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Exercise 2: Reduced density matrix and entanglement entropy

a) $|\psi\rangle = \sum_{i=1}^{2}\sum_{j=1}^{2} \varphi_{ij} |i\rangle |j\rangle$

$$|\psi\rangle_1 = |\uparrow\rangle_A |\downarrow\rangle_B$$

$$|\psi\rangle_2 = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right)$$

$$|\psi\rangle_3 = \frac{1}{2}\left(|\uparrow\rangle_A + |\downarrow\rangle_A \right)\left(|\uparrow\rangle_B + |\downarrow\rangle_B \right)$$

$$|\psi\rangle = \sum_{\sigma_1=\uparrow,\downarrow}\sum_{\sigma_2=\uparrow,\downarrow} \varphi_{\sigma_1,\sigma_2} |\sigma_1\rangle |\sigma_2\rangle$$

$$= \left(|\uparrow\rangle_A \; |\downarrow\rangle_A \right) \begin{pmatrix} \varphi_{\uparrow\uparrow} & \varphi_{\uparrow\downarrow} \\ \varphi_{\downarrow\uparrow} & \varphi_{\downarrow\downarrow} \end{pmatrix} \begin{pmatrix} |\uparrow\rangle_B \\ |\downarrow\rangle_B \end{pmatrix}$$

$$|\psi\rangle_1 = \left(|\uparrow\rangle_A \; |\downarrow\rangle_A \right) \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |\uparrow\rangle_B \\ |\downarrow\rangle_B \end{pmatrix}$$

$$|\psi\rangle_2 = \left(|\uparrow\rangle_A \; |\downarrow\rangle_A \right) \cdot \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{pmatrix} \begin{pmatrix} |\uparrow\rangle_B \\ |\downarrow\rangle_B \end{pmatrix}$$

$$|\psi\rangle_3 = \left(|\uparrow\rangle_A \; |\downarrow\rangle_A \right) \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} |\uparrow\rangle_B \\ |\downarrow\rangle_B \end{pmatrix}$$

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b) $\hat{\rho} \equiv Tr_B(|\psi\rangle\langle\psi|) = \sum_{j=1}^{2} \langle j|\psi\rangle\langle\psi|j\rangle$

with the matrix elements $\rho_{ii'} = \langle i|\hat{\rho}|i'\rangle = \sum_j \psi_{ij}\psi^*_{i'j}$

$$\hat{\rho}_A = (|\uparrow\rangle_A \;\; |\downarrow\rangle_A) \underbrace{\begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow} \end{pmatrix}}_{\rho_{\sigma_1\sigma_2}} \begin{pmatrix} \langle\uparrow|_A \\ \langle\downarrow|_A \end{pmatrix}$$

$$\hat{\rho}_{A1} = Tr_B(|\psi\rangle_1\langle\psi|_1) = \underbrace{\langle\uparrow_B|\psi_1\rangle}_{=\langle\uparrow_B|\downarrow_B\rangle|\uparrow_A\rangle}\langle\psi_1|\uparrow_B\rangle + \underbrace{\langle\downarrow_B|\psi_1\rangle}_{=\langle\downarrow_B|\downarrow_B\rangle|\uparrow_A\rangle}\underbrace{\langle\psi_1|\downarrow_B\rangle}_{=\langle\uparrow_A|\langle\downarrow_B|\downarrow_B\rangle}$$

$$= \underbrace{\langle\uparrow_B|\downarrow_B\rangle}_{=0}|\uparrow_A\rangle \qquad = \underbrace{\langle\downarrow_B|\downarrow_B\rangle}_{1}|\uparrow_A\rangle \qquad = \langle\uparrow_A|\underbrace{\langle\downarrow_B|\downarrow_B\rangle}_{1}$$

$$\hat{\rho}_{A1} = |\uparrow_A\rangle\langle\uparrow_A| \qquad \rho_{\sigma_1\sigma_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{\rho}_{A2} = Tr_B(|\psi\rangle_2\langle\psi_2|) = \langle\uparrow_B|\psi_2\rangle\langle\psi_2|\uparrow_B\rangle + \langle\downarrow_B|\psi_2\rangle\langle\psi_2|\downarrow_B\rangle$$

$\langle\uparrow_B|\psi_2\rangle = \frac{-1}{\sqrt{2}}\langle\uparrow_B|\uparrow_B\rangle|\downarrow_A\rangle = -\frac{1}{\sqrt{2}}\cdot|\downarrow_A\rangle \qquad \langle\psi_2|\uparrow_B\rangle = -\frac{1}{\sqrt{2}}\cdot\langle\downarrow_A|$

$\langle\downarrow_B|\psi_2\rangle = \frac{1}{\sqrt{2}}|\uparrow_A\rangle \qquad \langle\psi_2|\downarrow_B\rangle = \frac{1}{\sqrt{2}}\cdot\langle\uparrow_A|$

$$\hat{\rho}_{A2} = \frac{1}{2}|\downarrow_A\rangle\langle\downarrow_A| + \frac{1}{2}|\uparrow_A\rangle\langle\uparrow_A| \qquad \rho_{\sigma_1\sigma_2} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\hat{\rho}_{A3} = Tr_B(|\psi_3\rangle\langle\psi_3|) = \langle\uparrow_B|\psi_3\rangle\langle\psi_3|\uparrow_B\rangle + \langle\downarrow_B|\psi_3\rangle\langle\psi_3|\downarrow_B\rangle$$

$\langle\uparrow_B|\psi_3\rangle = \frac{1}{2}(|\uparrow\rangle_A + |\downarrow\rangle_A) \qquad \langle\psi_3|\uparrow_B\rangle = \frac{1}{2}(\langle\uparrow|_A + \langle\downarrow|_A)$

$\langle\downarrow_B|\psi_3\rangle = \frac{1}{2}(|\uparrow\rangle_A + |\downarrow\rangle_A) \qquad \langle\psi_3|\downarrow_B\rangle = \frac{1}{2}(\langle\uparrow|_A + \langle\downarrow|_A)$

$$\hat{\rho}_{A3} = \frac{1}{2}(|\uparrow\rangle_A + |\downarrow\rangle_A)(\langle\uparrow|_A + \langle\downarrow|_A) \qquad \rho_{\sigma_1\sigma_2} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

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Entanglement entropy S_e:

$$S_e = -\mathrm{Tr}_A[\hat{\varrho}_A \ln \hat{\varrho}_A] = -\sum_\alpha w_\alpha \ln w_\alpha$$

For state $|\psi_1\rangle$ $\quad \overline{\psi}_{\sigma_1\sigma_2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ eigenvalue: $w_1 = 1$, $w_2 = 0$

$$S_e = -w_1 \ln w_1 - w_2 \ln w_2 = \underbrace{-0\ln 0}_{\lim\limits_{x\to 0} x\cdot\ln x = 0} - 1\cdot\ln 1 = 0$$

For state $|\psi_2\rangle$ $\quad \overline{\psi}_{\sigma_1\sigma_2} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ $\quad w_1 = \frac{1}{2} \quad w_2 = \frac{1}{2}$

$$S_e = -\frac{1}{2}\ln\left(\frac{1}{2}\right) - \frac{1}{2}\ln\left(\frac{1}{2}\right) = \ln 2$$

For state $|\psi_3\rangle$ $\quad \overline{\psi}_{\sigma_1\sigma_2} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

$$\det\left(\overline{\psi}_{\sigma_1\sigma_2} - \lambda E\right) = \begin{vmatrix} \frac{1}{4}-\lambda & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4}-\lambda \end{vmatrix} = \left(\frac{1}{4}-\lambda\right)^2 - \frac{1}{16} = 0 \Rightarrow \lambda_1 = \frac{1}{2} \quad \lambda_2 = 0$$

$$\Rightarrow w_1 = \frac{1}{2} \quad w_2 = 0$$

$$S_e = -\frac{1}{2}\ln\left(\frac{1}{2}\right) - 0\ln 0 = \frac{1}{2}\ln 2$$

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