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Abstract. Without invoking a perturbative expansion, we define the cosmological curvature perturbation, and consider its behaviour assuming that the universe is smooth over a sufficiently large comoving scale. The equations are simple, resembling closely the first-order equations, and they lead to results which generalize those already proven in linear perturbation theory and (in part) in second-order perturbation theory. In particular, the curvature perturbation is conserved provided that the pressure is a unique function of the energy density.

Keywords: cosmological perturbation theory, inflation, physics of the early universe

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Contents

1. Introduction

In this note we define and study a non-linear generalization of the linear cosmological 'curvature perturbation' which is commonly denoted by ζ. Just as in linear theory, we find that the curvature perturbation is conserved during any era when the locally defined pressure is a unique function of the locally defined energy density, but is otherwise time dependent.

Before describing the calculation we mention the motivation which is actually twofold. The first motivation concerns the comparison of theory with observation. Observation has now established the existence of a nearly Gaussian and scale invariant spatial curvature perturbation, present already before comoving cosmological scales come inside the Hubble distance (enter the horizon). The curvature perturbation presumably is generated somehow by the perturbation of a scalar field, that perturbation in turn being created on each scale at horizon exit during inflation. Several mechanisms for achieving this have been proposed. In all of them it is supposed that the curvature perturbation, defined on suitable spacetime slices, is conserved (starting from the time when the creation mechanism ceases) until the approach of horizon entry. The conservation is supposed to hold by virtue of the condition, either proven or just assumed, that the pressure is a unique function of the energy density. This is the 'adiabatic pressure' condition, that in first-order perturbation theory leads to $\delta P/\delta\rho = \dot{P}/\dot{\rho}$.

The conservation of the curvature perturbation for adiabatic pressure has been demonstrated at linear order for an appropriate slicing of spacetime. That slicing can be taken to be the 'comoving' one (the one orthogonal to comoving worldlines) or the uniform-density one or the uniform-proper-expansion (uniform-Hubble) one; it does not matter which because in linear theory the three are known to coincide in the large scale limit [1, 2].

These results are all that one needs at the moment, but with better data in the future it will be necessary to go to second order. This is because primordial non-Gaussianity, which may well be big enough to be detected, is typically generated only at the second-order level. The study of second-order gauge invariant perturbations started only quite recently [3, 4] and has been hampered by its sheer complexity. The prospect of observing non-Gaussianity in the near future has led to renewed activity in this area [5]–[9] but the second-order approach has not yet been carried through to the point where it sufficiently generalizes the first-order large scale results.

As serious as its incompleteness is the fact that even on very large scales the second-order calculation is complicated, losing as a result the physical transparency of the first-order calculation. This brings us to our first motivation; we will provide a non-linear understanding of the large scale situation (applying therefore to all orders in the perturbative expansion) which follows the same lines as the first-order discussion and is not much more complicated. We will show explicitly how to recover powerful second-order results from this approach, without going through the complicated second-order formalism.

Our second motivation is more philosophical, concerning the possibility that inflation lasted for an exponentially large number of e-folds. In that case the expanding Universe which we observe is part of a huge region, which at the classical level was homogeneous before it left the horizon. When comparing the theory with observation, very distant parts of this huge region are irrelevant, because one can and should formulate the theory within a comoving box whose present size is just a few powers of ten bigger than the Hubble distance. Linear theory with the small second-order correction is then valid within the box, which is all that one needs for practical purposes. Still, it is of interest to understand the nature of the universe in very distant regions. In this connection it is important to recognize that a Gaussian 'perturbation' with a scale invariant spectrum actually becomes indefinitely large in an indefinitely large region. This means that one should try to understand the cosmological perturbations without invoking the perturbative expansion, which is precisely what we do here.

We work with the ADM formalism [10]. Historically, the ADM formalism was employed to study non-linearities already by Bardeen in [1], albeit in the context of perturbation theory. Non-perturbative studies of fluctuations in a scalar field during inflation followed in [11], also employing a gradient expansion (see also [12, 13]), and were recently extended to the multi-field case in [14]. In [15] the relation between the PPN formalism and linear perturbation theory was established using the ADM formalism and in [16] the formation of primordial black holes was studied numerically.

In addition, we should mention the existence of related work done by the Russian school, which is based on the construction of solutions of Einstein equations having locally Friedmann-like behaviour near the singularity, the so-called 'quasi-isotropic' approach. It was pioneered by Lifshitz and Khalatnikov in 1963 [17], generalized to the de Sitter case [18] and has been elaborated recently [19, 20].

This paper is organized as follows. In section 2 we define the metric of our spacetime, specify the assumptions for its validity, and describe the basic geometrical properties under our assumptions. In section 3 we show the conservation of a non-linear generalization of the curvature perturbation on very large scales. In section 4 we express our conserved quantity in a slice-independent manner and relate it to the ΔN formula derived in [21] (see also [22, 23]). We conclude in the final section. In appendix A we give the components of the relevant tensors and in appendix B we give the spatial traceless components of the Einstein equations.

2. Geometry and energy conservation

2.1. Metric

We use the standard $(3+1)$-decomposition of the metric (ADM formalism), which applies to any smooth spacetime [10]:

$$\mathrm{d}s^2 = -\mathcal{N}^2\,\mathrm{d}t^2 + \gamma_{ij}(\mathrm{d}x^i + \beta^i\,\mathrm{d}t)(\mathrm{d}x^j + \beta^j\,\mathrm{d}t), \tag{1}$$

where $\mathcal{N}$ is the lapse function, β^i the shift vector, and γ_{ij} the spatial three metric. (Greek indices will take the values $\mu,\nu = 0,1,2,3$, Latin indices $i,j = 1,2,3$. The spatial indices are to be raised or lowered by γ^{ij} or γ_{ij}.) In this $(3+1)$-decomposition, the unit timelike vector normal to the $x^0 = t =$ constant hypersurface n^μ has the components

$$n_\mu = [-\mathcal{N}, 0], \qquad n^\mu = \left[\frac{1}{\mathcal{N}}, -\frac{\beta^i}{\mathcal{N}}\right]. \tag{2}$$

We write the 3-metric, γ_{ij}, as a product of two terms,

$$\gamma_{ij} \equiv \mathrm{e}^{2\alpha}\tilde{\gamma}_{ij}, \tag{3}$$

where α and $\tilde{\gamma}_{ij}$ are functions of the spacetime coordinates (t, x^i), and $\det[\tilde{\gamma}_{ij}] = 1$. Because of the latter condition, the first factor is a locally defined scale factor which we denote by $\tilde{a}$,

$$\mathrm{e}^\alpha \equiv \tilde{a}. \tag{4}$$

We are interested though in the inhomogeneity of α, and so we factor out from e^α a global scale factor $a(t)$ and a perturbation ψ,

$$\mathrm{e}^\alpha = a(t)\mathrm{e}^{\psi(t,x^i)}. \tag{5}$$

We assume that ψ vanishes somewhere in the observable Universe (say at our location). This makes $a(t)$ the scale factor for our part of the universe, and ensures that ψ is a small perturbation throughout the observable Universe. (If we are surrounded by a super-large region within which ψ becomes large, then the appropriate scale factor for a generic observer should be $\tilde{a}$ evaluated at their own location, again making ψ small in their vicinity.)

In a similar way the matrix $\tilde{\gamma}_{ij}$ can be factored,

$$\tilde{\gamma} \equiv I e^{H}, \tag{6}$$

where I is the unit matrix. The condition $\det(\tilde{\gamma}) = 1$ ensures that the matrix H is traceless, which follows from the relation $\det(\exp(M)) = \exp(\mathrm{tr}(M))$, valid for any symmetric matrix M. In our part of the universe, with coordinates corresponding to the usual gauges, H_{ij} is a small perturbation.

2.2. Gradient expansion

Cosmological perturbation theory expands the exact equations in powers of the perturbations, keeping only terms of a finite order. In particular, first-order perturbation theory linearizes the exact equations. We will not use this perturbative approach, but instead will use the gradient expansion method [11, 16, 24], which is an expansion in the spatial gradient of these inhomogeneities. To be precise, we focus on some fixed time, and multiply each spatial gradient ∂_i by a fictitious parameter ϵ, and expand the exact equations as a power series in ϵ. Then we keep only the zero- and first-order terms and finally set $\epsilon = 1$.

In the perturbative approach the fictitious parameter ϵ would be multiplying the perturbations. Working to linear order in the ϵ of the gradient expansion obviously reproduces that subset of the linear perturbation theory expressions which can be derived by considering only those equations which have at least one spatial gradient acting on each perturbation. This is why our results will closely resemble those of linear perturbation theory.

The gradient expansion is useful when every quantity can be assumed to be smooth on some sufficiently large scale with coordinate size k^{-1}. If we are to model our actual universe with this smoothed universe, it is necessary to implement a smoothing procedure at the level of the field equations, either of Einstein gravity or of any alternative theory. However, smoothing the smaller scale inhomogeneities is a delicate issue, which is beyond the scope of the present paper. Here, we simply assume that there exists some kind of smoothing that can give a good approximation to the actual universe on coordinate scales greater than k^{-1}.[3] Focusing on the observable Universe, this corresponds to a comoving smoothing scale of physical size $a(t)/k$. Then, instead of introducing ϵ as a formal multiplier of the spatial gradients, we can make the identification

$$\epsilon \equiv k/aH, \tag{7}$$

where $H = \dot{a}/a$ is the Hubble parameter[4]. At a fixed time the limit $\epsilon \to 0$ corresponds to $k \to 0$. In Hubble units, the typical value of the gradient of a quantity f will be ϵf.

Our key physical assumption is that in the limit $\epsilon \to 0$, corresponding to a sufficiently large smoothing scale, the universe becomes *locally* homogeneous and isotropic (a FLRW universe). (By 'locally' we mean that a region significantly smaller than the smoothing

[3] In linear perturbation theory it is appropriate to use a Fourier expansion and then smoothing corresponds to dropping Fourier components with wavenumber bigger than k, but we have no use here for the Fourier expansion. Still, it is useful to keep the case of the Fourier expansion in mind.

[4] If we are surrounded by a super-large region, a should be replaced by the local scale factor $\tilde{a}(t, x^i)$, and H by the local Hubble parameter $\tilde{H}(t, x^i)$ that we define later.

scale, but larger than the Hubble scale, is to be considered.) The Hubble distance is the only geometric scale in the unperturbed universe. In the perturbed Universe there is in addition the scale $1/k$ under consideration, and possibly other scales provided by the stress–energy tensor. Unless one of the latter scales is bigger than $1/k$, local homogeneity and isotropy will be a good approximation achieved throughout the entire super-horizon era $k \ll aH$, and the results of this paper will be valid throughout that era. Because cosmological scales are so large, one expects this 'separate universe' hypothesis [25]–[27] to be valid for them, ensuring the maximum regime of applicability for our results.

An immediate consequence of our assumption is that the *locally measurable* parts of the metric should reduce to those of the FLRW. Thus there exists an appropriate set of coordinates with which the metric of any local region can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{8}$$

(We took this metric to be spatially flat, which is the expectation from inflation and agrees with observation; a small homogeneous curvature would make no difference.)

Let us see what this implies for the metric components. In the limit $\epsilon \to 0$, the above local metric should be globally valid. This implies that the metric component β_i vanishes in this limit, $\beta_i = \mathrm{O}(\epsilon)$.[5] It may be noted, however, that this is not really a necessary condition but rather a matter of choice of coordinates for convenience.

What about the quantity $\tilde{\gamma}_{ij}$? A homogeneous time-independent $\tilde{\gamma}_{ij}$ can be locally transformed away by choice of the spatial coordinates, but a homogeneous time-dependent $\tilde{\gamma}_{ij}$ is forbidden because it would not correspond to a FLRW universe. We therefore require $\dot{\tilde{\gamma}} = \mathrm{O}(\epsilon)$. In appendix B though, we show that $\dot{\tilde{\gamma}}$ decays like $\tilde{a}^{-3}$ in Einstein gravity if it is really linear in ϵ. Taking the usual view that decaying perturbations are to be ignored, we conclude that $\dot{\tilde{\gamma}}$ will be of second order in ϵ.

The conditions on the metric components are therefore

$$\beta_i = \mathrm{O}(\epsilon), \tag{9}$$

$$\dot{\tilde{\gamma}}_{ij} = \mathrm{O}(\epsilon^2). \tag{10}$$

There is no requirement on ψ and $\mathcal{N}$ since they are not locally observable. We note that in alternative theories of gravity, the assumption $\dot{\tilde{\gamma}} = \mathrm{O}(\epsilon^2)$ may not be as natural as in the Einstein case. Nevertheless, we assume this condition. In other words, we implicitly focus on a class of gravitational theories in which the condition $\dot{\tilde{\gamma}} = \mathrm{O}(\epsilon^2)$ is consistent with the field equations.

In view of equation (9), the line element simplifies, giving

$$\mathrm{d}s^2 = -\mathcal{N}^2\,\mathrm{d}t^2 + 2\beta_i\,\mathrm{d}x^i\,\mathrm{d}t + \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{11}$$

2.3. Energy conservation

By virtue of the separate universe assumption, the energy–momentum tensor will have the perfect fluid form

$$T_{\mu\nu} \equiv (\rho + P)u_\mu u_\nu + g_{\mu\nu}P, \tag{12}$$

[5] We adopt the traditional mathematics notation [28], according to which $f = \mathrm{O}(\epsilon^n)$ means that f falls like ϵ^n *or faster*.

where $\rho = \rho(x^\mu)$ is the energy density and $P = P(x^\mu)$ is the pressure.

First let us choose the spatial coordinates that comove with the fluid—that is, the threading of the spatial coordinates such that the threads $x^i =$ constant coincide with the integral curves of the 4-velocity u^μ (the comoving worldlines). Hence,

$$v^i = \frac{u^i}{u^0}\left(=\frac{\mathrm{d}x^i}{\mathrm{d}t}\right) = 0. \tag{13}$$

The components of the 4-velocity in these coordinates are

$$\begin{aligned} u^\mu &= \left[\frac{1}{\sqrt{\mathcal{N}^2 - \beta^k\beta_k}}, 0\right] = \left[\frac{1}{\mathcal{N}}, 0\right] + \mathrm{O}(\epsilon^2), \\ u_\mu &= \left[-\sqrt{\mathcal{N}^2 - \beta^k\beta_k}, \frac{\beta_i}{\sqrt{\mathcal{N}^2 - \beta^k\beta_k}}\right] = \left[-\mathcal{N}, \frac{\beta_i}{\mathcal{N}}\right] + \mathrm{O}(\epsilon^2). \end{aligned} \tag{14}$$

The expansion of u^μ in the comoving coordinates, $v^i = 0$, is given by

$$\theta \equiv \nabla_\mu u^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}u^\mu\right) = \frac{1}{\mathcal{N}\mathrm{e}^{3\alpha}}\partial_0\left(\mathcal{N}\mathrm{e}^{3\alpha}u^0\right) = \frac{1}{\mathcal{N}\mathrm{e}^{3\alpha}}\partial_t\left(\frac{\mathcal{N}\mathrm{e}^{3\alpha}}{\sqrt{\mathcal{N}^2 - \beta^i\beta_i}}\right). \tag{15}$$

Note that $\tilde{\gamma}_{ij}$ does not appear in the above expression because $\det\tilde{\gamma}_{ij} = 1$. The relation between the coordinate time $x^0 = t$ and the proper time τ along u^μ is

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = u^0 = \frac{1}{\sqrt{\mathcal{N}^2 - \beta^i\beta_i}}. \tag{16}$$

The energy conservation equation,

$$-u_\mu\nabla_\nu T^{\mu\nu} = \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\rho + (\rho + P)\theta\right] = 0\,, \tag{17}$$

reduces therefore to

$$\sqrt{\mathcal{N}^2 - \beta^k\beta_k}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\rho + (\rho + P)\theta\right] = \dot{\rho} + 3(\rho + P)\dot{\alpha} + \mathrm{O}(\epsilon^2) = 0, \tag{18}$$

where

$$\theta = \frac{3\dot{\alpha}}{\mathcal{N}} + \mathrm{O}(\epsilon^2). \tag{19}$$

It is important to note that the expansion of the hypersurface normal n^μ is given by

$$\theta_n \equiv \nabla_\mu n^\mu = \frac{3\dot{\alpha}}{\mathcal{N}} - \frac{1}{\mathcal{N}\,\mathrm{e}^{3\alpha}}\partial_i(\mathrm{e}^{3\alpha}\beta^i). \tag{20}$$

Thus θ and θ_n are equal to each other at linear order in ϵ. Note that the above argument uses only the energy conservation law, and hence applies to any gravitational theory as long as the energy conservation law holds.

Here let us point out a couple of immediate but important implications of the above. The first point is that the equivalence of θ and θ_n for the comoving threading readily implies the equivalence of θ and θ_n for any choice of threading for which $\beta^i = \mathrm{O}(\epsilon)$. This is because the change of the threading affects the numerical value of θ_n at a given world point (t, x^i) only in terms of $\mathrm{O}(\epsilon^2)$ as is clear from equation (20). The second point is that the above argument applies not only to the total fluid but also to any sub-component of the fluid, provided that it does not exchange energy with the rest, and that the comoving threading with respect to that component satisfies the condition $\beta^i = \mathrm{O}(\epsilon)$, that is, if the 3-velocity v^i remains $\mathrm{O}(\epsilon)$ for any threading with $\beta^i = \mathrm{O}(\epsilon)$.

Once we have the equivalence of θ and θ_n, it is convenient to introduce the notion of a local 'Hubble parameter' $3\tilde{H} \equiv \theta_n$,

$$\tilde{H} = \frac{1}{3}\theta_n = \frac{1}{\mathcal{N}}\left(\frac{\dot{a}}{a} + \dot{\psi}\right) + \mathrm{O}(\epsilon^2). \tag{21}$$

As is shown in appendix A, we then recover a local Friedmann equation once we appeal to the Einstein equations.

3. The evolution of the curvature perturbation

Now let us investigate the evolution of the curvature perturbation ψ. So far, we have not specified the choice of the time slicing. Below we shall consider some typical choices of the time slicing separately.

3.1. Uniform-density slicing

In this subsection we consider the uniform-density slicing, denoting ψ on this slicing by $-\zeta$. We shall need only the condition $\beta = \mathrm{O}(\epsilon)$, not the other condition $\dot{\tilde{\gamma}} = \mathrm{O}(\epsilon^2)$.

Following the linear treatment of [26], we avoid in this subsection the assumption of Einstein gravity. Instead we just consider some energy–momentum tensor $T_{\mu\nu}$, which satisfies $\nabla_\nu T^{\mu\nu} = 0$ corresponding to energy–momentum conservation. This is useful in two ways. First, it allows us to deal, if desired, with just one component of the cosmic fluid instead of the total. Second, our results will apply to the case (arising for instance in RSII cosmology [29]) where Einstein gravity is actually modified.

Throughout this paper we are working to first order in ϵ. We take the anisotropic stress of the fluid to be negligible at that order. In other words, the anisotropic stress is supposed to be of second order in ϵ. (This can be verified in specific cases, in particular if the fluid consists of a gas and/or scalar fields.) Then, because there is a unique local expansion rate $\tilde{H}$ to linear order in ϵ, the local energy conservation equation (18) has the unperturbed form to this order,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho = -3\tilde{H}(\rho + P) + \mathrm{O}(\epsilon^2). \tag{22}$$

Multiplying each side by $\mathcal{N}$ this becomes

$$\frac{\dot{a}}{a} + \dot{\psi} = -\frac{1}{3}\frac{\dot{\rho}}{\rho + P} + \mathrm{O}(\epsilon^2). \tag{23}$$

At each point this equation is valid independently of the slicing. Now let us go to the uniform-density time slicing, and denote ψ on this slicing by $-\zeta$. If, to first order in ϵ, P is a unique function of ρ (the 'adiabatic pressure' condition), then equation (23) shows that $\dot{\psi}$ is spatially homogeneous to first order. Since ψ is supposed to vanish at say our position, though the position can be chosen arbitrarily, this means that ψ on uniform-density slices is time independent to first order,

$$-\dot{\psi} = \dot{\zeta} = \mathrm{O}(\epsilon^2). \tag{24}$$

At this stage, as noted before, ζ can refer to the total cosmic fluid, or to a single component which does not exchange energy with the remainder.

3.2. The comoving and uniform-Hubble slicings

Now we invoke Einstein gravity, and show that the comoving and uniform-Hubble slicings coincide to first order in ϵ with the uniform-density slicing. By 'comoving slicing' we mean the one orthogonal to the comoving worldlines (it should be perhaps called the velocity-orthogonal slicing from the general relativity point of view, but the terminology 'comoving slicing' is usual in cosmology). As we have already decided to use the comoving worldlines as the threading this fixes the gauge completely, and we call it the comoving gauge.

By invoking the comoving slicing we are setting the vorticity of the fluid flow equal to zero, since the slicing exists if and only if that is the case. This is reasonable because vorticity is not generated from the vacuum fluctuation during inflation. Also, for a perfect fluid, there is a vorticity conservation law in arbitrary spacetime [30], which states that the magnitude of the vorticity vector is inversely proportional to $S\exp[\int \mathrm{d}P/(\rho+P)]$ along each fluid line, where S is the cross-sectional area of a congruence of the fluid orthogonal to the vorticity vector. Thus, the vorticity would become negligible a few Hubble times after horizon exit on each scale, even if it were somehow generated during inflation.

The Einstein equations are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{25}$$

where $T_{\mu\nu}$ refers now to the total fluid, and $G_{\mu\nu}$ and G are the Einstein tensor and Newton's constant, respectively. We consider its components in appendix A.3. The $(0,i)$ component gives in the comoving gauge

$$\partial_i \tilde{H} = \mathrm{O}(\epsilon^3), \tag{26}$$

and the $(0,0)$ component gives the local Friedmann equation

$$\tilde{H}^2 = \frac{8\pi G}{3}\rho + \mathrm{O}(\epsilon^2), \tag{27}$$

leading to $\partial_i\rho = \mathrm{O}(\epsilon^3)$. This makes the typical magnitudes of $\delta\rho$ and $\delta\tilde{H}$ go like ϵ^2. In other words, the uniform-density, uniform Hubble and comoving slices coincide to linear order in ϵ. These results are the same as those of linear perturbation theory.

Knowing that $\tilde{H}$ and ρ are both spatially homogeneous, we can learn about the lapse function by writing the energy conservation equation (22) in terms of coordinate time,

$$\frac{1}{\mathcal{N}}\dot{\rho} = -3\tilde{H}(\rho+P) + \mathrm{O}(\epsilon^2). \tag{28}$$

Since $\dot{\rho}$ and $\tilde{H}$ are spatially uniform we learn that $\mathcal{N}$ is of the form

$$\mathcal{N} = \frac{A(t)}{\rho(t)+P(t,x^i)} + \mathrm{O}(\epsilon^2), \tag{29}$$

where $A(t)$ can be any function which makes $\mathcal{N}$ positive definite. If the pressure is adiabatic, $\mathcal{N}$ is independent of position and we can choose $\mathcal{N} = 1$. In that case, the comoving gauge is also a synchronous gauge to first order in ϵ.

If the pressure is not adiabatic we can write on the uniform energy density slicing

$$\mathcal{N} = \frac{\rho(t) + P(t)}{\rho(t) + P(t, x^i)}. \tag{30}$$

This is the non-linear generalization of the known result in first-order perturbation theory,

$$\mathcal{N} = 1 - \frac{\delta P}{\rho + P}, \tag{31}$$

where δP is the pressure perturbation on uniform-density slices (the 'non-adiabatic' pressure perturbation)[6].

As shown in appendix B, the traceless part of the spatial components of the Einstein equations is $\mathrm{O}(\epsilon^2)$, while the trace part gives no additional information.

4. Gauge transformations and ΔN formula

As we are working to first order in ϵ, the threading is unique in the sense that all the threadings are equivalent to the comoving threading as discussed in section 2.3, leaving only the slicing to be fixed. In this section we consider the effect of a change of slicing on the curvature perturbation.

Let us define the number of e-foldings of expansion along an integral curve of the 4-velocity (a comoving worldline):

$$N(t_2, t_1; x^i) \equiv \frac{1}{3}\int_{t_1}^{t_2} \theta\,\mathcal{N}\,\mathrm{d}t = -\frac{1}{3}\int_{t_1}^{t_2} \mathrm{d}t\,\frac{\dot\rho}{\rho + P}\Big|_{x^i}, \tag{32}$$

where, for definiteness, we have chosen the spatial coordinates $\{x^i\}$ to be comoving with the fluid. The essential point to be kept in mind is that this definition is purely geometrical, independent of the gravitational theory one has in mind, and applies to any choice of time slicing.

From equation (21) we find

$$\psi(t_2, x^i) - \psi(t_1, x^i) = N(t_2, t_1; x^i) - \ln\left[\frac{a(t_2)}{a(t_1)}\right]. \tag{33}$$

Thus we have the very general result that the change in ψ, going from one slice to another, is equal to the difference between the actual number of e-foldings and the background value $N_0(t_2, t_1) \equiv \ln[a(t_2)/a(t_1)]$. One immediate consequence of this is that the number of e-foldings between two time slices will be equal to the background value, if we choose the 'flat slicing' on which $\psi = 0$. (This slicing is of course truly flat only if $\tilde\gamma_{ij} = \delta_{ij}$.) Thus the flat slicing is one of the *uniform integrated expansion* slicings [27].

[6] As we were preparing the present paper, a related one appeared [31] claiming that the spatial variation of the coordinate expansion rate in the non-adiabatic case could change the currently accepted predictions for observable quantities. We disagree with this conclusion, since that variation is already implicitly present in any correct formulation of cosmological perturbation theory.

Consider now two different time slicings, say slicings A and B, which coincide at $t = t_1$ for a given spatial point x^i of our interest (i.e., the 3-surfaces $\Sigma_A(t_1)$ and $\Sigma_B(t_1)$ are tangent to each other at x^i). Then the difference in the time slicing at some other time $t = t_2$ can be described by the difference in the number of e-foldings. From equation (33), we have

$$\begin{aligned}\psi_A(t_2, x^i) - \psi_B(t_2, x^i) &= N_A(t_2, t_1; x^i) - N_B(t_2, t_1; x^i) \\ &\equiv \Delta N_{AB}(t_2, x^i),\end{aligned} \tag{34}$$

where the indices A and B denote the slices A and B, respectively, on which the quantities are to be evaluated. As discussed in section 5, this generalizes, for large scales only, the known result of the first- [1] and second-order perturbation theory [3, 4, 7].

Now let us choose the slicing A to be such that it starts on a flat slice at $t = t_1$ and ends on a uniform-density slice at $t = t_2$, and take B to be the flat slicing all the time from $t = t_1$ to $t = t_2$. Then applying equation (34) to this case, we have

$$\psi_A(t_2, x^i) = N_A(t_2, t_1; x^i) - N_0(t_2, t_1) = \Delta N_F(t_2, t_1; x^i), \tag{35}$$

where $\Delta N_F(t_2, t_1; x^i)$ is the difference in number of e-foldings (from $t = t_1$ to $t = t_2$) between the uniform-density slicing and the flat slicing. This is a non-linear version of the ΔN formula that generalizes the first-order result of Sasaki and Stewart [21].

Now we specialize to the case $P = P(\rho)$. In this case, equation (33) reduces to

$$\psi(t_2, x^i) - \psi(t_1, x^i) = -\ln\left[\frac{a(t_2)}{a(t_1)}\right] - \frac{1}{3}\int_{\rho(t_1,x^i)}^{\rho(t_2,x^i)} \frac{\mathrm{d}\rho}{\rho + P}. \tag{36}$$

Thus, there is a conserved quantity, which is independent of the choice of time slicing, given by

$$-\zeta(x^i) \equiv \psi(t, x^i) + \frac{1}{3}\int_{\rho(t)}^{\rho(t,x^i)} \frac{\mathrm{d}\rho}{\rho + P}. \tag{37}$$

In the limit of linear theory, this reduces to the conserved curvature perturbation in the uniform-density, uniform-Hubble, or the comoving slicing,

$$-\zeta(x^i) = \mathcal{R}_c(x^i) = \psi(t, x^i) + \frac{\delta\rho(t, x^i)}{3(\rho + P)}. \tag{38}$$

Finally, we mention that the generalization of all the above results to the case of arbitrary threading, not restricted by the condition $\beta^i = \mathrm{O}(\epsilon)$, is formally trivial. Let us denote the general spatial coordinates by $\{X^i\}$. They are related to the comoving coordinates $\{x^i\}$ by a set of coordinate transformations, $x^i = F^i(t, X^i)$. Then all the equations above are valid for an arbitrary choice of threading on simply replacing the arguments x^i of all the functions by $F^i(t, X^i)$.

5. Applications to cosmological perturbation theory

In this section we make contact with cosmological perturbation theory, showing how our results both reproduce and extend known second-order results.

Let us begin with the first-order case. To first order in the perturbations the spatial metric becomes[7]

$$g_{ij} = a^2(t)[\delta_{ij}(1+2\psi) + H_{ij}]. \tag{39}$$

In this context it is known (see for instance [27]) that (always referring to the super-horizon regime) three slices coincide: uniform density, uniform proper expansion (uniform Hubble, the expansion being independent of the threading), and comoving slice. Also it is known that the perturbation ψ is independent of the threading. Finally, defining the curvature perturbation ζ as the value of $-\psi$ on this 'triple-coincidence' slicing, it is known that ζ is conserved as long as pressure is a unique function of energy density.

Going to second order (with an appropriate definition of ψ) only some of these statements have been verified. In particular, the conservation of ζ was shown by Sasaki and Shibata [16] in non-linear theory if P/ρ is constant and by Malik and Wands [7] in second-order perturbation theory setting $\tilde{\gamma}_{ij} = \delta_{ij}$ (i.e. ignoring the tensor), and by Salopek and Bond during single-component inflation. The main point of our paper has been to show that all of them are in fact valid (with again an appropriate definition of ψ).

Our finding is important because it ensures that the non-Gaussianity of the curvature perturbation can be calculated once and for all at the time of its creation, remaining thereafter constant until horizon entry [33]. This was the implicit assumption made by Maldacena [6] (see also [34]–[36], [14]); he calculated the non-Gaussianity of the curvature perturbation (to be precise, its bispectrum) a few Hubble times after horizon exit in a single-field model, defining it on the comoving slicing. It was also that of [37], who calculated the curvature perturbation just before curvaton decay, now on the uniform-density slicing, and that of [38] who calculated it at the end of inflation in a two-component inflation model with a straight inflaton trajectory (again on the uniform-density slicing).

The gauge transformations and gauge invariant expressions that we derived in the last section reproduce the second-order results. To see this, let us first consider two definitions of the curvature perturbation in the literature[8]. In all cases we shall employ the notation that a generic perturbation g is split into first- and second-order parts according to

$$g \equiv g_1 + \tfrac{1}{2} g_2. \tag{40}$$

One definition, used by Maldacena (introduced by Salopek and Bond in [11]) to calculate the non-Gaussianity generated by single-field inflation, coincides with our definition of ψ,

$$\mathrm{e}^{2\alpha} = a^2(t)\mathrm{e}^{2\zeta} = a^2(t)(1+2\zeta+2\zeta^2). \tag{41}$$

The other generalization, employed by Malik and Wands (based on [1, 4, 32]), is different and we denote it by ζ_{mw}; it is

$$\mathrm{e}^{2\alpha} = a^2(t)(1+2\zeta_{\mathrm{mw}}), \tag{42}$$

[7] The notation ψ is that of [32]; Kodama and Sasaki [2] denote the same quantity by $\mathcal{R}$.

[8] Another definition [5] is discussed elsewhere [33].

so that

$$\zeta_{\rm mw} = \zeta + \zeta^2, \tag{43}$$

or equivalently

$$\zeta_{2\rm mw} = \zeta_2 + 2(\zeta_1)^2. \tag{44}$$

Evaluated to first order in the perturbations, the gauge transformation equation (34) reduces to the known result [1]

$$\psi_A - \psi_B = H\Delta t, \tag{45}$$

where Δt is the time displacement between the slices. If one of the slicings has uniform density and the other is flat this gives a gauge invariant definition of ζ,

$$-\zeta = \psi + \frac{\mathcal{H}}{\rho_0'}\delta\rho, \tag{46}$$

where a prime denotes differentiation with respect to conformal time and $\mathcal{H} \equiv a'/a$, and the right-hand side is evaluated on a generic slicing.

To illustrate how things work at second order, we will just show how equation (34) reproduces the known second-order result [7][9]. Following along the lines of Lyth and Wands [27], we expand the integrated expansion to second order in a power series expansion centred on the flat slicing of equation (34),

$$\delta N = \frac{\partial N}{\partial\rho}\delta\rho + \frac{1}{2}\frac{\partial^2 N}{\partial\rho^2}\delta\rho^2. \tag{47}$$

Using equations (22) and (47) this gives the perturbed expansion at second order,

$$\delta N_2 = \frac{\mathcal{H}}{\rho_0'}\delta\rho_2 - 2\frac{\mathcal{H}}{\rho_0'^{\,2}}\delta\rho_1'\delta\rho_1 + \left(\mathcal{H}\frac{\rho_0''}{\rho_0'} - \mathcal{H}'\right)\left(\frac{\delta\rho_1}{\rho_0'}\right)^2, \tag{48}$$

where the right-hand side is evaluated on flat slices. The expression given in equation (48) is then related to $\zeta_{2\rm mw}$, if this is also evaluated on flat slices, by $\zeta_{2\rm mw} = -\delta N_2 + 2\zeta_1^2$, which coincides with equation (44) using (37), i.e. $\zeta_2 = -\delta N_2$.

6. Conclusions

Our central result is the existence of a conserved *non-perturbative* quantity ζ, corresponding to the scalar curvature perturbation in the perturbative case, which may be defined on the uniform-density, uniform-expansion or comoving slices since these coincide in the large scale limit. Locally, this statement follows from the equations of the coordinate-free approach as given for instance in [39] and reviewed in [40][10]. We have here preferred to employ the usual coordinate approach. This is because the coordinate approach is used in practice in almost all treatments of the evolution of perturbations during and after horizon entry, owing to the relative ease with which it handles the effect

[9] That this happens is stated without proof in [31], and for the special case of adiabatic pressure in [27].

[10] While finishing version 2 of the present paper, a related work on the evolution of perturbations appeared, using the covariant approach [41].

of particle collisions and free streaming. Using the coordinate approach has allowed us to make contact with existing second-order perturbation calculations, clarifying some previously mysterious connections between those of different authors. Also, on very, very large scales, it makes contact with the idea that the 'perturbations' are actually supposed to be random fields which can become arbitrarily large in an arbitrarily large region, thus providing in some sense a generalization of the stochastic description of scalar field evolution during inflation. Finally, the gradient expansion can be most easily implemented in the coordinate approach, in particular if we want to go to the next order in the gradient expansion.

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Appendix A: Tensor components

A.1. The metric tensor

The metric tensor is given by

$$g_{00} = -\mathcal{N}^2 + \beta^i \beta_i, \qquad g_{0i} = g_{i0} = \beta_i, \tag{A.1}$$

$$g_{ij} \equiv \gamma_{ij} = \mathrm{e}^{2\alpha} \tilde{\gamma}_{ij}, \tag{A.2}$$

and

$$g^{00} = -\frac{1}{\mathcal{N}^2}, \qquad g^{0i} = g^{i0} = \frac{\beta^i}{\mathcal{N}^2}, \tag{A.3}$$

$$g^{ij} = \gamma^{ij} - \frac{\beta^i \beta^j}{\mathcal{N}^2} = \frac{1}{\mathrm{e}^{2\alpha}} \tilde{\gamma}^{ij} - \frac{\beta^i \beta^j}{\mathcal{N}^2}, \tag{A.4}$$

where $\det[\tilde{\gamma}_{ij}] = 1$ and $\beta^i = \gamma^{ij} \beta_j$.

A.2. 3-geometry

The unit timelike vector normal to the hypersurface $t =$ constant is given by

$$n_\mu = [-\mathcal{N}, 0, 0, 0], \qquad n^\mu = \left[\frac{1}{\mathcal{N}}, -\frac{\beta^i}{\mathcal{N}} \right]. \tag{A.5}$$

This gives for the extrinsic curvature tensor [42]

$$K_{ij} = -n_{i;j} \tag{A.6}$$

$$= \frac{1}{2\mathcal{N}} \left[-\frac{\partial}{\partial t} \gamma_{ij} + \beta_{i|j} + \beta_{j|i} \right], \tag{A.7}$$

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which can be expressed as

$$K_{ij} = -\frac{\theta_n}{3}\gamma_{ij} + A_{ij}, \tag{A.8}$$

where $\gamma^{ij}A_{ij} = 0$ and θ_n is the expansion of n^μ,

$$\theta_n = \nabla_\mu n^\mu. \tag{A.9}$$

On large scales we have $A_{ij} = \mathrm{O}(\epsilon)$ by virtue of our assumptions, and equation (A.8) reduces to

$$K^i{}_j = -\frac{\theta_n}{3}\delta^i_j + \mathrm{O}(\epsilon^2) = \frac{\dot\alpha}{\mathcal{N}}\delta^i_j + \mathrm{O}(\epsilon^2) = -\frac{1}{\mathcal{N}}\left(\frac{\dot a}{a} + \dot\psi\right)\delta^i_j + \mathrm{O}(\epsilon^2). \tag{A.10}$$

In the case of a conformally flat 3-geometry, i.e., for $\tilde\gamma_{ij} = \delta_{ij}$, the intrinsic curvature on spatial 3-hypersurfaces is expressed as

$${}^{(3)}R = -\frac{2}{\mathrm{e}^{2\alpha}}\delta^{ij}(\psi_{,i}\psi_{,j} + 2\psi_{,ij}). \tag{A.11}$$

A.3. Einstein tensor

Here we give the components of the Einstein tensor to first order in ϵ, valid in an arbitrary gauge provided that the metric satisfies the conditions (9) and (10).

First let us consider the case of $\beta_i = 0$. Then, the time derivative $\partial/\partial t$ is along the normal vector $n_\mu = (-\mathcal{N}, 0)$. Then, for the $(0,0)$ and $(0,i)$ components of the Einstein tensor, we can apply the standard $(3+1)$ decomposition of the Ricci curvature tensor (i.e. the Gauss–Codacci equations), which are essentially the Hamiltonian and momentum constraint equations. They are given, for example, by equations (2.20) and (2.21) of [16]. In our notation, they are

$$G^0{}_0 = -\tfrac{1}{2}({}^{(3)}R + \tfrac{2}{3}K^2 - A^{ij}A_{ij}), \tag{A.12}$$

$$G^0{}_j = D_iA^i{}_j - \tfrac{2}{3}D_jK, \tag{A.13}$$

where D_i is the covariant derivative with respect to the metric γ_{ij}, and the extrinsic curvature K_{ij} is given by equation (A.8) with $\theta_n = -K$.

Since $A_{ij} = \mathrm{O}(\epsilon^2)$, and ${}^{(3)}R$ involves at least second derivatives of the metric tensor, these expressions reduce to

$$G^0{}_0 = -\tfrac{1}{3}K^2 + \mathrm{O}(\epsilon^2), \tag{A.14}$$

$$G^0{}_j = -\frac{1}{\mathcal{N}}\frac{2}{3}D_jK + \mathrm{O}(\epsilon^3), \tag{A.15}$$

where

$$K = -\frac{3}{\mathcal{N}}\left(\frac{\dot a}{a} + \dot\psi\right) + \mathrm{O}(\epsilon^2). \tag{A.16}$$

For the (i,j) components, they can be decomposed into the trace and traceless parts. Evaluation of the traceless part is slightly involved. We write down the corresponding components of the Einstein equations in appendix B, in which it is shown under our assumptions that they are $\mathrm{O}(\epsilon^2)$ and hence can be neglected. The trace part reduces to

$$G^i{}_i = \frac{2}{\mathcal{N}}\partial_t K - K^2. \tag{A.17}$$

Therefore

$$G^i{}_j = \frac{1}{3}\left(\frac{2}{\mathcal{N}}\partial_t K - K^2\right)\delta^i_j + \mathrm{O}(\epsilon^2). \tag{A.18}$$

It is easy to see that inclusion of $\beta_i = \mathrm{O}(\epsilon)$ does not change these results at all. The above results can be regarded as the $(3+1)$-decomposition with the hypersurface normal vector $n^\nu = (1/\mathcal{N}, 0)$. Thus, for a general choice of the spatial coordinates, we just have to perform the following replacements:

$$G^0{}_0 \to -n_\mu G^\mu{}_\nu n^\nu = G^0{}_0 + G^0{}_i\beta^i, \tag{A.19}$$

$$\mathcal{N}G^0{}_j \to -n_\mu G^\mu{}_\nu h^\nu{}_j = \mathcal{N}G^0{}_j, \tag{A.20}$$

$$G^i{}_j \to h^i{}_\mu G^\mu{}_\nu h^\nu{}_j = G^0{}_j\beta^i + G^i{}_j. \tag{A.21}$$

So, the difference is of the form $G^0{}_j\beta^i$, which is $\mathrm{O}(\epsilon^2)$.

To summarize, the components of the Einstein tensor on large scales are given by

$$G^0{}_0 = -\tfrac{1}{3}K^2 + \mathrm{O}(\epsilon^2), \tag{A.22}$$

$$G^0{}_j = -\frac{1}{\mathcal{N}}\frac{2}{3}D_j K + \mathrm{O}(\epsilon^2)\,, \tag{A.23}$$

$$G^i{}_j = \frac{1}{3}\left(\frac{2}{\mathcal{N}}\partial_t K - K^2\right)\delta^i_j + \mathrm{O}(\epsilon^2), \tag{A.24}$$

where K is given by equation (A.16).

Although not used in the present paper, it is useful to know the form of the $\mathrm{O}(\epsilon^2)$ corrections. The correction terms take a complicated form in general, but for the spatially conformally flat metric ($\tilde{\gamma}_{ij} = \delta_{ij}$) with $\beta^i = 0$, they take a relatively simple form. The components of the Einstein tensor in this case, valid to the accuracy of $\mathrm{O}(\epsilon^2)$, are given by

$$\begin{aligned}
G^0{}_0 &= -\tfrac{1}{3}K^2 + \gamma^{jk}(\psi_{,j}\psi_{,k} + 2\psi_{,jk}),\\
G^0{}_j &= -\frac{1}{\mathcal{N}}\frac{2}{3}K_{,j},\\
G^i{}_j &= \frac{1}{3}\left(\frac{2}{\mathcal{N}}\partial_t K - K^2\right)\delta^i_j + \gamma^{ik}\left[\frac{1}{\mathcal{N}}(\mathcal{N}_{,k}\psi_{,j} + \psi_{,k}\mathcal{N}_{,j} - \mathcal{N}_{,kj}) + \psi_{,k}\psi_{,j} - \psi_{,kj}\right].
\end{aligned} \tag{A.25}$$

A.4. Energy–momentum tensor

The 4-velocity is given by

$$u^0 = [\mathcal{N}^2 - (\beta_k + v_k)(\beta^k + v^k)]^{-1/2}, \tag{A.26}$$

$$u^i = u^0 v^i, \tag{A.27}$$

where v^i is the spatial velocity, and

$$u_0 = -u^0[\mathcal{N}^2 - \beta^k(\beta_k + v_k)], \tag{A.28}$$

$$u_i = u^0(v_i + \beta_i), \tag{A.29}$$

and $v_i = \gamma_{ij} v^j$. The components of the energy–momentum tensor are then given by

$$T^0{}_0 = -(u^0)^2(\rho + P)[\mathcal{N}^2 - \beta^k(v_k + \beta_k)] + P, \tag{A.30}$$

$$T^0{}_i = (u^0)^2(\rho + P)(v_i + \beta_i), \tag{A.31}$$

$$T^i{}_j = (u^0)^2(\rho + P)v^i(v_j + \beta_j) + \delta^i{}_j\, P. \tag{A.32}$$

We note that $T^0{}_0 = -\rho + \mathrm{O}(\epsilon^2)$ if β^i and v^i are both of $\mathrm{O}(\epsilon)$. If we choose our spatial coordinates to be comoving with the fluid, we have $v^i = 0$. Then the $(0, i)$ component of the Einstein equations tells us that $T^0{}_i = \mathrm{O}(\epsilon)$, which implies $\beta^i = \mathrm{O}(\epsilon)$. Hence the $(0, 0)$ component of the Einstein equations gives a local Friedmann equation at each spatial point x^i. In other words, as long as we are concerned with Einstein gravity, it is unnecessary to assume $\beta^i = \mathrm{O}(\epsilon)$, but the only condition we need to obtain the local Friedmann equation is the comoving condition $v^i = 0$ for the spatial coordinates.

Defining the projection tensor

$$h^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu, \tag{A.33}$$

i.e., projecting orthogonally to the velocity u^μ, we can write down the momentum conservation equation

$$h_{\lambda\nu}\nabla_\mu T^{\mu\nu} = 0, \tag{A.34}$$

which gives in components in the comoving gauge ($\beta^i = v^i = 0$)

$$(\rho + P)D_i \ln \mathcal{N} + D_i P = 0, \tag{A.35}$$

which reduces to the linear result in the comoving gauge on large scales. It may worth noting that this holds for general $\tilde{\gamma}_{ij}$.

The Raychauduri equation [43] in the comoving gauge ($\beta^i = v^i = 0$) and assuming zero vorticity is given by

$$\frac{1}{\mathcal{N}}\dot{\theta} + \frac{1}{3}\theta^2 + \frac{1}{2}(\rho + 3P) + \mathrm{O}(\epsilon^2) = 0. \tag{A.36}$$

With the identification of $\theta = 3\tilde{H}$, this is equal to the time derivative of the local Friedmann equation, which can also be obtained by combining the $(0, 0)$ component of the Einstein equations as given by equations (A.22) and (A.30), and the trace of the (i, j) component as given by equations (A.17) and (A.32).

Appendix B. Evolution equations for $\tilde{\gamma}_{ij}$

Here we write down the traceless part of the spatial components of the Einstein equations, that is, the evolution equations for $\tilde{\gamma}_{ij}$. They can be found, for example, in equations (2.11) and (2.12) of Shibata and Sasaki (SS) [16]. Their $\tilde{A}_{ij}$ is related to our A_{ij} by $\tilde{A}_{ij} = \mathrm{e}^{-2\alpha} A_{ij}$, or $\tilde{A}^i{}_j \equiv \tilde{\gamma}^{ij}\tilde{A}_{ij} = A^i{}_j$. Equation (2.11) of SS is

$$\partial_t \tilde{\gamma}_{ij} = -2\mathcal{N}\tilde{A}_{ij} + \pounds_\beta \tilde{\gamma}_{ij} - \tfrac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k, \tag{B.1}$$

where $\pounds_\beta$ is the Lie derivative along β^k, given for a second-rank tensor Q_{ij} by

$$\pounds_\beta Q_{ij} = \beta^k \partial_k Q_{ij} + Q_{ik}\partial_j\beta^k + Q_{kj}\partial_i\beta^k. \tag{B.2}$$

Thus, with the assumption that $\beta^k = \mathrm{O}(\epsilon)$, the assumption $\partial_t\tilde{\gamma}_{ij} = \mathrm{O}(\epsilon)$ is equivalent to $\tilde{A}_{ij} = \mathrm{O}(\epsilon)$.

Equation (2.12) of SS is

$$\begin{aligned}\partial_t \tilde{A}_{ij} = \mathcal{N}\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j\right) + \frac{1}{\mathrm{e}^{2\alpha}}\left[\mathcal{N}\left({}^{(3)}R_{ij} - \frac{\gamma_{ij}}{3}{}^{(3)}R\right) - \left(D_iD_j\mathcal{N} - \frac{\gamma_{ij}}{3}D^kD_k\mathcal{N}\right)\right] \\ + \pounds_\beta\tilde{A}_{ij} - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k - 8\pi G\frac{\mathcal{N}}{\mathrm{e}^{2\alpha}}\left(S_{ij} - \frac{\gamma_{ij}}{3}S^k{}_k\right),\end{aligned} \tag{B.3}$$

where S_{ij} is the spatial projection of the energy–momentum tensor,

$$S_{ij} = T_{ij}, \qquad S^k{}_k = \gamma^{k\ell}S_{k\ell}. \tag{B.4}$$

For a perfect fluid, or in the absence of anisotropic stress, we have

$$S_{ij} = (\rho + P)u_iu_j + \gamma_{ij}P. \tag{B.5}$$

Hence

$$S_{ij} - \frac{\gamma_{ij}}{3}S^k{}_k = (\rho+P)(u^0)^2\left[(v_i+\beta_i)(v_j+\beta_j) - \frac{\gamma_{ij}}{3}(v_k+\beta_k)(v^k+\beta^k)\right], \tag{B.6}$$

which is of second order in ϵ.

Assuming the anisotropic stress is negligible, equation (B.3) reduces to

$$\partial_t\tilde{A}_{ij} = \mathcal{N}K\tilde{A}_{ij} + \mathrm{O}(\epsilon^2) = -3\,\partial_t\alpha\,\tilde{A}_{ij} + \mathrm{O}(\epsilon^2). \tag{B.7}$$

Therefore, if $\tilde{A}_{ij} = \mathrm{O}(\epsilon)$, equation (B.7) has a decaying solution,

$$\tilde{A}_{ij} = \mathrm{e}^{-3\alpha}C_{ij}, \tag{B.8}$$

where $C_{ij} = \mathrm{O}(\epsilon)$ and $\partial_tC_{ij} = \mathrm{O}(\epsilon^2)$. Assuming that this decaying solution is absent (or ignorable), we have $\partial_t\tilde{\gamma}_{ij} = \mathrm{O}(\epsilon^2)$, which we assumed in the text.

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