Engineering Mathematics	Spring 2022

Problem Set 2

Instructor: Yongsoo Song — **Due on:** April 22, 2022

Please submit your answer through eTL. It should be a single PDF file (not jpg), either typed or scanned. **Please include your student ID and name (e.g. 2020-12345 YongsooSong.pdf).** You may discuss with other students the general approach to solve the problems, but the answers should be written in your own words. You should cite any reference that you used, and mention what you used it for.

Problem 1 (20 points)

The trace of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the sum of the diagonal entries, $\operatorname{tr} \mathbf{A} = a_{11} + \cdots + a_{nn}$.

(a) Suppose that $\mathbf{A}$ have (not necessarily distinct) eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that $\operatorname{tr} = \lambda_1 + \cdots + \lambda_n$.

(b) Show that traces of similar matrices are the same.

Problem 2 (20 points)

Shoe the following statements:

(a) Let $\mathbf{A}$ be an $m \times n$ real matrix. Then, $\langle \mathbf{Au}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{A}^T\mathbf{v} \rangle$ for any $\mathbf{u} \in \mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^m$.

(b) Let $\mathbf{A}$ be an $m \times n$ complex matrix. Then, $\langle \mathbf{Au}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{A}^*\mathbf{v} \rangle$ for any $\mathbf{u} \in \mathbb{C}^n$ and $\mathbf{v} \in \mathbb{C}^m$.

Problem 3 (20 points)

Let $\mathbf{A}$ be a square matrix. Show that if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors of $\mathbf{A}$ with distinct eigenvalues, then they are pairwise orthogonal.

Problem 4 (20 points)

Transform the following quadratic forms into the canonical forms, draw their graphs, and express $\mathbf{x} = (x_1, x_2)$ in terms of the new coordinate $\mathbf{y} = (y_1, y_2)$.

(a) $3x_1^2 + 22x_1x_2 + 3x_2^2 = 0$

(a) $-11x_1^2 + 84x_1x_2 + 24x_2^2 = 156$

Problem 5 (25 points)

A symmetric matrix $\mathbf{A}$ with real entries is called positive-definite if the quadratic form $Q = \mathbf{x}^T\mathbf{A}\mathbf{x}$ is positive for all $\mathbf{x} \neq \mathbf{0}$. Show that Q is positive definite if and only if the eigenvalues of $\mathbf{A}$ are all positive.

Problem 6 (70 points)

Solve the following ODEs:

(a) $y' = e^{2x-1} \cdot y^2$

(b) $xy' = x + y$

(c) $xy' = y + 3x^4 \cos^2(y/x)$, $y(1) = 0$.

(d) $(x^2 + y^2)dx - 2xydy = 0$

(e) $ydx + (y + \tan(x+y))\,dy = 0$ (set $F = \cos(x+y)$)

(f) $xy' + 4y = 8x^4$, $y(1) = 2$

(g) $y' + xy = xy^{-1}$, $y(0) = 3$

Problem 7 (25 points)

Solve the initial value problem $y' = x + y$, $y(0) = 0$. Apply the Picard iteration to the ODE and compare the results.