Engineering Mathematics 1 Problem Set 1

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Problem 1

(a) *False* Choosing two symmetric matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

gives

$$\mathbf{AB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \mathbf{BA} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

therefore $\mathbf{AB} \neq \mathbf{BA}$.

(b) *False* The example from (a) gives an example where $\mathbf{AB}$ is not symmetric.

(c) *False* Choosing $\mathbf{B} = \mathbf{I}_2$ and

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

satisfies

$$\mathbf{AB} = \mathbf{BA} = \mathbf{A},\ \mathbf{BC} = \mathbf{CB} = \mathbf{C}$$

but

$$\mathbf{AC} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ \mathbf{CA} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

therefore $\mathbf{AC} \neq \mathbf{CA}$.

(d) *True* Since $\mathbf{A}$ and $\mathbf{B}$ are upper triangular,

$$\mathbf{A}_{ij} = 0,\ \mathbf{B}_{ij} = 0 \ \ (\forall\, i, j \in \mathbb{N} \text{ s.t. } 1 \leq j < i \leq n)$$

Therefore, if $i > j$,

$$\begin{aligned} (\mathbf{AB})_{ij} &= \sum_{k=1}^{n} \mathbf{A}_{ik}\mathbf{B}_{kj} \\ &= \sum_{k=1}^{i-1} \mathbf{A}_{ik}\mathbf{B}_{kj} + \sum_{k=i}^{n} \mathbf{A}_{ik}\mathbf{B}_{kj} = 0 \end{aligned}$$

so $\mathbf{AB}$ is an upper triangular matrix.

Problem 2

Define $\mathbf{S}$ and $\mathbf{T}$ as

$$\mathbf{S}_{ij} = \mathbf{S}_{ji} = \frac{\mathbf{A}_{ij} + \mathbf{A}_{ji}}{2},\ \mathbf{T}_{ij} = \frac{\mathbf{A}_{ij} - \mathbf{A}_{ji}}{2},\ \mathbf{T}_{ji} = \frac{\mathbf{A}_{ji} - \mathbf{A}_{ij}}{2} \ \ (1 \leq i < j \leq n)$$

$$\mathbf{S}_{ii} = \mathbf{T}_{ii} = \frac{\mathbf{A}_{ii}}{2} \ \ (1 \leq i \leq n)$$

Then

$$\begin{aligned}
(\mathbf{S}+\mathbf{T})_{ij} &= \frac{\mathbf{A}_{ij} + \mathbf{A}_{ji}}{2} + \frac{\mathbf{A}_{ij} - \mathbf{A}_{ji}}{2} = \mathbf{A}_{ij} \ \ (1 \leq i < j \leq n) \\
(\mathbf{S}+\mathbf{T})_{ji} &= \frac{\mathbf{A}_{ij} + \mathbf{A}_{ji}}{2} + \frac{\mathbf{A}_{ji} - \mathbf{A}_{ij}}{2} = \mathbf{A}_{ji} \ \ (1 \leq i < j \leq n) \\
(\mathbf{S}+\mathbf{T})_{ii} &= \frac{\mathbf{A}_{ii}}{2} + \frac{\mathbf{A}_{ii}}{2} = \mathbf{A}_{ii} \ \ (1 \leq i \leq n)
\end{aligned}$$

Therefore $\mathbf{S} + \mathbf{T} = \mathbf{A}$.

Problem 3

We should prove the following:

$$\left(\mathbf{A}^k\right)_{ij} = 0 \ \ (k \geq 1,\ \forall\, i, j \in \mathbb{N} \text{ s.t. } i \geq j - k + 1)$$

When $k = 1$, $\mathbf{A}_{ij} = 1$ for all $i \geq j$.

Suppose the statement is true at $k = m$. Then

$$\left(\mathbf{A}^m\right)_{ij} = 0 \ \ (\forall\, i, j \in \mathbb{N} \text{ s.t. } i \geq j - m + 1)$$

If $i \geq j - m$,

$$\begin{aligned}
\left(\mathbf{A}^{m+1}\right)_{ij} &= \sum_{k=1}^{n} \left(\mathbf{A}^m\right)_{ik} \mathbf{A}_{kj} \\
&= \sum_{k=1}^{i+m-1} \left(\mathbf{A}^m\right)_{ik} \mathbf{A}_{kj} + \sum_{k=i+m}^{n} \left(\mathbf{A}^m\right)_{ik} \mathbf{A}_{kj} \\
&= 0
\end{aligned}$$

The statement is also true at $k = m+1$. Using mathematical induction, the statement is true is at all $k \geq 1$.

Therefore for $\mathbf{A}^n$, $\left(\mathbf{A}^n\right)_{ij} = 0$ for all $i \geq j - n + 1$. Since $j \leq n$, it satisfies for all i, j.

$\therefore \mathbf{A} = \mathbf{0}$

Problem 4

(a) Geometrically, 'rotating' two vectors and 'adding' them is equal to 'adding' them first and then 'rotating' them. Adding to vectors can be calculated by drawing a parallelogram with the vectors, but rotation of vectors similarly rotates the parallelogram too, so the result is equivalent. Also 'rotating' a vector and 'multiplying' it is equal to the opposite. Multiplying a scalar to a vector doesn't change its orientation but only its size, rotating a vector doesn't change its size but only is orientation. Thus the result is equivalent. Since rotation preserves both adding and scalar multiplication, it is a linear transformation.

(b) Rotations of the standard bases of $\mathbb{R}^2$ will give the corresponding matrix.

Let $\mathbf{x} = (x, y) = x\mathbf{e_1} + y\mathbf{e_2}$, then

$$L_\theta(\mathbf{x}) = L_\theta(x\mathbf{e_1} + y\mathbf{e_2}) = xL_\theta(\mathbf{e_1}) + yL_\theta(\mathbf{e_2})$$

since L_θ is a linear transformation.

Rotating $\mathbf{e_1} = (1, 0)$ and $\mathbf{e_2} = (0, 1)$ gives $L_\theta(\mathbf{e_1}) = (\cos\theta, \sin\theta),\ L_\theta(\mathbf{e_2}) = (-\sin\theta, \cos\theta)$.

Therefore

$$\begin{aligned} L_\theta(\mathbf{x}) &= x(\cos\theta, \sin\theta) + y(-\sin\theta, \cos\theta) \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

$\therefore \mathbf{A}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

(c) Since L_θ is a linear transformation and $\mathbf{A}_\theta$ is its corresponding matrix, the fundamental theorem of linear algebra gives:

$$(L_\alpha \circ L_\beta)(\mathbf{x}) = \mathbf{A}_\alpha \mathbf{A}_\beta \mathbf{x}$$

Since L_θ is the map of counterclockwise rotation, $L_{\alpha+\beta} = L_\alpha \circ L_\beta$. $\therefore L_{\alpha+\beta}(\mathbf{x}) = \mathbf{A}_\alpha \mathbf{A}_\beta \mathbf{x}$

Let $\mathbf{x} = (x, y)$, then

$$\begin{aligned} L_{\alpha+\beta}(\mathbf{x}) &= \mathbf{A}_{\alpha+\beta}\mathbf{x} \\ &= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \mathbf{A}_\alpha \mathbf{A}_\beta \mathbf{x} &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

Therefore,

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Problem 5

The rank of a matrix is equivalent to that of its row-equivalent matrices. We can find the row reduced echelon form of $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} c+2 & c+3 & \cdots & c+1+n \\ c+3 & c+4 & \cdots & c+2+n \\ \vdots & \vdots & \ddots & \vdots \\ c+n+1 & c+n+2 & \cdots & c+n+n \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} c+2 & c+3 & \cdots & c+1+n \\ 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ n-1 & n-1 & \cdots & n-1 \end{bmatrix} \rightarrow \begin{bmatrix} c+2 & c+3 & \cdots & c+1+n \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Since $(c+2,\ c+3,\ \cdots,\ c+1+n)$ and $(1,\ 1,\ \cdots,\ 1)$ are linearly independent, the rank of $\mathbf{A}$ is 2.

Problem 6

Let the column vectors of $\mathbf{B}$ as $\mathbf{B}_1,\ \mathbf{B}_2,\ \cdots,\ \mathbf{B}_n$. Then the columns of AB will be $\mathbf{AB}_1,\ \mathbf{AB}_2,\ \cdots,\ \mathbf{AB}_n$.

Let $\text{rank}(\mathbf{B}) = r$, then we can choose $I = \{i_1,\ i_2,\ \cdots,\ i_r\}$ such that $\mathbf{B}_{i_1},\ \mathbf{B}_{i_2},\ \cdots,\ \mathbf{B}_{i_r} \in \mathbf{B}$ are linearly independent. Then for $\mathbf{B}_k \in \mathbf{B}$, we can claim the following:

$$\forall k \notin I,\ \exists\, a_1,\ \cdots,\ a_r \ \text{ s.t. } \mathbf{B}_k = a_1\mathbf{B}_{i_1} + \cdots + a_r\mathbf{B}_{i_r}$$

$$\mathbf{AB}_k = a_1\mathbf{AB}_{i_1} + \cdots + a_r\mathbf{AB}_{i_r}$$

Therefore $\mathbf{AB}_{i_1},\ \cdots,\ \mathbf{AB}_{i_r},\ \mathbf{AB}_k$ are linearly dependent. $\therefore \text{rank}(\mathbf{AB}) \le r = \text{rank}(\mathbf{B})$

Then for any matrix $\mathbf{A}, \mathbf{B}$, $\text{rank}\big(\mathbf{B}^T\mathbf{A}^T\big) \le \text{rank}\big(\mathbf{A}^T\big)$. The rank theorem gives $\text{rank}(\mathbf{M}) = \text{rank}\big(\mathbf{M}^T\big)$ for any matrix $\mathbf{M}$. Therefore

$$\text{rank}(\mathbf{AB}) = \text{rank}\big((\mathbf{AB})^T\big) = \text{rank}\big(\mathbf{B}^T\mathbf{A}^T\big)$$
$$\le \text{rank}\big(\mathbf{A}^T\big) = \text{rank}(\mathbf{A})$$

Since $\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{A})$ and $\text{rank}(\mathbf{AB}) \leq \text{rank}(\mathbf{B})$,

$$\text{rank}(\mathbf{AB}) \leq \min\left\{\text{rank}(\mathbf{A}),\ \text{rank}(\mathbf{B})\right\}$$

If we take $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $\text{rank}(\mathbf{AB}) = 0$ and $\min\left\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\right\} = 1$, which satisfies $\text{rank}(\mathbf{AB}) < \min\left\{\text{rank}(\mathbf{A}),\ \text{rank}(\mathbf{B})\right\}$.

Problem 7

(a) For $\forall \mathbf{x} \in \ker \mathbf{A}^T\mathbf{A}$, $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{0}$. Then

$$\mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{0},\ (\mathbf{A}\mathbf{x})^T\mathbf{A}\mathbf{x} = \mathbf{0}$$

Since $\mathbf{A}\mathbf{x} \in \mathbb{R}^{m\times 1}$ and is a column vector,

$$(\mathbf{A}\mathbf{x})^T\mathbf{A}\mathbf{x} = \langle\mathbf{A}\mathbf{x},\ \mathbf{A}\mathbf{x}\rangle = \mathbf{0}$$

Therefore $\mathbf{A}\mathbf{x} = \mathbf{0}$, so $\mathbf{x} \in \ker \mathbf{A}$. $\therefore \ker \mathbf{A}^T\mathbf{A} \subseteq \ker \mathbf{A}$

For $\forall \mathbf{x} \in \ker \mathbf{A}$, $\mathbf{A}\mathbf{x} = \mathbf{0}$. Then $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{0}$, therefore $\mathbf{x} \in \ker \mathbf{A}^T\mathbf{A}$. $\therefore \ker \mathbf{A} \subseteq \ker \mathbf{A}^T\mathbf{A}$

Therefore $\ker \mathbf{A} = \ker \mathbf{A}^T\mathbf{A}$. The rank-nullity theorem gives:

$$\begin{aligned} \text{rank}\big(\mathbf{A}^T\mathbf{A}\big) &= n - \dim\left(\ker \mathbf{A}^T\mathbf{A}\right) \\ &= n - \dim\left(\ker \mathbf{A}\right) \\ &= \text{rank}(\mathbf{A}) \end{aligned}$$

(b)

Problem 8

(a) When $n = 1$, $\mathbf{X} = \begin{bmatrix} 1 \end{bmatrix}$, so $\det \mathbf{X} = 1$. The statement is true.

When $n = 2$, $\mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}$, so $\det \mathbf{X} = x_2 - x_1$. The statement is true.

Suppose the statement is true at $n = k$ ($k \geq 2$), then $\det \mathbf{X} = \prod_{1\leq i<j\leq k}(x_j - x_i)$. Now we can claim the following:

$$
\begin{aligned}
\det \mathbf{X}\Big|_{n=k+1} &= \det \begin{bmatrix} 1 & x_1 & \cdots & x_1^k \\ 1 & x_2 & \cdots & x_2^k \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{k+1} & \cdots & x_{k+1}^k \end{bmatrix} = \det \begin{bmatrix} 1 & x_1 & \cdots & x_1^k \\ 0 & x_2 - x_1 & \cdots & x_2^k - x_1^k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_{k+1} - x_1 & \cdots & x_{k+1}^k - x_1^k \end{bmatrix} \\
&= \det \begin{bmatrix} x_2 - x_1 & x_2^2 - x_1^2 & \cdots & x_2^k - x_1^k \\ x_3 - x_1 & x_3^2 - x_1^2 & \cdots & x_3^k - x_1^k \\ \vdots & \vdots & \ddots & \vdots \\ x_{k+1} - x_1 & x_{k+1}^2 - x_1^2 & \cdots & x_{k+1}^k - x_1^k \end{bmatrix} \\
&= \det \left(\begin{bmatrix} x_2 - x_1 & 0 & \cdots & 0 \\ 0 & x_3 - x_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{k+1} - x_1 \end{bmatrix} \begin{bmatrix} 1 & x_2 + x_1 & \cdots & \sum_{i=0}^{k-1} x_2^{k-1-i} x_1^i \\ 1 & x_3 + x_1 & \cdots & \sum_{i=0}^{k-1} x_3^{k-1-i} x_1^i \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{k+1} + x_1 & \cdots & \sum_{i=0}^{k-1} x_{k+1}^{k-1-i} x_1^i \end{bmatrix} \right) \\
&= \prod_{j=2}^{k+1} (x_j - x_1) \det \begin{bmatrix} 1 & x_2 + x_1 & \cdots & \sum_{i=0}^{k-1} x_2^{k-1-i} x_1^i \\ 1 & x_3 + x_1 & \cdots & \sum_{i=0}^{k-1} x_3^{k-1-i} x_1^i \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{k+1} + x_1 & \cdots & \sum_{i=0}^{k-1} x_{k+1}^{k-1-i} x_1^i \end{bmatrix} \\
&= \prod_{j=2}^{k+1} (x_j - x_1) \det \left(\begin{bmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{k-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{k+1} & x_{k+1}^2 & \cdots & x_{k+1}^{k-1} \end{bmatrix} \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{k-1} \\ 0 & 1 & x_1 & \cdots & x_1^{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \right) \\
&= \prod_{j=2}^{k+1} (x_j - x_1) \det \mathbf{X}\Big|_{n=k,\ 2\le i,\, j\le k+1} \\
&= \prod_{1=i<j\le k+1} (x_j - x_1) \prod_{2\le i<j\le k+1} (x_j - x_i) \\
&= \prod_{1\le i<j<k+1} (x_j - x_i)
\end{aligned}
$$

The statement is also true at $n = k+1$. Using mathematical induction, it is true at all $n \geq 1$.

(b) Let $p(x) = p_0 + p_1 x + \cdots + p_{n-1} x^{n-1}$, we should prove that a set $\{p_j : 0 \leq j < n\}$ exists and is unique.

$$
\begin{gathered}
p(x_1) = p_0 + p_1 x_1 + \cdots + p_{n-1} x_1^{n-1} = y_1 \\
\cdots \\
p(x_n) = p_0 + p_1 x_n + \cdots + p_{n-1} x_n^{n-1} = y_n
\end{gathered}
$$

Equations $p(x_i) = y_i$ $(1 \le i \le n)$ are like the above. It can be written as:

$$\begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Since all x_i's are distinct, $\det \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix} = \prod_{1 \le i < j \le k} (x_j - x_i) \ne 0.$

Therefore the linear system above has a unique and existing solution, therefore a set of solutions $\{p_j : 0 \le j < n\}$ exists and is unique.

Problem 9

Algorithm 1 Gauss-Jordan Elimination

function GAUSS-JORDAN ELIMINATION ($\mathbf{A}$: $n \times n+1$ augmented matrix)
  while $i = 1$ **to** n **do**
    while $j = i$ **to** n **do**
      if $\mathbf{A}_{ji} \ne 0$ **then**
        $\mathbf{A}_i \leftrightarrow \mathbf{A}_j$
        $\mathbf{A}_{ik} = \dfrac{\mathbf{A}_{ik}}{\mathbf{A}_{ii}}$ ▷ $k = 1,\ 2,\ \cdots,\ n+1$
        $\mathbf{A}_{lk} = \mathbf{A}_{lk} - \dfrac{\mathbf{A}_{li}}{\mathbf{A}_{ii}}\mathbf{A}_{ik}$ ▷ $l = i+1,\ i+2,\ \cdots,\ n$ and $k = 1,\ 2,\ \cdots,\ n+1$
        break
      end if
    end while
  end while
end function

There are several loops in this algorithm. First, a loop is used to gow through each row and change it, repeated n times. Next, a loop is used to find the next row with a nonzero element at its specified index, which has a complexity of $O(1)$, since the lower loops are only executed once when the nonzero element is found. Next, a loop is used to change the elements in each row, which have a complexity of $O(n^2)$ since it goes through all elements of the matrix. Therefore the overall complexity of the algorithm is $O(n^3)$.

Problem 10

All calculated bases from the process can be written as:

$$\mathbf{e}_i = \frac{\mathbf{v}_i - \langle \mathbf{v}_i, \mathbf{e}_1 \rangle \mathbf{e}_1 - \cdots - \langle \mathbf{v}_i, \mathbf{e}_{i-1} \rangle \mathbf{e}_{i-1}}{\|\mathbf{v}_i - \langle \mathbf{v}_i, \mathbf{e}_1 \rangle \mathbf{e}_1 - \cdots - \langle \mathbf{v}_i, \mathbf{e}_{i-1} \rangle \mathbf{e}_{i-1}\|}$$

Any inner product of two calculated bases are:

$$\begin{aligned} \langle \mathbf{e}_i, \mathbf{e}_j \rangle &= \left\langle \frac{\mathbf{v}_i - \langle \mathbf{v}_i, \mathbf{e}_1 \rangle \mathbf{e}_1 - \cdots - \langle \mathbf{v}_i, \mathbf{e}_{i-1} \rangle \mathbf{e}_{i-1}}{\|\mathbf{v}_i - \langle \mathbf{v}_i, \mathbf{e}_1 \rangle \mathbf{e}_1 - \cdots - \langle \mathbf{v}_i, \mathbf{e}_{i-1} \rangle \mathbf{e}_{i-1}\|}, \mathbf{e}_j \right\rangle \\ &= \frac{\langle \mathbf{v}_i, \mathbf{e}_j \rangle - \langle \mathbf{v}_i, \mathbf{e}_j \rangle}{\|\mathbf{v}_i - \langle \mathbf{v}_i, \mathbf{e}_1 \rangle \mathbf{e}_1 - \cdots - \langle \mathbf{v}_i, \mathbf{e}_{i-1} \rangle \mathbf{e}_{i-1}\|} \\ &= 0 \end{aligned}$$

Therefore all bases are orthogonal and their size are 1. Thus they are all orthonormal bases.

Let the dimension of vectors $\mathbf{v}_i$ each m. Then the process goes through n operations to earn n orthonormal basis vectors. In each process, up to $n-1$ calculations are made to get projection vectors. Calculating a projection vector requires calculating two inner products of the vectors, and since the dimension of each vector is m, it requires $2m$ operations. Therefore the overall complexity is equivalent to $O(2mn(n-1))$, which can be simplified as $O(mn^2)$.

References

[1] 'How to prove $\mathrm{Rank}(AB) \leq \min(\mathrm{Rank}(A), \mathrm{Rank}(B))$?', *Mathematics Stack Exchange*, 2016. Available: `https://math.stackexchange.com/q/48989`.

[2] Thomas Hughes, 'The Vandermonde Determinant, A Novel Proof', 2020. Available: `https://towardsdatascience.com/the-vandermonde-determinant-a-novel-proof-851d107bd728`.

[3] 'Gram–Schmidt process', *Wikipedia*. Available: `https://en.wikipedia.org/wiki/Gram-Schmidt_process`.

Usage of References

[1] : Problem 6
[2] : Problem 8
[3] : Problem 10