THE ASTROPHYSICAL JOURNAL, 201:1–19, 1975 October 1

THE PRIMORDIAL BLACK HOLE MASS SPECTRUM*

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Received 1975 January 31

ABSTRACT

We examine what mass spectrum of primordial black holes should result if the early universe consisted of small density fluctuations superposed on a Friedmann background. It is shown that only a certain type of fluctuation favors the formation of primordial black holes and that, consequently, their spectrum should always have a particular form. Since both the fluctuations which arise naturally and the fluctuations which are often invoked to explain galaxy formation are of the required type, primordial black holes could have had an important effect on the evolution of the universe. In particular, although primordial black holes are unlikely to have a critical density, big ones could have been sufficiently numerous to act as condensation nuclei for galaxies. Observational limits on the spectrum of primordial black holes place strong constraints on the magnitude of density fluctuations in the early universe and support the assumption that the early universe was nearly Friedmann rather than chaotic. Any model in which the early universe has a soft equation of state for a prolonged period is shown to be suspect, since primordial black holes probably form too prolifically in such a situation to be consistent with observation.

Subject headings: black holes — cosmology — galaxies

I. INTRODUCTION AND SUMMARY

In a previous paper (Carr and Hawking 1974) it was shown that black holes could have formed at very early stages in the history of the universe as a result of initial inhomogeneities. It was also shown that these "primordial" black holes would not have grown very much through accretion and so their masses today should be about the same as when they first formed. Recently, however, Hawking has made the striking prediction (Hawking 1974, 1975) that, because of quantum effects, any black hole should emit particles like a blackbody with a temperature inversely proportional to its mass. Despite the important conceptual change which Hawking's result introduces in the context of black holes in general, probably only a primordial black hole could be sufficiently small for the effect to be important. Hawking's prediction implies that any primordial black holes of less than 10^{15} g should have evaporated by now and raises the question of whether any primordial black holes could still exist.

This motivates a discussion of the expected mass spectrum of primordial black holes. (Henceforth a primordial black hole will be referred to as a pbh.) The main difficulty in trying to predict the pbh spectrum is that all pbh's probably form within the first second of the universe, when any cosmological model is highly dubious. This paper examines what pbh spectrum should result if one takes the simple view that the early universe consisted of small density fluctuations superposed on a Friedmann background. The small-fluctuation assumption is very strong (the universe may have been completely chaotic in its first second); but, as argued in § VI, it does seem to be supported by observational evidence. With such a model the pbh mass spectrum depends on only two features of the early universe: the equation of state, which determines how big a region must be when it stops expanding in order to collapse against the pressure forces, and the nature of the initial density fluctuations, which determines how likely a region is to stop expanding when it has this size. It turns out that if the equation of state is hard (as applies in all conventional models of the early universe), only fluctuations of a certain type favor pbh formation. Because of this, the pbh spectrum is predicted to always have a particular form. What is remarkable is that both the fluctuations which one might expect to arise naturally and the fluctuations which are often invoked to explain the existence of galaxies are of the type which favor pbh formation. This shows that, in principle, pbh's might exist over a large mass range.

An important feature of the predicted mass spectrum is that it only falls off as a power of the mass. This suggests that there should be at least some pbh's bigger than 10^{15} g and these should still exist today. If the initial density fluctuations are small, the fraction of the universe that goes into such pbh's at the time they form should be tiny. But because the mass in pbh's stays constant while the mass outside them is reduced (because of pressure) as the

* Supported in part by the UK Science Research Council and the US National Science Foundation [MPS75-01398].

universe expands, even a tiny fraction at early times could represent a substantial pbh density now. However, detailed analysis shows that pbh's are unlikely to have a critical density.

Another important feature of the predicted mass spectrum is that there could be an appreciable number density of big pbh's ("big" meaning more than $10^6\ M_\odot$). Although such big black holes might represent only a small fraction of the total mass in pbh's, each of them could by now have bound a large region of surrounding matter through its gravitational Coulomb effect. For example, Ryan (1972) has argued that a black hole of $10^7\ M_\odot$ could since decoupling have captured the mass of a typical galaxy ($10^{11}\ M_\odot$). It is therefore interesting to inquire whether big pbh's could be sufficiently numerous to form the seeds of some, or even most, galaxies. There are several attractive features in such a theory of galaxy formation, and these are discussed in a separate paper (Carr 1975). The viability of the theory depends crucially on the pbh mass spectrum. It turns out that, provided the universe had special (or perhaps even natural) initial conditions, the number density of big pbh's could be comparable to the galactic number density.

Observational data on the form of the pbh spectrum, if available, would give valuable information about the early universe. Pbh's are unique in this respect, since they alone could be expected to survive the dissipative effects which erase all other imprints of conditions in the first second of the universe. Although a knowledge of the pbh spectrum is not available (in fact there is no observational evidence that pbh's exist at all), several observations impose important limits on it. In particular the fraction of the universe in pbh's at early times must have been very small. This places strong constraints on the magnitude of any density fluctuations in the early universe and also supports the assumption that the early universe was not chaotic. Any model in which the early universe has a soft equation of state for a prolonged period (such as Hagedorn's "superbaryon" model) is also shown to be suspect since pbh's probably form too prolifically in such a situation to be consistent with observation.

The plan of this paper is as follows: § II discusses various models of the early universe; § III shows how initial inhomogeneities can develop into pbh's and how pbh's are likely to evolve once they have formed; § IV derives the mass spectrum of pbh's which form when the equation of state is hard; § V discusses the formation of pbh's when the equation of state is soft; § VI describes the observational limits on the pbh spectrum; and § VII assesses various models of the early universe in the light of these limits.

II. MODELS OF THE EARLY UNIVERSE

The early universe can be divided into three eras: 10^{-43}–10^{-23} s (the "prehadron" era), 10^{-23}–10^{-4} s (the "hadron" era), and after 10^{-4} s (the "posthadron" era). General relativity breaks down before 10^{-43} s because of quantum fluctuations in the metric, hadrons become acausal before 10^{-23} s, and the density of the universe exceeds nuclear density before 10^{-4} s. In all the models discussed it will be assumed that the universe is "$k = 0$" Friedmann apart from density fluctuations which are small on scales much bigger than the horizon. The "$k = 0$" assumption is always valid at early times, since the intrinsic curvature of the universe can be neglected then. From the point of view of predicting the pbh mass spectrum, only two features of the early universe are important: the equation of state in successive eras, and the nature of the initial fluctuations.

a) The Equation of State in Successive Eras

During the immediate posthadron era the universe is normally supposed to consist of a relativistic fluid of photons, neutrinos, and leptons with a small admixture of nucleons. The equation of state is $p = \mu/3$, where p is the pressure and μ is the density. Before 10^{-4} s there is spontaneous creation of hadrons, and the equation of state depends upon what picture of particle physics we adopt. We may adopt either the Elementary Particle or Composite Particle picture (henceforth referred to as "EP" and "CP," respectively). The EP picture assumes that all particles are made up of a small number of elementary constituents (such as quarks and various kinds of leptons). Before 10^{-4} s most of the particles are relativistic, so the universe may be treated as comprising a number of different kinds of blackbody radiation. Hence during the hadron era, $p = \mu/3$ continues to apply. The CP picture regards all particles as being made up of each other. There are no elementary particles, and during the hadron era the universe is supposed to consist of a large number of thermodynamically distributed ideal gases. Hagedorn (1970) has suggested a model in which, as $m \to \infty$, the number of particles species in the mass range $(m + dm, m)$ is

$$N(m)dm \to Am^{-B} \exp\left(\frac{m}{kT_m}\right)dm\,, \tag{2.1}$$

where A and B are constants and $T_m \approx 10^{12}$ K. Since the exponential factor balances the $\exp(-m/kT)$ factor which weights thermodynamic quantities, the temperature can never exceed T_m. The constant B is usually taken to be 5/2 or 3. As t decreases, the temperature of the universe increases asymptotically toward T_m and the energy of the universe tends to be concentrated in increasingly massive nonrelativistic particles (rather than in increasingly relativistic particles of fixed mass as in the EP model). In a baryon-symmetric universe this results in a soft equation of state of the form $p \sim \mu_* \log \mu/\mu^*$ for $B = 5/2$ and $p \sim \mu_*$ for $B = 3$. (Here $\mu_* \sim 10^{14}$ g cm^{-3} is the density at 10^{-4} s.) To a good approximation the universe can be regarded as pressureless before 10^{-4} s. In the

$B = 3$ baryon-symmetric model there is the possibility that most of the universe is contained in massive 10^{15} g grains at 10^{-23} s and that the slow decay of these grains results in the soft Hagedorn era extending well beyond 10^{-4} s (Carlitz *et al.* 1974). However, the soft era must not extend beyond 1 year if an equilibrium distribution of photons is to be established and not beyond a few minutes if helium is to be generated primordially. The massive grains also arise in the $B = 5/2$ baryon-symmetric model, but there they always decay by 10^{-4} s. If the universe is not baryon-symmetric, but always has an excess of baryons over antibaryons, the soft equation of state applies only if one permits the existence of particles with arbitrarily high baryon number. Otherwise the equation of state has the semihard form $p \sim \mu/\log \mu/\mu_*$.

Both the EP and CP pictures must be modified before 10^{-23} s because the size of the particle horizon is then less than 10^{-13} cm and hadrons are no longer causal units. In fact Hawking has pointed out that, in the CP picture, general relativity cannot be applied before 10^{-23} s because, if there are an infinite number of particle species, quantum fluctuations in the metric are important then. Thus, in the CP picture, 10^{-23} s should be regarded as the beginning of the classical universe. In the EP picture relativity is applicable in the prehadron era. Although any speculation about the state of matter in the prehadron era obviously transcends our present knowledge of physics, one might assume that the equation of state has the form $p = \gamma\mu$ where γ is a constant between 0 and 1.

b) *The Initial Fluctuations*

In an exact "$k = 0$" Friedmann universe every region has zero total energy; i.e., its negative gravitational energy exactly balances its positive kinetic energy of expansion. If energy fluctuations are introduced, some regions may have sufficiently negative energy to eventually stop expanding and collapse against the pressure forces. In the context of pbh's one is always considering fluctuations which are initially on a scale much bigger than the horizon, and, in such a situation, whether one regards a region as having negative energy because it is overdense or because it has an expansion deficit is a question of how one chooses surfaces of constant time. It will be convenient to choose a time coordinate such that, at some initial time, all energy fluctuations can be regarded solely as density perturbations. It is assumed that these density perturbations and nothing else are responsible for the formation of pbh's. Irregularities such as turbulence and vorticity are neglected since these act only on scales smaller than the horizon and are therefore unlikely to enhance pbh formation. (However, both turbulence and vorticity might prevent the formation of pbh's much smaller than the horizon when a soft equation of state would otherwise permit them.)

In principle the form of density fluctuations in the early universe could be arbitrary, but it is of particular interest to consider two types of fluctuations: (1) those which arise naturally in an initially homogeneous universe, and (2) those which are often postulated as part of the universe's initial conditions in order to explain its present structure. Whereas the second type of fluctuation is acausal, the first type is presumed to develop through causal mechanisms at some definite time. Since the fluctuations which develop earliest have most time to evolve into pbh's, it is crucial to specify when such natural fluctuations first arise. In fact, the natural fluctuations which are stressed in what follows all develop at about the beginning of the classical universe (10^{-43} s in the EP picture, 10^{-23} s in the CP picture). Although such fluctuations are, in a sense, part of the universe's initial conditions, they should still be regarded as "natural."

First consider the density fluctuations which arise naturally in the EP picture. Here one would expect a statistical fluctuation to develop in the number N of elementary particles in a region. For any causally connected region, this fluctuation should have the form $\Delta N \sim \sqrt{N}$. Hence on scales less than the horizon one might expect an overdensity $\delta \equiv \Delta m/m$ in regions of average mass m of the form $\delta \sim (m/m_*)^{-1/2}$ where m_* is the energy of the average elementary particle. In an initially homogeneous universe, a region can contain an excess of particles only at the expense of surrounding regions. Thus the statistical excess of particles in a region larger than the horizon can only derive from the shuffling of particles within horizon volumes at the surface of the region. This will be referred to as the "surface effect" and corresponds to $\Delta N \sim N^{1/3}$. Some authors (e.g., Carlitz *et al.* 1974) have assumed that this corresponds to a density fluctuation of the form $\delta \propto m^{-2/3}$. However, it is shown in Appendix A that this is erroneous. Instead, one must take the fluctuation due to the surface effect as $\delta \sim (m/m_0)^{-7/6}(m_*/m_0)^{1/2}$ where m_0 is the horizon mass at the time when the fluctuation develops. (This is consistent with a recent argument of Peebles [1974*a*] in which he also shows that it is incorrect to take the initial energy fluctuation as $\delta \propto m^{-2/3}$.)

It is not really meaningful to talk about EP fluctuations at times as early as 10^{-43} s because our normal concept of an elementary particle is not appropriate then. However, statistical density fluctuations are automatically built into cosmological models at 10^{-43} s because of quantum fluctuations in the metric. It can be shown (see, for example, Harrison 1970) that these give rise to density fluctuations of the form $\delta \sim \epsilon(m/m_0)^{-2/3}$ where ϵ is an arbitrary constant between 0 and 1 and $m_0 \sim 10^{-5}$ g, the horizon mass at 10^{-43} s. Such fluctuations should not be described as causal, but they are natural and they turn out to be very important in the context of pbh's.

The fluctuations which arise naturally in Hagedorn's model depend on whether or not the universe is baryon-symmetric. They also depend on the value of the parameter B, defined by equation (2.1). In the $B = 5/2$ baryon-symmetric model, matter starts off at 10^{-23} s in 10^{15} g grains, so the fluctuations are initially of order unity on the scale of the horizon. On scales larger than the horizon one then has $\delta \sim (m/m_0)^{-7/6}$, where $m_0 \sim 10^{15}$ g, due to the same "surface effect" which arises in the EP model. In the $B = 3$ baryon-symmetric model the initial fluctuations would be even larger, the initial grains having a mass $(\mu/\mu_*)m_0$ if thermodynamic equilibrium were maintained throughout the hadron era. However, Carlitz *et al.* argue that equilibrium should not exist throughout the hadron era because the initial grains decay too slowly. They assume that the initial grains have a mass of 10^{15} g (the maximum size consistent with causality), so the initial fluctuations have the same form as in the $B = 5/2$ model. The fluctuations in a universe which contains an excess of baryons over antibaryons are the same as those in a baryon-symmetric universe if one believes that particles can have arbitrarily high baryon number. However, if one only believes in particles with baryon number ± 1 or 0, the initial fluctuations are reduced by a factor of 10^{12}. (This factor is just the square root of the ratio of the initial particle number density in a baryon-symmetric universe to that in a universe with the observed local baryon excess.)

It has already been noted that, in the CP picture, quantum fluctuations in the metric should become important at 10^{-23} s. These fluctuations make the applicability of the Hagedorn model in the early hadron era very questionable. But they also introduce density fluctuations of the form $\delta \sim \epsilon(m/m_0)^{-2/3}$ where ϵ is an arbitrary constant between 0 and 1 (as before) and $m_0 \sim 10^{15}$ g. These fluctuations turn out to be more important for the formation of big pbh's than the Hagedorn type fluctuations.

Consider now the type of initial fluctuations which one might want to impose in order to explain the present structure of the universe (i.e., the existence of galaxies and clusters of galaxies). Ideally the required fluctuations might be those which arise naturally, but this may not be the case. Zel'dovich (1972) has pointed out that the fact that both galaxies and clusters of galaxies must have had an overdensity of order 10^{-4} when they first came within the horizon is consistent with initial fluctuations of the form $\delta \sim 10^{-4}(m/m_0)^{-2/3}$ where m_0 is the initial horizon mass. Zel'dovich claims that the same fluctuations, on a smaller mass scale, could have generated the entropy of the universe through dissipation. Such fluctuations could, of course, be regarded as natural if one interprets them as deriving from metric effects. But such an interpretation is not necessary.

Peebles, on the other hand, through considering correlations in the positions of galaxies, has argued that the most likely spectrum of fluctuations at decoupling was one of white-noise (Peebles 1974*b*). This is equivalent to $\delta \propto m^{-1/2}$. Unfortunately, Peebles's considerations apply only to fluctuations on the mass scale of galaxies and above. They say nothing about the fluctuations before decoupling, and these are the ones relevant for pbh formation. Nevertheless, Peebles's speculation is interesting in that one might argue for the natural existence of $\delta \propto m^{-1/2}$ fluctuations even on acausal scales due to the form of $\Delta N \sim \sqrt{N}$ statistical effect discussed earlier, viz., at the beginning of the universe the Creator sprinkles particles randomly over all scales! In fact it will be shown shortly that one cannot permit a $\sqrt{N}$ effect over all mass scales, so this type of picture can be rejected.

In general both natural and imposed fluctuations may be taken to have the form

$$\delta_* = \epsilon\left(\frac{m}{m_0}\right)^{-n}, \tag{2.2}$$

where ϵ and n are constants (at least over a limited mass range) and m_0 is the initial mass within the horizon. More precisely, it will be assumed that the initial overdensity in regions of initial mass m is normally distributed about zero with standard deviation given by equation (2.2). (The claim that a normal distribution applies is certainly justified when dealing with fluctuations which derive from natural statistical effects, but it is an assumption in more general circumstances.) There are two important features of natural fluctuations which are not shared by arbitrary imposed fluctuations. First, the natural fluctuations which have been discussed are all of the form (2.2) with $n = \frac{2}{3}$ or $n = 7/6$ over scales larger than the horizon. With imposed fluctuations, n could be arbitrary and it might be different over different mass ranges. Second, with natural fluctuations, there may be an imposed relationship between the value of ϵ and the mass of the initial grains. For example, a grain mass m_* gives rise to a surface effect with $\epsilon \sim (m_*/m_0)^{1/2}$, so $\epsilon \sim 1$ if and only if the grains have the horizon mass (in which case they are themselves black holes). There may also be a relationship between ϵ and the equation of state: in Hagedorn's picture either $\epsilon \sim 1$ and the equation of state is soft, or $\epsilon \sim 10^{-12}$ and the equation of state is hard. With imposed fluctuations there need be no connection between ϵ, m_*, and the equation of state.

An important point in the context of either natural or imposed fluctuations is that fluctuations of the form (2.2) with $n < \frac{2}{3}$ cannot be permitted over arbitrarily large mass scales. This is because any fluctuation with $\delta \sim (m/m_0)^{-2/3}$ will form a separate closed universe (this is justified in the next section) and therefore it is meaningless to talk about fluctuations which exceed $(m/m_0)^{-2/3}$. But any fluctuation with $n < \frac{2}{3}$ will exceed $(m/m_0)^{-2/3}$ for large enough values of m. This imposes definite limitations on the form of the initial fluctuations in our particular universe. In the first place the expected fluctuation can only have the form $\delta_* = \epsilon(m/m_0)^{-n}$ with $n < \frac{2}{3}$ for m less than some value of order $m_0\epsilon^{1/(n-2/3)}$. For values of m larger than this, the value of n must change to a value not less than $\frac{2}{3}$ (ϵ must also change). In the second place, whatever the value of n, for fixed m the distribution of δ cannot be exactly normal: there must be an upper cutoff in the distribution at $\delta \sim (m/m_0)^{-2/3}$. These considerations show that one can exclude, in particular, the existence of $\sqrt{N}$ fluctuations over arbitrarily large mass scales (unless ϵ is so small that $m_0\epsilon^{1/(n-2/3)}$ exceeds the mass of the universe). Since there is no natural reason why the $\sqrt{N}$ fluctuations should be cut off, one can probably reject $\sqrt{N}$ fluctuations on scales bigger than the horizon altogether.

Finally, it should be mentioned that Mészáros (1974) has argued that the formation of pbh's could itself give rise to appreciable density fluctuations. The idea is that, once some pbh's have formed, there will be a fluctuation in the number of them contained in any larger region. In fact, it is shown in Appendix B that this effect is unimportant on scales larger than the horizon and that, consequently, the fluctuations which Mészáros invokes cannot be important for the formation of later generations of pbh's.

III. THE EVOLUTION OF DENSITY PERTURBATIONS INTO PBH'S

It was shown in a previous paper (Carr and Hawking 1974) that an overdense region in a $k = 0$ Friedmann universe will overcome pressure and collapse to a black hole provided that the size of the region when it stops expanding, R_c, is bigger than the Jeans length, R_J, but not larger than the size of the particle horizon, R_h. The last condition ensures that the region is not a separate universe. When the equation of state is hard, $R_J \sim R_h$, so $R_c \sim R_h$ and the region falls inside its Schwarzschild radius at about the time it stops expanding. Thus there is no question of vorticity or turbulence preventing black hole formation. In the soft era of Hagedorn's model, $R_J \sim R_h(\mu/\mu_*)^{-1/2}$ so a pbh can be much smaller than the particle horizon at formation. However, were it much smaller than the horizon, the region forming it would have had to collapse some way before falling inside its Schwarzschild radius, and centrifugal and turbulent effects might prevent it from collapsing sufficiently.

In the following considerations it is assumed that gravity always dominates the evolution of an overdense region into a pbh. This assumption is very reasonable for a pbh of more than 10^{15} g: strong interactions can be neglected since the region is always bigger than 10^{-13} cm (the strong interaction range) until the pbh forms. For regions destined to become pbh's smaller than 10^{15} g, the assumption is less reasonable. Although such regions are always bigger than the particle horizon until they become pbh's, they are still presumably affected by the strong-interaction Coulomb field. On the other hand, the particles which mediate the strong interactions are themselves bigger than the particle horizon before 10^{-23} s (when these very small pbh's form), so our normal concepts of strong interaction break down. It will merely be assumed, without justification, that gravity dominates the evolution of regions smaller than 10^{15} g into pbh's. However, this assumption, and hence any comments about pbh's which form in the prehadron era (the ones which would have evaporated by now), is very questionable.

We now discuss how overdense gravity-dominated regions evolve into pbh's. In a Friedmann universe, regions separated by distances larger than the particle horizon behave just like parts of different Friedmann models. Thus the density difference between two regions, one of which has an energy perturbation and the other of which is unperturbed, is controlled by how the regions evolve separately (Peebles 1967). In the unperturbed region the metric may be written in the form

$$ds^2 = dt^2 - R^2[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{3.1}$$

where r and t are comoving coordinates, and where R evolves according to

$$\left(\frac{dR}{dt}\right)^2 = \frac{C}{R^{1+3f}} \Rightarrow R \propto t^{2/3(1+f)} . \tag{3.2}$$

Here C is a constant and f specifies the equation of state, $p = f\mu$. In the perturbed region the metric may be written as

$$ds^2 = d\tau^2 - S^2[(1 - \kappa r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)] , \tag{3.3}$$

where κ is the perturbed total energy per unit mass and the time coordinate τ is proper time as measured by comoving observers. One can also define another time t which defines spatial hypersurfaces orthogonal to the matter flow and which coincides with the time coordinate of equation (3.1) in the unperturbed region. It is convenient to choose the initial surfaces $t = t_0$ and $\tau = \tau_0$ to be the same and to choose the scale factors S and R such that $S_0 = R_0$ and $(dS/d\tau)_0 = (dR/dt)_0$ (where a subscript zero denotes values at t_0). Then initially the overdense region is comoving with the unperturbed background and the total energy perturbation is just the density perturbation. (It should be stressed that the density perturbation is a gauge-dependent quantity: it could be given an arbitrary value with some other choice of time coordinate.) The evolution of S in equation (3.3) is given by

$$\left(\frac{dS}{d\tau}\right)^2 = (C + \Delta C)\frac{1}{S^{1+3f}} - \kappa = C\left(\frac{1 + \delta_0}{S^{1+3f}} - \frac{\delta_0}{R_0^{1+3f}}\right) , \tag{3.4}$$

where δ_0 is the initial density contrast, $\Delta C/C$. It can be shown (Harrison 1970) that τ and t are related by

$$R(1 + \delta_0)^{f/(1+f)}d\tau = Sdt \tag{3.5}$$

and that the time and size of the overdense region when it begins to collapse are

$$t_c \sim t_0\delta_0^{-3(1+f)/[2(1+3f)]} , \tag{3.6}$$

$$S_c \sim R_0\delta_0^{-1/(1+3f)} . \tag{3.7}$$

Equations (3.6) and (3.7) allow the condition for pbh formation ($R_h > S_c > R_J$ at t_c) to be expressed in terms of conditions on δ_0 and R_0 at t_0. When the equation of state is hard, the condition for pbh formation can be written as

$$\alpha R_h > S_c > \beta R_J\,, \tag{3.8}$$

where α and $\beta \sim \sqrt{f}$ are constants of order unity. From equations (3.6) and (3.7) this requires

$$\alpha > \delta_0^{1/2}(R_0/t_0) > \beta\,, \tag{3.9}$$

a condition which is independent of the parameter which prescribes the equation of state. It can be shown that equation (3.9) is still the condition for pbh formation even if f changes in value during the region's expansion phase. Equation (3.9) can be written in terms of the initial mass of the region, m, as

$$\delta_{\max} \equiv \alpha^2\left(\frac{m}{m_0}\right)^{-2/3} > \delta_0 > \beta^2\left(\frac{m}{m_0}\right)^{-2/3} \equiv \delta_{\rm bh}\,. \tag{3.10}$$

The left-hand side of this inequality stresses that the initial overdensity must be less than $\sim(m/m_0)^{-2/3}$ if the region is not to form a separate universe. It is not meaningful to talk of fluctuations which exceed $(m/m_0)^{-2/3}$, and, as explained in § II, this imposes limits on the form of initial fluctuations in our particular universe. Equation (3.10) also demonstrates that the condition that a region evolve into a pbh in a hard era is (to an order of magnitude) the same as the condition that it form as a separate universe.

Condition (3.10) can be derived more directly from considerations of the gauge-invariant quantity

$$E \equiv \frac{1}{2}\left(\frac{dR}{d\tau}\right)^2 - \frac{GM}{R} \tag{3.11}$$

which represents the total (kinetic plus gravitational) energy per unit mass of a region of initial mass M and size R. Any region with E negative will have a 3-curvature $R^* \sim EM/R^2$. If the region over which E is negative is so large that $EM \sim 1$, the region closes up on itself. Density fluctuations δ correspond to energy fluctuations $|E| \sim \delta^{3/2}$, so we get the same condition for a separate universe as is indicated by equation (3.10). The fact that equation (3.10) is really an energy condition explains why it is independent of the equation of state.

When the equation of state is soft, the condition for pbh formation must be modified. In the Hagedorn model the Jeans length is

$$\begin{aligned} R_J &\approx R_h\left(\frac{t}{t_{\rm H}}\right)\left(\log\frac{\mu}{\mu_*}\right)^{1/2} && \text{if } B = 5/2 \text{ for } t < t_{\rm H} \\ &\approx R_h\left(\frac{t}{t_{\rm H}}\right) && \text{if } B = 3 \quad \text{for } t < t_{\rm H} \end{aligned} \tag{3.12}$$

where $t_{\rm H}$ denotes the end of the Hagedorn era. The logarithmic term goes from 10 to 1 as t goes from 10^{-23} to 10^{-4} s, so to a good approximation it can be neglected. Therefore the condition for pbh formation can be written as

$$\alpha > S_c/R_h > t_c/t_{\rm H}\,. \tag{3.13}$$

From equations (3.6) and (3.7) with $f = 0$, this is equivalent to

$$\alpha^2\left(\frac{m}{m_0}\right)^{-2/3} > \delta_0 > \left(\frac{t_0}{t_{\rm H}}\right)^{1/2}\left(\frac{m}{m_0}\right)^{-1/6}. \tag{3.14}$$

In fact, equations (3.6) and (3.7) do not apply exactly if the overdense region has fallen inside the particle horizon at t_c since boundary conditions (i.e., diffusion effects) must be taken into account. However, they should still give a good approximation for t_c and S_c if collapse is not prevented altogether.

Once a region has collapsed to a pbh, its further evolution is determined by two factors: the tendency of the black hole to evaporate due to Hawking's process, and the tendency of the black hole to grow through accretion. The first effect is negligible for any pbh of more than 10^{15} g, i.e., for any pbh formed after the prehadron era. The second effect is unlikely to cause a pbh to increase its mass by even one order of magnitude. This is a stronger statement than that made in a previous paper (Carr and Hawking 1974) that a pbh could not grow through accretion at the same rate as the universe, and it is justified by the following argument. Since accreted material will be crossing the Schwarzschild radius $R_{\rm S}$ at about the speed of light, the pbh will increase its mass at a rate

$$\frac{dM}{dt} \sim \mu R_{\rm S}^2 \sim \mu M^2 \sim t^{-2}M^2\,. \tag{3.15}$$

Integrating this equation gives

$$M \sim t \bigg/ \left[1 + \frac{t}{t_1}\left(\frac{t_1}{M_1} - 1\right)\right], \tag{3.16}$$

where M_1 is the mass of the pbh at any time t_1. Equation (3.15) is applicable only if the black hole is considerably smaller than the particle horizon at t_1, i.e., if $M_1 \sim \eta t_1$, where η is some fraction less than 1. Otherwise the expansion of the universe has to be taken into account. Thus as t tends to infinity, M tends to $M_1/(1 - \eta)$. Since η cannot be too close to 1, the black hole cannot increase its mass even by one order of magnitude.

In the case of a big pbh this conclusion should be treated with caution. A big pbh will tend to capture a lot of neighboring material through its gravitational Coulomb effect. The captured material will have some angular momentum and so will not necessarily fall into the black hole. Under these circumstances the density in the vicinity of the black hole might soon be much larger than the mean density in the universe, so equation (3.15) might considerably underestimate the accretion rate.

IV. THE MASS SPECTRUM OF PBH'S FORMED IN A HARD ERA

We first derive the spectrum of pbh's formed in eras when the equation of state is hard, since in this situation one does not have the complications which arise when pbh's can form smaller than the horizon. If we consider a spherical region at the beginning of the universe with initial radius R_0 much bigger than the particle horizon and initial mass $m \sim \mu_0 R_0^3$, then, from equation (3.10), the question of whether it evolves into a pbh is essentially one of comparing the expected fluctuation $\delta_* = \epsilon(m/m_0)^{-n}$ to $(m/m_0)^{-2/3}$. Effectively no pbh's are formed for values of m such that $\delta_* \ll (m/m_0)^{-2/3}$, but half the regions of mass m should evolve into pbh's if $\delta_* \sim (m/m_0)^{-2/3}$. The various possibilities are illustrated in Figure 1, which compares δ_* with $\delta_{\rm bh}$ and $\delta_{\rm max}$ for different values of n and ϵ. In the $n > \frac{2}{3}$ situation pbh formation is only favored if $\epsilon \sim 1$ and then only for $m \sim m_0$. In the $n = \frac{2}{3}$ situation whether pbh formation is favored depends sensitively on the value of ϵ: if $\epsilon \ll 1$, so that the δ_* line in Figure 1*b* is displaced some way below the $\delta_{\rm bh}$ line, pbh formation is unlikely on any mass scale; but if $\epsilon \sim 1$, so that the δ_* line is very close to or even above the $\delta_{\rm bh}$ line (but necessarily below $\delta_{\rm max}$), pbh formation is favored on all mass scales for which the $n = \frac{2}{3}$ fluctuation persists. In the $n < \frac{2}{3}$ situation no pbh's form for $m < m_0\epsilon^{1/(n-2/3)}$; when m exceeds $m_0\epsilon^{1/(n-2/3)}$, the form of the fluctuations must be modified. Therefore essentially no pbh's derive from $n < \frac{2}{3}$ fluctuations, since pbh formation is favored only in the mass range where such fluctuations are forbidden. Of course, one might choose to modify δ_* somewhere between the values of m at which it intersects $\delta_{\rm bh}$ and $\delta_{\rm max}$, and then there would be a small range of m in which pbh formation is favored. But in this case the formation of pbh's will be favored at higher values of m (where there is a new value of n) anyway, so including the effect of the $n < \frac{2}{3}$ part of the spectrum only slightly extends the pbh's mass range.

These qualitative considerations make it clear that only $n = \frac{2}{3}$ fluctuations favor the formation of pbh's over an extended mass range. Therefore, if pbh's exist over an extended mass range, their spectrum will always be that which derives from the $n = \frac{2}{3}$ situation. It is interesting that both metric fluctuations and the fluctuations which Zel'dovich postulates to explain galaxies have the $n = \frac{2}{3}$ form. Thus, in principle, one might expect the existence of some pbh's over a large mass range. How many pbh's form depends on the value of ϵ. With arbitrary initial fluctuations, there would be several mass ranges in which the fluctuations have the $n = \frac{2}{3}$ form (Fig. 1*d*). In each such range the spectrum will be cut off at the points where the value of n changes from $\frac{2}{3}$.

In deriving the mass spectrum itself, the probability that a spherical region of initial mass m has a density contrast in the range $(\delta + d\delta, \delta)$ is assumed to be (cf. Press and Schechter 1974)

$$P(\delta, m)d\delta = (2\pi)^{-1/2}\delta_*(m)^{-1} \exp\left(-\frac{\delta^2}{2\delta_*(m)^2}\right)\theta\left[\alpha\left(\frac{m}{m_0}\right)^{-2/3} - \delta\right]d\delta\,. \tag{4.1}$$

The θ term (θ denotes the Heaviside function) represents the fact that the distribution is cut off above $\delta_{\rm max}$. The remaining terms correspond to a normal distribution about zero with standard deviation $\delta_*(m)$. Using equations (2.2) and (3.10), the probability that a region of initial mass m becomes a pbh is

$$P(m) = \int_{\Xi\Psi}^{\Upsilon\Psi} (2\pi)^{-1/2} \exp(-x^2)dx\,, \tag{4.2}$$

where $\Psi \equiv (m/m_0)^{n-2/3}$, $\Xi \equiv \beta^2/\epsilon\sqrt{2}$, and $\Upsilon \equiv \alpha^2/\epsilon\sqrt{2}$. When $n < \frac{2}{3}$,

$$P(m) \sim \epsilon\left(\frac{m}{m_0}\right)^{2/3-n} \exp\left[-\frac{\beta^4}{2\epsilon^2}\left(\frac{m}{m_0}\right)^{2n-4/3}\right] \tag{4.3}$$

in the permitted mass range $m_0 < m < m_0\epsilon^{1/(n-2/3)}$, so there is an exponential lower cutoff at $m \sim m_0(\epsilon/\beta^2)^{1/(n-2/3)}$. When $n > \frac{2}{3}$, $P(m)$ also has the form (4.3), so there is an exponential upper cutoff at $m_0(\epsilon/\beta^2)^{1/(n-2/3)}$. If $\epsilon < \beta^2 \sim f$, the upper cutoff is below m_0 so that $P(m) \sim 0$ for all values of m. But if $\epsilon > \beta^2$, pbh's can form for $m \sim m_0$.

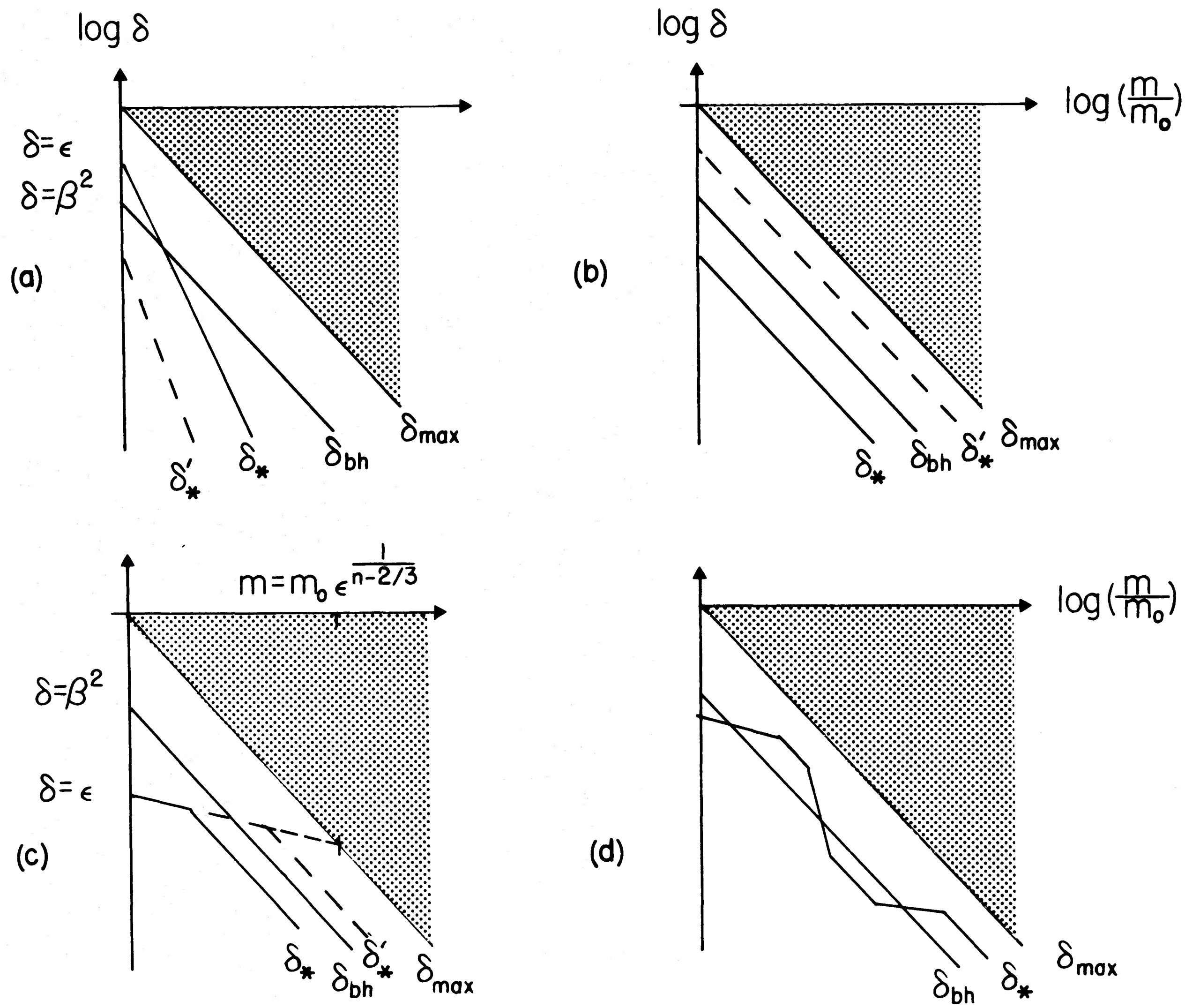

FIG. 1.—The diagrams compare the expected fluctuation, δ_*, with the fluctuation required to form a pbh in a hard era, δ_{bh}, and the fluctuation required to form a separate universe, δ_{max}. Fluctuations in the shaded regions are forbidden. (*a*) The $n > \frac{2}{3}$ case. (*b*) The $n = \frac{2}{3}$ case. (*c*) The $n < \frac{2}{3}$ case (where the form of δ_* must change before it enters the forbidden region). (*d*) The case where the fluctuations are imposed arbitrarily.

These results reflect the qualitative features of Figures 1*a* and 1*c*. In the interesting case $n = \frac{2}{3}$, equation (4.2) gives

$$P(m) \sim \int_{\Xi}^{\Upsilon} \exp(-x^2)dx \sim \epsilon \exp\left(-\frac{\beta^4}{2\epsilon^2}\right) \qquad \text{if } \epsilon \ll 1 . \tag{4.4}$$

Thus $P(m)$ is constant and is very small if ϵ is much less than $\beta^2 \sim f$. Essentially this is because, with $n = \frac{2}{3}$ fluctuations, regions always have an expected overdensity ϵ when they first come within the horizon.

Once a pbh has formed, nothing precludes a larger region which contains it from forming a pbh at some later time. To determine the final pbh spectrum we require the probability that a region with initial mass in the range $(m + dm, m)$ will become an ultimate black hole, i.e., a black hole which does not later become part of some larger black hole. Writing this probability as $\Pi(m)dm$, $P(m)$ and $\Pi(m)$ are related by

$$\int_m^\infty \Pi(m')dm' = P(m) + \int_m^\infty \text{prob} \begin{pmatrix} \text{ultimate pbh of mass in} \\ (m' + dm', m') \text{ contains } m \end{pmatrix} dm' . \tag{4.5}$$

When $P(m) \ll 1$, the probability that there is at least one ultimate pbh of mass within $(m' + dm', m')$ containing a particular black hole of mass m can be shown to be

$$\mathrm{prob}\,(m', m)dm' = \frac{\Pi(m')}{m'}(m'^{1/3} - m^{1/3})^3 dm' \qquad \text{if } m < m' \ll \frac{m}{P(m)}\,. \tag{4.6}$$

Hence equation (4.5) can be written as

$$P(m) = \int_m^\infty \Pi(m')\left[3\left(\frac{m}{m'}\right)^{1/3} - 3\left(\frac{m}{m'}\right)^{2/3} + \left(\frac{m}{m'}\right)\right]dm'\,. \tag{4.7}$$

When $n = \frac{2}{3}$, this gives

$$\Pi(m) \sim m^{-1}\epsilon \exp(-\beta^4/2\epsilon^2)\,. \tag{4.8}$$

When $n \neq \frac{2}{3}$, $\Pi(m)$ has the same exponential cutoffs as $P(m)$. The one situation in which equation (4.6) and the consequent equations do not apply is when $n = \frac{2}{3}$ and $\epsilon \sim 1$. In this case $P(m) \sim \frac{1}{2}$ for all m and half the mass in the universe is always in pbh's. It will be shown in § VI that this situation is not consistent with observation and therefore it will not be discussed further.

Now the mass of a region when it forms a pbh (i.e., the mass of the region when it begins to collapse, m_c) is related to its initial mass, m, by

$$m_c \sim \mu_c R_c^3(1 + \delta_c) \sim \mu_c R_c^3 \sim m\delta_0^{3f/(1+3f)}\,. \tag{4.9}$$

While the region is expanding, some of its mass is redshifted away (as it is for the rest of the universe); but once the black hole has formed, its mass will stay constant. Since $\delta_0 \sim (m/m_0)^{-2/3}$ for pbh formation, the final mass of all pbh's deriving from regions of initial mass m is

$$m_c \sim m^{(1+f)/(1+3f)} m_0^{2f/(1+3f)}\,. \tag{4.10}$$

This equation must be modified if f changes in value during the region's expansion phase. For example, in the Hagedorn model f changes from 0 to $\frac{1}{3}$ at the end of the Hagedorn era, t_{H}; in the EP picture f may change from some value γ to $\frac{1}{3}$ at 10^{-23} s. For pbh's which form after the transition in f, we have

$$\begin{aligned} m_c &\sim m^{(1+f)/(1+3f)} m_0^{2f/(1+3f)}\left(\frac{t_{\mathrm{H}}}{10^{-23}\,\mathrm{s}}\right)^{1/3} && \text{in Hagedorn's model} \\ &\sim m^{(1+f)/(1+3f)} m_0^{2f/(1+3f)}\,\mathrm{dex}\left[\frac{20(1-3\gamma)}{3(1+\gamma)}\right] && \text{in EP model}\,. \end{aligned} \tag{4.11}$$

The present number density of pbh's in the mass range $(m_c + dm_c, m_c)$ is

$$n(m_c)dm_c = \Pi[m(m_c)]\mu_0 F m_c^{-1} dm_c\,, \tag{4.12}$$

where $F \equiv [R(t_0)/R_{\mathrm{now}}]^3$ is the ratio of the number density now to what it would have been at t_0. Two other interesting functions are the present cumulative number density and cumulative mass density of all pbh's with mass greater than m_c. These are, respectively,

$$N(m_c) = \int_{m_c}^\infty \Pi[m(m_c)]\mu_0 F m_c^{-1} dm_c \tag{4.13}$$

and

$$M(m_c) = \int_{m_c}^\infty \Pi[m(m_c)]\mu_0 F dm_c\,. \tag{4.14}$$

These formulae only apply for $m_c > 10^{15}$ g since pbh's smaller than 10^{15} g would no longer exist according to Hawking. Substituting for Π, and dropping the subscript c (so that m is now the final mass of the pbh), we find that in the $n = \frac{2}{3}$ case

$$n(m) \sim \mu_0 F m_0^{-2}\epsilon \exp\left(-\frac{\beta^4}{2\epsilon^2}\right)\left(\frac{m}{m_0}\right)^{-(1+3f)/(1+f)-1}, \tag{4.15}$$

$$N(m) \sim \mu_0 F m_0^{-1}\epsilon \exp\left(-\frac{\beta^4}{2\epsilon^2}\right)\left(\frac{m}{m_0}\right)^{-(1+3f)/(1+f)}, \tag{4.16}$$

$$M(m) \sim \mu_0 F\epsilon \exp\left(-\frac{\beta^4}{2\epsilon^2}\right)\left(\frac{m}{m_0}\right)^{-2f/(1+f)}. \tag{4.17}$$

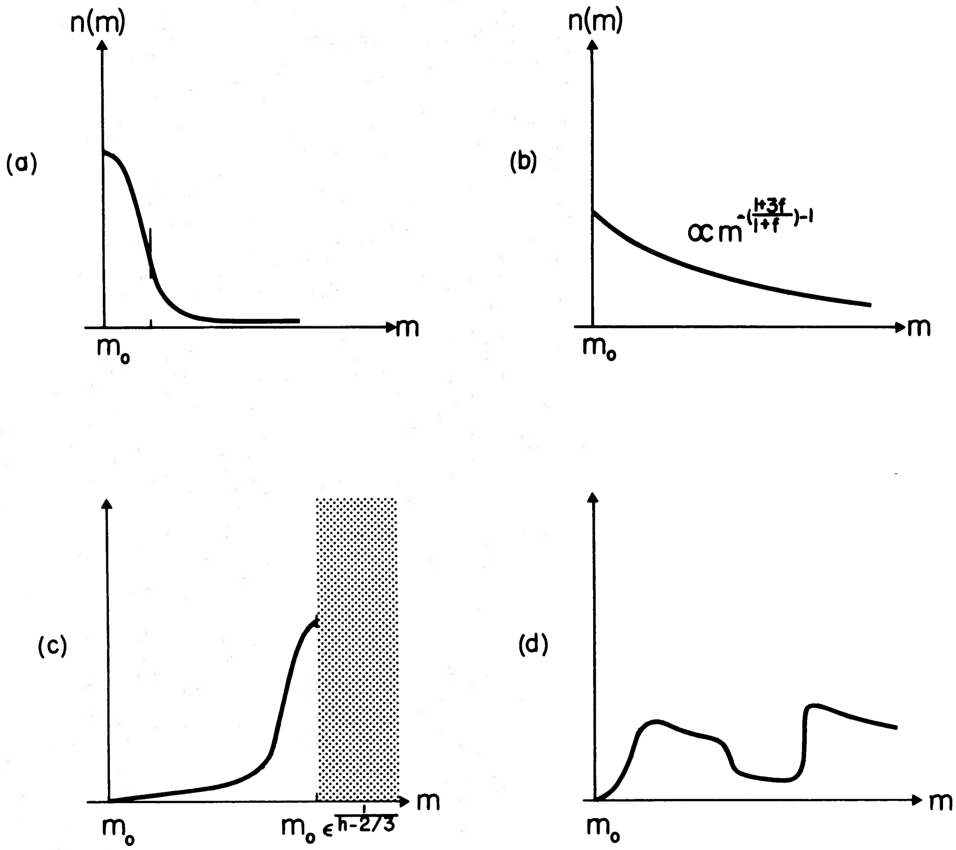

FIG. 2.—The diagrams show the form of the pbh spectrum for the four situations represented in Fig. 1. In (*a*) ($n > \frac{2}{3}$) the spectrum has an upper cutoff which cannot be much larger than m_0. In (*b*) ($n = \frac{2}{3}$) the spectrum falls off as a power of *m*. In (*c*) ($n < \frac{2}{3}$) the spectrum has a lower cutoff which is only slightly below the forbidden region. In (*d*) (arbitrary fluctuations) there are several mass ranges in which $n = \frac{2}{3}$ and the spectrum falls off like a power law in these ranges. (The power law will always be the same unless the equation of state changes.)

For arbitrary fluctuations, like those shown in Figure 1*d*, $N(m)$ and $M(m)$ may have several contributions of the form (4.16) and (4.17), and these must be summed. The form of the spectrum, $n(m)$, is illustrated in Figure 2 for the various situations shown in Figure 1.

The derivation of the above equations is based on statistical arguments and really only applies for pbh's considerably more massive than m_0. The number density of pbh's which are not much larger than m_0 depends on the nature of the particles or grains which exist at t_0. In the context of the EP picture, Sarfatt (1972) has proposed that everything must start off at 10^{-43} s in 10^{-5} g black holes because only a 10^{-5} g black hole is smaller than the particle horizon and bigger than its Compton wavelength then. Since 10^{-5} g black holes would explode almost immediately, there is no reason why the universe should not have started off like this. The initial 10^{15} g grains which characterize Hagedorn's model are also marginally black holes. Indeed, the slowly decaying grains of Carlitz *et al.* (1973) could be interpreted as 10^{15} g black holes which are evaporating by the Hawking process.

The application of the above equations to pbh's which form in the prehadron era is very dubious, for the reasons discussed in § III. Therefore, only four features of the mass spectrum will be discussed: the parameters

$$\Omega_B \equiv \frac{M(10^{15}\ \mathrm{g})}{\mu_{\mathrm{cr}}}\,, \qquad \Omega_G \equiv \frac{M(10^{39}\ \mathrm{g})}{\mu_{\mathrm{cr}}}\,, \qquad m_{\max}\,, \quad m_{\min} \tag{4.18}$$

If small black holes explode, Ω_B gives the present density of pbh's in units of critical density, $\mu_{\mathrm{cr}} \sim 10^{-29}$ g cm^{-3}. Ω_G gives the present density of pbh's bigger than $10^6\ M_\odot$, this being an important parameter for pbh seed theory of galaxy formation referred to in the Introduction. The parameters $m_{\max}$ and $m_{\min}$ prescribe the mass range in which the spectrum applies. They are arbitrary in the context of imposed fluctuations. In the context of natural $n = \frac{2}{3}$

fluctuations the spectrum has a natural upper cutoff at the value of m which corresponds to a present cumulative number density of one per universe, i.e.,

$$m_{\max} \approx \operatorname{dex}\left[\frac{1+f}{1+3f}(40 + \log_{10}\Omega_B) - 18\right] M_\odot . \tag{4.19}$$

The spectrum has no natural lower cutoff except m_0. In the EP model $t_0 \sim 10^{-43}$ s, $m_0 \sim 10^{-5}$ g, and $\mu_0 \sim 10^{94}$ g cm^{-3}. Assuming $f = \gamma$ in the prehadron era and $f = \frac{1}{3}$ thereafter until decoupling, equation (3.2) gives

$$F \sim \operatorname{dex}\left(-66 - \frac{40}{1+\gamma}\right) .$$

Therefore from equation (4.17)

$$\Omega_B \sim 10^{17}\epsilon \exp\left(-\frac{\beta^4}{2\epsilon^2}\right) \quad \text{if } n = \tfrac{2}{3} , \tag{4.20}$$

$$\Omega_G \sim 10^{5}\epsilon \exp\left(-\frac{\beta^4}{2\epsilon^2}\right) \quad \text{if } n = \tfrac{2}{3} . \tag{4.21}$$

If $n \neq \frac{2}{3}$, both factors are zero. In the CP model $t_0 \sim 10^{-23}$ s, $m_0 \sim 10^{15}$ g, $\mu_0 \sim 10^{54}$ g cm^{-3}, and equation (3.2) gives

$$F \sim 10^{-75}\left(\frac{t_{\rm H}}{10^{-4}}\right)^{-1/2} ,$$

where $t_{\rm H}$ is the end of the soft Hagedorn era in seconds. From equation (4.17)

$$\Omega_B \sim 10^{8}\left(\frac{t_{\rm H}}{10^{-4}}\right)^{-1/2}\epsilon \exp\left(-\frac{\beta^4}{2\epsilon^2}\right) \quad \text{if } n \geq \tfrac{2}{3} , \tag{4.22}$$

$$\Omega_G \sim 10^{4}\epsilon \exp\left(-\frac{\beta^4}{2\epsilon^2}\right) \quad \text{if } n = \tfrac{2}{3} \text{ and } t_{\rm H} < 10 . \tag{4.23}$$

If $n \neq \frac{2}{3}$, Ω_G is zero. It should be appreciated that, if $n = \frac{2}{3}$, the fraction of the universe that goes into pbh's of any mass is always of order $\epsilon \exp(-\beta^4/2\epsilon^2)$ at the time the pbh's form. Although this factor may be very small, Ω_B and Ω_G may be appreciable, since μ_0 is large ($\mu_0 \sim 10^{94}$ g cm^{-3} in the EP model, 10^{54} g cm^{-3} in the CP model). The dependence of Ω_B and Ω_G on the parameter f stems only from the fact that regions lose mass through redshift before they form pbh's. For pbh's formed in a soft era, $M(m)$ would have only a logarithmic dependence on m. However, when $f = 0$, pbh's may be smaller than the horizon at formation, so equation (4.17) may not be applicable.

V. THE FORMATION OF PBH'S IN A SOFT ERA

When the equation of state is soft, the Jeans length is much smaller than the horizon size, so the considerations of the previous section do not apply to pbh's which form after decoupling or during the soft era of the Hagedorn model. Even in these situations centrifugal and turbulent effects might prevent the formation of pbh's much smaller than the horizon, in which case the spectrum would be described by equation (4.15) with $f = 0$. However, if dissipation is sufficiently efficient, pbh's might form as small as the Jeans length, and we now examine what consequences this would have. In the Hagedorn situation the consequences will be shown to be inconsistent with observation, so the spectrum will not be derived in detail. The situation in which pbh's form after decoupling is not considered, since several extra processes are involved then.

From equation (3.14) the question of whether a region of mass m evolves into a pbh in the Hagedorn era is one comparing δ_* to $(t_0/t_{\rm H})^{1/2}(m/m_0)^{-1/6}$. The various situations which favor pbh formation are shown in Figure 3.

i) If $n > \frac{2}{3}$ and $\epsilon > (t_0/t_{\rm H})^{1/2}$, pbh formation is favored in the mass range $1 < m/m_0 < [\epsilon(t_{\rm H}/t_0)^{1/2}]^{1/(n-1/6)}$.

ii) If $n = \frac{2}{3}$ and $\epsilon > (t_0/t_{\rm H})^{1/2}$, pbh formation is favored in the range $1 < m/m_0 < \epsilon^2(t_{\rm H}/t_0)$; if $\epsilon \sim 1$, it is also favored for $m/m_0 > \epsilon^2(t_{\rm H}/t_0)$.

iii) If $\frac{2}{3} > n > 1/6$ and $\epsilon > (t_{\rm H}/t_0)^{n-2/3}$, pbh formation is favored in the range $1 < m/m_0 < \epsilon^{1/(n-2/3)}$ but the form of δ_* must be modified when $m/m_0 > \epsilon^{1/(n-2/3)}$ to avoid forming separate universes.

iv) If $n < 1/6$ and $(t_{\rm H}/t_0)^{-1/2} > \epsilon > (t_{\rm H}/t_0)^{n-2/3}$, pbh formation is favored in the range $[\epsilon(t_{\rm H}/t_0)^{1/2}]^{1/(n-1/6)} < m/m_0 < \epsilon^{1/(n-2/3)}$.

v) If $n < 1/6$ and $\epsilon > (t_{\rm H}/t_0)^{-1/2}$, pbh formation is favored in the range $1 < m/m_0 < \epsilon^{1/(n-2/3)}$.

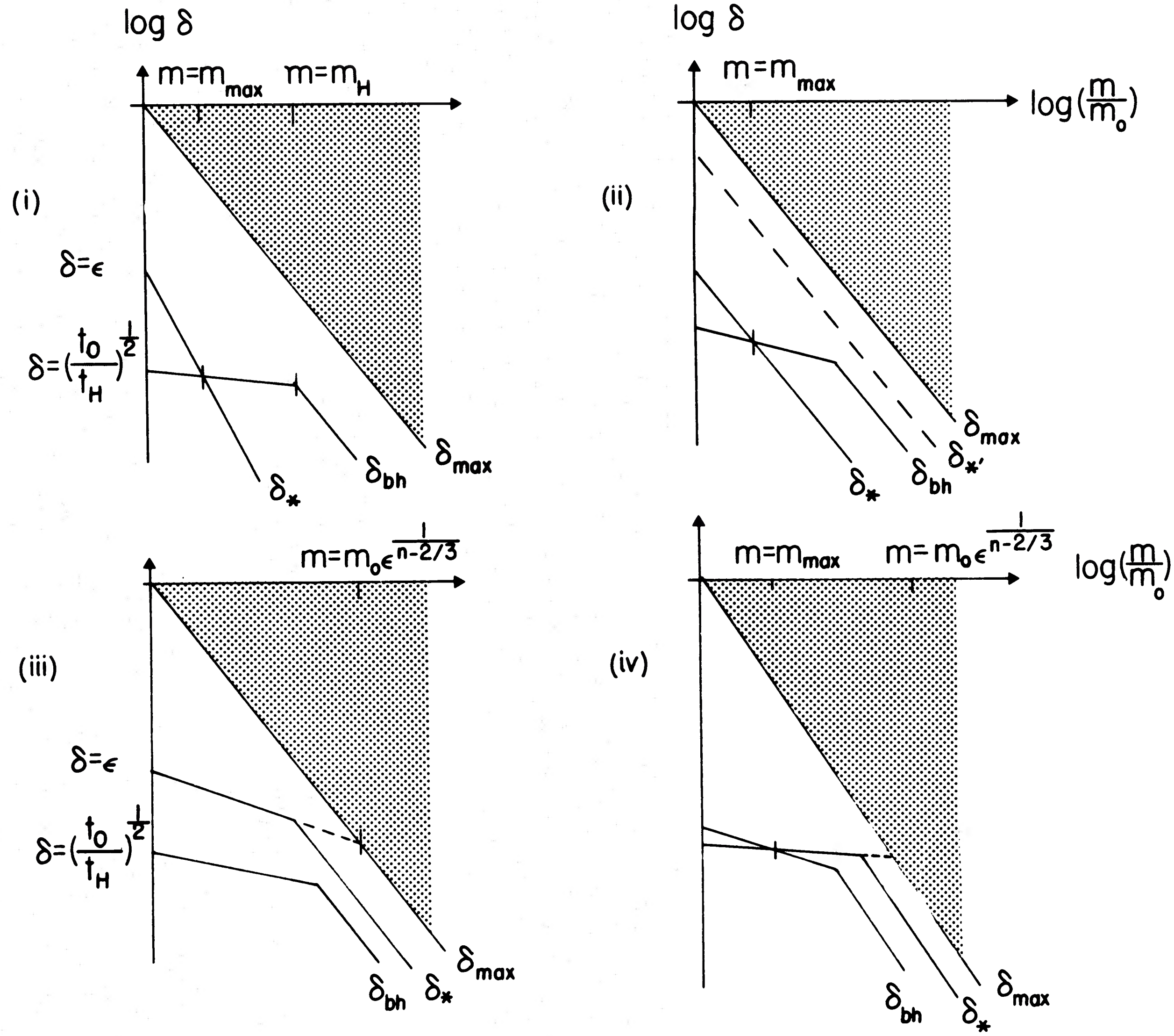

FIG. 3.—The diagrams compare the expected fluctuation, δ_*, with the fluctuation required to form a pbh in a soft era, δ_{bh}, and the fluctuation required to form a separate universe, δ_{max}. Fluctuations in the shaded regions are forbidden. At $m = m_H$ (the horizon mass at t_H, the end of the Hagedorn era) the equation of state goes hard. The four situations shown all favor pbh formation in some mass range, and they correspond to the first four situations discussed in the text.

Unless ϵ is so large that pbh formation is favored in the hard post-Hagedorn era, the spectrum always has an exponential upper cutoff at

$$m_{max} \sim m_0 \left[10\epsilon \left(\frac{t_H}{t_0} \right)^{1/2} \right]^{(n-1/6)} . \tag{5.1}$$

If the Hagedorn era is prolonged, m_{max} can be very large. For example, in Zel'dovich's picture ($n = \frac{2}{3}$, $\epsilon \sim 10^{-4}$) with $t_H \sim 1$ year, $m_{max} \sim 10^6\ M_\odot$. It is no longer true that only $n = \frac{2}{3}$ fluctuations favor pbh formation. Indeed, the most important feature of the above considerations is that, if $\epsilon > (t_0/t_H)^{1/2}$, half the mass of the universe should go into pbh's whatever the value of n. In § VI it is shown that such a situation is inconsistent with observation unless the radiation era is postponed almost until decoupling. Apart from this possibility, one would have to draw one of three conclusions: (1) that the universe never had a soft equation of state; (2) that ϵ is so small [$<(t_0/t_H)^{1/2} \sim 10^{-10}$ if $t_H \sim 10^{-4}$ s] that no pbh's form anyway; or (3) that some mechanism prevents pbh's forming much smaller than the particle horizon when m is less than m_{max}. The first possibility is consistent with the CP picture provided the universe is baryon-asymmetric and provided there do not exist superbaryons. The second possibility is unlikely because the existence of galaxies virtually demands the existence of fluctuations with ϵ

exceeding 10^{-10}. Whatever mechanisms play a role in possibility (3), it is very ad hoc to suppose that they will become ineffective only when $m \sim m_{\max}$, so the most natural conclusion is that they prevent (3.14) from ever being applicable. If this is not the case, then models which involve prolonged soft periods must be rejected; if it is the case, then the spectrum of pbh's formed in the Hagedorn era is described by equation (4.15) with $f = 0$.

VI. OBSERVATIONAL LIMITS ON THE PBH MASS SPECTRUM

Although there is no observational evidence that pbh's exist at all, several observations place limits on their spectrum. Some of these limitations are summarized in Figure 4.

1. Measurements of the universe's deceleration parameter (Sandage 1961) indicate that the mean density of pbh's today, μ_B, cannot exceed the critical density, or equivalently, 10^4 times the present energy density of the background radiation, μ_R. As we go back in time, μ_B increases like R^{-3} while μ_R increases like R^{-4}. Thus at times sufficiently early that the total density of the universe, μ_u, is dominated by radiation

$$\frac{\mu_B(t)}{\mu_u(t)} = 10^4 \Omega_B \left(\frac{R(t)}{R_{\text{now}}}\right) \sim 10^{-6} \Omega_B \left(\frac{t}{1\ \text{s}}\right)^{1/2} . \tag{6.1}$$

This equation holds until about $t = 1$ s, before which particle pairs are created. The fraction of the universe in pbh's at earlier times depends on the equation of state in the hadron era. At 10^{-23} s,

$$\begin{aligned} \mu_B/\mu_u &\sim 10^{-17}\Omega_B && \text{in EP model} \\ &\sim 10^{-8}\Omega_B \left(\frac{t_{\rm H}}{10^{-4}}\right)^{1/2} && \text{in Hagedorn's model}\,, \end{aligned} \tag{6.2}$$

so the pbh's which exist now could only represent a tiny fraction of the universe at early times. This supports the assumption that the early universe was nearly Friedmann. Also, unless $t_{\rm H} \sim 10^{12}$ s (so that the radiation era is virtually eliminated), one can reject any model in which half the universe goes into pbh's at early times.

2. According to Hawking (1974), a black hole of mass M will emit all types of particles with sufficiently small mass like a blackbody of temperature $T \sim 10^{27} M^{-1}$ K. This means that any pbh should evaporate in a time $\tau \sim 10^{-28} M^3$, which will be less than the age of the universe (10^{17} s) if M is less than about 10^{15} g. In the EP picture the evaporation will terminate in an explosion, releasing about 10^{30} ergs in the last 0.1 s. In the CP picture, where many more species of particle are emitted in the final moments, the explosion may be much more powerful, releasing 10^{35} ergs in the last 10^{-23} s. Hawking's prediction has two observational consequences. First, pbh's of about 10^{15} g could be exploding today. Each explosion should appear as a hard gamma-ray burst, which could be detectable if the pbh was within 10 pc. Thus the observed (or nonobserved) frequency of such bursts puts an upper limit on the number density of 10^{15} g pbh's. Although gamma-ray bursts have been observed (at a frequency of one every few months), they do not appear to have the characteristics expected of black hole explosions, since the gamma rays are too soft. A more important limitation derived from a second consequence of Hawking's

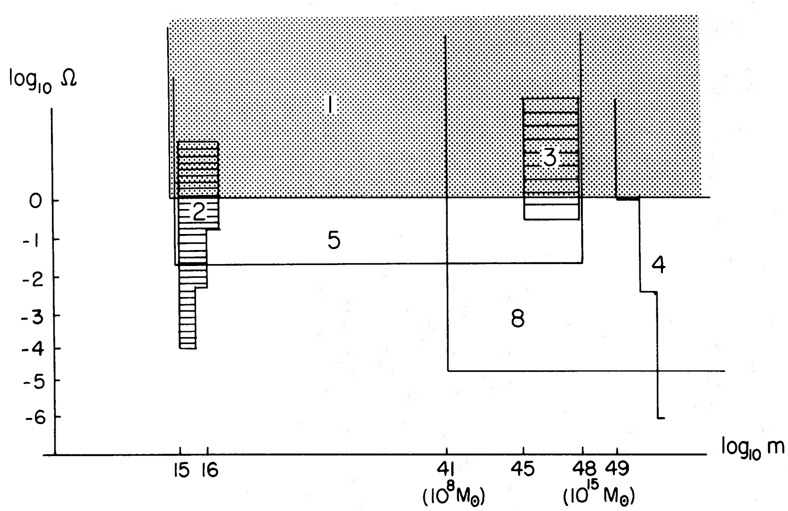

FIG. 4.—The diagram shows the observational limits on the density of pbh's in various mass ranges. The density is shown in units of critical density, and the mass in grams. The numbers refer to the text. Some of the limits discussed in the text are not shown because they do not have a simple interpretation in terms of this diagram.

prediction: the energy density of gamma rays produced by the evaporation of 10^{15} g pbh's should have the same order of magnitude as the density of the 10^{15} g pbh's themselves. Therefore the fact that the background gamma radiation in the 30–100 MeV range has a density of only about 10^{-8} times the critical density suggests that pbh's with mass around 10^{15} g must have a mean density less than 10^{-37} g cm^{-3}. This has also been pointed out by Chapline (1975). One might avoid this conclusion by arguing that only a small fraction of the energy released in each pbh evaporation is channelled into gamma rays. However, since both the total radiation background and the cosmic-ray background have a density of only 10^{-4} times the critical density, 10^{15} g pbh's would still need to have a density of less than 10^{-33} g cm^{-3}. Observations of the background radiation density also place a limit, though a weaker one, on the density of pbh's bigger than 10^{15} g: pbh's with mass around m must have an average density less than $10^{-33}(m/10^{15}\text{ g})^3$.

3. Press and Gunn (1973) have shown that the number density of large black holes is limited by the absence of image-doubling of distant sources. This is because light from a distant source would be bent by the gravitational field of any intervening condensed object. The probability that a given source will be imaged into two similar images is proportional to the density of the condensed objects, Ω_c, and is high if $\Omega_c \sim 1$. Thus the fraction of distant sources (such as quasars) with strong image-doubling should give a direct measure of Ω_c. Furthermore, the angular separation between the images is a measure of the mass of the scatterer [$\theta_{\text{sep}} \sim 10^{-6}(m/M_\odot)^{1/2}$ arcsec for sources at redshift ~ 2]. In principle this provides a direct way of measuring the mass spectrum of large pbh's. In practice one would only be able to detect pbh's in the mass range 10^4–10^{15} $M_\odot$ because θ_{sep} would be too small to resolve for m less than 10^4 $M_\odot$ and too large for the images to be identified for m greater than 10^{15} $M_\odot$. Observations already indicate that black holes in the 10^{12}–10^{15} $M_\odot$ range must have a density less than 0.2 the critical density.

4. Press and Gunn (1973) also point out that, if there were an appreciable density of pbh's bigger than 10^{15} $M_\odot$, one would expect our Galaxy to have a large peculiar velocity due to its gravitational interaction with the nearest one. Observational limits on the 24-hour anisotropy in the background radiation already indicate that the peculiar velocity of the Sun and hence our Galaxy cannot exceed about 300 km s^{-1}. Unless we are located in a special place, this precludes pbh's of mass around m having a density of more than $(m/10^{16}\ M_\odot)^{-2}$ times the critical density.

5. Gott *et al.* (1974) have argued that if all the matter in the universe is distributed as visible galaxies, then it must have a density of less than 0.04 the critical density (Gott *et al.* 1974). This is because the density enhancement found in small groups of galaxies has little effect on the local value of Hubble's constant. Thus pbh's could have a critical density only if they were distributed more uniformly than visible galaxies. However, unless pbh's have very large peculiar velocities [see (6)] or are bigger than 10^{15} $M_\odot$, it is hard to see how they could avoid being clustered like galaxies. It is therefore unlikely that pbh's have a critical density.

6. Van den Bergh (1969) has sought evidence for big black holes in the Virgo cluster by looking for unexplained tidal distortion in the galaxies there. His findings indicate that there are no black holes bigger than 10^{10} $M_\odot$ in the Virgo cluster. Even if a pbh which forms at a redshift Z has an initial peculiar velocity of order the velocity of light, its peculiar velocity today would only be of order Z^{-1} times the velocity of light. Therefore, there is no reason to expect pbh's of less than 10^{15} $M_\odot$ to have present peculiar velocities more than the cluster virial velocities ($\sim 10^8$ cm s^{-1}). Van den Bergh's result thus suggests that there cannot be many pbh's in the 10^{10} to 10^{15} $M_\odot$ range, since otherwise there should be some clustered inside Virgo.

7. The existence of many large black holes at decoupling would produce appreciable fluctuations in the microwave background on small angular scales, contrary to observation. If decoupling takes place at a redshift of about 1500, a black hole of mass m would then have an angular diameter (as observed today) of about $3° \times \Omega_u^{1/2} \times (m/10^{17}\ M_\odot)$. However, observations (Boynton and Partridge 1973) indicate that there cannot be any gross anisotropies in the microwave background on angular scales greater than 1″. This implies that there cannot be many pbh's bigger than 10^{15} $M_\odot$.

8. As pointed out in the Introduction, a pbh of only 10^7 $M_\odot$ may by now have captured the mass of a typical galaxy. Indeed, if the pbh seed theory of galaxy formation is viable (in particular, if stars can form before reheating), any big pbh should produce a galactic type object. Thus the number density of pbh's bigger than, say, 10^8 $M_\odot$ could not exceed the galactic number density. Equivalently, their mass density would have to be less than 10^{-5} times the critical density.

9. Zel'dovich and Sunyaev (1969) have argued that the injection of a significant amount of heat into the universe after some latest time, $t_{\max}$, would distort the blackbody spectrum of the microwave background, contrary to observation. Since one would expect dissipation to accompany the formation and accretion of large pbh's, this precludes the existence of many pbh's of more than a certain mass. If the density of the universe not in black holes is of the order of the critical density, then $t_{\max} \approx 10^9$ s. If it is much less than the critical density, $t_{\max}$ is much smaller. However, $t_{\max}$ must exceed 1 s, since electron-positron pairs are created before 1 s. Thus the formation of pbh's of less than 10^5 $M_\odot$ could not have caused distortion in the blackbody spectrum of the microwave background. There could still be pbh's bigger than 10^5 $M_\odot$, provided they are relatively infrequent.

10. In the EP picture, pbh's with mass in the range $10^9 t_{\max}{}^{1/3}$ to 10^{15} g explode after the time $t_{\max}$ which arose in the Zel'dovich-Sunyaev argument, and they should therefore have an effect on the background radiation. Pbh's with mass between 10^{13} and 10^{15} g will explode after decoupling, so photons from their explosions should

maintain their original temperature (apart from a redshift effect). Observations of the background spectrum therefore limit the number of pbh's which could ever have existed with mass in the range $10^9 t_{\max}{}^{1/3}$ to 10^{15} g. It turns out that the biggest effect on the background radiation comes from pbh's of about 10^{15} g, and the limitation which observations place on the density of these pbh's has already been considered in limitation (2).

The above observational constraints indicate that the chances of observing pbh's directly are very remote. Since $\Omega_B < 1$, pbh's of mass around m cannot have a number density exceeding $10^{10}(m/10^{15}\text{ g})^{-1}$ per cubic parsec. One would not notice a small black hole unless it hit the Earth (if then), and, even if the pbh's are bound within galaxies and $\Omega_B \sim 1$, this would only happen every $10^5(m/10^{15}\text{ g})$ years. Probably the only way to detect pbh's directly would be to see small ones exploding. Present detectors should allow us to notice a black hole explosion within 10 pc in Hagedorn's model, so limitation (2) does not rule out the possibility of noticeable explosions occurring quite frequently provided 10^{15} g pbh's are clustered within galaxies. However, in the EP picture, one could not expect to detect gamma rays from 10^{15} g pbh explosions with any reasonable frequency.

VII. ASSESSMENT OF COSMOLOGICAL MODELS

We now examine what constraints the observational limits discussed in the last section place on models of the early universe. In particular, we inquire whether any model consistent with these constraints could permit (1) pbh's to have a critical density ($\Omega_B \sim 1$) or (2) big pbh's to have the galactic number density ($\Omega_G \sim 10^{-6}$). For convenience, a model of the early universe will be denoted as $(f_1 f_2 f_3, n, \epsilon)$, where the f's prescribe the equation of state ($p = f\mu$) in the prehadron (if existent), hadron, and posthadron eras, and where n and ϵ are defined by equation (2.2).

Consider first models in which the fluctuations are natural (i.e., in which n and ϵ are the same over every mass range).

1. *The EP model with metric fluctuations*, $(\gamma\frac{1}{3}\frac{1}{3}, \frac{2}{3}, \epsilon)$. Since the spectrum decreases monotonically with mass, limitation (2) requires $\Omega_B < 10^{-4}$. This implies

$$\epsilon \exp\left(-\frac{\beta^4}{2\epsilon^2}\right) < 10^{-21} \Rightarrow \epsilon < \tfrac{1}{10}\beta^2 \sim \tfrac{1}{10}\gamma\,. \tag{7.1}$$

For $m > 10^{15}$ g, $M(m) \propto m^{-1/2}$ so the spectrum falls off too fast for Ω_G to be as large as 10^{-6}. In fact, from equation (4.19),

$$\begin{aligned} m_{\max} &\sim 10^2 \Omega_B{}^{2/3} M_\odot \qquad \text{if } \gamma = 1 \\ &\sim 10^9 \Omega_B{}^{2/3} M_\odot \qquad \text{if } \gamma = \tfrac{1}{3}\,, \end{aligned} \tag{7.2}$$

so no $10^6\ M_\odot$ pbh's are formed anyway. Ω_B depends very sensitively on ϵ. If $\epsilon \sim 1$, too much mass goes into pbh's to be consistent with observation; but if $\epsilon < 10^{-2}$, no pbh's form at all.

2. *The EP model with "surface" fluctuations*, $(\gamma\frac{1}{3}\frac{1}{3}, \frac{7}{6}, \epsilon)$. In this model pbh's only form if $\epsilon \sim 1$, which corresponds to the initial grains having a mass of 10^{-5} g. The spectrum has an exponential cutoff at about $10^{-3}\epsilon^2$ g, so there is no possibility that either $\Omega_B \sim 1$ or $\Omega_G \sim 10^{-6}$. However, this model is interesting since, if $\epsilon = 1$, it corresponds to one in which the whole universe starts off in black holes. Since the black holes would explode almost immediately, they could have generated the entropy of the universe.

3. *The baryon-asymmetric CP model without superbaryons*, $(\gamma\frac{1}{3}, n, \epsilon)$. If one permits only particles with baryon number ± 1 or 0, then $p \sim \mu/\log(\mu/\mu_*)$. Thus the parameter γ (although not strictly defined) is of order unity except at very early times. The fluctuations may be either "surface" type or "metric" type. For surface fluctuations $n = 7/6$ and $\epsilon \sim 10^{-12}$, so pbh's do not form. For metric fluctuations $n = \frac{2}{3}$ and limitation (2) precludes $\Omega_B \sim 1$ or $\Omega_G \sim 10^{-6}$. However, as in case (2), small pbh's could have had an important entropy generating effect.

4. *The baryon-symmetric or superbaryon CP model with surface fluctuations*, $(0\gamma, \frac{7}{6}, \epsilon \sim 1)$. In this model $\gamma = 0$ or $\frac{1}{3}$, according to whether or not the Hagedorn era is extended beyond 10^{-4} s. Section V showed that pbh's form too prolifically to be consistent with observation if they can form as small as the Jeans length during a soft era, so it is assumed that the spectrum is described by (4.15) with $f = 0$. In this case the formation of 10^{15} g pbh's is favored, since $\epsilon \sim 1$. However, the spectrum has an upper cutoff at a mass $10^{17}\epsilon^2$ g, so there cannot be any big pbh's. Limitation (2), $\Omega_B < 10^{-4}$, would seem to require

$$\epsilon \exp\left(-\frac{\beta^4}{2\epsilon^2}\right) < 10^{-12}\left(\frac{t_{\rm H}}{10^{-4}}\right)^{1/2}. \tag{7.3}$$

On the other hand, limitation (2) may not be applicable in this situation because, with initial grains of mass 10^{15} g, it is not inevitable that any pbh's would have formed small enough to have evaporated by now. Even if limitation (2) does apply, the 10^{15} g pbh's could have had an important entropy-generating effect.

5. *The baryon-symmetric or superbaryon CP model with metric fluctuations*, $(0\gamma, \frac{2}{3}, \epsilon)$. It is assumed that the spectrum is described by equation (4.15) with $f = 0$, as in model (4). If limitation (2) applies, pbh's cannot have a

critical density. However, $M(m)$ only falls off logarithmically for pbh's which form in the hadron era. From equations (4.11) and (4.23)

$$\begin{aligned}\Omega_G &\sim 10^{-4}\left(\frac{t_{\mathrm{H}}}{10^{-4}}\right)^{1/2}\Omega_B \qquad \text{if } t_{\mathrm{H}} < 10\ \mathrm{s} \\ &\sim 10^{-2}\Omega_B \qquad\qquad\qquad \text{if } t_{\mathrm{H}} > 10\ \mathrm{s}\,,\end{aligned} \tag{7.4}$$

so big pbh's could have a galactic number density if the Hagedorn era is extended beyond 10^{-4} s. This conclusion would not apply if one took limitation (2) in its strongest form, $\Omega_B < 10^{-8}$.

In the context of models with natural fluctuations, we can draw three conclusions: (1) if Hawking's theory is correct, pbh's which exist today cannot have a critical density; (2) big pbh's can have the galactic number density only if one believes in metric fluctuations and a soft hadron era; (3) because the fraction of the universe in pbh's at early times depends so sensitively on ϵ, it is not surprising that this fraction is very small.

Finally, consider models of the early universe in which the initial fluctuations are arbitrary (i.e., in which n and ϵ may be different over different mass ranges). The considerations of § III showed that, if the equation of state is hard, pbh's will form only over mass ranges in which $n = \frac{2}{3}$. In the EP picture, $f = \frac{1}{3}$ at the time when pbh's bigger than 10^{15} g form, so from equation (4.15) the spectrum of pbh's which exist today should always go down like $m^{-5/2}$. If δ_* has the form shown in Figure 1*d*, pbh's might exist in several mass ranges. Limitations (3) and (5) of § VI would seem to preclude $\Omega_B \sim 1$ but not $\Omega_G \sim 10^{-6}$. With special initial conditions, therefore, big pbh's could have the galactic number density.

The authors thanks S. W. Hawking, P. Mészáros, and M. J. Rees for many helpful discussions.

APPENDIX A

THE "SURFACE EFFECT"

It is reasonable to suppose that the early universe is grainy if one looks at it on a sufficiently fine scale. For example, Hagedorn's model postulates initial grains of 10^{15} g, and grains of some form must exist in the Elementary Particle model. Even if these grains were initially distributed exactly uniformly, one would expect them to develop small-scale motions. Therefore grains sufficiently near the surface of any region (but necessarily within one horizon distance of the surface) would be free to shuffle in and out of the region. It is easy to see that this "surface effect" produces statistical fluctuations in the number of grains, N, in regions of some specified size of the form $\Delta N \propto N^{1/3}$. Several authors (e.g., Carlitz *et al.* 1974), inferring that there is an expected density fluctuation of the form $\Delta M/M \propto M^{-2/3}$, and assuming that this fluctuation grows like $t^{2/3}$ (as applies for any small spherically symmetric perturbation), have concluded that by a time t the surface effect will have bound regions of mass $M_B \propto t$. Recently, however, Peebles (1974*a*) has pointed out that this energy argument is incorrect. He shows that if one analyzes the problem in terms of the power spectrum of the initial density perturbation and uses linear perturbation theory, then M_B only grows like $t^{4/7}$. He proves that the power spectrum analysis is valid provided one considers regions much bigger than the intergrain separation and provided the power spectrum does not fall off too rapidly with wavenumber. However, it is observed that the energy argument would be applicable provided the initial overdensity could be taken as $\delta \propto M^{-7/6}$ and not $\delta \propto M^{-2/3}$. For such an overdensity, growing like $t^{2/3}$, gives $M_B \propto t^{4/7}$ in accordance with Peebles. The following considerations show why it is in fact appropriate to take $\delta \propto M^{-7/6}$.

First, it is noted that the surface effect arises only because the grains develop peculiar velocities. These peculiar velocities themselves contribute to the total energy of a region and hence to the form of the energy fluctuation (although only as a second-order effect). Suppose that at time t the grains in some spherically symmetric comoving volume V have developed peculiar velocities $\boldsymbol{u}_i$ (where the suffix i labels the grains) which are superposed on their Hubble flow. Then the corresponding kinetic energy contribution is

$$T \equiv \frac{1}{2}\int_V \mu u^2 d^3\boldsymbol{r} = \sum_i \frac{1}{2} m_i u_i^2\,. \tag{A1}$$

There will also be a nonlinear potential energy fluctuation

$$F \equiv \frac{1}{2}\int_V\!\!\int \frac{(\mu(\boldsymbol{r}_1) - \bar{\mu})(\mu(\boldsymbol{r}_2) - \bar{\mu})}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}\, d^3\boldsymbol{r}_1 d^3\boldsymbol{r}_2 \sim \frac{(\Delta M)^2}{V^{1/3}}\,, \tag{A2}$$

where $\bar{\mu}$ is the mean density of the universe. It can be shown (Zel'dovich 1965) that the time development of T and F are related by

$$\frac{dT}{dt} + 2HT = \frac{dF}{dt} + HF \qquad \text{where } H = \frac{1}{R}\frac{dR}{dt}\,. \tag{A3}$$

(This virial-type equation is equivalent to the Layzer-Irvine equation used by Peebles.) In particular, equation (A3) implies that one can start from a state which is perfectly homogeneous at time t_0 (for example, one in which the grains initially touch), with $F = T = 0$, and evolve into a state with $F \neq 0$ and $T \neq 0$. This spontaneous birth of peculiar velocity and density perturbations reflects the fact that the Friedmann model is unstable to small perturbations. If one does start with $F = T = 0$, (A3) implies that at some later time t,

$$F = T + \frac{1}{R(t)} \int_{t_0}^{t} T(t')dR(t') . \tag{A4}$$

Thus, in a state which evolves from initial homogeneity, the peculiar velocity term T is always smaller than the potential energy term F. The fact that T and F do not exactly balance results in a negative contribution to the total energy of a region, and this explains why one expects bound systems to develop in a grainy universe. But this is a nonlinear effect, and the considerations of Peebles show that it corresponds to $\delta \propto M^{-7/6}$: by conservation of mass and momentum, the Fourier transform of the density fluctuation, δ_k, goes like k^2, so that the power spectrum

$$\sum_{k<V^{-1/3}} |\delta_k|^2$$

goes like $V^{-7/3}$, implying $M_B \propto t^{4/7}$.

The surface effect only seems to produce a statistical fluctuation $\delta \propto M^{-2/3}$ in the mass of a region because the region is taken to have a sharp boundary. One can see this by expressing the mass in a spherical volume $V \sim L^3$ as an integral over all space:

$$M = \int \mu(\boldsymbol{x})\phi(\boldsymbol{x})d^3\boldsymbol{x} , \tag{A5}$$

where the "weight" function $\phi(\boldsymbol{x})$ is 1 for $\boldsymbol{x}$ inside V, and 0 for $\boldsymbol{x}$ outside V. Since ϕ has the Fourier transform

$$\begin{aligned} \phi_k = \frac{1}{V} \int_V \phi(x) \exp(i\boldsymbol{k}\cdot\boldsymbol{x})d^3x &\sim 1 \qquad \text{for } k < L^{-1} \\ &\sim k^{-2}L^{-2} \qquad \text{for } k > L^{-1} , \end{aligned} \tag{A6}$$

and δ has the Fourier transform $\delta_k \propto k^2$, the fluctuation in the mass in V is

$$(\Delta M)^2 = V^2 \int |\delta_k|^2 |\phi_k|^2 d^3k \tag{A7}$$

$$= \int_0^{1/L} 4\pi k^6 V^2 dk + \int_{1/L}^{1/d} 4\pi k^2 V^{2/3} dk . \tag{A8}$$

The second integral is cut off at the value of k corresponding to the grain separation d, since $\delta_k \propto k^2$ does not apply for $k > d^{-1}$. The first term on the right-hand side of equation (A8) is proportional to $M^{-1/3}$ and therefore gives a $M^{-7/6}$ contribution to δ. However, the second term is proportional to $M^{2/3}$ so, as pointed out by Peebles, the biggest contribution to ΔM comes from its Fourier components with $k > L^{-1}$. Essentially, this is because ϕ_k only falls off slowly with increasing k. But, as Zel'dovich has pointed out, this feature is merely a consequence of the fact that the weight function ϕ is discontinuous at the surface of V, i.e., of the fact that V has been given a sharp boundary. If one smears out the boundary by taking the weight function to have the form

$$\phi(x) = \exp\left[-\frac{(\boldsymbol{x} - \boldsymbol{x}_0)^2}{L^2}\right] , \tag{A9}$$

where $\boldsymbol{x}_0$ is the center of V, then

$$\phi_k \sim \exp(-k^2L^2) , \tag{A10}$$

and ϕ_k falls off fast enough for the $k > L^{-1}$ contributions to (A8) to be negligible. Thus, smearing out the boundary of V reduces the expected mass fluctuation to $\delta \propto M^{-7/6}$ in agreement with Peebles. The justification for smearing out the boundary is presumably that the assumption of a sharp boundary is not consistent with the assumption of spherical symmetry on which the growth law, $\delta \propto t^{2/3}$, depends. One can, in effect, restore spherical symmetry by regarding the boundary of V as fuzzy.

APPENDIX B

THE "CLUSTERING" EFFECT

Recently Mészáros (1974) has focused attention on the fact that the formation of pbh's could generate density fluctuations (leading to their clustering) because of statistical fluctuations in their number density. Although such fluctuations would be very small initially (because only a small fraction of the universe could have been in pbh's at early times), they would arise very early and so might be important for galaxy formation. The nature of this "clustering" effect depends on whether or not the overdense regions which form pbh's are all initially surrounded by underdense regions. Both types of situations are possible. For example, a pbh will be surrounded by an underdense region in the context of the $n = 7/6$ surface effect. It will not be in the context of $n = \frac{2}{3}$ metric fluctuations. It will be shown that, in either situation, pbh formation cannot induce extra density fluctuations on scales larger than the horizon. Mészáros's effect cannot therefore affect the pbh spectrum, even if it is important for galaxy formation.

Consider regions of comoving volume V containing a number N of pbh's of mass m. (For simplicity the fact that there could be a range of values for m is neglected.) This number will have some mean value $\bar{N}(V)$; but, since there should be no correlation between pbh's forming in different parts of the universe, one might expect a statistical fluctuation ΔN in the value of N. Consider first the situation in which every pbh is initially surrounded by an underdense region. In this case the fluctuation in N will not induce a mass fluctuation until after the time t_1 when the pbh's form. After t_1 the equation of state ($p = f\mu$) will have a soft contribution: the mass in pbh's will remain constant, while the mass outside them will be reduced because of pressure. Thus at time $t > t_1$ the total mass in a region of mass M_1 at t_1 is

$$M = Nm + (M_1 - Nm)\left(\frac{t}{t_1}\right)^{-2f/(1+f)} \tag{B1}$$

with an expected fluctuation

$$\Delta M = m\Delta N\left[1 - \left(\frac{t}{t_1}\right)^{-2f/(1+f)}\right] \approx m\Delta N \qquad \text{if } t \gg t_1 \text{ and } f \neq 0\,. \tag{B2}$$

However, this will not introduce a density fluctuation because a region with a statistical excess of pbh's will expand faster than average because it has a softer equation of state than average. (Since $\mu \propto t^{-2}$, this extra expansion must exactly cancel the ΔM effect.) On the other hand, the region will constitute a temperature fluctuation since its radiation content, M_r, is less than average. When the region falls inside the Jeans length ($\sim$ the horizon size), oscillations in the photon gas will damp out this temperature fluctuation, inducing in the process a density perturbation

$$\delta \sim \frac{\Delta M_r}{M_r} \sim \frac{m\Delta N}{M_1}\left(\frac{t}{t_1}\right)^{2f/(1+f)}\,.$$

(Mészáros has shown that the pbh's, while not partaking in the oscillations of the photons, will be frozen into their overall expansion until matter-radiation equilibrium.) Thus the fluctuation in the number of pbh's in a region will not induce a density fluctuation until the region has fallen inside the horizon. Therefore, if the equation of state is hard ($f \neq 0$), these fluctuations cannot play a role in the formation of later generations of pbh's. If the equation of state is soft, the pbh clustering effect does not exist anyway.

Consider now the situation in which a pbh is not necessarily surrounded by an underdense region. This may occur either because the pbh's are fed into the initial conditions of the universe, or, more naturally, through metric fluctuations. As before, one might expect a fluctuation in the number of pbh's in regions of comoving volume V. However, because one does not have to wait until the pbh's form for a density fluctuation to develop (the fluctuation exists from the beginning of the universe), there is no sense in which pbh formation introduces "extra" density fluctuations. The fluctuation in the number of pbh's in V is inherent in the form of the initial density fluctuation $\delta_*(M)$. Thus if $\delta_* \propto M^{-2/3}$, one cannot consistently assume that there will be a $\sqrt{N}$ fluctuation in the number of pbh's in a region because then δ_* would not have the $n = \frac{2}{3}$ form for M sufficiently large. (In fact, it was shown in § III that one cannot permit $\sqrt{N}$ fluctuations for arbitrarily large N since they would form separate universes.) There could still be a $\sqrt{N}$ fluctuation on scales smaller than the horizon because of the pbhs' peculiar velocities. This might cause pbh's to cluster, but, if the equation of state is hard, it could not affect pbh formation.

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