time is $O(n)$, which is linear to the number of elements.

Exercise 5.10

(a) Let X_i the number of balls in bin i. X_i is distributed over the Poisson distribution of mean 1. Therefore

$$\Pr(X_i = 1) = \frac{e^{-1}1^j}{1!} = \frac{1}{e}$$

and the event where all bins have one balls each can be viewed in the Poisson persective as the intersection of events $X_i = 1$ for all i. Therefore the probability is bounded by $e\sqrt{n}$ multiplied to the Poisson probability, which gives

$$e\sqrt{n} \times \Pr\left(\bigcap_{i=1}^{n} (Y_i = 1)\right) = \frac{\sqrt{n}}{e^{n-1}}$$

as the upper bound.

(b) The probability can be calculated as the following.

$$\frac{n}{n} \times \frac{n-1}{n} \times \cdots \times \frac{1}{n} = \frac{n!}{n^n}$$

(c) Let Y a Poisson random variable with parameter n, then the probability that Y takes on the value n in

$$\Pr(Y = n) = \frac{e^{-n}n^n}{n!}$$

Therefore the two results from (a) and (b) approximately differ by this probability.

Theorem 5.6 explains this. The difference between the Poisson approximation and the actual case is only the initial work on setting the number of balls as n in the Poisson point of view; disregarding this difference makes the two distributions equivalent. Therefore this differnce gives exactly the value calculated above $\left(= \frac{e^{-n}n^n}{n!}\right)$ as the differnce factor between two probabilities.

Exercise 5.12

(a) The probability for a ball to land in a bin by itself is $\left(1 - \frac{1}{n}\right)^{b-1}$. If we start with b balls at a round, then the number of remaining balls after that round is $b\left(1 - \left(1 - \frac{1}{n}\right)^{b-1}\right)$.

(b) From (a), we obtain

$$\begin{aligned} x_{j+1} &= x_j\left(1 - \left(1 - \frac{1}{n}\right)^{x_j - 1}\right) \\ &\leq x_j\left(1 - \left(1 - \frac{x_j - 1}{n}\right)\right) = \frac{x_j^2 - x_j}{n} \\ &\leq \frac{x_j^2}{n}. \end{aligned}$$

It can be inductively calculated that the number of balls remaining is at most $\frac{n}{2}$. Therefore, for any i,

$$\begin{aligned} x_{i+k} &\leq \frac{x_{i+k-1}^2}{n} \leq \frac{x_{i+k-2}^4}{n^3} \leq \frac{x_{i+k-3}^8}{n^7} \\ &\leq \cdots \leq \frac{x_i^{2^k}}{n^{2^k - 1}} \\ &\leq \frac{\left(\frac{n}{2}\right)^{2^k}}{n^{2^k - 1}} \leq \frac{n}{2^{2^k}} \end{aligned}$$

so if $k = O(\log \log n)$, we obtain $x_{i+k} \leq 1$, therefore the total number of rounds needed is $i + k = O(\log \log n)$ since i is an arbitrary constant.

Exercise 5.14

(a) We can calculate each as the following.

$$\Pr(Z = \mu + h) = \frac{e^{-\mu}\mu^{\mu+h}}{(\mu+h)!}, \ \ \Pr(Z = \mu - h - 1) = \frac{e^{-\mu}\mu^{\mu-h-1}}{(\mu-h-1)!}$$

This gives that $\Pr(Z = \mu + h) \geq \Pr(Z = \mu - h - 1)$ is equivalent to

$$\mu^{2h+1} \geq \frac{(\mu+h)!}{(\mu-h-1)!}.$$

This is true in that

$$\begin{aligned}
\frac{(\mu+h)!}{(\mu-h-1)!} &= (\mu+h)(\mu+h-1)\cdots(\mu-h) \\
&= (\mu+h)(\mu-h)(\mu+(h-1))(\mu-(h-1))\cdots(\mu+1)(\mu-1)\mu \\
&= \left(\mu^2-h^2\right)\left(\mu^2-(h-1)^2\right)\cdots\left(\mu^2-1^2\right)\mu \\
&\leq \left(\mu^2\right)^h\mu = \mu^{2h+1}.
\end{aligned}$$

(b) From $\Pr(Z = \mu + h) \geq \Pr(Z = \mu - h - 1)$ when $0 \leq h \leq \mu - 1$, we obtain the following.

$$\begin{aligned}
\Pr(Z \geq \mu) = \sum_{z=\mu}^{\infty} \Pr(Z = z) &= \sum_{h=0}^{\infty} \Pr(Z = \mu + h) \\
&\geq \sum_{h=0}^{\mu-1} \Pr(Z = \mu + h) \\
&\geq \sum_{h=0}^{\mu-1} \Pr(Z = \mu - h - 1) \\
&= \sum_{z=0}^{\mu-1} (Z = z) = \Pr(Z < \mu)
\end{aligned}$$

Since $\Pr(Z \geq \mu) + \Pr(Z < \mu) = 1$, we obtain $\Pr(Z \geq \mu) \geq \frac{1}{2}$.

Exercise 5.16

(a) $X_1X_2\cdots X_k = 1$ if and only if $X_1 = \cdots = X_k = 1$, and 0 if any of the X_is are 0. Therefore

$$\begin{aligned}
\mathbf{E}[X_1X_2\cdots X_k] &= \Pr(X_1 = 1 \wedge \cdots \wedge X_k = 1) \\
&= \Pr(\text{The first } k \text{ bins are empty.}) \\
&= \frac{(n-k)^n}{n^n} = \left(1 - \frac{k}{n}\right)^n.
\end{aligned}$$

Similarly, $Y_1Y_2\cdots Y_k = 1$ if and only if $Y_1 = \cdots = Y_k = 1$. Since all Y_is are independent, we obtain

$$\begin{aligned}
\mathbf{E}[Y_1Y_2\cdots Y_k] &= \Pr(Y_1 = 1 \wedge \cdots \wedge Y_k = 1) \\
&= \Pr(Y_1 = 1)\Pr(Y_2 = 1)\cdots\Pr(Y_k = 1) \\
&= \left(1 - \frac{1}{n}\right)^{kn}.
\end{aligned}$$

Since $1 - \frac{k}{n} \leq \left(1 - \frac{1}{n}\right)^k$ for all positive integers n and k, we get

$$\mathbf{E}[X_1X_2\cdots X_k] \leq \mathbf{E}[Y_1Y_2\cdots Y_k]$$

(b) Considering the Taylor series of e^x, we obtain the following.

$$\mathbf{E}\left[e^{tX}\right] = \mathbf{E}\left[\sum_{m=0}^{\infty} \frac{(tX)^m}{m!}\right] = \sum_{m=0}^{\infty} \frac{t^m}{m!}\mathbf{E}[X^m]$$
$$\mathbf{E}\left[e^{tY}\right] = \mathbf{E}\left[\sum_{m=0}^{\infty} \frac{(tY)^m}{m!}\right] = \sum_{m=0}^{\infty} \frac{t^m}{m!}\mathbf{E}[Y^m]$$

Therefore $\mathbf{E}\left[e^{tX}\right] \le \mathbf{E}\left[e^{tY}\right]$ is equivalent to $\mathbf{E}[X^m] \le \mathbf{E}[Y^m]$ for all $m \ge 0$. This is true in that

$$\begin{aligned}
\mathbf{E}[X^m] &= \mathbf{E}\left[\left(\sum_{i=1}^{n} X_i\right)^m\right] \\
&= \mathbf{E}\left[\sum_{c_1+\cdots+c_n=m} \left(\frac{m!}{c_1!\cdots c_n!}\prod_{i=1}^{n} X_i^{c_i}\right)\right] = \sum_{c_1+\cdots+c_n=m} \left(\frac{m!}{c_1!\cdots c_n!}\mathbf{E}\left[\prod_{i=1}^{n} X_i^{c_i}\right]\right) \\
&= \sum_{c_1+\cdots+c_n=m} \left(\frac{m!}{c_1!\cdots c_n!}\mathbf{E}\left[\prod_{i=1}^{n} X_i\right]\right) \quad (\because X_i\text{s are independent and identically distributed.}) \\
&\le \sum_{c_1+\cdots+c_n=m} \left(\frac{m!}{c_1!\cdots c_n!}\mathbf{E}\left[\prod_{i=1}^{n} Y_i\right]\right) \quad (\because \text{(a)}) \\
&= \mathbf{E}[Y^m] \quad (\because \text{Similar to } X).
\end{aligned}$$

(c) For any t,

$$\begin{aligned}
\Pr(X \ge (1+\delta)\mathbf{E}[X]) &= \Pr\left(e^{tX} \ge e^{t(1+\delta)\mathbf{E}[X]}\right) \\
&\le \frac{\mathbf{E}\left[e^{tX}\right]}{e^{t(1+\delta)\mathbf{E}[X]}} \quad (\because \text{Markov's inequality}) \\
&\le \frac{\mathbf{E}\left[e^{tY}\right]}{e^{t(1+\delta)\mathbf{E}[X]}} \quad (\because \text{(b)}) \\
&= \frac{e^{\mathbf{E}[Y]\left(e^t-1\right)}}{e^{t(1+\delta)\mathbf{E}[X]}} \\
&= \left(\frac{e^{\left(e^t-1\right)}}{e^{t(1+\delta)}}\right)^{\mathbf{E}[X]}. \quad (\because \mathbf{E}[X] = \mathbf{E}[Y])
\end{aligned}$$

Substitute $t = \ln(1+\delta)$, then

$$\Pr(X \ge (1+\delta)\mathbf{E}[X]) \le \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mathbf{E}[X]}.$$