Engineering Mathematics 2
Seoul National University

Homework 8
2021-16988 Jaewan Park

Exercise 8.1

Since X and Y are uniform random variables, we have the following density functions for $x, y \in [0, 1]$.

$$f_X(x) = 1, \quad f_Y(y) = 1$$

The functions differ by whether $X + Y$ is larger than 1 or not. If $0 \leq X + Y \leq 1$, we obtain

$$\begin{aligned} f_{X+Y}(z) &= \int_0^z f_X(x) f_Y(z-x)dx = \int_0^z 1 \cdot 1dx = z \\ F_{X+Y}(z) &= \int_{-\infty}^z f_{X+Y}(z)dz = \int_0^z zdz = \frac{z^2}{2}. \end{aligned}$$

If $1 < X + Y \leq 2$, we obtain

$$\begin{aligned} f_{X+Y}(z) &= \int_{z-1}^1 f_X(x) f_Y(z-x)dx = \int_{z-1}^1 1 \cdot 1dx = 2 - z \\ F_{X+Y}(z) &= \int_{-\infty}^z f_{X+Y}(z)dz = \int_0^1 zdz + \int_1^z (2-z)dz = 2z - \frac{z^2}{2} - 1. \end{aligned}$$

Exercise 8.4

Let X and Y the minute we each arrive at. Both are uniform random variables over $[0, 60]$, and we should find the probability $\Pr(|X - Y| \leq 15)$. We have the following density functions for $x, y \in [0, 60]$.

$$f_X(x) = \frac{1}{60}, \quad f_Y(y) = \frac{1}{60}$$

The density and distribution functions for $X - Y$ differ by whether $X - Y$ is larger than 0 or not. If $-60 \leq X - Y \leq 0$, we obtain

$$\begin{aligned} f_{X-Y}(z) &= \int_0^{60+z} f_X(x) f_Y(x-z)dx = \frac{60+z}{3600} \\ F_{X-Y}(z) &= \int_{-\infty}^z f_{X-Y}(z)dz = \int_{-60}^z \frac{60+z}{3600}dz = \frac{z}{60} + \frac{z^2}{7200} + \frac{1}{2}. \end{aligned}$$

If $0 \leq X - Y \leq 60$, we obtain

$$\begin{aligned} f_{X-Y}(z) &= \int_z^{60} f_X(x) f_Y(x-z)dx = \frac{60-z}{3600} \\ F_{X-Y}(z) &= \int_{-\infty}^z f_{X-Y}(z)dz = \int_{-60}^0 \frac{60+z}{3600}dz + \int_0^z \frac{60-z}{3600}dz = \frac{z}{60} - \frac{z^2}{7200} + \frac{1}{2}. \end{aligned}$$

Therefore

$$\Pr(|X-Y|\le 15) = \Pr(X-Y\le 15) - \Pr(X-Y\le -15) = F_{X+Y}(15) - F_{X+Y}(-15) = \frac{7}{16}.$$

Exercise 8.9

Let X a uniform random variable in $(0,1)$ and Y an exponentially distributed random variablle with parameter λ. Then the connection between the two can be written as

$$Y = -\frac{\log X}{\lambda}.$$

This can be shown by retrieving the distribution function of Y, which is

$$\begin{aligned} F_Y(y) = \Pr(Y\le y) = \Pr\left(-\frac{\log X}{\lambda}\le y\right) &= \Pr\bigl(X\ge e^{-\lambda y}\bigr) = 1 - F_X\bigl(e^{-\lambda y}\bigr) \\ &= \begin{cases} 0 & (y<0) \\ 1-e^{-\lambda y} & (y\ge 0) \end{cases}. \end{aligned}$$

Exercise 8.19

Let S the set of buses we want to take as soon as they arrive. Also let W_i the wait time of the ith line and W the wait time until the first bus's arrival, then $W_i \sim \exp(1/\mu_i)$ and $W = \min_{i\in S} W_i \sim \exp\bigl(\sum_{i\in S}(1/\mu_i)\bigr)$. Therefore

$$\mathbf{E}[W] = \frac{1}{\sum_{i\in S}(1/\mu_i)}.$$

The probability of W_i being the mininum is $\frac{1/\mu_i}{\sum_{i\in S}(1/\mu_i)}$, so each elements of S will be chosen as the first bus with this probability. Now let T the travel time of the chosen bus, then

$$\mathbf{E}[T] = \sum_{i\in S} t_i \cdot \frac{1/\mu_i}{\sum_{j\in S}(1/\mu_j)} = \frac{\sum_{i\in S}(t_i/\mu_i)}{\sum_{i\in S}(1/\mu_i)}$$

Therefore the total expected time to cross town when we choose a bus from S is as the following.

$$\mathbf{E}[W] + \mathbf{E}[T] = \frac{1}{\sum_{i\in S}(1/\mu_i)} + \frac{\sum_{i\in S}(t_i/\mu_i)}{\sum_{i\in S}(1/\mu_i)} = \frac{1+\sum_{i\in S}(t_i/\mu_i)}{\sum_{i\in S}(1/\mu_i)}$$

Our concern is to choose S that minimizes the above probability. If we name the bus lines $1,2,\cdots,n$ in order of increasing t_i, we can restrict the range of possible tries of S to one of $\{\{1\},\{1,2\},\cdots,\{1,2,\cdots,n\}\}$. If S contains a specific bus line, lines with shorter travel time than that should also be in S.