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Foliation dependence of black hole apparent horizons in spherical symmetry

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Numerical studies of gravitational collapse to black holes make use of apparent horizons, which are intrinsically foliation dependent. We expose the problem and discuss possible solutions using the Hawking-Hayward quasilocal mass. In spherical symmetry, we present a physically sensible approach to the problem by restricting to spherically symmetric spacetime slicings. In spherical symmetry, the apparent horizons enjoy a restricted gauge independence in any spherically symmetric foliation, but physical quantities associated with them, such as surface gravity and temperature, are fully gauge dependent. The widely used comoving and Kodama foliations, which are of particular interest, are discussed in detail as examples.

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I. INTRODUCTION

Realistic black holes interact with their environment and are, therefore, dynamical. The gravitational collapse leading to black hole formation is also a highly dynamical process. In time-dependent situations, the event horizons (which are null surfaces) familiar from the study of stationary black holes [1–3] are replaced in practice by apparent horizons, which can have timelike, lightlike, or spacelike nature (e.g., Refs. [4–7]). These are defined as the locus of vanishing expansion of a null geodesic congruence emanating from a spacelike compact 2-surface $\mathcal{S}$ with spherical topology.

In contrast with the event horizon, which is a global concept defined using the global structure of spacetime, the apparent horizon is a *quasilocal* concept. In numerical studies of collapse, it is more practical to track apparent horizons, rather than event horizons which require the knowledge of the entire future history of the spacetime[1]; [10,12,13]. A significant fraction of research in numerical relativity aims at predicting with high precision the waveforms of gravitational waves generated in the merger of compact-object binary systems or in stellar collapse to form black holes. These waveforms enter data banks for use in the laser interferometric detection of gravitational waves. Comparison with templates played a crucial role in the recent observations of gravitational waves from black hole mergers by the LIGO Collaboration [14,15]. These numerical works also use apparent horizons.

Apparent horizons suffer from a drawback: in general, they are foliation dependent because they depend on the choice of the 2-surface $\mathcal{S}$, which is chosen to lie in some hypersurface $\mathcal{H}$ that is a surface of simultaneity for some family of observers $\mathcal{O}$. This is a problem since the existence of a horizon is ultimately the defining feature of a black hole,[2] and this means that, in dynamical situations, the defining feature of a black hole used in practice depends on the observer. *A priori*, the situation seems actually worse: even the *existence* of a black hole seems to depend on the observer, as epitomized by the fact that the Schwarzschild spacetime (the prototypical black hole geometry) has no apparent horizons in certain foliations [17,18], giving the impression that cosmic censorship is violated. The basic idea of Ref. [17] is this: the Schwarzschild-Kruskal geometry admits "angular horizons." For example, the north pole of a 2-sphere inside the Schwarzschild black hole cannot send light signals to events with angular coordinate θ larger than a critical value, for example, at the south pole of a 2-sphere. These north and south poles are spacelike-related. Hence, it

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[1]See, however, Refs. [8–11].

[2]In Rindler's words, a horizon is "a frontier between things observable and things unobservable" [16].

is possible (as shown in Ref. [17]; see also Ref. [18]) to construct spacelike Cauchy surfaces which interpolate between trajectories of the North pole, which come close to the $r = 0$ singularity, and trajectories of the South pole, which remain outside the black hole. The causal past of any Cauchy surface in a slicing so constructed contains no trapped surfaces, yet it comes arbitrarily close to the singularity. Needless to say, these slicings are highly non-spherically symmetric and certainly contrived, but the problem of principle remains that foliations containing no apparent horizons exist even in the Schwarzschild geometry.

Is the situation really that bad? Pragmatically, the problem would be ameliorated, if not solved, if there were preferred foliations of spacetime in a practical sense, that is, if the spacetime slicing was somehow fixed by physical arguments such as symmetries. Fixing the spacetime foliation is ultimately equivalent to fixing a family of observers, and this is already a practical necessity in certain problems related with black hole physics. For example, the temperature of a black hole calculated with quantum field theory (for a scalar field) in curved space depends on the vacuum state, which in turn depends on which observer is chosen since the particle number operator is not invariant under the change of frame.

How do we fix the spacetime slicing in a physically meaningful way? We do not consider here Lorentz violating theories which involve a preferred frame in their formulation, restricting ourselves to general relativity, for which there is no preferred frame or coordinate system involved in the specification of the theory. In a practical sense, however, in the presence of a spacetime symmetry, a foliation which respects this symmetry is preferred from both a geometric and physical point of view: the invariance of the fundamental theory is broken by the specific geometry of the solution. This is just one more example of how broken symmetries occur in solutions to the equations of physics. For example, in Friedmann-Lemaître-Robertson-Walker universes, preferred observers are those associated with spatial homogeneity and isotropy, i.e., those who see the cosmic microwave background as homogeneous and isotropic around them (apart from the well-known tiny temperature fluctuations with $\delta T/T \sim 10^{-5}$). Similarly, we argue that in the presence of spherical symmetry, foliations which preserve spherical symmetry are naturally more convenient and that, *with this restriction*, the gauge-dependence problem of apparent horizons is circumvented in practice, but the thermodynamics of these horizons remains fully gauge dependent. The reason for this restricted gauge independence can be summarized in the fact that, in spherical symmetry, the areal radius R is a geometrically well-defined quantity and the apparent horizons of *any* foliation respecting spherical symmetry are located by the scalar equation[3]

$$\nabla_c R \nabla^c R = 0. \tag{1.1}$$

This scalar equation for a scalar quantity is certainly covariant and gauge independent (however, using R as a coordinate would be a gauge choice).

The restricted gauge independence of apparent horizons is verified for the important examples of the Kodama and comoving gauges, which are probably the ones most used in the literature. The argument applies also to the computation of the Misner-Sharp-Hernandez quasilocal mass M_{MSH} which is widely used in spherical collapse and in the thermodynamics of apparent horizons.[4] When evaluated on an apparent horizon, this quantity satisfies the scalar equation [21]

$$R_{\mathrm{AH}} = 2M_{\mathrm{MSH}}(R_{\mathrm{AH}}). \tag{1.2}$$

More generally, studying black holes from the dynamical point of view requires specifying the black hole mass, one of its fundamental parameters. In the different context of black hole (or horizon) thermodynamics, the black hole mass plays the role of internal energy in the first law of thermodynamics. Most of the literature on black holes assumes that this mass is defined by the Hawking-Hayward quasilocal mass [22,23] computed at the apparent horizon. In spherical symmetry, the Hawking-Hayward quasilocal mass reduces to the Misner-Sharp-Hernandez mass [21,24] used in studies of spherical collapse.

Our goal in this article is not to prove new theorems with full rigor; instead, we adopt a more pragmatic approach (in the philosophy of Ref. [25]) to the issue of apparent horizons and look into possible ways to alleviate the problems. While rigorous mathematical results (see Refs. [26–32] for already established examples) are ultimately desirable, a more pragmatic approach may be convenient to move forward in practical applications, leaving the problems of principle to be attacked in the future. In the next section, we briefly review the dual role that the Hawking-Hayward quasilocal mass plays as the internal energy in the first law of black hole thermodynamics and as the physical mass in dynamical black hole spacetimes. We discuss how this quasilocal construct could select a foliation, but this choice is useless for practical purposes. To proceed, we restrict to spherical symmetry in Sec. III, and we discuss two important examples, the comoving (Landau) and the Kodama gauges, and the explicit relation between them. We show how the apparent horizons coincide, as geometric surfaces, but have different thermodynamic interpretations according to the comoving and Kodama observers. We use metric signature $-+++$ and units in which the speed of light and Newton's constant are unity (and, when discussing thermodynamics, we use

[3]We refer to black hole apparent horizons, but most of our considerations apply to cosmological apparent horizons as well.

[4]The Misner-Sharp-Hernandez mass is also used for cosmological black holes [7,19] and coincides with the Lemaître mass used in Lemaître-Tolman-Bondi geometries [20].

geometrized units in which also the reduced Planck constant and the Boltzmann constant are unity), and we otherwise follow the notation of Wald's book [1].

II. HAWKING-HAYWARD MASS AND APPARENT HORIZONS

A. Hawking-Hayward mass and a preferred foliation

In a spacetime $(\mathcal{M}, g_{ab})$ which is a solution of the Einstein equations, let $\mathcal{S}$ be a two-dimensional, spacelike, embedded, compact, and orientable surface which is chosen to lie in some hypersurface $\mathcal{H}$ that is a surface of simultaneity for some family of observers $\mathcal{O}$. The four-dimensional spacetime metric g_{ab} induces a 2-metric h_{ab} on $\mathcal{S}$, with $\mathcal{R}^{(h)}$ being the Ricci scalar of h_{ab}. One can consider the congruences of ingoing $(-)$ and outgoing $(+)$ null geodesics $k^a_\pm$ at the surface $\mathcal{S}$. Denote with $\theta_{(\pm)}$ and $\sigma^{(\pm)}_{ab}$ the expansion scalars and the shear tensors of these null geodesic congruences, respectively. The Hawking quasilocal mass associated with the 2-surface $\mathcal{S}$ is the integral quantity [22]

$$M_{\mathrm{HH}} = \frac{1}{8\pi G}\sqrt{\frac{A}{16\pi}}\int_{\mathcal{S}}\mu\bigg(\mathcal{R}^{(h)} + \theta_{(+)}\theta_{(-)} - \frac{1}{2}\sigma^{(+)}_{ab}\sigma^{ab}_{(-)} - 2\omega_a\omega^a\bigg), \tag{2.1}$$

where μ is the volume 2-form on $\mathcal{S}$ and A is the area of $\mathcal{S}$. The last term $-2\omega_a\omega^a$ in the integral was added by Hayward [23] to complete the definition. Here, ω^a is the anholonomicity, that is, the projection onto $\mathcal{S}$ of the commutator of the null normal vectors to $\mathcal{S}$. If this term disappears, as in spherical symmetry, M_{HH} reduces to the Hawking quasilocal prescription [22]. Consistent with spherical symmetry, we will assume that the surface $\mathcal{S}$ is a topological 2-sphere.

Clearly, the choice of the surface $\mathcal{S}$ is essential in both the construction (2.1) and in the computation of M_{HH}. The value of this quantity depends on the choice of the unit normal n^a to $\mathcal{S}$.

B. Fixing the foliation: Method 1 (observer 4-velocity)

One possible way to fix the foliation is to identify the (timelike) 4-velocity u^a of an "observer" at $\mathcal{S}$ with n^a (remember that $\mathcal{S}$ is spacelike). A foliation of spacetime is associated with a family of observers. A possible (maybe even natural) choice for this observer is picking the one that "sees" the hypersurface $\mathcal{H}$ containing $\mathcal{S}$ and the matter on it at rest, implying $\mathcal{S}$ will also be at rest. If the unit normal n^a to the surface $\mathcal{S}$ has only its time component different from zero, then the surface $\mathcal{S}$ is at rest in these coordinates. One can then foliate the three-dimensional space $\mathcal{H}$ by carrying points of the surface $\mathcal{S}$ along the normal as done in the construction of Gaussian normal coordinates (e.g., Ref. [1]). There will be some kind of coordinate singularity as one approaches the center of $\mathcal{S}$.

The Hawking-Hayward mass M_{HH} should be the time component of a timelike 4-vector P^a describing the energy and 3-momentum of a certain region of spacetime enclosed by the compact surface $\mathcal{S}$ [33]. Therefore, the zero component P^0 of this vector should be gauge dependent. As, in special and general relativity, the "mass of a particle" is identified with its rest mass, the zero component of its 4-momentum in the frame in which the particle is at rest, it would be natural to identify the Hawking-Hayward quasilocal mass associated with a compact spacelike 2-surface $\mathcal{S}$ with the energy (2.1) seen by an observer for which $\mathcal{S}$ is at rest. A different observer in motion with respect to $\mathcal{S}$ would ascribe a different value to the mass of this surface. If it were accepted that the Hawking-Hayward mass describes the mass energy seen by an observer at rest with respect to the surface $\mathcal{S}$, then there would be no room for considering spacetime foliations defined by different observers $u^a \neq n^a$.

Can this be the "right" procedure to fix the foliation? The definition of Hawking-Hayward mass (2.1) applies to *spacelike* (compact) 2-surfaces. In practice, in black hole studies, the surface $\mathcal{S}$ is chosen to be the two-dimensional intersection of the black hole apparent horizon (a three-dimensional world tube, generated by a vector field t^a) with a time slice (a spacelike 3-surface $\mathcal{H}$).[5] Three-dimensional apparent horizons can be spacelike, null, or timelike, but their intersections with hypersurfaces of constant time $\mathcal{H}$ are two-dimensional spacelike surfaces, and hence the definition applies. There are two independent null normal vectors $k^a_\pm$ to the surface $\mathcal{S}$. There is also a spacelike normal s^a lying in the time slice $\mathcal{H}$ containing $\mathcal{S}$ and a timelike normal vector n^a to $\mathcal{S}$ that is normal to $\mathcal{H}$. This works even when the apparent horizon is null or spacelike, as it does not depend on its causal nature. The identification $u^a = n^a$ as the normal to $\mathcal{S}$ leads to the interpretation of M_{HH} as "the mass seen by this observer." But if we also demand that our observer be at rest with respect to the apparent horizon $\mathcal{S}$, then we must have $u^a \propto t^a$, with t^a the tangent to the three-dimensional horizon world tube. The latter can be null or spacelike, in which case the above interpretation of M_{HH} becomes moot. Moreover, in practical calculations of collapse, the apparent horizon is not at rest in the reference frame of interest.[6] We conclude that this method does not seem suitable.

However, it must make sense also for other observers, who do not see $\mathcal{S}$ at rest, to speak of the mass of a black hole. As an alternative, it is possible to consider a foliation with $u^a \neq n^a$ specified in some different way, as we do in the next section for the special case of spherical symmetry. Once a foliation is fixed, one cannot use another slicing in the same calculation. In general spacetimes, one does not know how to fix the

[5]Note that, in the literature, the terminology "apparent horizon" may refer to both the three-dimensional world tube and its two-dimensional cross section.

[6]This is the case, for example, when comoving coordinates are used while the apparent horizon is not comoving.

foliation, but this is possible in spherical symmetry, to which we devote the rest of this work.

III. SPHERICAL SYMMETRY

The assumption of spherical symmetry facilitates the analytical study of black holes; it allows modelling them with a set of appropriate tools such as the areal radius, the Misner-Sharp-Hernandez mass, the Kodama vector field, and the Hayward-Kodama surface gravity.

Therefore, let us specialize the previous discussion to spherical symmetry. In a spherically symmetric spacetime, the Hawking-Hayward mass reduces to the Misner-Sharp-Hernandez mass $M_{\rm MSH}$ [34]. If R is the areal radius of the spherical geometry, then $M_{\rm MSH}$ is defined by [21,34]

$$1-\frac{2M_{\rm MSH}(R)}{R}=\nabla_c R\nabla^c R, \tag{3.1}$$

and the apparent horizons (when they exist) are the roots of the equation

$$\nabla_c R\nabla^c R=0, \tag{3.2}$$

in any foliation respecting the spherical symmetry, so that

$$R_{\rm AH}=2M_{\rm MSH}(R_{\rm AH}) \tag{3.3}$$

at the apparent horizons. This relation generalizes the one holding for the mass and radius of the Schwarzschild event horizon.

Before proceeding, one should note that the areal radius R is a geometric quantity [21] which is independent of the foliation $\{\mathcal{H}\}$ which determines $\{\mathcal{S}\}$, and Eq. (3.2) locating the apparent horizons is a *scalar* equation. Therefore, all *spherically symmetric* foliations will produce the same apparent horizons (but this is not true of nonspherical foliations, as demonstrated by the example of Ref. [17]). In other words, these horizons enjoy a restricted gauge independence within the subset of spherically symmetric gauges. Similarly, the Hawking-Hayward mass is gauge dependent in general, but in spherical symmetry, it becomes gauge independent because it is defined by the scalar equation (3.1).

In spherical symmetry, one is usually interested in computing the Hawking-Hayward/Misner-Sharp-Hernandez quasilocal mass when the surface $\mathcal{S}$ coincides with a 2-sphere of symmetry with unit normal n^a. Usually, one needs to compute the Hawking-Hayward mass *on the apparent horizon*, which depends on the foliation chosen. A crucial point of our discussion is that it is natural to consider spacetime foliations which are spherically symmetric, and it seems contrived to do otherwise and not take into account the undeniable simplifications brought about by the symmetries of the spacetime under study. Then, a foliation will be identified with a family of observers u^a which have only time and radial components, $u^\mu=(u^0,u^1,0,0)$ in coordinates adapted to the symmetry. The 2-sphere $\mathcal{S}$ can have radial motion in this foliation, i.e., $n^\mu=(n^0,n^1,0,0)$ with $n_c n^c=-1$ in these coordinates.[7]

In the following, we discuss explicitly two families of observers which have the prescribed spherically symmetric form $u^\mu=(u^0,u^1,0,0)$: the comoving (Landau) gauge and the Kodama foliation. These are only examples (albeit important ones) intended to illustrate the restricted gauge invariance, which has already been discussed in general.

A. Fixing the foliation: Method 2 (comoving gauge)

Numerical studies of the spherical collapse of a fluid to a black hole have often employed the comoving gauge; that is, the timelike 4-vector u^a describing the foliation is identified with the 4-velocity of the collapsing fluid.[8] This procedure fixes the spacetime foliation in a way alternative to the previous one.

The energy momentum tensor of a perfect fluid[9] has the form

$$T^{ab}=(\rho+p)u^a u^b+pg^{ab}, \tag{3.4}$$

where the fluid 4-velocity $(u^t,u^t v^i)$ is normalized as $u_c u^c=-1$ and v^i is its 3-velocity. Misner and Sharp [24] write their equation in spherically symmetric form, setting the shift $N^i=0$ and with 3-velocity $v^i=0$. In other words, they fix the foliation by assuming that the normal to the hypersurfaces $t=\text{const}$ coincides with the fluid 4-velocity (comoving gauge).[10] The line element in comoving gauge is

$$ds^2=-e^{2\phi}dt^2+e^\lambda dr^2+R^2 d\Omega_{(2)}^2, \tag{3.5}$$

where ϕ, λ, and R (the areal radius) are all functions of t and r and $d\Omega_{(2)}^2=d\theta^2+\sin^2\theta d\varphi^2$ is the line element on the unit 2-sphere. (See the Appendix for examples of known spacetimes expressed in the comoving gauge.) It can be shown that there is a special slicing (t',r') such that the energy flux vanishes relative to the fluid 4-velocity; thus,

$$u_a T^{ab}(g_{bc}+u_b u_c)=0 \tag{3.6}$$

holds in this frame (called the Landau or energy frame). Here, $g_{bc}+u_b u_c\equiv\gamma_{bc}$ is the Riemannian metric on the 3-space, and by the Einstein field equations, u^a is then a Ricci eigenvector so the Landau gauge is uniquely defined geometrically. (Choosing the Kodama gauge, as discussed in Sec. III B, does not give zero heat flux for the matter fluid,

[7] In particular, if the surface $\mathcal{S}$ is at rest in this foliation, $n^\mu=(1/\sqrt{|g_{00}|},0,0,0)$ in adapted coordinates.

[8] In general, the latter is not parallel to the Kodama vector.

[9] We recall that a perfect fluid is a fluid with no dissipative effects (no shear and no viscosity).

[10] This is possible because spherical symmetry implies the vorticity is zero.

which shows that the Kodama gauge is different from the energy frame.)

Consider a general 2-sphere of symmetry $\mathcal{S}$, for which one writes the associated Misner-Sharp-Hernandez mass. This surface is either at rest or expanding/contracting radially with respect to the fluid. Choosing the comoving gauge eliminates by *fiat* the question of foliation dependence of the apparent horizons. Apparent horizons (which are 2-spheres once we restrict to spherically symmetric foliations) are almost never comoving and thus move radially with respect to observers comoving with the fluid. The comoving observer with 4-velocity u^c does not coincide with the unit normal n^c to the surface $\mathcal{S}$ at the apparent horizon.

Let us now introduce some tools adapted to the comoving observer, in order to make contact with the literature (we will use them again in Sec. IV). Following Ref. [35], we define the derivatives with respect to proper time and proper radial distance as measured by the comoving observers

$$D_t \equiv \mathrm{e}^{-\phi}\partial_t, \qquad D_r \equiv \mathrm{e}^{-\lambda/2}\partial_r. \tag{3.7}$$

Applying these derivative operators to the areal radius, one obtains

$$U \equiv D_t R = \mathrm{e}^{-\phi}\frac{\partial R}{\partial t} \equiv \mathrm{e}^{-\phi}\dot{R}, \tag{3.8}$$

$$\Gamma \equiv D_r R = \mathrm{e}^{-\lambda/2}\frac{\partial R}{\partial r} \equiv \mathrm{e}^{-\lambda/2}R'. \tag{3.9}$$

The apparent horizons of the geometry (3.5) are located by the roots of Eq. (3.2), which yields [35]

$$\Gamma^2 - U^2 = 0 \tag{3.10}$$

with $U = -\Gamma$ corresponding to black hole apparent horizons and $U = \Gamma$ corresponding to cosmological apparent horizons (for an expanding universe). Using the quantities U and Γ, the line element (3.5) is rewritten as

$$ds^2 = -\frac{\dot{R}^2}{U^2}dt^2 + \frac{R'^2}{\Gamma^2}dr^2 + R^2 d\Omega_{(2)}^2, \tag{3.11}$$

and the equation $\nabla_c R \nabla^c R = 0$ locating the apparent horizons, of course, reproduces Eq. (3.10).

The three-dimensional velocity v of an observer with respect to the matter is obtained from the derivative of the proper radial distance with respect to the proper time of comoving observers, given by [35]

$$\frac{dR}{d\tau} = \mathrm{e}^{-\phi}\left(\dot{R} + R'\frac{dr}{dt}\right) = U + \Gamma v. \tag{3.12}$$

In particular, one may evaluate the above at the horizon to get $v_H^{(C)}$, the 3-velocity of the horizon with respect to the comoving observer. Using the Einstein equations with (3.4), it yields [35]

$$v_H^{(C)} = \frac{1 + 8\pi R_{AH}^2 p}{1 - 8\pi R_{AH}^2 \rho}, \tag{3.13}$$

where the energy density and pressure are evaluated at the horizon location.

Finally, the 4-velocity of comoving observers has components

$$u_{(C)}^{\mu} = (\mathrm{e}^{-\phi}, 0, 0, 0) \tag{3.14}$$

in comoving coordinates (t, r, θ, φ). This will be compared with the 4-velocity of the Kodama observer—as introduced below—in Sec. IV.

B. Fixing the foliation: Method 3 (Kodama gauge)

1. Kodama vector field and Kodama gauge

In general relativity and with spherical symmetry, it is possible to introduce the Kodama vector field K^a [36], which is used in the black hole literature as a substitute for timelike Killing vectors, which do not exist in time-dependent geometries. By contracting the Einstein tensor G_{ab} with the Kodama vector, one obtains an energy current $J^a \equiv G^{ab}K_b$ which, surprisingly, is covariantly conserved, $\nabla^b J_b = 0$ [36] (a circumstance referred to as the "Kodama miracle" [37]). As shown in Ref. [34], the Misner-Sharp-Hernandez mass is the conserved Noether charge associated with the covariant conservation of the Kodama current J^a.

A third way of fixing the foliation involves using the related Kodama gauge. The line element of a spherically symmetric spacetime can always be written as follows in a gauge naturally using the areal radius R (a geometrical quantity defined in a covariant way) as the radial coordinate:

$$\begin{aligned} ds^2 &= g_{00}(T,R)dT^2 + g_{11}(T,R)dR^2 + R^2 d\Omega_{(2)}^2 \\ &\equiv h_{ab}dx^a dx^b + R^2 d\Omega_{(2)}^2. \end{aligned} \tag{3.15}$$

(See the Appendix for examples of known spacetimes expressed in the Kodama gauge.) Equation (3.2) locating the apparent horizons becomes simply

$$g^{RR}(T,R) = 0. \tag{3.16}$$

The Kodama vector is defined in a gauge-independent way as [36]

$$K^a = \epsilon^{ab}\nabla_b R \tag{3.17}$$

(where ϵ_{ab} is the volume form of the 2-metric h_{ab}), and it lies in the 2-space (t, R) orthogonal to the 2-spheres of symmetry. Its components in coordinates (T, R, θ, φ) are

$$K^a = \frac{1}{\sqrt{|g_{TT}|g_{RR}}}\left(\frac{\partial}{\partial T}\right)^a. \tag{3.18}$$

K^a identifies a preferred notion of time and becomes null on the apparent horizons, where $K_c K^c = 0$, while it is timelike outside a black hole apparent horizon and spacelike inside [36]. This behavior is parallel to that of the timelike Killing field in stationary spacetimes with a Killing horizon. Reference [37] advocates the gauge (3.15) on the basis of the simplicity it brings to the discussion of spherical apparent horizons. The gauge (3.15) using the areal radius R as radial coordinate seems motivated by the spherical symmetry and will be referred to as "Kodama gauge"[11] since the Kodama vector is naturally associated with this gauge. We will now see that the characteristics of the Hawking-Hayward mass in spherical symmetry support the choice of R as the radial coordinate, and thus physically motivate the choice of the Kodama gauge.

2. Hawking-Hayward mass

There is a wide consensus on the relevance of the dynamical horizon (which is a spacelike apparent horizon [38]) in nonstationary situations. For spherically symmetric dynamical horizons, the variation of energy constructed with the Ashtekar-Krishnan energy flux, introduced for dynamical horizons [38] and computed for apparent horizons, is the variation of the Hawking-Hayward/Misner-Sharp-Hernandez mass. The infinitesimal form of the area law is

$$\frac{dR}{2G} = dE_R, \tag{3.19}$$

where we have restored Newton's constant G for convenience. κ_R is the effective surface gravity of Ref. [38][12] associated with the R-foliation as

$$\kappa_R \equiv \frac{1}{2R} \tag{3.20}$$

so that the infinitesimal form of the law is recast into the familiar form

$$\frac{\kappa_R dA}{8\pi G} = dE_R, \tag{3.21}$$

where A is the area of a generic cross section. Any other foliation leads to the law

$$\frac{\kappa_r dA}{8\pi G} = dE_r \tag{3.22}$$

provided that we define the effective surface gravity κ_r of the r-foliation as

$$\kappa_r = \frac{dr}{dR}\kappa_R, \tag{3.23}$$

and

$$dE_r = \frac{dr}{dR} dE_R. \tag{3.24}$$

One can choose different foliations for the apparent horizon in the spherically symmetric case, but the energy appearing in the area law reduces to the Hawking-Hayward mass only if the areal radius R is chosen as the radial coordinate [38]. This gauge dependence can generically appear in other types of quasilocal masses such as the Brown-York mass [19] which are candidates for the role of mass for astrophysical systems [39].

Next, the surface $\mathcal{S}$ appearing in the definition of the Hawking-Hayward mass is often identified with an apparent horizon (this is true, in particular, when identifying the Hawking-Hayward mass M_{HH} with the internal energy of a black hole in the first law of horizon thermodynamics [40–43]). Let us say that the foliation is now fixed by the Kodama gauge prescription given above. In general, the unit normal to the apparent horizon $\mathcal{S}$ has components $n^\mu = (n^0, n^1, 0, 0)$ in the gauge (3.15), which are constrained by the normalization

$$g_{00}(n^0)^2 + g_{11}(n^1)^2 = -1 \tag{3.25}$$

with $n^1 \neq 0$, so the surface $\mathcal{S}$ is not at rest in this gauge but is expanding or contracting radially. The Hawking-Hayward mass is computed in the frame in which $u^a \propto K^a$, that is, for the observer which has as time the Kodama time [37], and according to which the apparent horizon is not necessarily at rest. One can compute the mass M_{HH} on 2-spheres of symmetry outside the apparent horizon and then take the limit to this apparent horizon (in which, however, K^a becomes a null vector [36]).

As shown by Hayward [34], M_{MSH} is the conserved Noether charge associated with the conservation of the Kodama energy current J^a, so M_{HH} is the "Newtonian" mass in a frame in which there is no spatial flow of Kodama energy. This is not the frame in which $\mathcal{S}$ is at rest; in general, the apparent horizon is not at rest but is in radial motion in the "Kodama foliation," since $n^1 \neq 0$ in these coordinates.

In the Kodama gauge, the Kodama current has components

$$J^\mu = (G^{00}K_0, G^{10}K_0, 0, 0). \tag{3.26}$$

The component J^1 is proportional to G^{TR} which, using the Einstein equations, is clearly proportional to a radial flow of "energy" T^{TR}.

The Kodama foliation is therefore tightly linked to the widely used Misner-Sharp-Hernandez mass, which is one more argument in favor of its relevance, especially for thermodynamical considerations.

[11]This terminology is not standard.
[12]The latter is different from the Hayward case, but they have the same stationary limit.

3. Kodama gauge and horizon thermodynamics

In discussions of the thermodynamics of apparent horizons, one way[13] to compute the Hawking temperature consists of the so-called tunneling method, which uses the Kodama time as a preferred notion of time and the corresponding energy of a particle [40–43,47,48]. An adiabatic approximation expressing the fact that the apparent horizons evolve slowly with respect to a "background" time scale of the dynamical spacetime is needed, but is not usually stated explicitly in the literature [7,49]. Clearly, fixing the foliation by making use of the areal radius R and of the Kodama vector K^a does not, *per se*, prove that the Hawking temperature derived with the tunneling method is physical, nor that the first law of thermodynamics for apparent horizons based on this prescription is consistent and correct. The proof of these statements must come from somewhere else. The procedure presented here is only a physically plausible way of fixing the foliation for apparent horizons.

The tunneling method, used to derive the surface gravity and temperature of apparent horizons using the Kodama time, implicitly fixes the foliation and therefore selects apparent horizons. In other words, this method selects the family of observers $u^a_{(K)}$ associated with this foliation as those with 4-velocity proportional to the Kodama vector K^a [37] [see Eqs. (4.9) and (4.10) below]. However, the temperature of the black hole apparent horizon depends on the observer, and this fact makes the entire thermodynamics of apparent horizons fully gauge dependent, a property which, although intuitive *a posteriori*, is not usually noted even if there is debate on which is the "correct" temperature of apparent horizons.

IV. RELATION BETWEEN COMOVING AND KODAMA GAUGES

Let us consider now the problem of finding the transformation from the comoving gauge (3.5) to the Kodama gauge (3.15). As shown below, the transformation cannot be written in a completely explicit way. In order to find the coordinate transformation, rewrite the line element (3.5) in terms of the areal radius $R(t,r)$, using the fact that

$$dr = \frac{dR - \dot{R}dt}{R'}. \tag{4.1}$$

Substituting Eq. (4.1) into the line element (3.5) yields

$$ds^2 = -\left(\mathrm{e}^{2\phi} - \frac{\dot{R}^2}{R'^2}\mathrm{e}^{\lambda}\right)dt^2 + \frac{\mathrm{e}^{\lambda}}{R'^2}dR^2 - \frac{2\dot{R}\mathrm{e}^{\lambda}}{R'^2}dtdR + R^2 d\Omega^2_{(2)}. \tag{4.2}$$

[13]In fact, this is the only method thus far which is able to compute explicitly surface gravity and temperature of time-dependent apparent horizons in general (but spherical) spacetimes [40–43]. Other methods have, thus far, produced results only for particular spacetime metrics [6,44–46].

The $dtdR$ cross-term can be eliminated by introducing a new time coordinate $T(t,R)$ such that

$$dT = \frac{1}{F}(dt + \beta dR), \tag{4.3}$$

where $F(t,R)$ is an integrating factor guaranteeing that dT is an exact differential and $\beta(t,R)$ is a function to be determined. By substituting into the line element, one obtains

$$\begin{aligned} ds^2 = & -\left(\mathrm{e}^{2\phi} - \frac{\dot{R}^2}{R'^2}\mathrm{e}^{\lambda}\right)F^2 dT^2 \\ & + 2F\left[\beta\left(\mathrm{e}^{2\phi} - \frac{\dot{R}^2}{R'^2}\mathrm{e}^{\lambda}\right) - \frac{\dot{R}}{R'^2}\mathrm{e}^{\lambda}\right]dTdR \\ & + \left[\frac{\mathrm{e}^{\lambda}}{R'^2} - \beta^2\left(\mathrm{e}^{2\phi} - \frac{\dot{R}^2}{R'^2}\mathrm{e}^{\lambda}\right) + 2\frac{\dot{R}}{R'^2}\mathrm{e}^{\lambda}\beta\right]dR^2 \\ & + R^2 d\Omega^2_{(2)}. \end{aligned} \tag{4.4}$$

By setting

$$\beta(t,R) = \frac{\dot{R}\mathrm{e}^{\lambda}}{R'^2(\mathrm{e}^{2\phi} - \mathrm{e}^{\lambda}\dot{R}^2/R'^2)}, \tag{4.5}$$

the line element (4.4) is diagonalized to the Kodama gauge

$$\begin{aligned} ds^2 = & -\left(\mathrm{e}^{2\phi} - \frac{\dot{R}^2}{R'^2}\mathrm{e}^{\lambda}\right)F^2 dT^2 + \frac{\mathrm{e}^{\lambda+2\phi}}{R'^2\mathrm{e}^{2\phi} - \dot{R}^2\mathrm{e}^{\lambda}}dR^2 \\ & + R^2 d\Omega^2_{(2)} \\ \equiv & \; g_{TT}(T,R)dT^2 + g_{RR}(T,R)dR^2 + R^2 d\Omega^2_{(2)}. \end{aligned} \tag{4.6}$$

The metric components g_{TT} and g_{RR} are given implicitly as functions of T and R. The integrating factor F (which in general is not unique) must satisfy the equation

$$\frac{\partial}{\partial R}\left(\frac{1}{F}\right) = \frac{\partial}{\partial t}\left(\frac{\beta}{F}\right), \tag{4.7}$$

which cannot be solved analytically except in trivial situations.

In terms of the quantities U and Γ of Eqs. (3.8) and (3.9), the Kodama line element (4.6) reads

$$ds^2 = -\mathrm{e}^{2\phi}\left(1 - \frac{U^2}{\Gamma^2}\right)F^2 dT^2 + \frac{dR^2}{\Gamma^2 - U^2} + R^2 d\Omega^2_{(2)}. \tag{4.8}$$

The apparent horizons are located by the scalar equation $\nabla_c R \nabla^c R = 0$, which in the Kodama gauge reduces to $g^{RR} = 0$ and, of course, again gives $U = \pm\Gamma$. Therefore, we see explicitly that the apparent horizons in the comoving gauge coincide with the apparent horizons in the Kodama gauge. As already remarked, this conclusion is not surprising since it is just an example of the general consideration that the areal radius R is a geometric quantity defined in an invariant way in spherical symmetry and the scalar equation $\nabla_c R \nabla^c R = 0$, which is gauge invariant, locates the apparent horizons in any spherically symmetric gauge. However, the 3-velocity of an apparent horizon with respect to the comoving observer (i.e., to matter) is different from its 3-velocity with respect to the Kodama observers. The 4-velocity of Kodama observers has components

$$u^{\mu}_{(K)} = \left(\frac{e^{-\phi}}{F\sqrt{1 - U^2/\Gamma^2}}, 0, 0, 0 \right) \tag{4.9}$$

in Kodama coordinates (T, R, θ, φ). Therefore, this vector is parallel to, but does not coincide with, the Kodama vector (3.18)

$$K^{\mu} = \left(\frac{\Gamma e^{-\phi}}{F}, 0, 0, 0 \right). \tag{4.10}$$

The comoving observers have 4-velocity given by

$$u^{\mu'}_{(C)} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} u^{\mu}_{(C)} = \frac{\partial x^{\mu'}}{\partial t} e^{-\phi}, \tag{4.11}$$

where $x^{\mu} \to x^{\mu'}$ is the coordinate transformation from comoving to Kodama coordinates and Eq. (3.14) has been used. We have

$$dT = \frac{1}{F}(dt + \beta dR) = \frac{\beta \dot{R} + 1}{F} dt + \frac{\beta R'}{F} dr \tag{4.12}$$

and, using Eq. (4.5),

$$\frac{\partial T}{\partial t} = \frac{1}{F}\left[1 + \frac{\dot{R}^2 e^{\lambda}}{R'^2 (e^{2\phi} - e^{\lambda} \dot{R}^2 / R'^2)} \right] = \frac{\Gamma^2}{F(\Gamma^2 - U^2)}. \tag{4.13}$$

Equation (4.11) and

$$u^{1'}_{(C)} = \frac{\partial R}{\partial t} e^{-\phi} \equiv U \tag{4.14}$$

then give

$$u^{\mu'}_{(C)} = \left(\frac{e^{-\phi}}{F(1 - U^2/\Gamma^2)}, U, 0, 0 \right) \tag{4.15}$$

in Kodama coordinates.

The scalar product between Kodama and comoving 4-velocities is

$$u^{a}_{(K)} u^{(C)}_{a} = -\frac{1}{\sqrt{1 - U^2/\Gamma^2}} = -\gamma(v_{\mathrm{rel}}), \tag{4.16}$$

where v_{rel} is the instantaneous three-dimensional relative velocity between comoving and Kodama observers and $\gamma(v_{\mathrm{rel}})$ is the corresponding Lorentz factor. This (radial) 3-velocity has magnitude

$$|v_{\mathrm{rel}}| = \left| \frac{U}{\Gamma} \right|, \tag{4.17}$$

which depends on both time and radial location. At the apparent horizon, where $U = -\Gamma$, one has $|v_{\mathrm{rel}}| = 1$. The relative velocity of the Kodama observer with respect to the comoving observer is always equal to $\pm c$ at the horizon (i.e., to ± 1 since we are setting $c = 1$). This is logical since the Kodama vector is null at the apparent horizon.

Although the apparent horizons as determined by comoving and Kodama observers are the same, these surfaces will be perceived as different by these two families of observers because they are accelerated with respect to one another. If a temperature can be meaningfully assigned to time-dependent apparent horizons, the vacuum state of a field on the dynamical spacetime will be different with respect to Kodama and comoving observers. Moreover, the temperatures determined by these observers will be Doppler shifted with respect to each other because the apparent horizon has different velocities relative to these observers.

In order to illustrate that, consider the velocity of the apparent horizon in the Kodama gauge. We have

$$(v^{(K)}_{\mathrm{H}})^2 = \frac{g_{00}^2}{g_{11}^2} \left(\frac{dR}{dT} \right)^2 = \frac{e^{-2\phi} \Gamma^2}{F^2 (\Gamma^2 - U^2)^2} \left(\frac{dR}{dT} \right)^2. \tag{4.18}$$

Since

$$\begin{aligned} \frac{dT}{dR} &= \frac{1}{F}\left(\frac{dt}{e^{\phi} U dt + e^{\lambda/2} \Gamma dr} + \beta \right) \\ &= \frac{1}{F}\left(\frac{e^{-\phi}}{U + \Gamma v} + \frac{U e^{-\phi}}{\Gamma^2 - U^2} \right) \\ &= \frac{e^{-\phi}}{F} \frac{\Gamma(\Gamma + Uv)}{(U + \Gamma v)(\Gamma^2 - U^2)}, \end{aligned} \tag{4.19}$$

we have at the horizon

$$\begin{aligned} (v^{(K)}_{\mathrm{H}})^2 &= \frac{e^{-2\phi}\Gamma^2}{F^2(\Gamma^2 - U^2)^2} \frac{F^2 e^{2\phi} (\Gamma^2 - U^2)^2}{\Gamma^2} \left[\frac{U(1-v)}{-U(1-v)} \right]^2 \\ &= 1, \end{aligned} \tag{4.20}$$

which is expected since, again, the Kodama observer is null at the horizon. However, the relative velocity $v^{(C)}_{H}$ of the horizon with respect to the comoving observer, given by Eq. (3.13), is in general not unity. The horizon indeed has a different velocity for the two observers.

The surface gravities $\kappa_{(K)}$ and $\kappa_{(C)}$ of Kodama and comoving observers will also differ, and the horizon temperatures are given by $T_{(i)} = \kappa_{(i)}/(2\pi)$ (in geometrized units).[14] Therefore, even though the previous

[14]We can attribute to the surface gravity of an observer a "geometric temperature" which can appear in a thermodynamic law. Hence, the geometric temperatures of these observers are different. However, we note that the Kodama vector temperature has two important properties: 1) it reduces to the Killing temperature for the case of an event horizon, and 2) it is a key quantity in the tunneling method.

considerations alleviate in practice the foliation-dependence problem of apparent horizons, the issue of associating physically meaningful temperatures and thermodynamics with these horizons is not addressed by our considerations above, and it does not seem possible to alleviate the gauge dependence of the thermodynamics of apparent horizons.

V. CONCLUSIONS

Nowadays, apparent and trapping horizons are the common choice to describe black hole boundaries in numerical simulations of gravitational collapse and in works computing the waveforms of gravitational waves emitted during dynamical events in order to build banks of templates for the interferometric detection of gravitational waves. The recent LIGO observations of gravitational waves from binary black hole mergers and the extraction of the orbital parameters of the binary [14,15] rely heavily on such waveforms and, therefore, on apparent horizons. Apparent and trapping horizons are much easier to locate than the teleological event horizons. Therefore, it is hard to overemphasize the importance of apparent horizons in gravitational physics [7]. However, apparent horizons depend on the spacetime slicing, as exemplified by the fact that one can even find foliations of the Schwarzschild spacetime which do not contain apparent horizons [17,18]. The foliation dependence of apparent horizons is a serious problem since in dynamical situations the apparent horizon takes on the role of black hole boundary. The very existence of a black hole in dynamical spacetimes is therefore questioned if the existence of apparent horizons depends on the foliation, and realistic black holes are indeed dynamical due to the interaction with their environment and, ultimately, also because of Hawking radiation.

It seems that, at least in the presence of spherical symmetry as used in many studies of gravitational collapse, there should be a reasonable (although restricted) way out of this foliation-dependence problem. We restrict to this symmetry. Furthermore, it is natural to adopt a spherical foliation. Then, an apparent horizon is located by the scalar equation $\nabla_c R \nabla^c R = 0$, where the areal radius R is a gauge-independent quantity, which makes the apparent horizons independent of the (spherically symmetric) foliation. This fact is reassuring since all observers associated with spherical foliations will agree on the existence of a black hole (identified with the region inside the same apparent horizon). Thus, although restricting to spherically symmetric foliations does not, by all means, imply full gauge independence, a restricted gauge invariance *within this family of foliations* arises. Having established this fact, we then examined two examples, the comoving and the Kodama gauges. Although these are just examples, they are important because of their widespread use. The comoving gauge (used mostly in numerical simulations) and the Kodama gauge (used mostly in the thermodynamics of apparent horizons and in Hawking radiation studies) are obvious candidates for a spherical foliation. Our discussion about the relation between the comoving and Kodama foliations will hopefully facilitate communication between the different communities working in these areas. We have checked explicitly that the apparent horizon is the same in these two gauges. However, due to their relative radial motion, comoving and Kodama observers perceive the same apparent horizon differently. While our discussion sheds some light on the foliation dependence in the important case of spherical symmetry, the general situation of dynamical nonspherical geometries remains an open problem. In this case, the full Hawking-Hayward quasilocal energy (2.1) (including the square of the anholonomicity) will have to be taken into account.

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APPENDIX: EXAMPLES OF COMOVING/ KODAMA GAUGES

1. Schwarzschild black hole

For an example, consider the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dT^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2 d\Omega^2_{(2)}, \tag{A1}$$

which, under this usual form, is naturally expressed in the Kodama gauge. Therefore, an observer at rest in the usual Schwarzschild coordinates is a Kodama observer, i.e., an observer who remains on a given sphere of symmetry of areal radius R (what we usually call an "accelerated observer"). But the line element can be rewritten in the comoving gauge, using Lemaître coordinates, as

$$ds^2 = -dt^2 + \frac{dr^2}{[3(r-t)/(4M)]^{2/3}} + (2M)^{2/3}\left[\frac{3}{2}(r-t)\right]^{4/3} d\Omega^2_{(2)} \tag{A2}$$

with $R = (2M)^{1/3}[\frac{3}{2}(r-t)]^{2/3}$. An observer at rest in these coordinates is in free fall toward the black hole. Using the definition of the apparent horizon, it can be shown that the latter is located at $R = 2M$ in both frames and that we have $M_{\mathrm{HH}} = M_{\mathrm{MSH}} = M$.

2. Friedmann-Lemaître-Robertson-Walker cosmology

The usual form of the Friedmann-Lemaître-Robertson-Walker line element

$$ds^2 = -dt^2 + \frac{a^2(t)}{1-kr^2}dr^2 + a^2(t)r^2 d\Omega_{(2)}^2, \qquad \text{(A3)}$$

with $a(t)$ the scale factor and k the spatial curvature, is naturally written in comoving coordinates. By defining a new radial coordinate $R = a(t)r$, one obtains the so-called pseudo–Painlevé-Gullstrand[15] line element

$$ds^2 = -\left(1 - \frac{H^2R^2}{1-kR^2/a^2}\right)dt^2 - \frac{2HR}{1-kR^2/a^2}dtdR + \frac{dR^2}{1-kR^2/a^2} + R^2 d\Omega_{(2)}^2, \qquad \text{(A4)}$$

where $H = \dot{a}/a$ is the Hubble parameter. However, although we have chosen the areal radius as our new radial coordinate, we have not yet obtained the Kodama gauge of Eq. (3.15). That is because integral curves of the vector ∂_t and of the Kodama vector do not coincide [37] (we do not have $K \propto \partial_t$). One can get rid of the cross-term by introducing a new time coordinate as in Eq. (4.3),

$$ds^2 = -\left(1 - \frac{H^2R^2}{1-kR^2/a^2}\right)F^2 dT^2 + \frac{dR^2}{1-kR^2/a^2-H^2R^2} + R^2 d\Omega_{(2)}^2, \qquad \text{(A5)}$$

with a, H, and F implicit functions of T and R (see Ref. [7]). Now, we indeed have that $K \propto \partial_T$, and we can refer to T as the "Kodama time" of Ref. [37]. An observer at rest in these coordinates is a Kodama observer, who remains on a given sphere of symmetry. The very common line elements presented in this Appendix are widely used and often preferred for their physical relevance. For the same reason, the apparent horizons uniquely defined in these gauges should be given a preferred status with respect to apparent horizons defined in nonsymmetric gauges.

[15]The name comes from the similarity of this metric with the Painlevé-Gullstrand form of the Schwarzschild metric. However, in the latter, the dR^2 term appears with a unit coefficient, which is not the case here, whence the term "pseudo." See Ref. [7] for the true Painlevé-Gullstrand form of the Friedmann-Lemaître-Robertson-Walker spacetimes.

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