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Growth or decay of cosmological inhomogeneities as a function of their equation of state

G. L. Comer*
Département d'Astrophysique Relativiste et de Cosmologie, Centre National de la Recherche Scientifique, Observatoire de Paris, 92195 Meudon, France

Nathalie Deruelle
Département d'Astrophysique Relativiste et de Cosmologie, Centre National de la Recherche Scientifique, Observatoire de Paris, 92195 Meudon, France
and Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, England

David Langlois
Département d'Astrophysique Relativiste et de Cosmologie, Centre National de la Recherche Scientifique, Observatoire de Paris, 92195 Meudon, France

Joe Parry
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, England
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We expand Einstein's equations in the synchronous gauge in terms of a purely space-dependent, "seed," metric. The (nonlinear) solution accurately describes a universe inhomogeneous at scales larger than the Hubble radius. We show that the inhomogeneities grow or decay, as time increases, depending on the equation of state for the matter (supposed to be a perfect fluid). We then consider the case when matter is a scalar field with an arbitrary potential. Finally we discuss the generality of the model and show that it is an attractor for a class of generic solutions of Einstein's equations.

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I. INTRODUCTION

Although the Friedmann-Robertson-Walker (FRW) models give a satisfactory representation of the geometry of the Universe, it is nevertheless essential to be able to describe inhomogeneities. Indeed, on one hand, we must describe small scales where the Universe is obviously not homogeneous, and on the other hand, we would like to be able to answer the long-standing question of whether this homogeneity on large scales could arise from a "chaotic" geometry.

The linearized theory of perturbations on FRW backgrounds, though a powerful and thoroughly used tool, remains, by definition, limited. In this paper, we follow an alternative approach introduced by Lifschitz and Khalatnikov in 1960 (see [1] and references therein), and build iteratively nonlinear solutions of Einstein's equations which describe inhomogeneous universes.

In a synchronous reference frame (where time is the proper length along the geodesics perpendicular to an initial spacelike hypersurface), the line element is of the form

$$ds^2 = -dt^2 + \gamma_{ij}(x^k,t)dx^i dx^j \qquad (i,j = 1,2,3).$$

At each point x^k one can define a local scale factor a and a local "Hubble" time H^{-1} by

$$a^2 \equiv \gamma^{1/3}\,, \qquad H \equiv \dot{a}/a,$$

where $\gamma \equiv \det(\gamma_{ij})$ and $\dot{a} \equiv \partial a/\partial t$. The Hubble time is thus the characteristic proper time on which the metric evolves at point x^k: $\dot{\gamma}_{ij} \sim H\gamma_{ij}$. The characteristic comoving length on which the metric varies is denoted L: $\partial_i\gamma_{jk} \sim L^{-1}\gamma_{jk}$. On scales much less than L the three-metric depends on time only and space is almost homogeneous and flat, albeit not necessarily isotropic.

Suppose now that the geometry of the Universe is, at some time, such that there exists a synchronous reference frame in which all spatial gradients are small compared to time derivatives, more precisely, where we have

$$\frac{1}{a}\partial_i\gamma_{jk} \ll \dot{\gamma}_{jk} \qquad \Longleftrightarrow \qquad aL \gg H^{-1}. \tag{1.1}$$

Such an assumption, which means that the characteristic scale of spatial variation is much bigger than the Hubble radius, is the so-called "long wavelength approximation" (see [2] and references therein). This approximation is also "anti-Newtonian" [3] in the sense that the characteristic speed of the problem, $v \equiv aL/H^{-1}$, is much bigger

*Permanent address: Department of Science and Mathematics, Parks College of Saint Louis University, Cahokia, IL 62206.

than the light velocity.

It is important to note that the above assumption, for a given L, cannot always be valid since the comoving Hubble radius, that is $(Ha)^{-1}$, varies in general with time. In the case of standard cosmology ($\ddot{a} < 0$), it increases with time and the approximation is valid for any scale, sufficiently near the initial singularity. This is why approximation (1.1) is at the root of the Belinski-Lifschitz-Khalatnikov analysis of singularities (see, e.g., [4]). On the other hand, when matter violates the strong energy condition ($\ddot{a} > 0$), this approximation becomes valid for any scale at sufficiently late times.

Neglecting all spatial gradients in the Einstein equations is therefore a way to study inhomogeneities much larger than the Hubble radius. But it can also be used as a starting point to solve Einstein's equations iteratively. The first order then corresponds to the long wavelength approximation. The following order, where the gradient terms are built out of the long wavelength solution, can be seen either as a finer approximation to the exact solution if the metric satisfies condition (1.1), or as belonging to a series which can be hoped to converge towards the exact solution. What we have in mind here is an expansion similar to that of $\exp x = 1 + x + x^2/2! + \cdots$, where including the extra orders is a refinement if $x \ll 1$ [the equivalent of condition (1.1)], but is a necessity if $x > 1$, in which case a good approximation can be obtained provided one goes to a sufficiently high order. If our expansion scheme behaves similarly, it will then also be suited to describe inhomogeneities smaller than the Hubble radius. We shall, however, leave to a future work a study of its convergence in that case.

More precisely, the iteration scheme consists in writing the three-metric as a sum of spatial tensors, built out of a first order, quasi-isotropic, "seed" metric, the number of gradients increasing with the order. The coefficients, depending on time, are determined, order by order, by Einstein's equations. The solution so obtained does not possess enough arbitrary functions to be generic, because the first order metric has been chosen to be quasi-isotropic. But the most general first order solution, studied by Tomita [3], which contains the right number of arbitrary functions, tends towards our special solution, as we shall show.

In this paper we shall use this iteration scheme to extend the results of linear perturbation theory (see, e.g., [5] and references therein) and show that, when matter is a perfect fluid which violates the strong energy condition, nonlinear inhomogeneities larger than the Hubble radius "freeze out" as the Hubble radius shrinks to a point. Inside it the geometry becomes homogeneous, isotropic, and flat. Inhomogeneities smaller than the Hubble radius (assuming that the iteration scheme is suited to describe them as well) behave similarly: they decay and freeze out as they leave the Hubble radius. In the case of standard matter, on the other hand, the inhomogeneities thaw out when entering the Hubble radius and then grow with time. We also examine the case when matter is a scalar field with an arbitrary potential and we generalize the analysis of Muller, Schmidt, and Starobinski for power-law inflation [6]. We then compare and contrast this iteration scheme to the long wavelength approximation scheme of Croudace, Parry, Salopek and Stewart [7]. In the cases they consider, that is, dust and a scalar field, both approaches yield the same results.

II. THE EXPANSION SCHEME FOR A PERFECT FLUID

We place ourselves in a synchronous reference frame, where the metric takes the form

$$ds^2 = -dt^2 + \gamma_{ij}(t, x^k)dx^i dx^j. \tag{2.1}$$

(Throughout this paper latin letters will denote spatial indices and greek letters spacetime indices.) The matter is taken to be a perfect fluid with pressure p and energy density ρ, moving along a congruence of timelike curves represented by a unit vector field u^μ, and characterized by the stress-energy tensor

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu}, \tag{2.2}$$

with the further restriction that $p/\rho = \Gamma - 1 = \text{const.}$ The index Γ is supposed to be less than 2, so that the sound speed is less than the light velocity, but can be less than 1, thus allowing for an "inflationary" perfect fluid. The Einstein equations in a synchronous reference frame (see, e.g., Ref. [8]) can be written as

$$ {}^{(3)}R_i^j + \frac{1}{2\sqrt{\gamma}} \frac{\partial}{\partial t} \left(\sqrt{\gamma} K_i^j\right)$$

$$= \frac{2\dot{K} + K_k^l K_l^k}{4\,(2 - 3\Gamma - 2\Gamma u^k u_k)} \left[2\Gamma u^j u_i + \delta_j^i (2 - \Gamma)\right], \tag{2.3}$$

$$\kappa\rho = \frac{2\dot{K} + K_k^l K_l^k}{2\,(2 - 3\Gamma - 2\Gamma u^k u_k)}, \tag{2.4}$$

$$\kappa\rho\Gamma u_i = -\frac{1}{2\sqrt{1 + u^k u_k}} \left(K_{i;j}^j - K_{,i}\right), \tag{2.5}$$

where $K_{ij} \equiv \dot{\gamma}_{ij}$ is the extrinsic curvature (a dot denotes the derivative with respect to time, a semicolon the covariant derivative with respect to γ_{ij}); γ is the determinant of the three-metric γ_{ij}, all the indices are raised with the inverse metric γ^{ij}, $K \equiv K_i^i$, and $\kappa \equiv 8\pi G$.

If the terms quadratic in the three-velocity can be neglected, the above equations decouple, the first involving only the three-metric, the other two giving respectively the energy density and the three-velocity as functions of the spatial metric.

We now consider these equations order by order in the gradient expansion. At first order, we keep only terms linear in the spatial gradients. We therefore ignore the Ricci tensor ${}^{(3)}R_i^j$, which is quadratic in $\partial_i \gamma_{jk}$, as well as the terms involving $u_i u^j$ since (2.5) tells us that the three-velocity is at least first order in the spatial gradients. The traceless part of (2.3) then becomes

$$\frac{1}{2\sqrt{\gamma}}\frac{\partial}{\partial t}\left[\sqrt{\gamma}\left(K_i^j-\frac{1}{3}K\delta_i^j\right)\right]=0. \tag{2.6}$$

This equation is readily integrated into

$$K_i^j-\frac{1}{3}K\delta_i^j=\frac{1}{\sqrt{\gamma}}S_i^j, \tag{2.7}$$

where S_i^j is a time-independent traceless matrix, which, however, can be spatially dependent. Furthermore, defining $A=\gamma^{1/3}$ so that $K=3\dot{A}/A$, (2.7) can be rewritten as

$$K_i^j=\frac{\dot{A}}{A}\delta_i^j+A^{-3/2}S_i^j. \tag{2.8}$$

The function A (the square of the "scale factor" a) satisfies

$$4\frac{\partial}{\partial t}\left(\frac{\dot{A}}{A}\right)+3\Gamma\left(\frac{\dot{A}}{A}\right)^2+\frac{2-\Gamma}{2}|S|^2A^{-3}=0, \tag{2.9}$$

where $|S|^2\equiv S_i^jS_j^i$, which is obtained by taking the trace of (2.3). All the equations determining the first order metric are then exhausted and have been treated in all generality.

Now, as will be shown in Sec. IV, the first term on the right-hand side of (2.8) always becomes dominant at sufficiently late times over the second term, which represents a local anisotropy. From now on in this section, we therefore restrict ourselves to the computationally simpler case $S_i^j=0$. Then Eq. (2.8) immediately yields

$$\gamma_{ij}=Ah_{ij}, \tag{2.10}$$

where h_{ij} is a time-independent symmetric matrix, while A, solution of the truncated differential equation (2.9), is of the form

$$A(t)=\tilde{A}(t-t_0)^{4/3\Gamma}. \tag{2.11}$$

At sufficiently late time, the *a priori* space-dependent function $t_0(x)$ can be neglected (assuming it is bounded), and, from now on in this section, the solution at lowest order for the metric is chosen to be

$${}^{(1)}\gamma_{ij}=A_0(t)h_{ij}\quad\text{with}\quad A_0(t)=t^{4/3\Gamma}. \tag{2.12}$$

($\tilde{A}$ has been absorbed in the matrix h_{ij}.) At each point the time evolution of the approximate three-metric is thus that of the corresponding flat FRW metric. At the same approximation, the expressions for the energy density and the three-velocity follow from (2.4) and (2.5):

$${}^{(1)}\rho=\frac{4}{3\Gamma^2}\frac{1}{t^2}\quad\text{and}\quad{}^{(1)}u_i=0. \tag{2.13}$$

The next, third, order is obtained by taking into account the terms neglected previously, in particular the Ricci tensor of the three-metric. It is consistent with the approximation to take the Ricci tensor of ${}^{(1)}\gamma_{ij}$ in place of ${}^{(3)}R_i^j$, and convenient to write it in terms of the Ricci tensor, denoted R_{ij}, of h_{ij}, seen as the components of a metric, to be called "seed" metric (concretely, this is the first order spatial metric when $t=1$). We will thus look for corrections to the metric of the form

$${}^{(3)}\gamma_{ij}=A_0(t)\left\{f_2(t)R_{ij}+\frac{1}{3}\left[g_2(t)-f_2(t)\right]Rh_{ij}\right\}, \tag{2.14}$$

where $R_i^j\equiv h^{ik}R_{kj}$ and $R\equiv R_i^i$. [The form (2.14) is not the most general correction which is allowed by the Einstein equations, but we postpone this discussion to Sec. IV.] Substituting $\gamma_{ij}={}^{(1)}\gamma_{ij}+{}^{(3)}\gamma_{ij}$ in the Einstein equation (2.3) and keeping the terms with two gradients, we find that f_2 and g_2 must satisfy the equations

$$\ddot{f}_2+\frac{3}{2}\frac{\dot{A}_0}{A_0}\dot{f}_2=-\frac{2}{A_0},\qquad \ddot{g}_2+\frac{3}{2}\Gamma\frac{\dot{A}_0}{A_0}\dot{g}_2=\frac{2-3\Gamma}{2A_0}, \tag{2.15}$$

the solutions of which can be written in integral form:

$$\begin{aligned}f_2^{(g)}&=-2\int dtA_0^{-3/2}\int dt'A_0^{1/2},\\ g_2^{(g)}&=\frac{2-3\Gamma}{2}\int dtA_0^{-3\Gamma/2}\int dt'A_0^{\frac{3\Gamma}{2}-1},\end{aligned} \tag{2.16}$$

where the index (g) indicates that the general solution of the homogeneous equation corresponding to (2.15) is included. In the case of a perfect fluid, where $A_0=t^{4/3\Gamma}$, they can be written explicitly as $f_2^{(g)}=f_2^{(h)}+f_2$, $g_2^{(g)}=g_2^{(h)}+g_2$ with

$$f_2=-\frac{9\Gamma^2}{9\Gamma^2-4}t^{2-\frac{4}{3\Gamma}},\qquad g_2=-\frac{1}{4}\frac{9\Gamma^2}{9\Gamma-4}t^{2-\frac{4}{3\Gamma}}, \tag{2.17}$$

and $f_2^{(h)}=\beta_2+\beta_2't^{1-2/\Gamma}$, $g_2^{(h)}=\gamma_2+\gamma_2't^{-1}$, where β_2, β_2', γ_2, and γ_2' are arbitrary constants. Now, as will be shown in Sec. IV and in the Appendix, these homogeneous solutions can be ignored at late times so that we take, as our third order metric, (2.14) with f_2 and g_2 given by (2.17).

At the following order (fifth), the calculations, although straightforward in principle, become more complicated in practice, because the Ricci tensor of the perturbed metric has to be computed. The explicit calculations are given in the Appendix. We plan to use them in future work to study the convergence properties of the series.

For a qualitative analysis of the evolution of inhomogeneities it is, however, enough to notice that the solution to any order n is of the form

$$\gamma_{ij}=t^{\alpha}\left[h_{ij}+\sum_{p=1}^{n}t^{p\beta}C_{ij}^{(p)}\right], \tag{2.18}$$

where $\alpha=4/3\Gamma$, $\beta=2-4/3\Gamma$, and $C_{ij}^{(p)}$ are spatial tensors, which are of order $2p$ in the spatial gradients, that is, $O(L^{-2p})$. This can be shown easily by recurrence. Indeed, consider the equations at the order $(n+1)$, assuming that all terms up to the order n have been determined. Then K_i^j is of the form

$$K_i^j = t^{-1}\left[\alpha\delta_i^j + \sum_{p=1}^{n+1} t^{p\beta} C_i'^{(p)j}\right].$$

After mutiplication by t^2, we can review the powers of the miscellaneous source terms in the "master" equation (2.3): the terms coming from the previous orders are of the form $\sum_{p=0}^{n+1} t^{p\beta} C_i'^{(p)j}$; the terms coming from the curvature parts are of the form $t^{2-\alpha}\sum_{q=0}^{n} t^{p\beta} C_i'^{(p)j}$; as for the terms coming from the stress-energy tensor, they are of one or the other type. These remarks thus show that the three-metric is of the form (2.18).

We are now in a position to address the main question: in which cases do the corrective terms grow or decay? First we note that the $(n+1)$th terms are *a priori* small compared with the nth ones if $t^\beta L^{-2} \langle\langle 1 \Leftrightarrow L\rangle\rangle (aH)^{-1}$, that is, if the inhomogeneity is much bigger than the Hubble radius. We then look at the time powers in the expansion of the three-metric (2.18). As can easily be seen the corrective terms die off with time if $\beta < 0$, that is

$$\Gamma < 2/3. \tag{2.19}$$

This condition corresponds to a negative effective gravitational mass (given by $\rho + 3p$). The perfect fluid is therefore of an "inflationary" kind and violates the strong energy condition. The behavior of the metric perturbations is given by

$$\frac{\gamma_{ij}(x)-\gamma_{ij}(x_0)}{\gamma_{ij}(x_0)} \approx \frac{h_{ij}(x)-h_{ij}(x_0)}{h_{ij}(x_0)} + \sum_{p=1}^{n} t^{p\beta}\frac{C_{ij}^{(p)}(x)-C_{ij}^{(p)}(x_0)}{h_{ij}(x_0)}, \tag{2.20}$$

where x_0 is a fiducial point. The first term is constant in time while the correcting terms are decreasing. Furthermore, the comoving scale of the Hubble radius shrinks as $t^{\beta/2}$ and therefore space becomes completely homogeneous, isotropic, and flat within the Hubble radius, the three-metric being $\gamma_{ij} = t^\alpha h_{ij}(x_0)$. To complete our argument, we also have to consider the evolution of the three-velocity. Its form

$$u_i \sim t^{2-\alpha} t^{-1} \sum_{q=0}^{n} t^{q\beta} U_i^{(p)},$$

where $U_i^{(p)}$ is a space-dependent vector, shows immediately that it tends to zero with time when the condition (2.19) on Γ is satisfied. We have thus extended the well-known conclusion of the linear theory that the metric perturbations freeze out when outside the Hubble radius to the case of nonlinear inhomogeneities in an "inflationary" perfect fluid.

If now the inhomogeneities we consider are not large compared with the Hubble radius the corrective terms are not small *a priori* but the series, if enough terms are included, may still describe accurately their evolution towards freeze out.

When matter satisfies the strong energy condition, that is, $\Gamma > 2/3$, the conclusions are the opposite, since, if the first term in (2.20) is still constant, the curvature terms all increase with time, therefore meaning that the metric perturbations thaw out as they enter the Hubble radius. This result is consistent with linear theory conclusions, which it generalizes to more strongly inhomogeneous spacetimes.

III. EXPANSION SCHEME FOR A SCALAR FIELD

With the boost of inflationary models, scalar fields have become a standard type of matter in the context of cosmology. In this section, we wish to show that the expansion scheme illustrated above with a perfect fluid as matter, applies as well to the case of a scalar field.

A scalar field ϕ is characterized by the stress-energy tensor

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi + V(\phi)\right), \tag{3.1}$$

where $V(\phi)$ is its potential. The Einstein equations, in a synchronous reference frame, read

$$\frac{1}{2}\frac{\partial K}{\partial t} + \frac{1}{4}K_i^j K_j^i = \kappa\left[\dot\phi^2 + V(\phi)\right], \tag{3.2}$$

$${}^{(3)}R_i^j + \frac{1}{2\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}K_i^j\right) = \kappa\left(\partial_i\phi\partial^j\phi + V\delta_i^j\right), \tag{3.3}$$

and

$$-\frac{1}{2}\left(K_{i;j}^j - K_{,i}\right) = \kappa\dot\phi\partial_i\phi. \tag{3.4}$$

At first order, assuming once more that the three-metric is approximated by $\gamma_{ij} = A(x,t)h_{ij}(x)$, i.e., that the anisotropy matrix S in (2.8) is zero, the local scale factor and scalar field behave as in the corresponding flat FRW spacetime, since the two first Einstein equations (3.2) and (3.3) reduce to, with the identification $H \equiv \frac{1}{2}\frac{\dot A}{A}$,

$$3H^2 = \frac{2}{\kappa}\left(\frac{\partial H}{\partial\phi}\right)^2 + \kappa V(\phi) \tag{3.5}$$

and

$$\frac{\partial H}{\partial\phi} = -\frac{1}{2}\kappa\dot\phi \tag{3.6}$$

(which imply the Klein-Gordon equation $\ddot\phi + 3H\dot\phi + dV/d\phi = 0$), while the last Einstein equation (3.4) gives

$$\frac{1}{2}\partial_i H = -\kappa\dot\phi\partial_i\phi. \tag{3.7}$$

The first order differential equation (3.5) gives $H(\phi,\tilde\phi)$, where $\tilde\phi$ is an integration "constant," *a priori* depending on space, and hence

$$A = \exp\left(-\kappa \int d\phi \frac{H}{\partial H/\partial\phi}\right). \tag{3.8}$$

The second equation, (3.6), gives ϕ as a function of $[t - t_0(x)]$, where $t_0(x)$ is another integration "constant." Now the last Einstein equation, (3.7)—the "momentum constraint," tells us that $\tilde{\phi}$ cannot depend on space. As for $t_0(x)$, we shall restrict it to be constant, as in the previous section. This implies that A and ϕ depend only on time and we shall henceforth denote them $A_0(t)$ and $\phi_0(t)$.

An explicit solution cannot be found for a general potential, but if it is exponential,

$$V(\phi) = \tilde{V} \exp\left(-\sqrt{\frac{2\kappa}{p}}\phi\right),$$

we have $A_0 = t^{2p}$, $H = p/t$, and $\phi_0 = \sqrt{\frac{2p}{\kappa}} \ln\left(\frac{\kappa \tilde{V}}{p(3p-1)} t\right)$. More generically, in an inflationary regime where $\ddot{\phi} \simeq 0$, we have $A_0 \simeq \exp(2Ht)$ and $\dot{\phi}_0$ constant and small.

At third order the spatial metric and the scalar field are expanded as

$$\gamma_{ij} = A_0(t)h_{ij} + A_2(t)Rh_{ij} + B_2(t)R_{ij}, \tag{3.9}$$

$$\phi = \phi_0(t) + \phi_2(t)\tilde{R}. \tag{3.10}$$

The Einstein equations (3.3) and (3.4) then yield (with $a_2 \equiv A_2/A_0$, $b_2 \equiv B_2/A_0$, and $V' \equiv dV/d\phi$)

$$\ddot{b}_2 + 3H\dot{b}_2 + 2/A_0 = 0, \tag{3.11}$$

$$\ddot{a}_2 + 6H\dot{a}_2 + H\dot{b}_2 - 2\kappa V_0'\phi_2 = 0, \tag{3.12}$$

and

$$\frac{1}{4}\dot{b}_2 + \dot{a}_2 = -\kappa\dot{\phi}_0\phi_2, \tag{3.13}$$

of which we choose the particular solutions

$$b_2 = -2\int dt\, A_0^{-3/2} \int dt'\, A_0^{1/2}\,,$$
$$a_2 = \frac{1}{2}\int dt\, \dot{\phi}_0^2 \int dt' \frac{\left(\dot{\phi}_0 A_0^{\frac{1}{2}}\right)^{\cdot}}{\dot{\phi}_0^3 A_0^{\frac{1}{2}}} \dot{b}_2, \tag{3.14}$$

ϕ_2 then being given by (3.13). The last Einstein equation, (3.2), must then be identically satisfied, which is indeed the case.

In the particular case of an exponential potential, the expressions (3.15) and (3.13) become

$$b_2 = \frac{t^{2-2p}}{p^2-1}, \qquad a_2 = \frac{t^{2-2p}}{2(p+1)(3-2p)},$$
$$\phi_2 = \frac{1}{2\sqrt{2p\kappa}} \frac{t^{2-2p}}{(p+1)(3-2p)}. \tag{3.15}$$

This third order solution agrees with the results of Muller *et al.* [6]. Furthermore, if we invert $\phi = \phi_0(t) + \phi_2(t)R$ to obtain $t = t_0(\phi) + t_2(\phi)R$, the metric becomes

$$\gamma_{ij} = A(\phi)h_{ij} + \frac{p^2}{p^2-1}H(\phi)^{-2}R_{ij} \tag{3.16}$$

and coincides with the expression given by Salopek and Stewart in Ref. [2].

The fifth order solution for a general potential is given in the Appendix. In the case of an exponential potential the time dependence of ${}^{(5)}\gamma_{ij}$ is $A_0(t)t^{4-4p}$.

When the potential is exponential the metric at any order is then of the form (2.18) with $\alpha = 2p$ and $\beta = 2-2p$. The conclusion drawn in the previous section therefore still holds, and coincides with that of Muller, Schmidt, and Starobinski [6]: inhomogeneities freeze out when outside the Hubble radius if $\beta < 0 \iff p > 1$, that is, for power-law inflation. The conclusion also holds in the more generic case when the condition for slow roll is satisfied since a_2 and b_2 then decrease quasiexponentially. A more detailed analysis of the evolution of inhomogeneities when the condition for slow roll is not satisfied everywhere and when they are not necessarily large compared to the Hubble radius, as well as a comparison with the numerical results of Goldwirth and Piran [9], is left for future work.

IV. GENERICNESS OF THE SOLUTION

Because the question we address is intended to be general, without any particular assumptions about the spacetime, we need to investigate the genericness of the inhomogeneous solution that was built in the previous sections. A property of this construction is that it is completely determined by the choice of the six "seed" metric components h_{ij}. Now the synchronous gauge does not completely fix the system of coordinates: four transformations of coordinates parametrized by space-dependent functions are still allowed, three pure spatial plus one mixing time and space. The latter is not allowed because it would destroy the pure time dependence of A_0 in (2.12) or (3.8). Therefore three physically independent functions of space parametrize our set of solutions. This is not enough. Indeed the maximal number of independent functions is 8 for a spacetime with perfect fluid as matter: the purely gravitational part, corresponding to the two tensorial degrees of freedom, is described by four functions, the matter density by one and the three-velocity by three (see Ref. [8]). In the case of the scalar field, the number of independent functions is 6, because the functions describing the matter are ϕ and $\dot{\phi}$. We shall for definiteness restrict our analysis to a perfect fluid. A pertinent question is thus to identify the five functions that were somehow fixed in the expansion sheme, to analyze their physical meaning, and to check if their presence does not modify the conclusions we reached concerning the evolution of the inhomogeneities.

The restrictions imposed on the solution were pointed out along the derivation in Sec. II: we assumed the vanishing of the traceless matrix S_i^j in (2.9), and later neglected $t_0(x)$ in (2.11). The first order solution when these assumptions are not made was studied by Tomita in the case of dust [3]. We here extend his analysis to more general perfect fluids and simplify some steps. The traceless part of the general equation determining the metric, Eq. (2.8), can be rewritten as

$$\frac{\partial}{\partial\tau}\left(\frac{\gamma_{ij}}{A}\right) = S_i^k\left(\frac{\gamma_{kj}}{A}\right), \tag{4.1}$$

if a new time τ is introduced such that $\frac{d\tau}{dt} = A^{-3/2}$. To this equation must be added the condition

$$S_i^k\gamma_{kj} = S_j^k\gamma_{ki}, \tag{4.2}$$

which expresses the symmetry of K_{ij}. Assuming that the matrix S is diagonalizable (this does not spoil its generality), it can be written

$$S = MDM^{-1}, \tag{4.3}$$

where $D = \mathrm{Diag}(r_1, r_2, r_3)$. As a consequence of S being traceless the eigenvalues r_a satisfy the condition $\sum r_a = 0$. The column vectors of the matrix M, denoted e^a with components e_i^a, are eigenvectors of S. If we call G the matrix with components γ_{ij}/A and denote $\tilde{G} = M^{-1}G(M^{-1})^T$ (T denotes the transpose matrix), it is easy to check that

$$\left(D\tilde{G}\right)^T = D\tilde{G}, \tag{4.4}$$

by using the matricial version of the condition (4.2): $(SG)^T = SG$. This implies, when the eigenvalues are distinct, that $\tilde{G}$ is diagonal. Equation (4.1) is then equivalent to

$$\frac{\partial}{\partial\tau}\tilde{G} = D\tilde{G}, \tag{4.5}$$

which can thus be decomposed into three independent differential equations. The solution for γ_{ij} finally is

$$\gamma_{ij} = e_i^a\gamma_{ab}e_j^b\,,$$
$$\gamma_{ab} = A\,\mathrm{Diag}[\lambda_1e^{r_1\tau}, \lambda_2e^{r_2\tau}, \lambda_3e^{r_3\tau}], \tag{4.6}$$

with $\lambda_1\lambda_2\lambda_3 = 1$, since $\gamma = \det(\gamma_{ab}) = A^3$. As for A, it is determined by the trace part of the equation satisfied by the three-metric, that is, Eq. (2.9). This equation admits a first integral,

$$\frac{\dot{A}}{A} = \frac{4}{3\Gamma}A^{-3/2}\sqrt{c_0A^{\frac{3}{2}(2-\Gamma)} + \frac{\sigma\Gamma}{2-\Gamma}}, \tag{4.7}$$

where $c_0(x)$ is an integration constant, and $\sigma \equiv 3\Gamma(2-\Gamma)|S|^2/32$. Equation (4.7) can be solved explicitly as a function of the time τ for all S_i^j:

$$A = \left[\sqrt{c_0}\sqrt{\frac{2-\Gamma}{\sigma\Gamma}}\sinh\left(\sqrt{\frac{\sigma(2-\Gamma)}{c_0\Gamma}}(\tau_0-\tau)\right)\right]^{\frac{4}{3(\Gamma-2)}}. \tag{4.8}$$

From the relation (2.4) giving the energy density as a function of the space metric and using the first integral (4.7), we find

$$\rho = \frac{4c_0}{3\Gamma^2}A^{-\frac{3}{2}\Gamma}. \tag{4.9}$$

Hence c_0 must be positive. (It is an easy side exercise to show that in vacuum $[c_0 = 0,\ A = (3|S|^2/8)^{1/3}(|t-t_0(x)|)^{2/3}]$ the solution (4.6), when imposed to be spherically symmetric and exact, reduces to the Schwarzschild metric in the Lemaitre coordinates.)

Similarly, the three-velocity can be obtained from (2.5):

$$A^{-\frac{3}{2}(\Gamma-1)}u_i = A^{3/2}\left(A^{-3/2}\sqrt{A^{\frac{3}{2}(2-\Gamma)} + \frac{\sigma\Gamma}{2-\Gamma}}\right)_{,i} - \frac{3\Gamma}{8}\partial_jS_i^j + \frac{3\Gamma}{16}\gamma^{kl}\partial_i\gamma_{lj}S_k^j. \tag{4.10}$$

Let us now count the independent free spatial functions characterizing this more general first order solution (4.6)–(4.10): the traceless matrix S_i^j has eight independent components; integration of A yields two functions (c_0, τ_0), while integration of (4.5) yields two independent functions [the λ_a in (4.6)]. The total is twelve, and this reduces to eight when the four transformations of coordinates are taken into account. This is the maximal number of parameters and this implies that the first order solution (4.6)–(4.10) is generic in the sense that it lies in a space with the same number of dimensions as superspace.

A natural continuation would be to go to the next order of the approximation scheme with the whole first order solution. This tactic is unfortunately rather heavy because it is preferable to work in a triad basis instead of the initial coordinate system. We refer the reader to Tomita [3] for this approach. Here we will adopt a more modest attitude, motivated by the fact that we are mainly interested by late time behavior.

Late time corresponds to the limit $\tau \to \tau_0$ for which $A \to \infty$. The asymptotic behavior of the metric (4.6) is then

$$\gamma_{ab} \to (t-t_0)^{\frac{4}{3\Gamma}}\mathrm{Diag}[\lambda_c] \quad\Longleftrightarrow\quad \gamma_{ij} \to (t-t_0)^{\frac{4}{3\Gamma}}h_{ij}(x) \tag{4.11}$$

with h_{ij} an arbitrary symmetric matrix. We see that the influence of the "anisotropic" matrix decreases with time (this is well known in the context of homogeneous cosmology, see, e.g., [10]) and we recover, in the late time limit where t_0 can also be neglected, the quasi-isotropic first order solution (2.12) considered in Sec. II.

To proceed, we do not neglect the anisotropy completely because we would lose the corresponding degrees of freedom, but consider it as a small perturbation. The solution $\gamma_{ij} = Ah_{ij}$ with $A = (t-t_0)^{4/3\Gamma}$ is thus completed by the pertubation $\delta\gamma_{ij}$, solution of

$$\delta\dot{\gamma}_{ij} = \frac{\dot{A}}{A}\delta\gamma_{ij} + A^{-1/2}S_i^kh_{kj}, \tag{4.12}$$

which is the first order correction to Eq (4.1). If, furthermore, we also consider that t_0 is a small quantity, we finally obtain

$$\gamma_{ij} = t^{4/3\Gamma}\left(h_{ij} + \frac{4}{3\Gamma}t_0 t^{-1}h_{ij} + t^{1-2/\Gamma}\tilde{S}_{ij}\right), \quad (4.13)$$

where $\tilde{S}_{ij}$ is a symmetric traceless tensor (with respect to h_{ij}).

Putting the last two terms of (4.13) on the same footing as the third order term (2.14), we end up with two more types of perturbations in addition to the curvature term [(2.14) and (2.17)]. In fact, these extra perturbative terms are of the same type as those which arise when solving the homogeneous equation corresponding to (2.15) and can be classified in three categories: terms of the form $t^{1-2/\Gamma}\left(l_i^j - \frac{1}{3}l\delta_i^j\right)$ (where l_{ij} is an arbitrary tensor); terms of the form $t^{-1}l\delta_i^j$; and terms of the form const $\times\, l_i^j$.

The terms of the third type can be reabsorbed in the "seed" metric h_{ij}. They correspond to a mere renormalization of the zeroth order approximation, and we can ignore them, assuming that we take the "renormalized" "seed" metric from the beginning. The two first types of terms (considered in [6]) can either be eliminated by a suitable change of coordinates (see the Appendix), or decrease with time if $\Gamma < 2$, which is the case for all fluids with a sound velocity less than the speed of light.

We have hence shown that the special solutions with three independent free functions built in Sec. II are indeed attractors of a generic class of solutions, so that our conclusions about the growth or decay of inhomogeneities as a function of their equation of state are fairly general.

V. COMPARISON WITH THE HAMILTON-JACOBI METHOD

The Hamilton-Jacobi equation for general relativity is obtained using the Arnowitt-Deser-Misner (ADM) formalism in which the spacetime is foliated by a family of everywhere spacelike hypersurfaces. The metric is written in terms of a three-metric γ_{ij} and the lapse and shift functions N, N_i. The lapse and shift determine the foliation and are arbitrary; they are chosen towards the end of the calculation. The action is rewritten in the Hamiltonian formalism using these variables. Variation of the action with respect to the lapse and shift produces two constraints—the Hamiltonian and momentum constraints. Variation of the action with respect to the field variables and their conjugate momenta gives two sets of "evolution equations." New canonical variables can be introduced by means of a generating functional $\mathcal{S}$. The conjugate momenta are then expressed in terms of functional derivatives of $\mathcal{S}$. The Hamiltonian and momentum constraints then become functional differential equations for $\mathcal{S}$. The next step is to solve half the evolution equations using the solution for the generating functional. The other half are solved automatically by our solution. This method gives a particular solution to the problem, and not the most general solution.

The momentum constraint is solved by choosing $\mathcal{S}$ to be invariant under infinitesimal coordinate transformations. The Hamiltonian constraint, which is now called the Hamilton-Jacobi equation, is harder to solve and an approximation method is used. The generating functional is expanded in a series of terms increasing in the number of spatial gradients they contain. For example, ${}^{(3)}R$, the Ricci scalar of the three-metric has two spatial gradients. This expansion is in essence the same as the "direct approach" presented above. The Hamilton-Jacobi equation is then solved at each order. Using diffeomorphism invariance of the generating functional a form for the generating functional can be guessed at each order. Substituting the ansatz at each order into the Hamilton-Jacobi equation gives partial differential equations which may easily be solved. For example, at the first order for gravity coupled to a scalar field ϕ a simple ansatz is

$$\mathcal{S} = -\frac{2}{\kappa}\int d^3x\gamma^{1/2}H(\phi)\ ,$$

and substitution of this into the Hamilton-Jacobi equation gives (3.5). Salopek and Stewart [12] give a simple ansatz for a perfect fluid. More generally, choosing $\mathcal{S}$ to be of this type, with H dependent on some scalar fields, is equivalent to ignoring the anisotropy S_j^i, that is, to choosing particular long wavelength solutions such as (3.5) or (2.12) (see Ref. [11] for a more general ansatz).

A foliation must be chosen before solving the evolution equations. For gravity interacting with a pressure-free collisionless perfect fluid (dust) with velocity potential χ, a foliation in which χ is constant on each hypersurface is chosen. χ can then be chosen as the time variable, and if the shift N_i is set to zero, this causes the lapse N to be 1, i.e., the synchronous gauge. The metric evolution equation for dust at fourth order, Eqs. (A6) in the Appendix with $\Gamma = 1$, is in perfect agreement with the result calculated using the Hamilton-Jacobi approach as shown in Croudace *et al.* [7], Eq. (3.25).

Hamilton-Jacobi methods can also give the general solution at the first order as outlined in Sec. IV in this paper by choosing a more complicated ansatz and solving the momentum constraint explicitly. In particular, Salopek and Stewart [12] have used this method to find the general solution for dust. One advantage of the Hamilton-Jacobi formalism is that one can choose the foliation in the course of the calculation. This simplifies the algebra, for example, in the case of a single scalar field including two spatial gradients, Eq. (3.16). Another advantage is that the generating functional $\mathcal{S}$ may be instantly written down at each order using a recursion relation derived by Parry, Salopek, and Stewart [13]. They give the recursion relation in two special cases: one dust field interacting with gravity, one scalar field interacting with gravity. In both cases they ignore the decaying modes. The remaining calculation is the substitution into the metric evolution equation, and its integration, which is straightforward.

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APPENDIX: THE SOLUTION TO THE FIFTH ORDER IN THE GRADIENTS

As explained in the text, we look for solutions of Einstein's equations whose three-metric (in the t = const slices of a synchronous reference frame) can be expanded as a sum of spatial tensors of increasing order in the gradients of a "seed," with coefficients whose time dependence is determined by Einstein equations. The line element is thus of the form

$$\begin{aligned} ds^2 &= -dt^2 + \gamma_{ij}(t,x^k)dx^i dx^j \; ; \quad \gamma_{ij} = {}^{(1)}\gamma_{ij} + \delta\gamma_{ij} \, , \\ \delta\gamma_{ij} &= {}^{(3)}\gamma_{ij} + {}^{(5)}\gamma_{ij} + \cdots , \\ {}^{(1)}\gamma_{ij} &= A_0(t)h_{ij}(x^k) \, , \\ {}^{(3)}\gamma_{ij} &= A_2(t)Rh_{ij} + B_2(t)R_{ij} \, , \\ {}^{(5)}\gamma_{ij} &= A_4(t)R^2h_{ij} + B_4(t)RR_{ij} + C_4(t)R_{lm}R^{lm}h_{ij} \\ &\quad + D_4(t)R_{im}R^m_j + E_4(t)\,\Box\, Rh_{ij} + F_4(t)\nabla_i\nabla_j R \\ &\quad + g_4(t)\,\Box\, R_{ij} . \end{aligned} \tag{A1}$$

The "seed" $h_{ij}(x^k)$ is a set of six functions of space which can be reduced to three by a suitable change of spatial coordinates. The metric γ_{ij} will therefore not be generic (eight arbitrary functions would be required if matter is a perfect fluid, six if it is a scalar field). However, as shown in Sec. IV, it will be an attractor of a class of generic solutions.

The correction $\delta\gamma_{ij}$ is chosen to be a tensor built from gradients of h_{ij} alone and hence is of the form above where R_{ij}, the Ricci tensor of the seed h_{ij} is quadratic in $\partial_i h_{jk}$, and where terms such as R^2 or $\nabla_i\nabla_j R$, with ∇_i the covariant derivative with respect to h_{ij}, are quartic in $\partial_i h_{jk}$. All indices are raised with the inverse metric h^{ij}; $R \equiv h^{ij}R_{ij}$; $\Box \equiv h^{ij}\nabla_i\nabla_j$. The Riemann tensor does not appear as it reduces to the Ricci tensor in three dimensions, and, because of the Bianchi identity, terms in $\nabla_i\nabla_l R^l_j (= \frac{1}{2}\nabla_i\nabla_j R)$ and $\nabla_l\nabla_m R^{lm} (= \frac{1}{2}\,\Box\, R)$ do not appear in ${}^{(5)}\gamma_{ij}$. The coefficients A_0, A_2, B_2, A_4, etc. are functions of time to be determined by Einstein's equations.

Writing Einstein's equations [Eqs. (2.3)–(2.5) in text] requires computing (up to the fifth order in the gradients) (a) the inverse metric γ^{ij} (using the definition: $\gamma_{ik}\gamma^{kj} = \delta^j_i$), (b) the intrinsic curvature K^i_j ($\equiv \gamma^{ik}\dot{\gamma}_{kj}$, where $\dot{\gamma}_{ij} \equiv \partial\gamma_{ij}/\partial t$), as well as $K \equiv K^i_i$, $\dot{K}^i_j$, $\dot{K}$, KK^i_j, and $K^i_j K^j_i$, (c) the covariant derivative $K^i_{j;l}$ of K^i_j with respect to γ_{ij}, which is at least third order in the gradients [the calculation uses the relation $K^i_{j;l} = \nabla_l K^i_j + (\delta\Gamma)^i_{lm}K^m_j - (\delta\Gamma)^m_{lj}K^i_m$ with $(\delta\Gamma)^i_{lm} = \frac{1}{2A_0}(\nabla_l\delta\gamma_{mp} + \nabla_m\delta\gamma_{pl} - \nabla_p\delta\gamma_{lm})$], and (d) the Ricci tensor ${}^{(3)}R_{ij}$ built with the metric γ_{ij}. It is at least second order in the gradients. [This uses the relation ${}^{(3)}R_{ij} = R_{ij} + \delta R_{ij}$ with $\delta R_{ij} = \nabla_k(\delta\Gamma)^k_{ij} - \nabla_j(\delta\Gamma)^k_{ik}$.]

The stress-energy tensor [Eq. (2.2) or (3.1) in the text] is also expanded in spatial gradients. If matter is a perfect fluid, with energy density ρ, three-velocity u_i, and pressure $p = (\Gamma - 1)\rho$ where the index Γ is a constant ($\Gamma = 1$ for dust, $\Gamma = 4/3$ for radiation, etc.), we set

$$\begin{aligned} \rho &= \rho_0(t) + \rho_2(t)R + \rho_4(t)R^2 + \mu_4(t)R_{lm}R^{lm} \\ &\quad + \epsilon_4(t)\,\Box\, R + \cdots , \\ u_i &= u_3(t)\nabla_i R + u_5(t)R\nabla_i R + v_5(t)R^l_i\nabla_l R \\ &\quad + w_5(t)R^{lm}\nabla_i R_{lm} + x_5(t)R^l_m\nabla_l R^m_i \\ &\quad + g_5(t)\nabla_i\,\Box\, R + \cdots \end{aligned} \tag{A2}$$

where the time dependence of the coefficients ρ_0, ρ_2, u_3, etc. will be determined by Einstein's equations. If matter is a scalar field $\phi(t, x^k)$ with potential $V(\phi)$, we write

$$\begin{aligned} \phi &= \phi_0(t) + \phi_2(t)R + \phi_4(t)R^2 + \psi_4(t)R_{lm}R^{lm} \\ &\quad + \xi_4(t)\,\Box\, R + \cdots \end{aligned} \tag{A3}$$

from which the expansion of the stress-energy tensor [Eq. (3.1) in the text] is easily obtained.

We can then solve Einstein's equations order by order. In the case of a perfect fluid Einstein's equations first determine the metric [because $u_i u^j$ is sixth order, see Eqs. (2.3)–(2.5) in the text]. At order one Eq. (2.3) in the text reduces to $2\dot{H} + 3\Gamma H^2 = 0$ (with $2H \equiv \dot{A}_0/A_0$) whose general solution is $A_0 = \tilde{A}(t - t_0)^{4/3\Gamma}$. Since we restrict A_0 to be a function of t alone, the constant t_0 can be set to zero, and the constant $\tilde{A}$ can be absorbed in the seed h_{ij}. Hence the solution [Eq. (2.2) in the text] is

$$H = \frac{2}{3\Gamma t} , \qquad A_0(t) = t^{\frac{4}{3\Gamma}} . \tag{A4}$$

At order 3 Einstein's equations reduce to Eq. (2.15) in the text [with $B_2 \equiv A_0 f_2$, $A_2 \equiv \frac{1}{3}A_0(g_2 - f_2)$] whose general solution is $B_2^{(g)} = B_2 + B_2^{(h)}$, $A_2^{(g)} = A_2 + A_2^{(h)}$, where the particular solutions B_2 and A_2 are [cf. Eq. (2.17) in the text]

$$\begin{aligned} B_2 &= -\frac{9\Gamma^2}{9\Gamma^2 - 4}t^2 \, , \\ A_2 &= -\frac{9\Gamma^2}{9\Gamma^2 - 4}\,\frac{1}{9\Gamma - 4}\left(\frac{3}{4}\Gamma^2 - 3\Gamma + 1\right)t^2 \, , \end{aligned} \tag{A5}$$

and where the homogeneous solutions, referred to as "decaying modes" in Ref. [7], are $B_2^{(h)} = t^{4/3\Gamma}(\beta_2 + \beta_2' t^{1-2/\Gamma})$ and $A_2^{(h)} = t^{4/3\Gamma}(\alpha_2 + \alpha_2' t^{-1} - \frac{\beta_2'}{3}t^{1-2/\Gamma})$, where $\beta_2, \beta_2', \alpha_2$, and α_2' are arbitrary constants. Now, as shown in Sec. IV, we note the following.

(1) The terms in β_2 and α_2, having the same time dependence as $A(t)$, can be absorbed in a redefinition of the seed: $h_{ij} \to h_{ij} + \beta_2 R_{ij} + \alpha_2 R h_{ij}$.

(2) The term in α_2' can be eliminated by a suitable change of coordinates. [Indeed let us perform the in-

finitesimal transformation $x'^\mu = x^\mu + \xi^\mu$ with $\xi^0 = T_2(t)R, \xi^i = L_3(t)\nabla_i R$. It is easy to show that the frame remains synchronous if $T_2(t) =$ const and $L_3 =$ const $+ T_2 \int dt\, A_0^{-1}$, and that the three-metric γ_{ij} transforms into $\gamma'_{ij} = \gamma_{ij} - \dot{A}_0 R T_2 h_{ij} = \gamma_{ij} - \frac{4}{3\Gamma} T_2 R t^{4/3\Gamma - 1} h_{ij}$. Therefore choosing $T_2 = 3\Gamma\alpha'_2/4$ eliminates the aforementioned term.]

(3) Finally the term in β'_2 can be absorbed in the correction a generic zeroth order seed, such that ${}^{(0)}K^i_j = 2H\delta^i_j + S^i_j/A^{3/2}$ with S^i_j small [see Eq. (4.13) in text], would have made to γ_{ij}. This correction decreases as time increases (if $\Gamma < 2$, that is for all fluids with sound velocity less than the speed of light), which is in keeping with the general argument in the text about metric (A1) being an attractor of a class of generic solutions. Hence the homogeneous solutions can be ignored and the metric at order 3 is (1.1) with A_2 and B_2 given by (A5).

At order 5 Einstein's equations [(2.3)–(2.5) in the text] equate to zero a sum of terms (in $\Box R_{ij}$, $\nabla_i\nabla_j R$, $R^l_i R_{jl}$, etc.), the coefficients of which we consequently set equal to zero. As examples, the coefficient of $\Box R_{ij}$ (the simplest) is $\frac{1}{2}(-b_2/A_0 + \ddot{g}_4 + 3H\dot{g}_4)$, where $a_2 \equiv A_2/A_0$ etc., and that of $R^2\delta^i_j$ (the most complicated) is

$$\left(a_4 - \frac{1}{2}a_2^2\right)^{\cdot\cdot} + 3H\Gamma\left(a_4 - \frac{1}{2}a_2^2\right)^{\cdot} + \frac{b_2}{A_0} - (\Gamma - 2)\frac{a_2}{2A_0}$$
$$-(1-\Gamma)\left[H\dot{b}_4 - H(a_2 b_2)^{\cdot} + \frac{1}{2}\dot{a}_2\dot{b}_2\right]$$
$$+\frac{3}{4}\Gamma\dot{a}_2^2 - \frac{1}{8}(2-\Gamma)\dot{b}_2^2.$$

This yields seven linear second order differential equations for $A_4, ..., G_4$. Here again the homogeneous solutions can be ignored and we are left with the particular solutions. They are all of the type $Y_4 = \bar{y}t^{4-4/3\Gamma}$, Y_4 standing collectively for $(A_4, ..., G_4)$ and the $\bar{y}$ being given by

$$\bar{g} = -\frac{81\Gamma^4}{4(3\Gamma+2)(3\Gamma-2)^2(9\Gamma-2)}, \qquad \bar{f} = -\frac{81\Gamma^4(3\Gamma^2 - 12\Gamma + 4)}{16(3\Gamma+2)(3\Gamma-2)^2(9\Gamma-2)(9\Gamma-4)},$$
$$\bar{d} = \frac{81\Gamma^4(15\Gamma+2)}{2(3\Gamma+2)^2(3\Gamma-2)^2(9\Gamma-2)}, \qquad \bar{b} = \frac{81\Gamma^4(9\Gamma^3 - 171\Gamma^2 + 54\Gamma + 8)}{4(3\Gamma+2)^2(3\Gamma-2)^2(9\Gamma-2)(9\Gamma-4)},$$
$$\bar{e} = \frac{81\Gamma^4(-3\Gamma^3 + 6\Gamma^2 + 12\Gamma - 8)}{16(3\Gamma-2)^2(3\Gamma+2)(15\Gamma-8)(9\Gamma-2)}, \qquad \bar{c} = \frac{162\Gamma^4(-12\Gamma^2 + 3\Gamma + 2)}{(3\Gamma+2)^2(3\Gamma-2)^2(15\Gamma-8)(9\Gamma-2)},$$
$$\bar{a} = \frac{81\Gamma^4\left(-\frac{729}{64}\Gamma^7 + \frac{405}{16}\Gamma^6 - \frac{1701}{16}\Gamma^5 + 1638\Gamma^4 - \frac{7737}{4}\Gamma^3 + 639\Gamma^2 + 21\Gamma - 28\right)}{(3\Gamma+2)^2(3\Gamma-2)^2(9\Gamma-4)^2(15\Gamma-8)(9\Gamma-2)}. \tag{A6}$$

[In the case of dust ($\Gamma = 1$), they reduce to $\bar{g} = -81/140$, $\bar{f} = \bar{e} = 81/560$, $\bar{d} = 1377/350$, $\bar{b} = -81/35$, $\bar{c} = -162/175$, $\bar{a} = 81 \times 89/(80 \times 140)$.]

The metric being thus determined, the evolution of the matter variables is obtained from the remaining Einstein equations. The energy density ρ and three-velocity u_i up to order 2 and 3 are, respectively,

$$\kappa\rho_0 = \frac{4}{3\Gamma^2}t^{-2}, \qquad \kappa\rho_2 = \frac{3\Gamma}{2(9\Gamma-4)}t^{\frac{-4}{3\Gamma}},$$
$$u_3 = \frac{27\Gamma^3(1-\Gamma)}{8(3\Gamma+2)(9\Gamma-4)}t^{3-\frac{4}{3\Gamma}}. \tag{A7}$$

Three checks on the coefficients (A6) can be made, which are a good indication that they must be correct. First, they agree, in the case of dust, with the result obtained in [7] (see Sec. V for a comparison of the two methods). Second, if the seed is taken to be that of a maximally symmetric space ($R_{ij} = \frac{1}{3}Rh_{ij}$ with $R =$ const), the metric becomes $\gamma_{ij} = h_{ij}[A + (A_2 + B_2/3)R + (A_4 + B_4/3 + C_4/3 + D_4/9)R^2 + \cdots]$ and must agree with the expansion in R of the exact Friedmann-Robertson-Walker solution $\gamma_{ij} = \bar{A}(t)h_{ij}$ where $\bar{A}(t)$, solution of the Friedmann equation $2(\bar{A}^{\cdot}/\bar{A})^{\cdot} + 3\Gamma(\bar{A}^{\cdot}/\bar{A})^2 = (2 - 3\Gamma)R/6\bar{A}$, reads $\bar{A}(t) = t^{4/3\Gamma} - \frac{3\Gamma^2}{4(9\Gamma-4)}Rt^2 + \frac{9\Gamma^4(-9\Gamma^2+18\Gamma-8)}{64(15\Gamma-8)(9\Gamma-4)^2}t^{4-4/3\Gamma} + \cdots$. We checked that, at least for $\Gamma = 0, \infty, 2$, and 3, the two metrics indeed coincide. Finally we checked that the fifth order part of the three-velocity u_i is zero in the case of dust, in agreement with the fact that in a synchronous reference frame, dust, which follows geodesics, can be at rest.

When matter is a scalar field the resolution of Einstein's equations [(3.2)–(3.4) in text] proceeds along similar lines. The derivation of orders 1 and 3 was shown in the text, the metric being (A1) with $A_0(t), A_2(t)$, and $B_2(t)$ given by (3.8) and (3.15) in the text and the scalar field being (A3) with $\phi_0(t)$ and $\phi_2(t)$ given by Eqs. (3.6) and (3.8) in text.

At order 5 Einstein's equations split respectively into 7, 5, and 3 equations for the 7+3=10 metric and scalar field coefficients. Five equations must then be identically satisfied and serve as a check on the solution. The result is, with again $g_4 \equiv G_4/A_0$, etc., and ignoring once more the homogeneous solutions,

$$g_4 = \int dt\, A_0^{-3/2} \int dt'\, A_0^{1/2} b_2, \quad d_4 = -4g_4 + \frac{1}{2} b_2^2, \quad f_4 = \int dt\, A_0^{-3/2} \int dt'\, A_0^{1/2} a_2\,,$$

$$b_4 = -\frac{1}{2} d_4 - f_4 + a_2 b_2 - \frac{3}{2} \int dt\, a_2 \dot{b}_2, \quad \kappa \xi_4 = -\frac{1}{\dot{\phi}_0} \left(\dot{e}_4 + \frac{1}{4} \dot{g}_4 \right)\,,$$

$$e_4 = \int dt\, \dot{\phi}_0^2 \int \frac{dt'}{\dot{\phi}_0^2} \left[\frac{a_2}{A_0} - H \dot{f}_4 + \frac{1}{2} \dot{g}_4 \left(H + \frac{\ddot{\phi}_0}{\dot{\phi}_0} \right) \right], \quad \kappa \psi_4 = -\frac{1}{\dot{\phi}_0} \left(\dot{c}_4 - \frac{5}{4} \dot{g}_4 + \frac{1}{8} b_2 \dot{b}_2 \right)\,,$$

$$c_4 = \int dt\, \dot{\phi}_0^2 \int dt'\, \frac{1}{\dot{\phi}_0^2} \left[\frac{2b_2}{A_0} - \left(\frac{7}{2} H + \frac{5}{2} \frac{\ddot{\phi}_0}{\dot{\phi}_0} \right) \dot{g}_4 + \frac{1}{4} b_2 \dot{b}_2 \left(3H + \frac{\ddot{\phi}_0}{\dot{\phi}_0} \right) \right]\,,$$

$$\kappa \phi_4 = -\frac{1}{\dot{\phi}_0} \left\{ (\dot{a}_4 - \dot{a}_2 a_2) + \frac{3}{8} \left[\dot{b}_4 - (a_2 b_2)^{\cdot} \right] + \frac{1}{8} a_2 \dot{b}_2 - \frac{1}{8} \dot{g}_4 + \frac{1}{2} \kappa \dot{\phi}_2 \phi_2 \right\}\,,$$

$$a_4 = \frac{1}{2} a_2^2 + \int dt\, \dot{\phi}_0^2 \int \frac{dt'}{\dot{\phi}_0^2} \left\{ -\frac{b_2}{A_0} + \left[\dot{b}_4 - (a_2 b_2)^{\cdot} \right] \left(\frac{5}{4} H + \frac{3}{4} \frac{\ddot{\phi}_0}{\dot{\phi}_0} \right) - \frac{3}{2} \dot{a}_2^2 - \frac{1}{2} \dot{a}_2 \dot{b}_2 \right\}$$
$$+ \int dt \dot{\phi}_0^2 \int \frac{dt'}{\dot{\phi}_0^2} \left[\left(3H + \frac{\ddot{\phi}_0}{\dot{\phi}_0} \right) \left(\frac{-\dot{g}_4}{4} + \frac{a_2 \dot{b}_2}{4} + \kappa \dot{\phi}_2 \phi_2 \right) + \kappa \phi_2^2 \left(\frac{-\dddot{\phi}_0}{\dot{\phi}_0} + \frac{3\dot{\phi}_0^2}{2} - \frac{3H\ddot{\phi}_0}{\dot{\phi}_0} \right) \right]\,.$$

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