Engineering Mathematics 2
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Homework 9
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Exercise 9.2

We have $X \sim \mathcal{N}(0,1)$, and its density function is $f_X(x) = \dfrac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right)$. Therefore when $n \geq 2$,

$$\begin{aligned}
\mathbf{E}[X^n] &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} x^n dx \\
&= \left[-\frac{1}{\sqrt{2\pi}} e^{-x^2/2} x^{n-1}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} (n-1) e^{-x^2/2} x^{n-2} dx \\
&= (n-1)\mathbf{E}\left[X^{n-2}\right].
\end{aligned}$$

For even $n \geq 2$,

$$\mathbf{E}[X^n] = (n-1)\mathbf{E}\left[X^{n-2}\right] = \cdots = (n-1)(n-3)\cdots 1\mathbf{E}\left[X^0\right] = (n-1)(n-3)\cdots 1 \geq 1$$

For odd $n \geq 3$,

$$\mathbf{E}[X^n] = -(n-1)\mathbf{E}\left[X^{n-2}\right] = \cdots = (-1)^{(n-1)/2}(n-1)(n-3)\cdots 2\mathbf{E}\left[X^1\right] = 0$$

Since $\mathbf{E}[X] = 0$, we can say $\mathbf{E}[X^n] = 0$ for all odd $n \geq 1$.

Exercise 9.3

We can calculate the covariance as the following.

$$\begin{aligned}
\mathbf{Cov}(Y_i, Y_j) &= \mathbf{E}[(Y_i - \mathbf{E}[Y_i])(Y_j - \mathbf{E}[Y_j])] = \mathbf{E}[(a_{i1}X_1 + \cdots + a_{in}X_n)(a_{j1}X_1 + \cdots + a_{jn}X_n)] \\
&= \mathbf{E}\left[\sum_{1 \leq p,q \leq n} a_{ip}a_{jq}X_pX_q\right] = \sum_{1 \leq p,q \leq n} a_{ip}a_{jq}\mathbf{E}[X_pX_q] \\
&= \sum_{k=1}^{n} a_{ik}a_{jk}\mathbf{E}\left[X_k^2\right] = \sum_{k=1}^{n} a_{ik}a_{jk}
\end{aligned}$$

Exercise 9.4

(a) For n datapoints of X and Y, let $\mathbf{u} = \mathbf{X} - \mathbf{E}[X]$, $\mathbf{v} = \mathbf{Y} - \mathbf{E}[Y]$ where $\mathbf{X}$ and $\mathbf{Y}$ are the vectors of the datapoints. Then we obtain the following.

$$\mathbf{Cov}(X,Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])] = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \mathbf{E}[X])(Y_i - \mathbf{E}[Y]) = \frac{1}{n-1}\mathbf{u} \cdot \mathbf{v}$$

$$\mathbf{Var}(X) = \mathbf{E}\big[(X - \mathbf{E}[X])^2\big] = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \mathbf{E}[X])^2 = \frac{1}{n-1}\|\mathbf{u}\|^2$$

$$\mathbf{Var}(Y) = \mathbf{E}\big[(Y - \mathbf{E}[Y])^2\big] = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \mathbf{E}[Y])^2 = \frac{1}{n-1}\|\mathbf{v}\|^2$$

From the Cauchy-Schwarz inequality we know that $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$, so

$$|\mathbf{Cov}(X,Y)| \le \sqrt{\mathbf{Var}(X)}\sqrt{\mathbf{Var}(Y)}.$$

Therefore

$$|\rho_{XY}| = \frac{|\mathbf{Cov}(X,Y)|}{\sigma_X\sigma_Y} \le 1.$$

(b) Since $\mathbf{Cov}(X,Y) = \mathbf{E}[XY] - \mathbf{E}[X]\mathbf{E}[Y]$, if X and Y are independent then $\mathbf{Cov}(X,Y) = 0$ and $\rho_{XY} = 0$.

(c) Let X be a random variable that is either -1 or 1 with probability 0.5, and Y a random variable that is 0 if $X = -1$ and either -1 or 1 with probability 0.5 if $X = 1$. Then both X and Y have 0 mean, and

$$\mathbf{E}[XY] = (-1)\times 0\times 0.5 + 1\times(-1)\times 0.25 + 1\times 1\times 0.25 = 0.$$

Therefore $\mathbf{Cov}(X,Y) = 0$ and $\rho_{XY} = 0$.

Exercise 9.6

The following Python code gives us the result, and the upper bound is approximately 0.02.

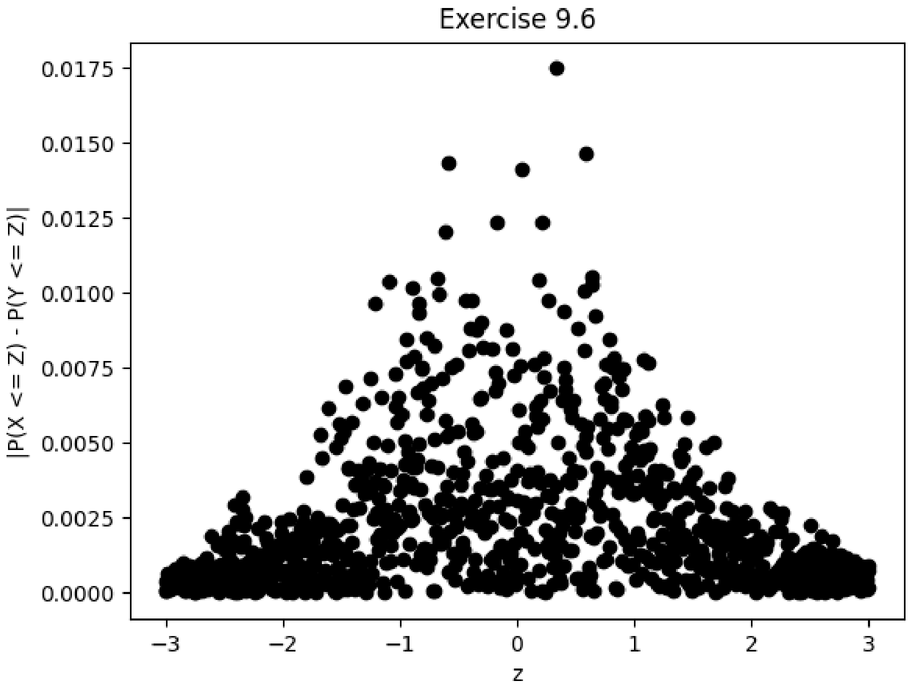
```
from statistics import NormalDist
import numpy as np
import matplotlib.pyplot as plt

N = 10000

plt.title("Exercise 9.6")
plt.xlabel("z")
plt.ylabel("|P(X <= Z) - P(Y <= Z)|")

for z in np.linspace(-3, 3, 1000):
    x_le_z_count = 0
    for _ in range(N):
        x_le_z_count += int((sum(np.random.uniform(0, 1, 12)) - 6) <= z)
    p_x_le_z = x_le_z_count / N
    p_y_le_z = NormalDist(0, 1).cdf(z)
    plt.scatter(z, abs(p_x_le_z - p_y_le_z), color='black')

plt.savefig("plot.png")
```



Exercise 9.14

(a) We can compare the distribution functions as the following.

$$\Pr(Y \le y) = \Pr(XZ \le y) = \frac{1}{2}\Pr(X \le y) + \frac{1}{2}\Pr(X \ge -y) = \frac{1}{2}\Pr(X \le y) + \frac{1}{2}\Pr(X \le y) = \Pr(X \le y)$$

Therefore Y has the same distribution as X.

(b) Since X and Y both follow standard normal distributions, we know in the case of the following,

$$\begin{aligned}\Pr(X \le -2, Y \le -1) &= \Pr(X \le -2, XZ \le -1)\\ &= \frac{1}{2}\Pr(X \le -2, X \le -1) + \frac{1}{2}\Pr(X \le -2, X \ge 1)\\ &= \frac{1}{2}\Pr(X \le -2) \ne \Pr(X \le -2)\Pr(Y \le -1).\end{aligned}$$

the joint distribution functions do not match. Therefore X and Y are dependent.

(c) Let $B \sim Ber\left(\frac{1}{2}\right)$ random variable, then $Y = X(2B-1)$. If X and Y are jointly normal, $X+Y$, which is a linear combination of the two, should also be normally distributed. Since $X + Y = 2BX$,

$$\Pr(X + Y \le k) = \Pr(2BX \le k) = \frac{1}{2}\Pr(2X \le k) + \frac{1}{2}\Pr(0 \le k)$$

so $X + Y$ is a combination of $2X \sim \mathcal{N}(0, 4)$ and a fixed point 0. Therefore $X + Y$ is not normally distributed.

(d) The distribution and density functions of XY are

$$
\begin{aligned}
F_{XY}(k) &= \Pr(XY \le k) = \Pr\big(X^2 Z \le k\big) = \frac{1}{2}\Pr\big(X^2 \le k\big) + \frac{1}{2}\Pr\big(X^2 \ge -k\big) \\
&= \begin{cases} F_X\left(\sqrt{k}\right) & (k \ge 0) \\ 1 - F_X\big(\sqrt{-k}\big) & (k < 0) \end{cases} \\
f_{XY}(k) &= \frac{d}{dk} F_{XY}(k) = \frac{1}{2\sqrt{|k|}} f_X\left(\sqrt{|k|}\right).
\end{aligned}
$$

Therefore the correlation coeffecient is

$$
\begin{aligned}
\rho_{XY} &= \frac{\mathbf{Cov}(X,Y)}{\sigma_X \sigma_Y} = \mathbf{E}[XY] \\
&= \int_{-\infty}^{\infty} k \cdot \frac{1}{2\sqrt{|k|}} f_X\left(\sqrt{|k|}\right) dk = \int_{-\infty}^{0} -\frac{1}{2}\sqrt{-k} f_X\left(\sqrt{-k}\right) dk + \int_{0}^{\infty} \frac{1}{2}\sqrt{k} f_X\left(\sqrt{k}\right) dk \\
&= 0.
\end{aligned}
$$