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Biological Physics II

Problem Set 1

Please hand in your solutions before 12:00 noon on Wedenesday, April 21, 2021.

Note: For the numerical exercises in this course, use any programming language/software of your choice.

1. Classical mechanics

1+4+3 = 8 points

Consider a particle with position x and momentum p, and obeying a Hamiltonian

$$H = \frac{p^2}{2} + U(x).$$

a) Write down the equations of motion of this system as two coupled first-order differential equations. Do you expect this system to exhibit a stable fixed point? Explain qualitatively.

b) Consider the potential $U(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$. Numerically simulate $x(t)$ and $p(t)$ with the initial condition $x(0) = 0$, $p(0) = 1$, over the time interval $t = 0$ to $t = 15$. Plot $x(t)$ and $p(t)$. Do you observe a fixed point in the results? Plot the orbit of the system in the $x - p$ plane. Derive an exact expression for the curve you obtain. (Hint: Energy conservation.)

c) Now add a friction force term $-p$ to the equation of motion, and repeat the numerical exercise in the previous part. Explain in words why you see a fixed point in this case.

2. Insect population dynamics

3+2+5+6+3+3=22 points

An insect population has size N, the evolution of which is governed by the equation

$$\frac{dN}{dt} = RN\big(1 - \frac{N}{K}\big) - p(N). \tag{2.1}$$

The first term is a logistic growth term, where R is the population growth rate at small population size, and K is the carrying capacity of the environment. The second term $p(N)$ is the population decrease rate due to predation, and is given by

$$p(N) = \frac{BN^2}{A^2 + N^2}. \tag{2.2}$$

All parameters are assumed to be positive.

a) By an appropriate transformation, the equation (2.1) can be written in terms of dimensionless variables x, τ and dimensionless parameters r, k as

$$\frac{dx}{d\tau} = rx\big(1 - \frac{x}{k}\big) - \frac{x^2}{1+x^2}. \tag{2.3}$$

Find the transformation between the original variables and parameters in equation (2.1) and those in (2.3).

b) We now focus on analyzing (2.3). Show that this equation has a fixed point $x^* = 0$ for all parameter values. We call this the trivial fixed point. Prove that it is unstable. Show analytically that for fixed r and sufficiently small k, the system has only one non-trivial fixed point. Is it stable or unstable?

c) As k is increased, the number of non-trivial fixed points changes through bifurcations. To understand the bifurcation structure of the full model, prove that at a bifurcation point, r and k must have the parametric form

$$r = \frac{2y^3}{(1+y^2)^2};\ k = \frac{2y^3}{y^2-1}. \tag{2.4}$$

d) Numerically plot the curve generated by (2.4) in the $r - k$ plane. You should find that the curve separates the plane into three regions. What is the number of non-trivial fixed points in each of these regions, and what are their stabilities? Explain how you obtain your answers. For the region with three non-trivial fixed points, sketch the flow diagram on the x-axis. (Notice that two of these three non-trivial fixed points are stable; the smaller value is called the *refuge* level of the insect population, while the larger value is called the *outbreak* level.)

e) Consider the case $r = 0.4$, $k = 40$, which belongs to the region with three non-trivial fixed points. Numerically simulate $x(\tau)$ with the two different initial conditions $x(0) = 2$ and $x(0) = 2.3$. In both cases, let the simulations run for enough time such that the system nearly reaches a fixed point. Plot both the trajectories in the same graph. Estimate the refuge and outbreak levels in this system from the plots.

f) Consider a small variation of the problem, where the parameter r is allowed to change slowly in time. Assume that the change is quasistatic, i.e the population is always at a fixed point of (2.3) at any value of r. Starting with the parameters $r = 0.4$, $k = 40$ and the population at the refuge level for these parameters, implement an algorithm to increase r quasistatically up to 0.6. Plot x as a function of r. Find the value of r at which the population jumps from the refuge level to the outbreak level. Explain the jump in terms of bifurcations.