1. Moment generating function

3+4+5= 12 pts

Let $t \geq 0$ be a continuous random variable with probability density function (PDF) $P_t(t)$. Then its moment generating function (MGF) is given by $G_t(z) = \langle e^{tz} \rangle$, where $\langle \cdot \rangle$ denotes average with respect to the distribution $P_t(t)$.

a) Let the Taylor series of the MGF be

$$G_t(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k. \tag{1.1}$$

The k-th moment of t is defined as $\langle t^k \rangle$. Show that it is equal to a_k.

a) $\frac{d^k}{dz^k} G_t(z)\Big|_{z=0} = \langle \frac{d^k}{dz^k}(e^{tz}) \rangle = \langle t^k \rangle$

$G_t(z) = \langle e^{tz} \rangle = \langle \sum_{k=0}^{\infty} \frac{t^k}{k!} z^k \rangle = \sum_{k=0}^{\infty} \frac{\langle t^k \rangle}{k!} z^k$

$a_k = \langle t^k \rangle$

b)

b) Let $t_1,\ t_2, \ldots,\ t_n$ be n independent random variables, with the respective MGFs $G_{t_1}(z_1),\ G_{t_2}(z_2), \ldots, G_{t_n}(z_n)$. Let $T_n = t_1 + t_2 + \cdots + t_n$. Show that the MGF of T_n is

$$G_{T_n}(z) = \prod_{j=1}^{n} G_{t_j}(z). \tag{1.2}$$

Now let the variables t_j, $j = 1, 2, \ldots, n$ be consecutive time intervals in a Poisson process, where each waiting time is identically distributed with rate constant r. Then T_n is the time required for n events to occur. Express the MGF of T_n. Determine the first and second moments of T_n by using the result of part a).

$G_{t_1}(z) = \langle e^{t_1 z} \rangle$

. . . .

$G_{t_n}(z) = \langle e^{t_n z} \rangle$

$\prod_{j=1} G_{t_j}(z) = \langle e^{t_1 z} \rangle \langle e^{t_2 z} \rangle \cdots \langle e^{t_n z} \rangle = \langle e^{(t_1 + t_2 + \cdots t_n) z} \rangle = \langle e^{T_n \cdot z} \rangle = G_{T_n}(z)$

$\frac{dG_{T_n}}{dz}\Big|_{z=0} = \langle T_n \rangle \qquad \frac{d^2 G_{T_n}}{dz^2}\Big|_{z=0} = \langle T_n^2 \rangle$

c) Numerically simulate a Poisson process with rate constant $r = 1$ and find the distribution of T_n for $n = 5$, $n = 10$ and $n = 20$. Use a bin size of about 0.05 together with 10^6 - 10^7 realizations of the process for a smooth histogram. Argue that the PDF of T_n for large n should approach a Gaussian distribution. Analytically calculate the mean and variance of T_{20} from the formulas in the previous part, and use these to plot the Gaussian approximation to the distribution of T_{20} and plot it in the same figure as the numerical result.

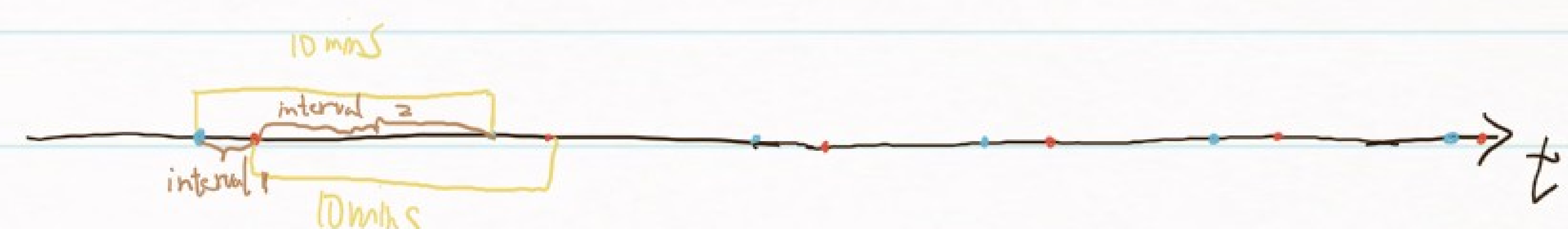
Poisson process $P_{T_n} = \frac{(rT_n)^n}{n!} e^{-rT_n}$ where rate $r = 1$, $n = 5, 10, 20$

$P_{T_5} = \frac{T_5^5}{5!} \times e^{-T_5} \qquad P_{T_{10}} = \frac{T_{10}^{10}}{10!} \times e^{-T_{10}} \qquad P_{T_{20}} = \frac{T_{20}^{20}}{20!} \times e^{-T_{20}}$

mean of $T_{20} = rT_{20}$

variance of $T_{20} = rT_{20}$

(d) we can use a line to represent the schedule of train



blue dots represents the west train arrive

red dots represents the east train arrive

according to this time line, Prob(interval 1) < Prob(interval 2)

which means I am more likely to get west train

If interval 2 is 4 times the length of interval 1, I will find that I visit visit Alice about 4 times as frequently as Bob.

2. Gillespie algorithm.

a) $P_{ji}(\tau_{ji}) = r_{ji} e^{-r_{ji}\tau_{ji}}$

$CDF(\tau_{ji}) = \int_0^{\tau_{ji}} (r_{ji} e^{-r_{ji}\tau_{ji}})\, d\tau_{ji} = -e^{-r_{ji}\tau_{ji}} \Big|_0^{\tau_{ji}} = 1 - e^{-r_{ji}\tau_{ji}}$

$CDF(\tau) = \int_0^{\tau} r_{ki} e^{-r_{ki}\tau}\, d\tau = -e^{-r_{ki}\tau} \Big|_0^{\tau}$, where $\tau = \min\{\tau_{1i}, \tau_{2i}, \dots \tau_{ki}\}$

b) $P_i(\tau) = R_i e^{-R_i \tau}$

$N = 1$ at $t = 0$