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Biological Physics II

Problem Set 2

Please hand in your solutions before 12:00 noon on Wedenesday, May 5, 2021.

2. Predator-prey Dynamics

3+8+5+7+5+2=30 points

Consider a prey population of size N and a predator population of size P, with the dynamics given by

$$\begin{aligned} \frac{dN}{dt} &= N\big(r(1-\frac{N}{k}) - k\frac{P}{N+p}\big) \\ \frac{dP}{dt} &= sP(1-k\frac{P}{N}), \end{aligned} \tag{2.1}$$

where all parameters are positive.

a) Find a transformation that allows us to rewrite the above equation in terms of dimensionless variables and parameters as

$$\frac{du}{d\tau} = u(1-u) - \frac{auv}{u+d}; \quad \frac{dv}{d\tau} = bv(1-\frac{v}{u}). \tag{2.2}$$

b) Find the fixed points of Eq (2.2) with positive $u,\ v$. You should find only one. Write down the condition involving a, b, d that makes the fixed point stable[1]. Using this, show that the fixed point is always stable for $a < \frac{1}{2}$.

c) For $a = 1.5$, numerically plot the boundary line in the $b-d$ plane that separates the regions of stability and instability of the fixed point. Indicate the two regions. What is the value of d in this case above which the fixed point is always stable? Find it analytically and check it with your numerical plot.

d) For $a = 1.5, b = 0.2, d = 0.1$, use the numerical result in the previous part to show that the fixed point is not stable. Numerically find the (approximate) fixed point (u^*, v^*). Sketch with pen and paper the nullclines of the system (2.2) for these parameters. Mark the fixed point, and draw arrows on the nullclines to indicate the local direction of flow. This system has a limit cycle with the fixed point lying within it. Using the arrows on the nullclines, determine whether the cycle is traversed in the anticlockwise or clockwise direction.

[1] From this exercise onwards, we will reserve the word *stable* for fixed points where all eigenvalues of the Jacobian have negative real parts, consistent with the lecture notes.

e) For the parameters in the above part, numerically integrate (2.2) with any initial condition (where $u, v > 0$) and for sufficient time such that convergence to the limit cycle can be observed. Plot $u(\tau)$, $v(\tau)$ as well as the orbit in the $u-v$ plane. Numerically average the values of u and v on the limit cycle, and compare the values to the numerically obtained value of the fixed point. Do they agree?

f) Based on the above observations, do you expect the system (2.2) to have an underlying integrable structure like the Lotka-Volterra equation? Explain in words.

g) **Bonus problem:** In part d), you were told that the system has a limit cycle in some parameter range. Here we *prove* its existence. The first step is to find a confined region in the $u-v$ space with a boundary Ω such that the flow everywhere on Ω points inwards into the region. This indicates that there must be an attractor inside the region. The Poincaré-Bendixson theorem[2] states that in two dimensions, such an attractor can only be a fixed point or a limit cycle. Since the fixed point in this case is unstable, there must be a limit cycle attractor within the region. The problem then boils down to constructing an appropriate closed curve Ω. Formally, you need to show that $\hat{n} \cdot \hat{f} > 0$ on your chosen Ω, where $\hat{n}$ is the inward normal vector to the curve Ω, and $\hat{f} = (\frac{du}{d\tau}, \frac{dv}{d\tau})$.
Hint: The region enclosed by Ω must contain the fixed point. In this problem, it helps if Ω is a polygon. Moreover, choose as many of its sides to be horizontal or vertical as possible such that the direction of flow can be checked easily from the sign of the right hand side of the equations in (2.2).

[2] Look into the books by Strogatz or Hofbauer and Sigmund for more details on the theorem. For the present exercise, however, the information given in the question should be sufficient.