1. The Ricker Model

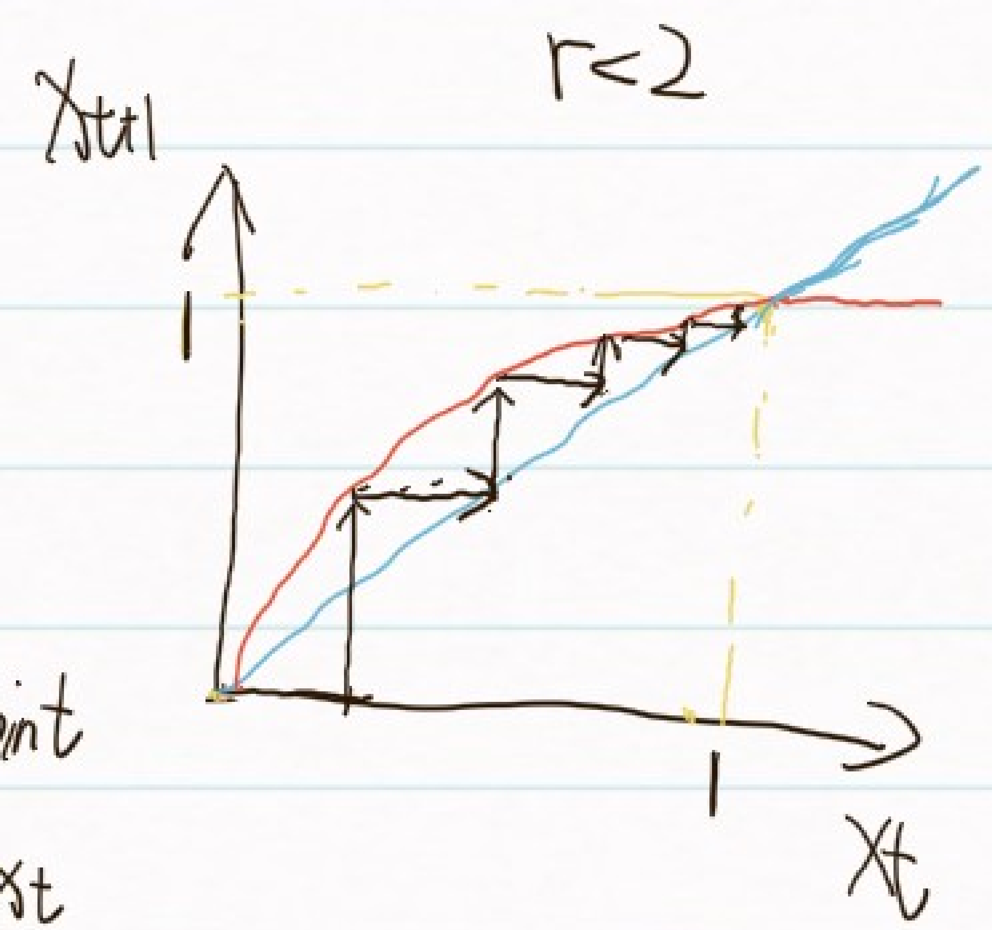
$$N_{t+1} = N_t e^{r(1-\frac{N_t}{K})} \quad (1)$$

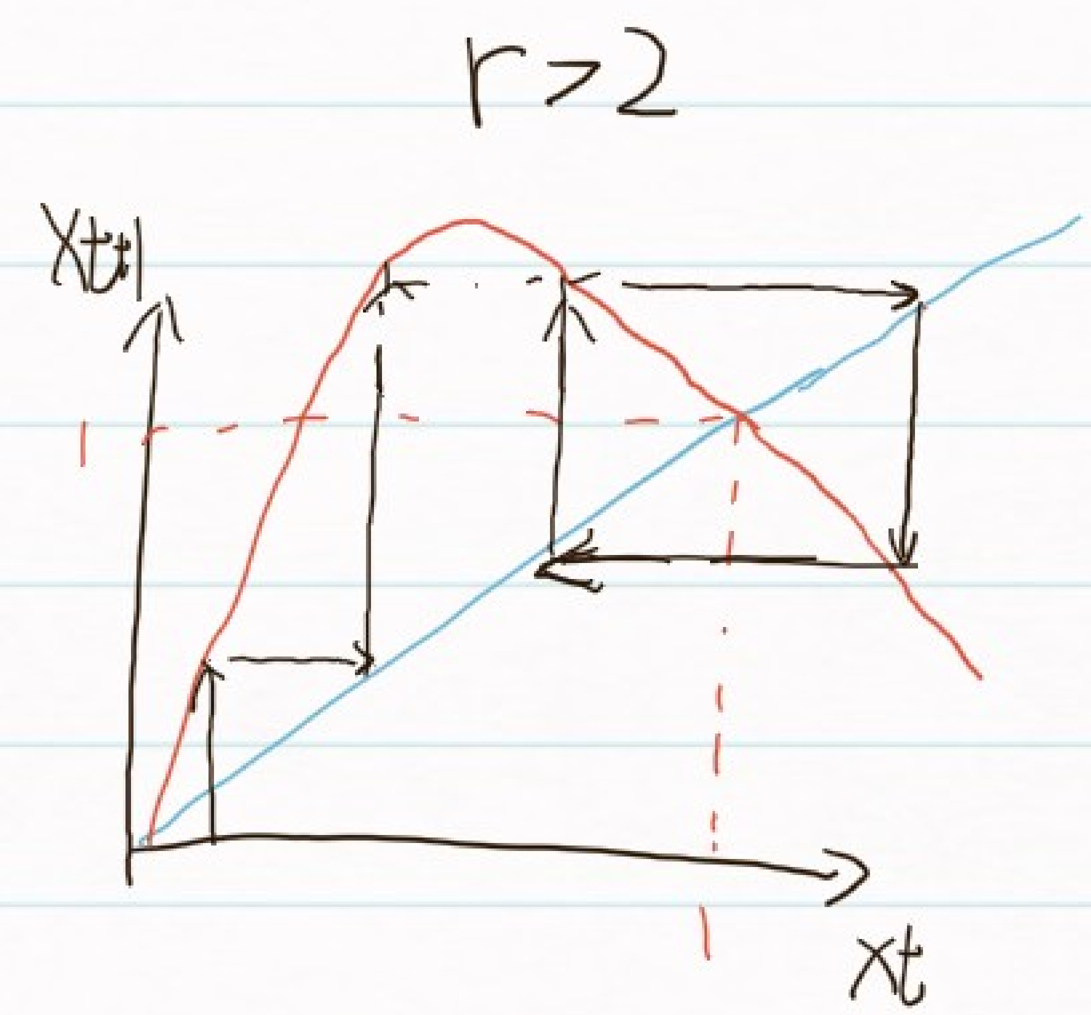
a) $x_{t+1} = x_t \cdot e^{r(1-x_t)}$

(1) divide by $K \Rightarrow \frac{N_{t+1}}{K} = \frac{N_t}{K} e^{(1-\frac{N_t}{K})}$

replace $\frac{N_t}{K} = x_t$, $\frac{N_{t+1}}{K} = x_{t+1} \Rightarrow x_{t+1} = x_t \cdot e^{r(1-x_t)}$

transformation





consider fixed point

$$\begin{cases} x_{t+1} = x_t \\ x_{t+1} = x_t \cdot e^{r(1-x_t)} \end{cases} \Rightarrow x_t = x_t \cdot e^{r(1-x_t)}$$

when $x_t = 0$, there is a trivial fixed point

at $x=0$:

assume $x_t = \varepsilon > 0$ $\quad x_{t+1} \approx \varepsilon \cdot e^{r(1-\varepsilon)} > \varepsilon = x_t$

here is the reason why unstable

if $x_t \neq 0$ $\quad x_t = x_t e^{r(1-x_t)} \Rightarrow e^{r(1-x_t)} = 1 \Rightarrow r(1-x_t) = 0$

$x_t = 1$ where is another fixed point

at $x=1$

assume $x_t = 1+\varepsilon$ $\quad x_{t+1} = (1+\varepsilon) \cdot e^{-r\varepsilon} < 1+\varepsilon = x_t$

$$x'_{t+1} = e^{r(1-x_t)} - r x_t \cdot e^{r(1-x_t)} = (1-r x_t) e^{r - r x_t}$$

at $x_t = 1$ $\quad |x'_{t+1}| = |(1-r)| < 1 \Rightarrow$ stable fixed point $0 < r < 2 \Rightarrow r_l = 0,\ r_u = 2$

at $x_t = 0$ $\quad |x'_{t+1}| = |e^r|$ always greater than $1 \Rightarrow x_t = 0$ is unstable FP

b) $V_t = (x_t - x^*)^2 \quad V_{t+1} < V_t$

$$V_{t+1} - V_t = (x_{t+1} - x^*)^2 - (x_t - x^*)^2 = (x_{t+1} - x_t)(x_{t+1} + x_t - 2x^*)$$

$$= x_t(e^{r(1-x_t)} - 1)(x_t(1 + e^{r(1-x_t)}) - 2x^*) \quad \text{where } x^* = 1$$

$$= x\underbrace{(e^{r(1-x)} - 1)}\,[\underbrace{x(1 + e^{r(1-x)}) - 2}]$$

assume $x = 1 + \varepsilon$, $\varepsilon > 0$

$$e^{r(1-x)} - 1 < 0$$

$$x(1 + e^{r(1-x)}) - 2 = 1 + \varepsilon + (1+\varepsilon)e^{-r\varepsilon} - 2$$

$$= \varepsilon - 1 + (1+\varepsilon)(1 - r\varepsilon) = \varepsilon - 1 + 1 + \varepsilon - r\varepsilon = (2 - r)\varepsilon > 0 \quad \text{due to } 0 < r < 2$$

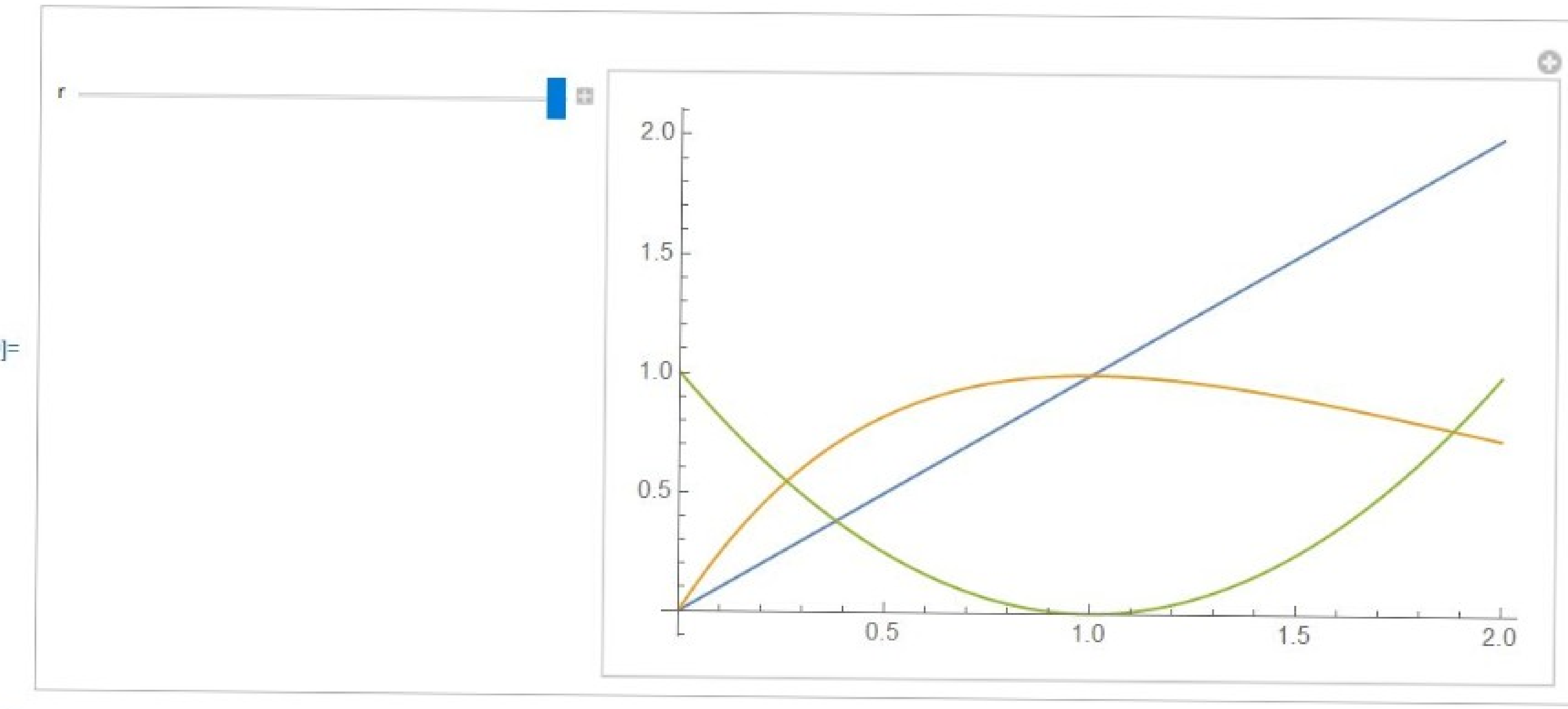
$\therefore V_{t+1} - V_t < 0$

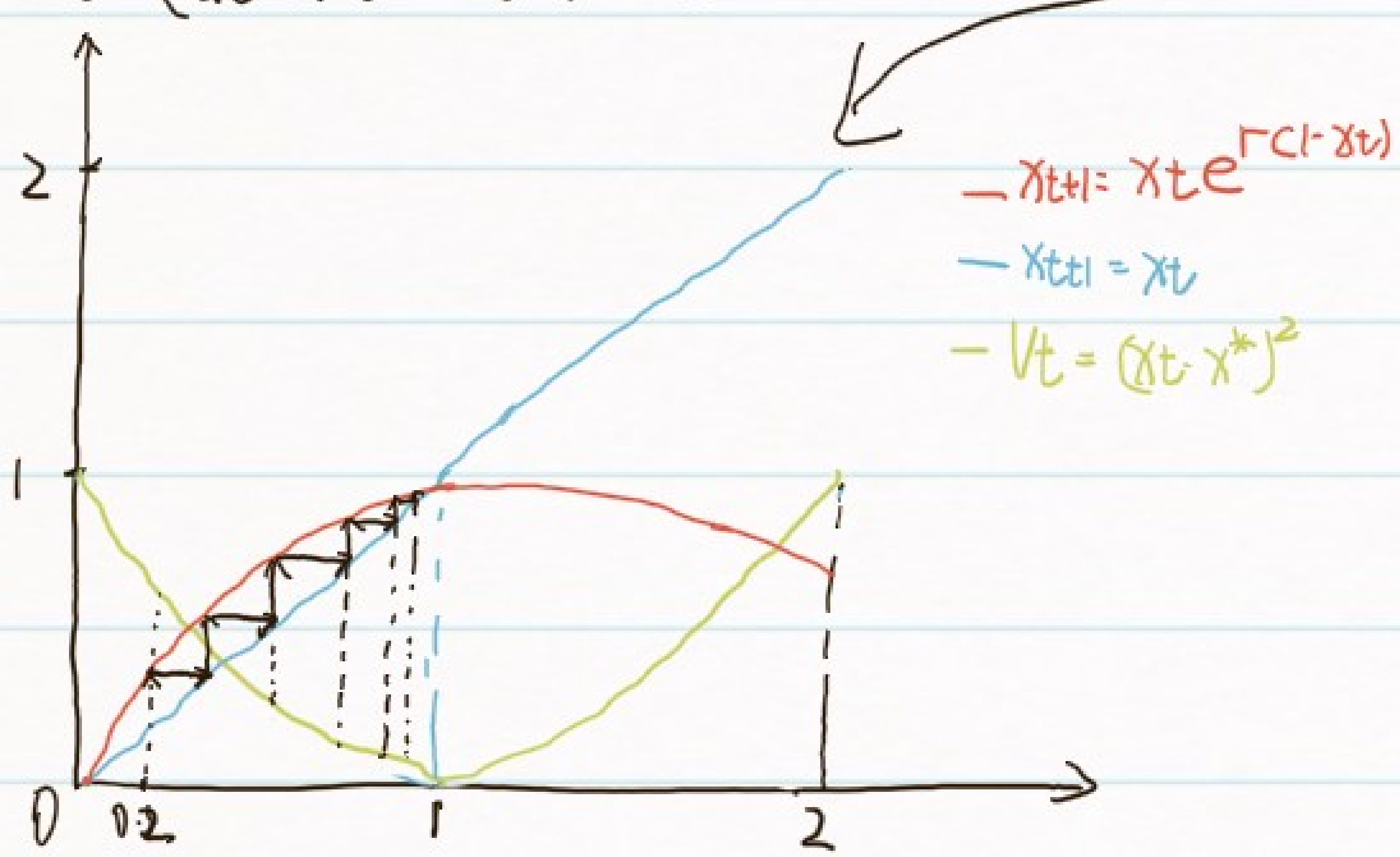
$$V_t = (x_t - x^*)^2 = x_t^2 + 1 - 2x_t$$

```
In[29]:= Manipulate[Plot[{x, x * Exp[r (1 - x)], x^2 - 2 x + 1}, {x, 0, 2}], {r, 0, 1}]
```

[交互式操作 [绘图 [指数形式

Out[29]=





t	0	1	2	3	4	5
x_t	0.2	0.445	0.775	0.971	0.9996	0.9999999
V_t	0.64	0.308	0.05	0.0009	1.93×10^{-7}	9.37×10^{-15}

bioset4.py - C:/Users/Xiongxiao Wang/AppData/Local/Programs/Python/...

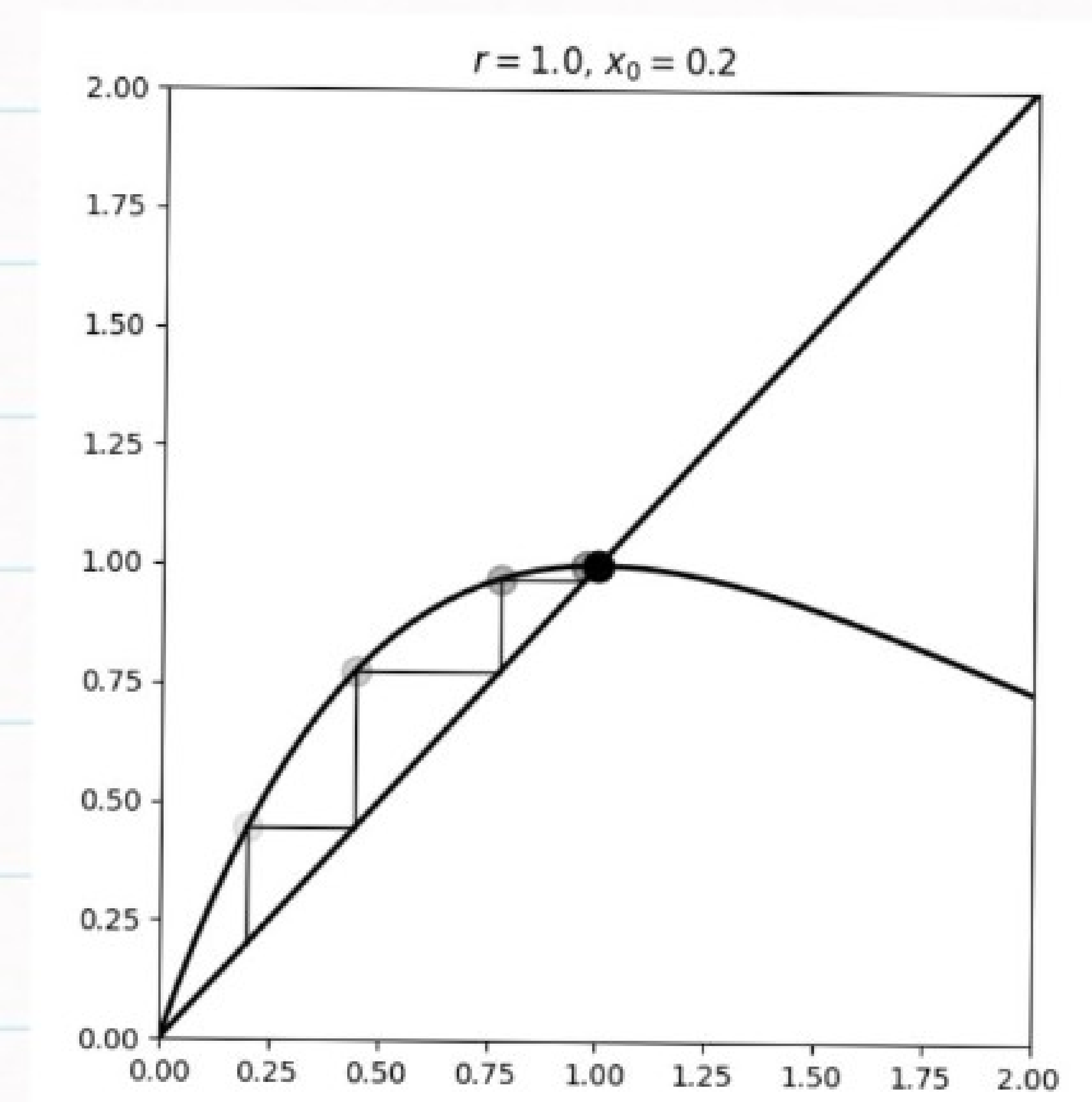
File Edit Format Run Options Window Help

```
import math
x=0.2;
for n in range (0,100):
    v=(x-1)**2;
    print(n, x, v)
    x=x*math.exp(1-x);
```

IDLE Shell 3.9.5

File Edit Shell Debug Options Window Help

```
Python 3.9.5 (tags/v3.9.5:0a7dcbd, May  3 2021, 17:27:52) [MSC v.1928 64 bit (AMD64)] on win32
Type "help", "copyright", "credits" or "license()" for more information.
>>>
= RESTART: C:/Users/Xiongxiao Wang/AppData/Local/Programs/Python/Python39/bioset4.py
0 0.2 0.6400000000000001
1 0.445108185698493 6 0.307904925578817 55
2 0.775268313845492 4 0.050504330761848085
3 0.970625653328974 3 0.00086285224234959 72
4 0.999560031507474 4 1.9357227441523897e-07
5 0.999999903185469 6 9.373053295132665e-15
6 0.9999999999999953 2.1742978700154138e-29
7 1.0 0.0
8 1.0 0.0
9 1.0 0.0
10 1.0 0.0
11 1.0 0.0
12 1.0 0.0
13 1.0 0.0
14 1.0 0.0
15 1.0 0.0
16 1.0 0.0
17 1.0 0.0
18 1.0 0.0
19 1.0 0.0
20 1.0 0.0
21 1.0 0.0
22 1.0 0.0
23 1.0 0.0
24 1.0 0.0
25 1.0 0.0
```



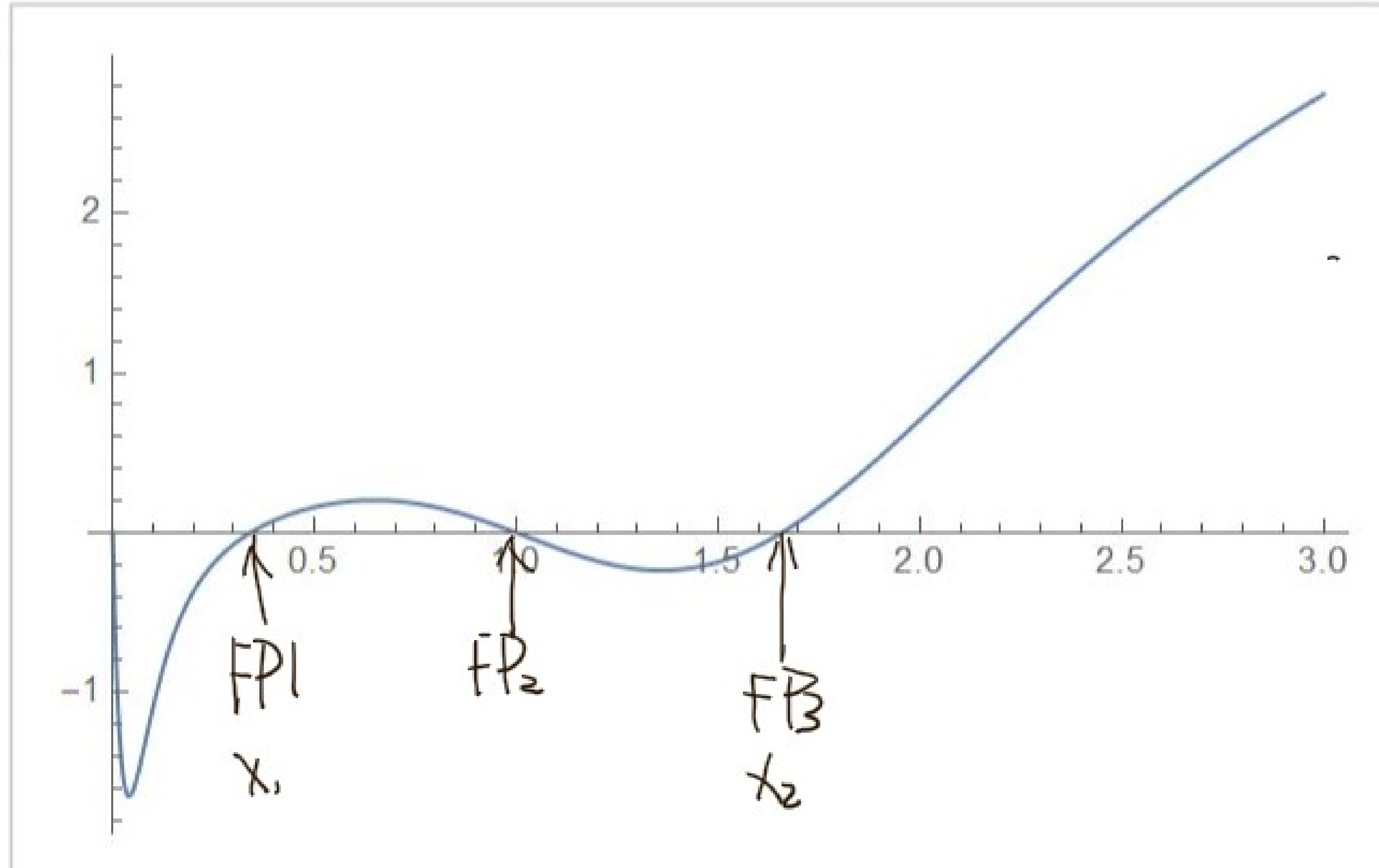
d)

In[52]:=
```
f[x_, r_] := x * Exp[r (1 - x)]
```
指数形式

```
Manipulate[Plot[x - f[f[x, r], r], {x, 0, 3}], {r, 0, 2.4}]
```
交互式操作 绘图

Out[53]=

r



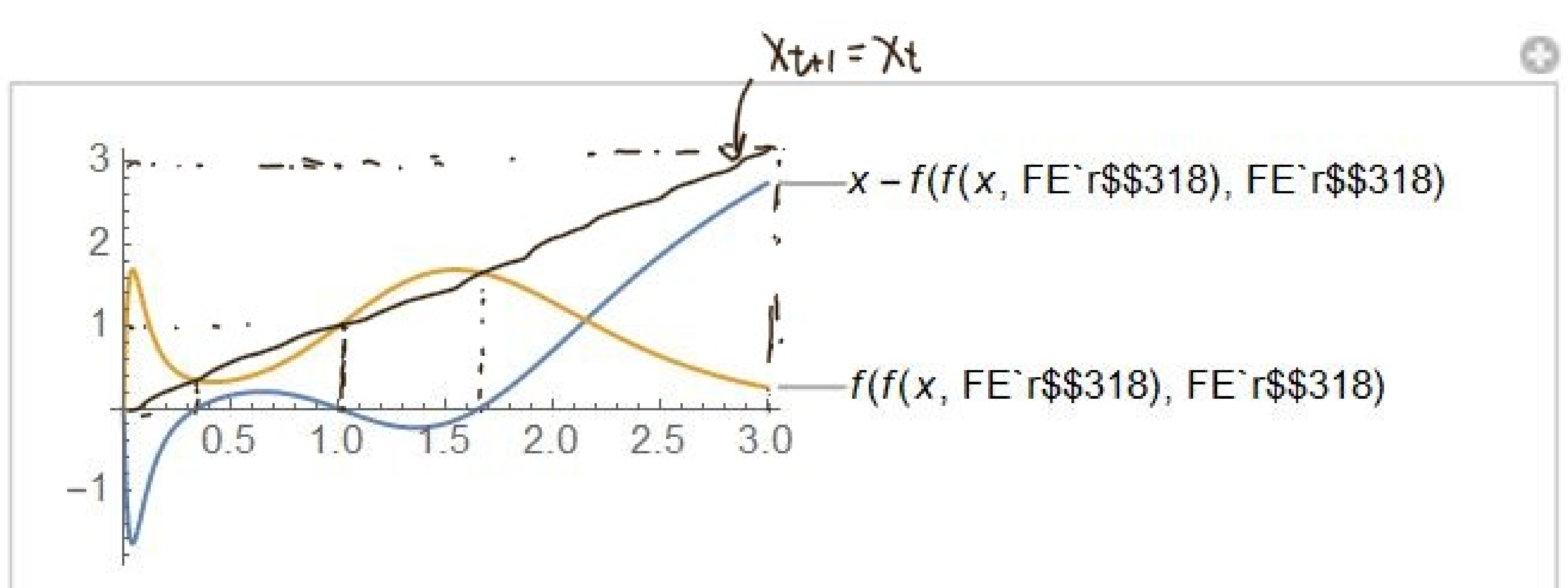
$x_1 \approx 0.341$ $x_2 \approx 1.658$

In[59]:=
```
f[x_, r_] := x * Exp[r (1 - x)]
```
指数形式

```
Manipulate[Plot[{x - f[f[x, r], r], f[f[x, r], r]}, {x, 0, 3}, PlotLabels → "Expressions"], {r, 0, 2.4}]
```
交互式操作 绘图 数据绘制标签

Out[60]=

r



initial condition $x_0 = 0.2$

bioset4.py - C:/Users/Xiongxiao Wang/AppData/L

File Edit Format Run Options Window Help

```python
import math
def f(x_):
    f1=x_*math.exp(2.4*(1-x_));
    return f1
x=0.2;
for n in range (0,100):
    print(n,x);
    x=f(f(x));
```

IDLE Shell 3.9.5

File Edit Shell Debug Options Window Help

```
4.py
0 0.2
1 0.5692144778275476
2 0.3786714983194463 5
3 0.32720227098581883
4 0.3500558640238458
5 0.33712264573311956
6 0.3438350430310286
7 0.34016257037906306
8 0.34212006034954007
9 0.34106132073317763
10 0.34162956970191993
11 0.34132329786633325
12 0.3414880013722872
13 0.34139932173905707
14 0.3414470376002283
15 0.34142135413278824
16 0.34143517587466593
17 0.3414277368513517
18 0.341431740402434
19 0.34142958569863 28
20 0.34143074533791 99
21 0.34143012122674987
22 0.3414304571182413
23 0.3414302763437619
24 0.34143037363520845
25 0.3414303212736562
26 0.34143034945425405
27 0.34143033428766295
```

$\uparrow$ t $\uparrow$ g_t

$\leftarrow x_1^*$

$x_0 = 1.2$

bioset4.py - C:/Users/Xiongxiao Wang/AppData/Lo

File Edit Format Run Options Window Help

```python
import math
def f(x_):
    f1=x_*math.exp(2.4*(1-x_));
    return f1
x=1.2;
for n in range (0,100):
    print(n,x);
    x=f(f(x));
```

IDLE Shell 3.9.5

File Edit Shell Debug Options Window Help

```
0 1.2
1 1.377438187930333
2 1.6130983229820666
3 1.6784257288915043
4 1.6470437887202105
5 1.664483179992611
6 1.6553117894420997
7 1.6603000234658376
8 1.6576319277389653
9 1.6590724390303804
10 1.658298522420481
11 1.6587154259206844
12 1.6584911645550824
13 1.6586118929736204
14 1.658546927293722
15 1.65858189408186
16 1.658563076007963
17 1.6585732039863748
18 1.6585677532496494
19 1.658570686815244
20 1.6585691079974574
21 1.6585699577071664
22 1.658569500400135
23 1.6585697465195428
24 1.6585696140599322
25 1.658569685348729
26 1.6585696469816276
27 1.6585696676305195
28 1.6585696565174382
```

$\uparrow$ t $\uparrow$ g_t

x_2^*

$x_0 = 2.5$

bioset4.py - C:/Users/Xiongxiao Wang/AppData/Local/Programs/Python/...

File Edit Format Run Options Window Help

```python
import math
def f(x_):
    f1=x_*math.exp(2.4*(1-x_));
    return f1
x=2.5;
for n in range (0,100):
    print(n,x);
    x=f(f(x));
```

IDLE Shell 3.9.5

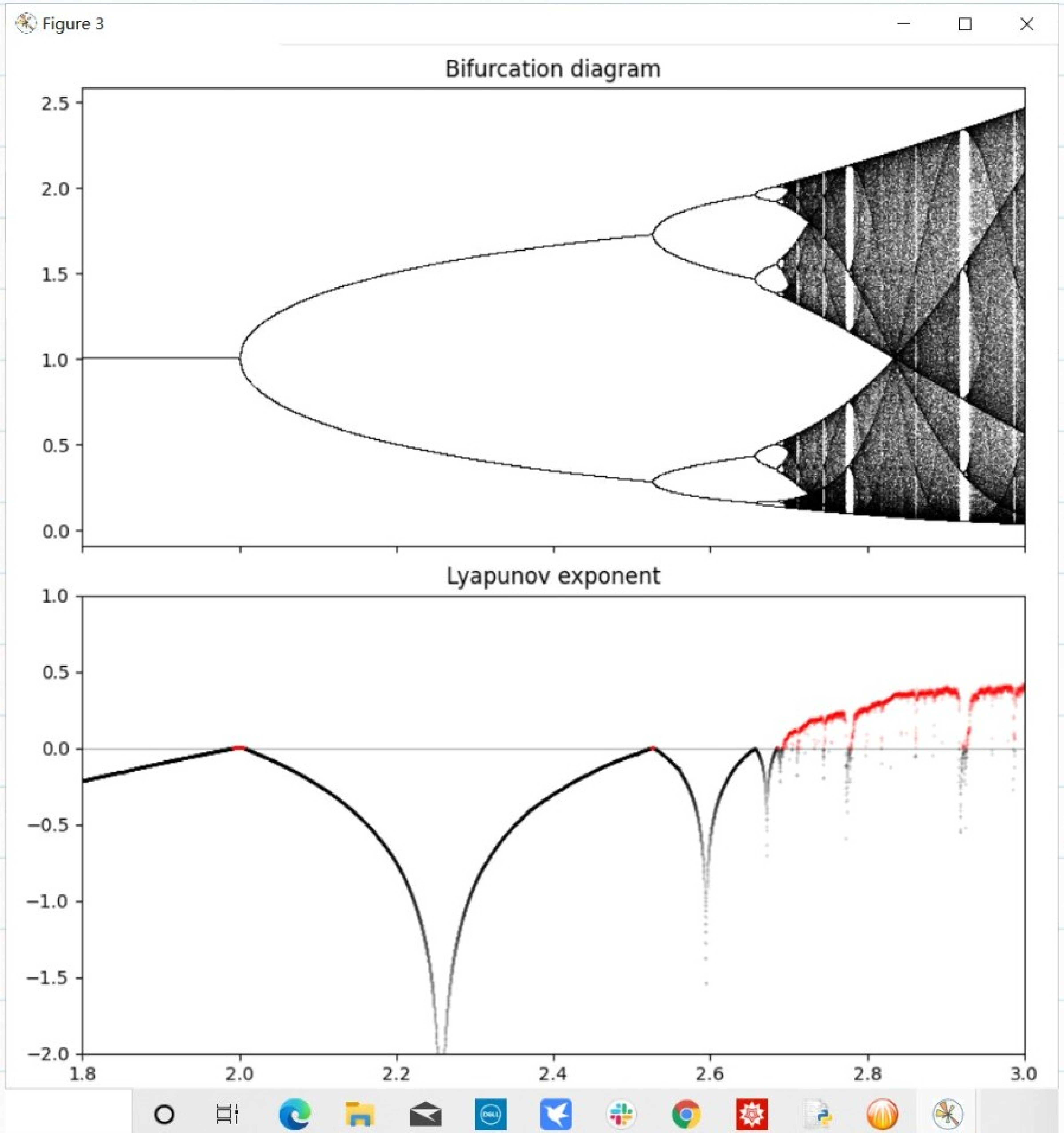
File Edit Shell Debug Options Window Help

```
0 2.5
1 0.6391273154085874
2 0.4366682743192862
3 0.3239196386016153
4 0.35233452178255303
5 0.3360932466424732
6 0.3444356220639931
7 0.33985412510808927
8 0.342290120776675 1
9 0.3409710035649877
10 0.3416785223778305
11 0.3412970525328385
12 0.34150215543911255
13 0.3413917125506424
14 0.34145113524683246
15 0.34141914951610924
16 0.3414363625876722
17 0.3414270982304267
18 0.34143208412067133
19 0.34142940071698386
20 0.3414308448951191
21 0.34143006764621636
22 0.34143048595501024
23 0.3414302608240603
24 0.34143038198780734
25 0.3414303167783521
26 0.3414303518735943
27 0.34143033298559194
28 0.3414303431509749
```

get close to x_1^*

$\uparrow$ t $\uparrow$ g_t

e)




```
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 6),
                               sharey=True)
plot_system(1, .2, 10, ax=ax1)

n = 10000
r = np.linspace(1.8, 3.0, n)
iterations = 1000
last = 100
x = 1e-5 * np.ones(n)
lyapunov = np.zeros(n)
fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(8, 9),
                               sharex=True)
for i in range(iterations):
    x = f(r, x)
    # We compute the partial sum of the
    # Lyapunov exponent.
    lyapunov += np.log(abs((1-r*x)*np.exp(r-r*x)))
    # We display the bifurcation diagram.
    if i >= (iterations - last):
        ax1.plot(r, x, ',k', alpha=.25)
ax1.set_xlim(1.8, 3.0)
ax1.set_title("Bifurcation diagram")

# We display the Lyapunov exponent.
# Horizontal line.
ax2.axhline(0, color='k', lw=.5, alpha=.5)
# Negative Lyapunov exponent.
ax2.plot(r[lyapunov < 0],
         lyapunov[lyapunov < 0] / iterations,
         '.k', alpha=.5, ms=.5)
# Positive Lyapunov exponent.
ax2.plot(r[lyapunov >= 0],
         lyapunov[lyapunov >= 0] / iterations,
         '.r', alpha=.5, ms=.5)
ax2.set_xlim(1.8, 3.0)
ax2.set_ylim(-2, 1)
ax2.set_title("Lyapunov exponent")
plt.tight_layout()
plt.show()
```