Engineering Mathematics 2
Seoul National University

Homework 2
2021-16988 Jaewan Park

Exercise 2.9

(a) The probability of $\max(X_1, X_2)$ or $\min(X_1, X_2)$ to be a specific value is the following.

$$\mathbf{Pr}[\max(X_1, X_2) = n] = \frac{2n-1}{k^2}$$

$$\mathbf{Pr}[\min(X_1, X_2) = n] = \frac{2k-2n+1}{k^2}$$

Therefore

$$\mathbf{E}[\max(X_1, X_2)] = \sum_{n=1}^{k} n \cdot \frac{2n-1}{k^2} = \frac{k(k+1)(4k-1)}{6k^2}$$

$$\mathbf{E}[\min(X_1, X_2)] = \sum_{n=1}^{k} n \cdot \frac{2k-2n+1}{k^2} = \frac{k(k+1)(2k+1)}{6k^2}$$

(b) Since $\mathbf{E}[X_1] = \mathbf{E}[X_2] = \displaystyle\sum_{n=1}^{k} \frac{1}{k} n = \frac{k+1}{2}$,

$$\mathbf{E}[\max(X_1, X_2)] + \mathbf{E}[\min(X_1, X_2)] = \mathbf{E}[X_1] + \mathbf{E}[X_2] = k+1$$

(c) Since $\max(X_1, X_2) + \min(X_1, X_2) = X_1 + X_2$,

$$\mathbf{E}[\max(X_1, X_2)] + \mathbf{E}[\min(X_1, X_2)] = \mathbf{E}[\max(X_1, X_2) + \min(X_1, X_2)] = \mathbf{E}[X_1 + X_2] = \mathbf{E}[X_1] + \mathbf{E}[X_2]$$

Exercise 2.15

Let X the random variable of the number of total flips until the kth head, and X_i the random variable of the number of flips from the $(i-1)$th head to the ith head. Then X_i has a geometric distribution with probability p. Therefore $\mathbf{E}[X_i] = \dfrac{1}{p}$. Then

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^{k} X_i\right] = \sum_{i=1}^{k} \mathbf{E}[X_i] = \frac{k}{p}.$$

Exercise 2.18

We should prove that the item stored in the memory at each step is uniform over the items that have appeared. At the first step, it is trivial that the first and only item is uniform over the items that have appeared. Suppose the item in the memory at the kth step satisfies this condition. Then at the $(k+1)$th step, with a probability of $\dfrac{k}{k+1}$ the item will stay in the memory. Since the previous item is uniformly distributed over the first k items, the first k items can be in the memory at the $(k+1)$th step with a uniform probability of $\dfrac{1}{k+1}$. On the other hand, with a probability of $\dfrac{1}{k+1}$ the item in the memory will be transferred at the $(k+1)$th step. Therefore the $(k+1)$ items all have the same probability $= \dfrac{1}{k+1}$ to be in the memory at the $(k+1)$th step. Using mathematical induciton, we can claim that item stored in the memory is uniformly distributed over the items that have appeared until that step.

Exercise 2.22

Define X_{ij} the random variable that has the value 1 when a_i and a_j are inverted, and 0 otherwise. Then

$$\mathbf{Pr}[X_{ij} = 1] = \mathbf{Pr}[X_{ij} = 0] = \frac{1}{2}$$

since there are an equal number of possible orderings where the two are inverted or not. Therefore the expectation of X_{ij} is $\frac{1}{2}$. Now let X the random variable of the number of total inversions needed on a list, then X is equal to the sum of X_{ij}s without repeats. Therefore

$$\begin{aligned}
\mathbf{E}[X] &= \mathbf{E}\left[\sum_{1\le i<j\le n} X_{ij}\right] \\
&= \sum_{1\le i<j\le n} \mathbf{E}[X_{ij}] = \sum_{1\le i<j\le n} \frac{1}{2} \\
&= \frac{1}{2}\cdot\frac{n(n-1)}{2} = \frac{n(n-1)}{4}.
\end{aligned}$$

Exercise 2.27

From $\mathbf{Pr}[X = x] = \dfrac{6}{\pi^2 x^2}$ we obtain the following.

$$\begin{aligned}
\mathbf{E}[X] &= \sum_{x=1}^{\infty} x\mathbf{Pr}[X = x] \\
&= \sum_{x=1}^{\infty} \frac{6}{\pi^2 x} = \infty
\end{aligned}$$

Therefore the expectation diverges.

Exercise 2.32

(a) For E_i to happen, the ith candidate should actually be the best candidate, i should be larger than m, and the second best candidate should be among the first m candidates. Therefore

$$\mathbf{Pr}[E_i] = \begin{cases} \dfrac{1}{n} \times \dfrac{m}{i-1} & (i > m) \\ 0 & (\text{otherwise}) \end{cases}$$

and we finally obtain the following.

$$\begin{aligned}
\mathbf{Pr}[E] &= \mathbf{Pr}\left[\sum_{i=1}^{n} E_i\right] = \sum_{i=1}^{n} \mathbf{Pr}[E_i] \\
&= \frac{m}{n} \sum_{i=m+1}^{n} \frac{1}{i-1}
\end{aligned}$$

(b) We can bound $\sum_{i=m+1}^{n} \frac{1}{i-1}$ like the following:

$$\int_{m+1}^{n+1} \frac{1}{x-1}dx \le \sum_{i=m+1}^{n} \frac{1}{i-1} \le \int_{m}^{n} \frac{1}{x-1}dx$$

Therefore

$$\frac{m}{n}(\ln n - \ln m) \leq \sum_{i=m+1}^{n} \frac{1}{i-1} \leq \frac{m}{n}(\ln (n-1) - \ln (m-1)).$$

(c) Let $f(m) = \dfrac{m}{n}(\ln n - \ln m)$, then

$$\frac{df}{dm} = \frac{\ln n - \ln m - 1}{n}, \quad \frac{d^2 f}{dm^2} = -\frac{1}{mn}$$

thus $\dfrac{df}{dm} = 0$ when $m = \dfrac{n}{e}$, and $f(m)$ is at its maximum since $\left.\dfrac{d^2 f}{dm^2}\right|_{m=\frac{n}{e}} = -\dfrac{e}{n^2} < 0$. Since the above inequality holds for all m and n, we can find the lower bound of $\mathbf{Pr}[E]$.

$$\mathbf{Pr}[E] \geq \sup\left\{\frac{m}{n}(\ln n - \ln m)\right\} = \frac{1}{e}$$