CIRprocessOptimalPairsTrading

December 15, 2023

In this report, we provide an extensive summary on the progress that was made during this quarter. There will be snippets of code included, as well as the appropriate results

Problem Identification

There were the major goals of this independent study.

```
1. Understand the theory behind Optimal Pairs Trading when assets follow a given SDE
2. Implement a rudimentary version of a Pairs-Trading algorithm in Python
```

The beginning of the quarter was primarily spent identifying what final product I wanted to present by the end of the quarter. I had spent much time understanding Advanced Probability Theory via MATH 521/522/523 at UW, and I even participated in an *Applications of Stochastic Calculus* reading group in the spring. I wanted to take what I learned from those classes/seminars and translate it into a meaningful application related to what I learned from the CFRM curriculum. I made sure to include the fact that I wanted to work on my programming skills so I wanted the final product to have some code snippets that could be deployed to live data.

I decided on Optimal Pairs Trading because I had learned about Optimal Control and I wanted to investigate how it could be used in finance. This is when I came across the book- *Optimal Mean Reversion Trading: Mathematical Analysis and Practical Applications*[1] which was perfect for the quarter. My goal was then to take a section from this book and translate it into working python code. Specifically, I wanted to implement the *Maximum Likelihood Estimation* algorithm and derivation for optimal entry and exit times from scratch. With my goals clearly identified, I then wished to identify the resources (IDEs, libraries) needed to deploy these algorithms using real world data

Resources

One of the platforms I used extensively this quarter was *QuantConnect*. I was able to effectively watch the free courses on Algorithmic Trading 101, and play around with their free data sets that were on the cloud. I was also able to play around with the existing trading algorithms on the platform, and back test them to different pairs of stocks (via pairstrading lab). I spent a week or 2 familiarizing myself with the platform and understand how you can deploy various algorithms to FOREX, Equities and Futures data. Furthermore, I was also able to read over many research papers and new publications on the platform, which is also where I came across the article that I primarily based on work on: https://www.quantconnect.com/research/15294/optimal-pairs-trading/p1 [2]

In this article, the author implements a Pairs Trading algorithm, where they assume that the portfolio follows an OU process. They implement the MLE algorithm from scratch and derive the optimal entry and exit times appropriately. I referred to the code in this article extensively, especially when implementing the portfolio management part of the code. Reading this article also

gave me a better direction on where I wanted to the project to go. Since someone had already implemented Chapter 2 of your book, I wanted to try something more ambitious and complicated. Specifically, I wanted to implement the same Optimal Pairs Trading algorithm, but with assuming that the underlying portfolio follows a CIR process (Chapter 4). This means that I would have to go through the text and translate all the formulas into Python code, but where we assume that volatility is stochastic. I first implemented the MLE routine:

Maximum Likelihood for CIR process

As stated in [1] page 82, given portfolio values (y_i), the model parameters can be found by maximizing the average log-likelihood function:

$$l(\theta,\mu,\sigma,\{y_i\}) = \frac{1}{n}\sum_{i=1}^{n} \ln f^{CIR}(y_i|y_{i-1};\theta,\mu,\sigma)$$

$$= 2\ln(\tilde{\sigma}) - \frac{1}{n\tilde{\sigma}^2}\sum_{i=1}^{n}(y_i + y_{i-1}e^{-\mu\Delta t}) + \frac{1}{n}\sum_{i=1}^{n}(\frac{q}{2}\ln(\frac{y_i}{y_i e^{-\mu\Delta t}}) + \ln I_q(\frac{2}{\tilde{\sigma}^2}\sqrt{y_i y_{i-1}e^{-\mu\Delta t}}))$$

where $\tilde{\sigma} = \sigma^2\frac{1-e^{-\mu\Delta t}}{2\mu}, q = \frac{2\mu\theta}{\sigma^2} - 1$ and $I_q(z)$ is the modified Bessel function of the first kind and order q. Below is the code that is used to maximize the likelihood function, using SciPy to get the special functions and optimization routine:

[ ]:
```
import pandas as pd
import scipy.optimize as so
import scipy.special as sp
import numpy as np


def loglikelihood(params,*args):
    theta,mu,sigma = params

    y, dt = args
    n = len(y)


    sigma_tilde = np.sqrt(sigma**2*((1 - np.e**(-mu*dt))/(2*mu)))
    q = 2*mu*theta/sigma**2 - 1
    sum_1 = 0
    sum_2 = 0

    for i in range(1,n):
        sum_1 += y[i] + y[i - 1]*np.e**(-mu*dt)

    for i in range(1,n):

        sum_2 += q/2*np.log(y[i]/(y[i - 1]*np.e**(-mu*dt))) + np.log(sp.
↪iv(q,2*np.sqrt(y[i]*y[i - 1]*np.e**(-mu*dt))/sigma_tilde**2))
```

```
    log_likelihood = -2*np.log(sigma_tilde) - 1/(n*sigma_tilde**2)*sum_1 + (1/
↪n)*sum_2

    return -log_likelihood

def get_coefficients(y,dt,tol=1e-10):

    theta_init = np.mean(y)
    initial_guess = (theta_init, 0.01, 0.001)
    bounds = ((1e-5, None), (1e-5, None), (1e-5, np.sqrt(2*theta_init*0.01)))
      # initial guesses for theta, mu, sigma
    result = so.minimize(loglikelihood, initial_guess, args=(y, dt),␣
↪bounds=bounds)
    theta, mu, sigma = result.x

    return theta, mu, sigma, result.fun
```

note that here, we are minimizing the negative of the likelihood, which is equivalent to maximization. Note that here, y refers to the hypothetical portfolio values, where $P_t = S_t^{(1)} - \beta S_t^{(2)}$, for 2 stocks $S_t^{(1)}, S_t^{(2)}$ and allocation β. As described in [1], we compute the data for $\beta = 0.1, \cdots, 1$ and determine which portfolio has the highest likelihood to determine the optimal β. It should be noted that we did not test for stationarity of the 2 stocks, which should have been prior to MLE. Here, we assume that S_1 follows gold futures and S_2 follows crude oil futures (both are using closing data).

[ ]:
```
def compute_portfolio_values(S_1, S_2, B):
        return S_1 - B * S_2
```

This code was taken from [2]:

[ ]:
```
def argmax_B_alloc(ts_A, ts_B, dt):


        def compute_coefficients(x):
            portfolio_values = compute_portfolio_values(ts_A, ts_B, x)
            return get_coefficients(portfolio_values, dt)

        vectorized = np.vectorize(compute_coefficients)
        linspace = np.linspace(.001, 1, 100)
        res = vectorized(linspace)
        index = res[3].argmax()

        return res[0][index], res[1][index], res[2][index], linspace[index]
```

With this routine, we are able to implement the MLE routine for CIR process. However, upon running the above code, we ran into some issues. While the code compiles, we get values of infinity for the likelihood. I have verified that the formulas are indeed correct but I suspect that it has to do with the other functions that were used for [2]. One of the problems is that there is a dynamic bound

associated with σ to ensure that it never becomes negative. I assume that SciPy's optimization library takes care of this but I was unable to verify this. Next, upon debugging, I noticed that the infinite term was coming from the Modified Bessel Function component in my MLE routine. I initially decided to apply a soft-bound on the volatility but this did not work. I intend to investigate this behavior and figure out why this occured. To circumvent this issue (and not spend too much time on the MLE step), I utilized the PyMLE library: https://github.com/jkirkby3/pymle [3]

Preparing Time Series

[ ]:
```
import yfinance as yf

gld = yf.download('GC=F', start='2020-01-01', end='2022-12-31', progress=False)
crude = yf.download('CL=F', start='2020-01-01', end='2022-12-31',␣
 ↪progress=False)
silver = yf.download('SI=F', start='2020-01-01', end='2022-12-31',␣
 ↪progress=False)
copper = yf.download('HG=F', start='2020-01-01', end='2022-12-31',␣
 ↪progress=False)
palladium = yf.download('PA=F', start='2020-01-01', end='2022-12-31',␣
 ↪progress=False)
platinum = yf.download('PL=F', start='2020-01-01', end='2022-12-31',␣
 ↪progress=False)
#corn = yf.download('C=F', start='2020-01-01', end='2022-12-31', progress=False)
ts = compute_portfolio_values(gld['Adj Close'], crude['Adj Close'], 1)
```

Check Stationarity

[ ]:
```
from statsmodels.tsa.stattools import adfuller

def check_stationarity(timeseries):
    dftest = adfuller(timeseries, autolag='AIC')
    dfoutput = pd.Series(dftest[0:4], index=['Test Statistic','p-value','#Lags␣
 ↪Used','Number of Observations Used'])
    for key,value in dftest[4].items():
        dfoutput['Critical Value (%s)'%key] = value
    print(dfoutput)

check_stationarity(ts)
```

```
Test Statistic                  -3.332660
p-value                          0.013480
#Lags Used                       0.000000
Number of Observations Used    755.000000
Critical Value (1%)             -3.439041
Critical Value (5%)             -2.865376
Critical Value (10%)            -2.568813
dtype: float64
```

[ ]:
```
from pymle.models import CIR
import pymle.models.OrnsteinUhlenbeck as OU
from pymle.core.TransitionDensity import ExactDensity, KesslerDensity
from pymle.fit.AnalyticalMLE import AnalyticalMLE
import numpy as np
import matplotlib.pyplot as plt

# ============================
# Set the true model (CIR) params, to simulate the process
# ============================

# ============================
# Fit maximum Likelihood estimators
# ============================
# Choose some initial guess for params fit
param_bounds = [(0.0001, 1), (0.0001, 30), (0.0001, 1)]
goldCrudeModelsCIR = []
goldCrudeModelsOU = []
#silverCrudeModels = []
#goldSilverModels = []
#copperCrudeModels = []

dt = 1/len(gld)
# Choose some initial guess for params fit

for i in np.linspace(.001, 0.5, 100):
    model = CIR()
    model2 = OU()
    goldCrude = compute_portfolio_values(gld['Close'], crude['Close'], i)
    guess = np.array([0.5, np.mean(goldCrude), 0.10])
    #silverCrude = compute_portfolio_values(silver['Close'], crude['Close'], i)
    #goldSilver = compute_portfolio_values(gld['Close'], silver['Close'], i)
    #copperCrude = compute_portfolio_values(copper['Close'], crude['Close'], i)

    exact_est = AnalyticalMLE(goldCrude, param_bounds, dt,␣
 ↪density=ExactDensity(model)).estimate_params(guess)
    goldCrudeModelsCIR.append(exact_est)
    exact_est2 = AnalyticalMLE(goldCrude, param_bounds, dt,␣
 ↪density=ExactDensity(model2)).estimate_params(guess)
    goldCrudeModelsOU.append(exact_est2)
    #exact_est = AnalyticalMLE(silverCrude, param_bounds, dt,␣
 ↪density=ExactDensity(model)).estimate_params(guess)
    #silverCrudeModels.append(exact_est)
    #exact_est = AnalyticalMLE(goldSilver, param_bounds, dt,␣
 ↪density=ExactDensity(model)).estimate_params(guess)
    #goldSilverModels.append(exact_est)
```

```
    #exact_est = AnalyticalMLE(copperCrude, param_bounds, dt,␣
↪density=ExactDensity(model)).estimate_params(guess)
    #copperCrudeModels.append(exact_est)
```

```
Initial Params: [5.00000000e-01 1.79019763e+03 1.00000000e-01]
Initial Likelihood: 645.5567775700779
```

```
c:\Users\skvn3\anaconda3\Lib\site-
packages\scipy\optimize\_hessian_update_strategy.py:182: UserWarning: delta_grad
== 0.0. Check if the approximated function is linear. If the function is linear
better results can be obtained by defining the Hessian as zero instead of using
quasi-Newton approximations.
  warn('delta_grad == 0.0. Check if the approximated '
```

```
`gtol` termination condition is satisfied.
Number of iterations: 44, function evaluations: 148, CG iterations: 75,
optimality: 4.15e-07, constraint violation: 0.00e+00, execution time: 0.29 s.
Final Params: [3.69561514e-01 2.76072431e+01 1.26208050e-02]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.79019763e+03 1.00000000e-01]
Initial Likelihood: -52015.310656696165
`gtol` termination condition is satisfied.
Number of iterations: 15, function evaluations: 52, CG iterations: 22,
optimality: 7.17e-07, constraint violation: 0.00e+00, execution time: 0.042 s.
Final Params: [1.04527773e-04 2.99998726e+01 9.99970901e-01]
Final Likelihood: -51581.17111371555
Initial Params: [5.00000000e-01 1.78985897e+03 1.00000000e-01]
Initial Likelihood: 645.6500600468917
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 56, CG iterations: 17,
optimality: 3.58e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.03470117e-04 2.99887922e+01 1.93404211e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78985897e+03 1.00000000e-01]
Initial Likelihood: -52006.59719075144
`gtol` termination condition is satisfied.
Number of iterations: 63, function evaluations: 224, CG iterations: 103,
optimality: 5.51e-07, constraint violation: 0.00e+00, execution time: 0.16 s.
Final Params: [ 0.05279122 28.44922883  0.13258586]
Final Likelihood: -52153.55235631512
Initial Params: [5.00000000e-01 1.78952032e+03 1.00000000e-01]
Initial Likelihood: 645.743140657635
`gtol` termination condition is satisfied.
Number of iterations: 24, function evaluations: 72, CG iterations: 26,
optimality: 5.09e-07, constraint violation: 0.00e+00, execution time: 0.16 s.
Final Params: [8.99972748e-02 2.99814535e+01 1.05028728e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78952032e+03 1.00000000e-01]
Initial Likelihood: -52008.29664613461
```

```
`gtol` termination condition is satisfied.
Number of iterations: 31, function evaluations: 112, CG iterations: 36,
optimality: 7.95e-07, constraint violation: 0.00e+00, execution time: 0.098 s.
Final Params: [0.9999994  0.00184985 0.99999934]
Final Likelihood: -51110.06205613357
Initial Params: [5.00000000e-01 1.78918166e+03 1.00000000e-01]
Initial Likelihood: 645.8360266323045
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 56, CG iterations: 21,
optimality: 5.15e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.36355248e-04 2.99838889e+01 3.30072548e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78918166e+03 1.00000000e-01]
Initial Likelihood: -52020.409022845706
`gtol` termination condition is satisfied.
Number of iterations: 21, function evaluations: 64, CG iterations: 26,
optimality: 2.43e-07, constraint violation: 0.00e+00, execution time: 0.047 s.
Final Params: [1.00267671e-04 2.99922840e+01 9.99999906e-01]
Final Likelihood: -51538.965608708226
Initial Params: [5.00000000e-01 1.78884301e+03 1.00000000e-01]
Initial Likelihood: 645.9287143478077
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 68, CG iterations: 30,
optimality: 7.37e-07, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [0.00081805 0.11734836 0.01432009]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78884301e+03 1.00000000e-01]
Initial Likelihood: -52042.93432088404
`gtol` termination condition is satisfied.
Number of iterations: 33, function evaluations: 124, CG iterations: 44,
optimality: 4.50e-07, constraint violation: 0.00e+00, execution time:  0.1 s.
Final Params: [0.99999698 0.00543043 0.99999579]
Final Likelihood: -51137.66536404031
Initial Params: [5.00000000e-01 1.78850436e+03 1.00000000e-01]
Initial Likelihood: 646.0212051878855
`xtol` termination condition is satisfied.
Number of iterations: 74, function evaluations: 352, CG iterations: 123,
optimality: 5.34e+01, constraint violation: 0.00e+00, execution time: 0.58 s.
Final Params: [0.00262138 0.36916824 0.07522133]
Final Likelihood: 928.1054514044058
Initial Params: [5.00000000e-01 1.78850436e+03 1.00000000e-01]
Initial Likelihood: -52075.872540251
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 84, CG iterations: 41,
optimality: 5.32e-07, constraint violation: 0.00e+00, execution time: 0.06 s.
Final Params: [0.99999706 0.005013   0.9999958 ]
Final Likelihood: -51153.029363599315
Initial Params: [5.0000000e-01 1.7881657e+03 1.0000000e-01]
```

```
Initial Likelihood: 646.1135032778693
`xtol` termination condition is satisfied.
Number of iterations: 39, function evaluations: 136, CG iterations: 44,
optimality: 4.99e+01, constraint violation: 0.00e+00, execution time: 0.22 s.
Final Params: [1.42388588e-04 2.99825236e+01 7.33387109e-02]
Final Likelihood: 849.3358436119504
Initial Params: [5.0000000e-01 1.7881657e+03 1.0000000e-01]
Initial Likelihood: -52090.844447420095
`gtol` termination condition is satisfied.
Number of iterations: 22, function evaluations: 76, CG iterations: 31,
optimality: 1.91e-07, constraint violation: 0.00e+00, execution time: 0.051 s.
Final Params: [9.99999153e-01 6.76469408e-04 9.99999131e-01]
Final Likelihood: -51169.428793188665
Initial Params: [5.00000000e-01 1.78782705e+03 1.00000000e-01]
Initial Likelihood: 646.2056019620791
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 56, CG iterations: 19,
optimality: 2.57e-07, constraint violation: 0.00e+00, execution time:  0.1 s.
Final Params: [1.56378383e-03 2.99992917e+01 8.20065878e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78782705e+03 1.00000000e-01]
Initial Likelihood: -52081.16928487019
`xtol` termination condition is satisfied.
Number of iterations: 54, function evaluations: 244, CG iterations: 72,
optimality: 7.03e-06, constraint violation: 0.00e+00, execution time: 0.14 s.
Final Params: [1.00005442e-04 2.95872978e+01 9.99999992e-01]
Final Likelihood: -51471.76193565174
Initial Params: [5.00000000e-01 1.78748839e+03 1.00000000e-01]
Initial Likelihood: 646.297500296417
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 64, CG iterations: 26,
optimality: 2.91e-07, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [3.17109433e-03 3.52385431e+00 2.31278001e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78748839e+03 1.00000000e-01]
Initial Likelihood: -52065.40653667215
`gtol` termination condition is satisfied.
Number of iterations: 28, function evaluations: 84, CG iterations: 38,
optimality: 4.88e-07, constraint violation: 0.00e+00, execution time: 0.064 s.
Final Params: [1.00007852e-04 4.75102902e-01 9.99999993e-01]
Final Likelihood: -51460.1993176337
Initial Params: [5.00000000e-01 1.78714974e+03 1.00000000e-01]
Initial Likelihood: 646.3892036576909
`xtol` termination condition is satisfied.
Number of iterations: 52, function evaluations: 216, CG iterations: 72,
optimality: 7.05e+01, constraint violation: 0.00e+00, execution time: 0.37 s.
Final Params: [1.03236212e-04 2.18910195e+01 9.50285165e-02]
Final Likelihood: 757.9175363056643
```

```
Initial Params: [5.00000000e-01 1.78714974e+03 1.00000000e-01]
Initial Likelihood: -52055.951106138535
`xtol` termination condition is satisfied.
Number of iterations: 58, function evaluations: 256, CG iterations: 72,
optimality: 1.30e-04, constraint violation: 0.00e+00, execution time: 0.16 s.
Final Params: [1.00007219e-04 2.71147527e+01 9.99999992e-01]
Final Likelihood: -51450.78457121675
Initial Params: [5.00000000e-01 1.78681109e+03 1.00000000e-01]
Initial Likelihood: 646.480709142822
`gtol` termination condition is satisfied.
Number of iterations: 20, function evaluations: 56, CG iterations: 23,
optimality: 3.60e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.40321021e-04 9.46317588e+00 1.01051188e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78681109e+03 1.00000000e-01]
Initial Likelihood: -52048.41200041356
The maximum number of function evaluations is exceeded.
Number of iterations: 250, function evaluations: 1076, CG iterations: 383,
optimality: 4.28e-05, constraint violation: 0.00e+00, execution time: 0.66 s.
Final Params: [0.96640467 0.01666932 0.9999999 ]
Final Likelihood: -51217.43418783396
Initial Params: [5.00000000e-01 1.78647243e+03 1.00000000e-01]
Initial Likelihood: 646.5720191455629
`gtol` termination condition is satisfied.
Number of iterations: 21, function evaluations: 60, CG iterations: 22,
optimality: 6.76e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [2.85346710e-02 2.98928301e+01 6.11413935e-02]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78647243e+03 1.00000000e-01]
Initial Likelihood: -51998.07335265123
`xtol` termination condition is satisfied.
Number of iterations: 229, function evaluations: 1004, CG iterations: 365,
optimality: 1.75e-05, constraint violation: 0.00e+00, execution time: 0.61 s.
Final Params: [0.96699605 0.0068182  0.99999995]
Final Likelihood: -51225.08767025133
Initial Params: [5.00000000e-01 1.78613378e+03 1.00000000e-01]
Initial Likelihood: 646.663129093524
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 76, CG iterations: 32,
optimality: 6.54e-07, constraint violation: 0.00e+00, execution time: 0.14 s.
Final Params: [3.03037389e-03 1.45363304e+01 1.24680930e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78613378e+03 1.00000000e-01]
Initial Likelihood: -51960.40013419719
`gtol` termination condition is satisfied.
Number of iterations: 24, function evaluations: 76, CG iterations: 31,
optimality: 5.53e-07, constraint violation: 0.00e+00, execution time: 0.053 s.
Final Params: [1.00035252e-04 2.99347181e+01 9.99999971e-01]
```

```
Final Likelihood: -51435.45096677354
Initial Params: [5.00000000e-01 1.78579512e+03 1.00000000e-01]
Initial Likelihood: 646.7540392347727
`gtol` termination condition is satisfied.
Number of iterations: 29, function evaluations: 92, CG iterations: 29,
optimality: 2.85e-07, constraint violation: 0.00e+00, execution time: 0.18 s.
Final Params: [4.71968073e-02 2.99983666e+01 1.19156322e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78579512e+03 1.00000000e-01]
Initial Likelihood: -51956.379984596635
`gtol` termination condition is satisfied.
Number of iterations: 30, function evaluations: 104, CG iterations: 40,
optimality: 1.81e-09, constraint violation: 0.00e+00, execution time: 0.083 s.
Final Params: [9.99999668e-01 1.11876082e-04 9.99999615e-01]
Final Likelihood: -51227.71958344629
Initial Params: [5.00000000e-01 1.78545647e+03 1.00000000e-01]
Initial Likelihood: 646.8447571550093
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 68, CG iterations: 29,
optimality: 5.38e-07, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [0.00011924 0.08505562 0.05419468]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78545647e+03 1.00000000e-01]
Initial Likelihood: -52084.870011285224
`gtol` termination condition is satisfied.
Number of iterations: 97, function evaluations: 376, CG iterations: 138,
optimality: 3.75e-09, constraint violation: 0.00e+00, execution time: 0.25 s.
Final Params: [9.99999105e-01 1.26314478e-04 9.99999891e-01]
Final Likelihood: -51230.24192990152
Initial Params: [5.00000000e-01 1.78511782e+03 1.00000000e-01]
Initial Likelihood: 646.9352727923265
`gtol` termination condition is satisfied.
Number of iterations: 21, function evaluations: 56, CG iterations: 21,
optimality: 3.14e-07, constraint violation: 0.00e+00, execution time:  0.1 s.
Final Params: [6.28992325e-04 2.97347486e+01 4.08764776e-02]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78511782e+03 1.00000000e-01]
Initial Likelihood: -52107.80268307618
`gtol` termination condition is satisfied.
Number of iterations: 20, function evaluations: 72, CG iterations: 37,
optimality: 3.37e-07, constraint violation: 0.00e+00, execution time: 0.05 s.
Final Params: [0.99999684 0.00562146 0.999996  ]
Final Likelihood: -51233.9083879206
Initial Params: [5.00000000e-01 1.78477916e+03 1.00000000e-01]
Initial Likelihood: 647.0255897370355
`gtol` termination condition is satisfied.
Number of iterations: 30, function evaluations: 92, CG iterations: 36,
optimality: 2.04e-07, constraint violation: 0.00e+00, execution time: 0.17 s.
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Final Params: [0.0074269  4.84459426 0.00501226]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78477916e+03 1.00000000e-01]
Initial Likelihood: -52127.574481117495
`gtol` termination condition is satisfied.
Number of iterations: 18, function evaluations: 56, CG iterations: 24,
optimality: 8.45e-07, constraint violation: 0.00e+00, execution time: 0.039 s.
Final Params: [1.21153607e-04 2.99947166e+01 5.72838989e-04]
Final Likelihood: -52153.55235631512
Initial Params: [5.00000000e-01 1.78444051e+03 1.00000000e-01]
Initial Likelihood: 647.1157110408917
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 56, CG iterations: 19,
optimality: 2.57e-07, constraint violation: 0.00e+00, execution time:  0.1 s.
Final Params: [3.56055962e-03 2.99919626e+01 1.28671103e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78444051e+03 1.00000000e-01]
Initial Likelihood: -52152.59274046753
`gtol` termination condition is satisfied.
Number of iterations: 42, function evaluations: 156, CG iterations: 55,
optimality: 1.20e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [9.99999573e-01 5.00843931e-04 9.99999124e-01]
Final Likelihood: -51243.943211816484
Initial Params: [5.00000000e-01 1.78410185e+03 1.00000000e-01]
Initial Likelihood: 647.2056328459239
`gtol` termination condition is satisfied.
Number of iterations: 20, function evaluations: 52, CG iterations: 18,
optimality: 2.64e-07, constraint violation: 0.00e+00, execution time:  0.1 s.
Final Params: [1.53854263e-03 2.97823452e+01 8.02645713e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78410185e+03 1.00000000e-01]
Initial Likelihood: -52153.55235631512
`gtol` termination condition is satisfied.
Number of iterations: 26, function evaluations: 96, CG iterations: 32,
optimality: 3.34e-09, constraint violation: 0.00e+00, execution time: 0.066 s.
Final Params: [0.99999826 0.0031837  0.99999659]
Final Likelihood: -51248.94685196412
Initial Params: [5.0000000e-01 1.7837632e+03 1.0000000e-01]
Initial Likelihood: 647.2953556933476
`xtol` termination condition is satisfied.
Number of iterations: 35, function evaluations: 140, CG iterations: 45,
optimality: 2.32e+03, constraint violation: 1.75e+03, execution time: 0.22 s.
Final Params: [ 4.95389359e-01  1.78362946e+03 -7.47248474e-02]
Final Likelihood: 808.213717305362
Initial Params: [5.0000000e-01 1.7837632e+03 1.0000000e-01]
Initial Likelihood: -52136.06141107524
`gtol` termination condition is satisfied.
Number of iterations: 71, function evaluations: 272, CG iterations: 102,
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optimality: 1.18e-07, constraint violation: 0.00e+00, execution time: 0.18 s.
Final Params: [9.99999577e-01 3.66435665e-04 9.99998790e-01]
Final Likelihood: -51255.0959335574
Initial Params: [5.00000000e-01 1.78342455e+03 1.00000000e-01]
Initial Likelihood: 647.3848801393676
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 56, CG iterations: 22,
optimality: 6.21e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [2.95726184e-03 2.99805648e+01 6.37011733e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78342455e+03 1.00000000e-01]
Initial Likelihood: -52111.94497091658
`gtol` termination condition is satisfied.
Number of iterations: 21, function evaluations: 80, CG iterations: 38,
optimality: 8.08e-07, constraint violation: 0.00e+00, execution time: 0.052 s.
Final Params: [0.99999752 0.01448631 0.99996439]
Final Likelihood: -51262.48448081631
Initial Params: [5.00000000e-01 1.78308589e+03 1.00000000e-01]
Initial Likelihood: 647.4742076852383
`xtol` termination condition is satisfied.
Number of iterations: 76, function evaluations: 316, CG iterations: 114,
optimality: 3.93e+01, constraint violation: 0.00e+00, execution time: 0.52 s.
Final Params: [0.02590995 2.89091753 0.07456732]
Final Likelihood: 824.2816619096886
Initial Params: [5.00000000e-01 1.78308589e+03 1.00000000e-01]
Initial Likelihood: -52082.21337500255
`gtol` termination condition is satisfied.
Number of iterations: 39, function evaluations: 144, CG iterations: 53,
optimality: 9.88e-08, constraint violation: 0.00e+00, execution time: 0.098 s.
Final Params: [9.99999614e-01 2.84871892e-04 9.99998404e-01]
Final Likelihood: -51270.90555299571
Initial Params: [5.00000000e-01 1.78274724e+03 1.00000000e-01]
Initial Likelihood: 647.5633344078085
`xtol` termination condition is satisfied.
Number of iterations: 33, function evaluations: 132, CG iterations: 43,
optimality: 2.40e+03, constraint violation: 1.75e+03, execution time: 0.21 s.
Final Params: [ 4.91012480e-01  1.78261383e+03 -7.47029761e-02]
Final Likelihood: 810.8264807789215
Initial Params: [5.00000000e-01 1.78274724e+03 1.00000000e-01]
Initial Likelihood: -52025.650563322946
`gtol` termination condition is satisfied.
Number of iterations: 22, function evaluations: 76, CG iterations: 31,
optimality: 6.45e-08, constraint violation: 0.00e+00, execution time: 0.053 s.
Final Params: [9.99999646e-01 5.41686615e-04 9.99999590e-01]
Final Likelihood: -51280.55538683088
Initial Params: [5.00000000e-01 1.78240858e+03 1.00000000e-01]
Initial Likelihood: 647.6522623460471
`gtol` termination condition is satisfied.
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Number of iterations: 26, function evaluations: 76, CG iterations: 36,
optimality: 2.55e-07, constraint violation: 0.00e+00, execution time: 0.14 s.
Final Params: [3.16453050e-02 1.91872159e+01 2.38406336e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78240858e+03 1.00000000e-01]
Initial Likelihood: -52006.03741451504
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 48, CG iterations: 23,
optimality: 3.59e-07, constraint violation: 0.00e+00, execution time: 0.04 s.
Final Params: [9.97478093e-01 2.96890491e+01 2.54261169e-02]
Final Likelihood: -52153.55235631512
Initial Params: [5.00000000e-01 1.78206993e+03 1.00000000e-01]
Initial Likelihood: 647.740989784545
`gtol` termination condition is satisfied.
Number of iterations: 27, function evaluations: 84, CG iterations: 26,
optimality: 8.64e-07, constraint violation: 0.00e+00, execution time: 0.15 s.
Final Params: [8.03864321e-01 2.99930124e+01 2.05332848e-02]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78206993e+03 1.00000000e-01]
Initial Likelihood: -52023.37392857812
`gtol` termination condition is satisfied.
Number of iterations: 22, function evaluations: 76, CG iterations: 35,
optimality: 5.38e-07, constraint violation: 0.00e+00, execution time: 0.054 s.
Final Params: [9.99999867e-01 9.95629483e-04 9.99999319e-01]
Final Likelihood: -51303.350435815766
Initial Params: [5.00000000e-01 1.78173128e+03 1.00000000e-01]
Initial Likelihood: 647.8295217403528
`gtol` termination condition is satisfied.
Number of iterations: 14, function evaluations: 52, CG iterations: 19,
optimality: 3.66e-07, constraint violation: 0.00e+00, execution time: 0.097 s.
Final Params: [1.00285882e-04 2.99999817e+01 1.00206170e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78173128e+03 1.00000000e-01]
Initial Likelihood: -52077.6601055122
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 72, CG iterations: 25,
optimality: 4.21e-07, constraint violation: 0.00e+00, execution time: 0.05 s.
Final Params: [1.00063312e-04 2.99852919e+01 9.99999622e-01]
Final Likelihood: -51492.13634358118
Initial Params: [5.00000000e-01 1.78139262e+03 1.00000000e-01]
Initial Likelihood: 647.9178531613945
`xtol` termination condition is satisfied.
Number of iterations: 52, function evaluations: 268, CG iterations: 59,
optimality: 5.58e+01, constraint violation: 0.00e+00, execution time: 0.42 s.
Final Params: [1.43450037e-04 2.99809934e+01 7.47568665e-02]
Final Likelihood: 931.7525471994304
Initial Params: [5.00000000e-01 1.78139262e+03 1.00000000e-01]
Initial Likelihood: -52107.993615008294
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`gtol` termination condition is satisfied.
Number of iterations: 30, function evaluations: 112, CG iterations: 41,
optimality: 8.58e-07, constraint violation: 0.00e+00, execution time: 0.074 s.
Final Params: [9.99998201e-01 3.46781570e-04 9.99994226e-01]
Final Likelihood: -51323.94394841421
Initial Params: [5.00000000e-01 1.78105397e+03 1.00000000e-01]
Initial Likelihood: 648.005986561294
`gtol` termination condition is satisfied.
Number of iterations: 27, function evaluations: 84, CG iterations: 29,
optimality: 6.23e-07, constraint violation: 0.00e+00, execution time: 0.16 s.
Final Params: [ 0.11551831 17.23851459  0.05986157]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78105397e+03 1.00000000e-01]
Initial Likelihood: -52128.31893552451
`gtol` termination condition is satisfied.
Number of iterations: 25, function evaluations: 92, CG iterations: 41,
optimality: 5.63e-07, constraint violation: 0.00e+00, execution time: 0.064 s.
Final Params: [0.99999981 0.00176265 0.99999722]
Final Likelihood: -51333.79999990534
Initial Params: [5.00000000e-01 1.78071531e+03 1.00000000e-01]
Initial Likelihood: 648.0939163476846
`gtol` termination condition is satisfied.
Number of iterations: 26, function evaluations: 76, CG iterations: 31,
optimality: 2.48e-07, constraint violation: 0.00e+00, execution time: 0.14 s.
Final Params: [9.92227457e-04 2.97969016e+01 1.37772227e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78071531e+03 1.00000000e-01]
Initial Likelihood: -52134.575520950966
`gtol` termination condition is satisfied.
Number of iterations: 14, function evaluations: 56, CG iterations: 25,
optimality: 2.05e-07, constraint violation: 0.00e+00, execution time: 0.037 s.
Final Params: [ 0.99993421 29.99999954  0.99991465]
Final Likelihood: -51390.802009147315
Initial Params: [5.00000000e-01 1.78037666e+03 1.00000000e-01]
Initial Likelihood: 648.1816503427726
`gtol` termination condition is satisfied.
Number of iterations: 21, function evaluations: 56, CG iterations: 21,
optimality: 5.30e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.56253637e-02 1.67011017e+00 9.09527738e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78037666e+03 1.00000000e-01]
Initial Likelihood: -52120.24175224372
`gtol` termination condition is satisfied.
Number of iterations: 28, function evaluations: 104, CG iterations: 44,
optimality: 3.19e-08, constraint violation: 0.00e+00, execution time: 0.07 s.
Final Params: [0.99999877 0.00226131 0.99999632]
Final Likelihood: -51348.380559335565
Initial Params: [5.00000000e-01 1.78003801e+03 1.00000000e-01]
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Initial Likelihood: 648.2691827166441
`gtol` termination condition is satisfied.
Number of iterations: 32, function evaluations: 100, CG iterations: 30,
optimality: 3.60e-07, constraint violation: 0.00e+00, execution time: 0.19 s.
Final Params: [1.08384150e-02 2.71681248e+01 1.01066094e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.78003801e+03 1.00000000e-01]
Initial Likelihood: -52108.07704918682
`gtol` termination condition is satisfied.
Number of iterations: 24, function evaluations: 88, CG iterations: 31,
optimality: 6.80e-08, constraint violation: 0.00e+00, execution time: 0.06 s.
Final Params: [0.99999881 0.00221025 0.99999635]
Final Likelihood: -51353.46842151577
Initial Params: [5.00000000e-01 1.77969935e+03 1.00000000e-01]
Initial Likelihood: 648.3565131222888
`gtol` termination condition is satisfied.
Number of iterations: 29, function evaluations: 92, CG iterations: 39,
optimality: 3.04e-07, constraint violation: 0.00e+00, execution time: 0.17 s.
Final Params: [5.58774335e-03 1.01524097e+01 1.23306843e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77969935e+03 1.00000000e-01]
Initial Likelihood: -52089.019675032934
`gtol` termination condition is satisfied.
Number of iterations: 48, function evaluations: 180, CG iterations: 78,
optimality: 9.89e-09, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [9.99999770e-01 5.14648483e-04 9.99999278e-01]
Final Likelihood: -51359.45287863891
Initial Params: [5.0000000e-01 1.7793607e+03 1.0000000e-01]
Initial Likelihood: 648.4436512248726
`gtol` termination condition is satisfied.
Number of iterations: 20, function evaluations: 56, CG iterations: 20,
optimality: 2.37e-07, constraint violation: 0.00e+00, execution time:  0.1 s.
Final Params: [1.69884289e-04 1.70837712e+01 3.87731085e-02]
Final Likelihood: 0.0
Initial Params: [5.0000000e-01 1.7793607e+03 1.0000000e-01]
Initial Likelihood: -52060.026968725535
`gtol` termination condition is satisfied.
Number of iterations: 26, function evaluations: 96, CG iterations: 44,
optimality: 6.73e-07, constraint violation: 0.00e+00, execution time: 0.068 s.
Final Params: [0.99999882 0.00187769 0.99999628]
Final Likelihood: -51366.36279330391
Initial Params: [5.00000000e-01 1.77902205e+03 1.00000000e-01]
Initial Likelihood: 648.5305818992592
`xtol` termination condition is satisfied.
Number of iterations: 69, function evaluations: 328, CG iterations: 85,
optimality: 5.62e+01, constraint violation: 0.00e+00, execution time: 0.54 s.
Final Params: [6.98867684e-04 2.73531229e+01 7.44439605e-02]
Final Likelihood: 931.7187016331843
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Initial Params: [5.00000000e-01 1.77902205e+03 1.00000000e-01]
Initial Likelihood: -52037.38182399242
`gtol` termination condition is satisfied.
Number of iterations: 29, function evaluations: 112, CG iterations: 39,
optimality: 8.52e-07, constraint violation: 0.00e+00, execution time: 0.075 s.
Final Params: [0.99999315 0.01215639 0.99997991]
Final Likelihood: -51373.89791811499
Initial Params: [5.00000000e-01 1.77868339e+03 1.00000000e-01]
Initial Likelihood: 648.6173145142848
`gtol` termination condition is satisfied.
Number of iterations: 30, function evaluations: 100, CG iterations: 32,
optimality: 5.34e-07, constraint violation: 0.00e+00, execution time: 0.19 s.
Final Params: [4.93510671e-01 2.99832873e+01 8.65864062e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77868339e+03 1.00000000e-01]
Initial Likelihood: -52021.08424083609
`xtol` termination condition is satisfied.
Number of iterations: 48, function evaluations: 208, CG iterations: 66,
optimality: 1.79e-04, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [ 0.7338669  29.9967001   0.99999971]
Final Likelihood: -51123.38329572229
Initial Params: [5.00000000e-01 1.77834474e+03 1.00000000e-01]
Initial Likelihood: 648.703847289641
`gtol` termination condition is satisfied.
Number of iterations: 20, function evaluations: 56, CG iterations: 24,
optimality: 3.94e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.35268932e-04 9.06699869e+00 1.51741554e-02]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77834474e+03 1.00000000e-01]
Initial Likelihood: -52011.13421925578
`xtol` termination condition is satisfied.
Number of iterations: 49, function evaluations: 220, CG iterations: 61,
optimality: 7.26e-05, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [ 0.91233642 29.99740241  0.99999994]
Final Likelihood: -51462.678123204605
Initial Params: [5.00000000e-01 1.77800608e+03 1.00000000e-01]
Initial Likelihood: 648.7901787503066
`gtol` termination condition is satisfied.
Number of iterations: 36, function evaluations: 124, CG iterations: 44,
optimality: 4.41e-07, constraint violation: 0.00e+00, execution time: 0.23 s.
Final Params: [5.06273260e-03 2.99862034e+01 3.58050707e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77800608e+03 1.00000000e-01]
Initial Likelihood: -52007.5317592519
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 68, CG iterations: 29,
optimality: 8.04e-08, constraint violation: 0.00e+00, execution time: 0.048 s.
Final Params: [0.99999897 0.00185427 0.99999547]
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Final Likelihood: -51395.01628837758
Initial Params: [5.00000000e-01 1.77766743e+03 1.00000000e-01]
Initial Likelihood: 648.8763100942675
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 52, CG iterations: 16,
optimality: 2.56e-07, constraint violation: 0.00e+00, execution time: 0.094 s.
Final Params: [4.20516203e-02 2.99951876e+01 1.37246040e-02]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77766743e+03 1.00000000e-01]
Initial Likelihood: -52010.27686082405
`xtol` termination condition is satisfied.
Number of iterations: 48, function evaluations: 220, CG iterations: 64,
optimality: 3.44e-04, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [ 0.73294779 29.96537486  0.9999985 ]
Final Likelihood: -51124.776372883935
Initial Params: [5.00000000e-01 1.77732878e+03 1.00000000e-01]
Initial Likelihood: 648.9622441367553
`gtol` termination condition is satisfied.
Number of iterations: 27, function evaluations: 72, CG iterations: 37,
optimality: 2.40e-07, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [0.00048661 0.1962637  0.00102973]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77732878e+03 1.00000000e-01]
Initial Likelihood: -52019.369523972404
`gtol` termination condition is satisfied.
Number of iterations: 30, function evaluations: 108, CG iterations: 37,
optimality: 4.01e-07, constraint violation: 0.00e+00, execution time: 0.073 s.
Final Params: [9.99999855e-01 7.12372568e-04 9.99999357e-01]
Final Likelihood: -51412.6315089455
Initial Params: [5.00000000e-01 1.77699012e+03 1.00000000e-01]
Initial Likelihood: 649.0479764178299
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 68, CG iterations: 23,
optimality: 3.18e-07, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [6.05482824e-03 2.00302575e+01 1.40695722e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77699012e+03 1.00000000e-01]
Initial Likelihood: -52034.809748697444
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 88, CG iterations: 40,
optimality: 7.64e-08, constraint violation: 0.00e+00, execution time: 0.06 s.
Final Params: [0.9999953  0.00830399 0.99997898]
Final Likelihood: -51422.81424857288
Initial Params: [5.00000000e-01 1.77665147e+03 1.00000000e-01]
Initial Likelihood: 649.1335075439009
`gtol` termination condition is satisfied.
Number of iterations: 21, function evaluations: 60, CG iterations: 22,
optimality: 2.44e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
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Final Params: [0.00239058 0.03839735 0.00388858]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77665147e+03 1.00000000e-01]
Initial Likelihood: -52056.59753499752
`gtol` termination condition is satisfied.
Number of iterations: 15, function evaluations: 52, CG iterations: 21,
optimality: 4.64e-07, constraint violation: 0.00e+00, execution time: 0.037 s.
Final Params: [1.09994094e-04 2.99998839e+01 9.99986679e-01]
Final Likelihood: -51468.42994298412
Initial Params: [5.00000000e-01 1.77631281e+03 1.00000000e-01]
Initial Likelihood: 649.2188371660544
`xtol` termination condition is satisfied.
Number of iterations: 24, function evaluations: 96, CG iterations: 31,
optimality: 2.32e+03, constraint violation: 1.75e+03, execution time: 0.15 s.
Final Params: [ 4.90692714e-01  1.77617809e+03 -7.38543574e-02]
Final Likelihood: 797.533215443794
Initial Params: [5.00000000e-01 1.77631281e+03 1.00000000e-01]
Initial Likelihood: -52084.732882875054
`gtol` termination condition is satisfied.
Number of iterations: 15, function evaluations: 52, CG iterations: 20,
optimality: 6.79e-07, constraint violation: 0.00e+00, execution time: 0.041 s.
Final Params: [1.07122088e-04 2.99998302e+01 9.99983759e-01]
Final Likelihood: -51455.69032161209
Initial Params: [5.00000000e-01 1.77597416e+03 1.00000000e-01]
Initial Likelihood: 649.3039638063096
`gtol` termination condition is satisfied.
Number of iterations: 28, function evaluations: 88, CG iterations: 36,
optimality: 2.44e-07, constraint violation: 0.00e+00, execution time: 0.16 s.
Final Params: [1.90426560e-02 2.99621178e+01 3.99376830e-02]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77597416e+03 1.00000000e-01]
Initial Likelihood: -52119.21579232708
`gtol` termination condition is satisfied.
Number of iterations: 26, function evaluations: 96, CG iterations: 31,
optimality: 4.33e-08, constraint violation: 0.00e+00, execution time: 0.07 s.
Final Params: [0.9999986  0.00254143 0.99999408]
Final Likelihood: -51426.98327044523
Initial Params: [5.00000000e-01 1.77563551e+03 1.00000000e-01]
Initial Likelihood: 649.3888939704132
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 56, CG iterations: 20,
optimality: 5.49e-07, constraint violation: 0.00e+00, execution time:  0.1 s.
Final Params: [1.35677635e-04 2.99828140e+01 3.17725414e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77563551e+03 1.00000000e-01]
Initial Likelihood: -52145.851503866754
`gtol` termination condition is satisfied.
Number of iterations: 27, function evaluations: 100, CG iterations: 43,
```

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optimality: 7.43e-07, constraint violation: 0.00e+00, execution time: 0.068 s.
Final Params: [0.99999879 0.00230555 0.99999425]
Final Likelihood: -51425.51642864884
Initial Params: [5.00000000e-01 1.77529685e+03 1.00000000e-01]
Initial Likelihood: 649.4736198008277
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 64, CG iterations: 24,
optimality: 2.33e-07, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [0.00052417 0.18957098 0.05893655]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77529685e+03 1.00000000e-01]
Initial Likelihood: -52094.05945568739
`gtol` termination condition is satisfied.
Number of iterations: 22, function evaluations: 68, CG iterations: 21,
optimality: 3.89e-07, constraint violation: 0.00e+00, execution time: 0.052 s.
Final Params: [1.00055185e-04 2.99878297e+01 9.99999618e-01]
Final Likelihood: -51423.70131475656
Initial Params: [5.0000000e-01 1.7749582e+03 1.0000000e-01]
Initial Likelihood: 649.5581457170633
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 56, CG iterations: 20,
optimality: 4.92e-07, constraint violation: 0.00e+00, execution time:  0.1 s.
Final Params: [1.31638489e-04 2.99846114e+01 3.29477870e-03]
Final Likelihood: 0.0
Initial Params: [5.0000000e-01 1.7749582e+03 1.0000000e-01]
Initial Likelihood: -52080.669499593336
`gtol` termination condition is satisfied.
Number of iterations: 35, function evaluations: 96, CG iterations: 48,
optimality: 1.91e-07, constraint violation: 0.00e+00, execution time: 0.071 s.
Final Params: [1.00010334e-04 2.99936024e+01 9.99999978e-01]
Final Likelihood: -51415.1516167545
Initial Params: [5.00000000e-01 1.77461954e+03 1.00000000e-01]
Initial Likelihood: 649.6424676197136
`gtol` termination condition is satisfied.
Number of iterations: 22, function evaluations: 68, CG iterations: 31,
optimality: 7.54e-07, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [2.42923724e-02 1.64997342e+01 1.18877951e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77461954e+03 1.00000000e-01]
Initial Likelihood: -52091.87023513341
`xtol` termination condition is satisfied.
Number of iterations: 97, function evaluations: 404, CG iterations: 160,
optimality: 3.55e-04, constraint violation: 0.00e+00, execution time: 0.25 s.
Final Params: [0.27303509 0.05031851 0.99999996]
Final Likelihood: -51321.65994084462
Initial Params: [5.00000000e-01 1.77428089e+03 1.00000000e-01]
Initial Likelihood: 649.7265885535832
`gtol` termination condition is satisfied.
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Number of iterations: 20, function evaluations: 60, CG iterations: 24,
optimality: 8.96e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.00275886e-04 2.99719621e+01 1.02387991e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77428089e+03 1.00000000e-01]
Initial Likelihood: -52095.95525386988
`xtol` termination condition is satisfied.
Number of iterations: 157, function evaluations: 688, CG iterations: 230,
optimality: 3.37e-03, constraint violation: 0.00e+00, execution time: 0.41 s.
Final Params: [0.66280693 0.4604149  0.99999892]
Final Likelihood: -51105.72407562087
Initial Params: [5.00000000e-01 1.77394224e+03 1.00000000e-01]
Initial Likelihood: 649.8105104971831
`gtol` termination condition is satisfied.
Number of iterations: 17, function evaluations: 52, CG iterations: 20,
optimality: 9.16e-07, constraint violation: 0.00e+00, execution time: 0.099 s.
Final Params: [1.00485789e-04 2.99942745e+01 1.00394012e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77394224e+03 1.00000000e-01]
Initial Likelihood: -52080.2869311058
`gtol` termination condition is satisfied.
Number of iterations: 17, function evaluations: 56, CG iterations: 19,
optimality: 4.02e-07, constraint violation: 0.00e+00, execution time: 0.04 s.
Final Params: [1.00477607e-04 2.99994973e+01 9.99992964e-01]
Final Likelihood: -51380.97089371181
Initial Params: [5.00000000e-01 1.77360358e+03 1.00000000e-01]
Initial Likelihood: 649.894228455028
`gtol` termination condition is satisfied.
Number of iterations: 22, function evaluations: 64, CG iterations: 24,
optimality: 4.84e-07, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [1.00587153e-04 2.99236958e+01 1.20526949e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77360358e+03 1.00000000e-01]
Initial Likelihood: -52086.50451483117
`gtol` termination condition is satisfied.
Number of iterations: 17, function evaluations: 56, CG iterations: 21,
optimality: 4.19e-07, constraint violation: 0.00e+00, execution time: 0.041 s.
Final Params: [1.00470927e-04 2.99994758e+01 9.99991864e-01]
Final Likelihood: -51368.160553890106
Initial Params: [5.00000000e-01 1.77326493e+03 1.00000000e-01]
Initial Likelihood: 649.9777473815757
`gtol` termination condition is satisfied.
Number of iterations: 29, function evaluations: 88, CG iterations: 38,
optimality: 2.23e-07, constraint violation: 0.00e+00, execution time: 0.17 s.
Final Params: [0.02467819 0.1810977  0.01804162]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77326493e+03 1.00000000e-01]
Initial Likelihood: -52114.608005045826
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`gtol` termination condition is satisfied.
Number of iterations: 14, function evaluations: 52, CG iterations: 22,
optimality: 4.76e-07, constraint violation: 0.00e+00, execution time: 0.037 s.
Final Params: [1.05245729e-04 2.99999762e+01 9.99976575e-01]
Final Likelihood: -51356.8210093612
Initial Params: [5.00000000e-01 1.77292627e+03 1.00000000e-01]
Initial Likelihood: 650.0610620403402
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 56, CG iterations: 20,
optimality: 5.56e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.36941948e-04 2.99826095e+01 3.32449680e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77292627e+03 1.00000000e-01]
Initial Likelihood: -52153.55235631512
`xtol` termination condition is satisfied.
Number of iterations: 112, function evaluations: 484, CG iterations: 184,
optimality: 5.77e-03, constraint violation: 0.00e+00, execution time: 0.31 s.
Final Params: [0.66713409 3.93825622 0.99999995]
Final Likelihood: -51071.6248010837
Initial Params: [5.00000000e-01 1.77258762e+03 1.00000000e-01]
Initial Likelihood: 650.1441747071489
`gtol` termination condition is satisfied.
Number of iterations: 22, function evaluations: 64, CG iterations: 27,
optimality: 3.11e-07, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [1.00348811e-04 1.99350045e+01 1.02631022e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77258762e+03 1.00000000e-01]
Initial Likelihood: -52153.55235631512
The maximum number of function evaluations is exceeded.
Number of iterations: 250, function evaluations: 1056, CG iterations: 356,
optimality: 1.50e-03, constraint violation: 0.00e+00, execution time: 0.65 s.
Final Params: [ 0.13600872 18.35674414  0.99999898]
Final Likelihood: -51267.18801594389
Initial Params: [5.00000000e-01 1.77224897e+03 1.00000000e-01]
Initial Likelihood: 650.2270884281118
`gtol` termination condition is satisfied.
Number of iterations: 21, function evaluations: 60, CG iterations: 24,
optimality: 2.32e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [8.47487796e-03 1.43274065e+01 3.79331186e-02]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77224897e+03 1.00000000e-01]
Initial Likelihood: -52153.55235631512
`gtol` termination condition is satisfied.
Number of iterations: 16, function evaluations: 56, CG iterations: 21,
optimality: 9.20e-09, constraint violation: 0.00e+00, execution time: 0.047 s.
Final Params: [1.00236874e-04 2.99999977e+01 9.99996205e-01]
Final Likelihood: -51323.68123274083
Initial Params: [5.00000000e-01 1.77191031e+03 1.00000000e-01]
```

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Initial Likelihood: 650.3097943939549
`gtol` termination condition is satisfied.
Number of iterations: 24, function evaluations: 68, CG iterations: 34,
optimality: 3.56e-07, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [2.53488200e-04 1.62537734e+01 1.05925010e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77191031e+03 1.00000000e-01]
Initial Likelihood: -52153.55235631512
`xtol` termination condition is satisfied.
Number of iterations: 79, function evaluations: 360, CG iterations: 117,
optimality: 7.02e-04, constraint violation: 0.00e+00, execution time:  0.2 s.
Final Params: [ 0.6793773  29.8828501   0.99999954]
Final Likelihood: -51062.43018326229
Initial Params: [5.00000000e-01 1.77157166e+03 1.00000000e-01]
Initial Likelihood: 650.3923013720417
`xtol` termination condition is satisfied.
Number of iterations: 88, function evaluations: 424, CG iterations: 130,
optimality: 5.40e+01, constraint violation: 0.00e+00, execution time: 0.74 s.
Final Params: [1.07594908e-04 2.61563022e+01 7.42286488e-02]
Final Likelihood: 935.5108079367501
Initial Params: [5.00000000e-01 1.77157166e+03 1.00000000e-01]
Initial Likelihood: -52153.55235631512
`xtol` termination condition is satisfied.
Number of iterations: 79, function evaluations: 348, CG iterations: 102,
optimality: 9.26e-04, constraint violation: 0.00e+00, execution time:  0.2 s.
Final Params: [ 0.68014282 29.86456561  0.99999976]
Final Likelihood: -51062.48338113236
Initial Params: [5.000000e-01 1.771233e+03 1.000000e-01]
Initial Likelihood: 650.4746063351413
`gtol` termination condition is satisfied.
Number of iterations: 18, function evaluations: 52, CG iterations: 19,
optimality: 4.23e-07, constraint violation: 0.00e+00, execution time: 0.096 s.
Final Params: [1.45807127e-03 2.99985004e+01 1.35159183e-02]
Final Likelihood: 0.0
Initial Params: [5.000000e-01 1.771233e+03 1.000000e-01]
Initial Likelihood: -52153.55235631512
`xtol` termination condition is satisfied.
Number of iterations: 32, function evaluations: 156, CG iterations: 33,
optimality: 3.21e-04, constraint violation: 0.00e+00, execution time: 0.099 s.
Final Params: [ 0.68096851 29.99955384  0.99999929]
Final Likelihood: -51064.11175585363
Initial Params: [5.00000000e-01 1.77089435e+03 1.00000000e-01]
Initial Likelihood: 650.5567046984409
`gtol` termination condition is satisfied.
Number of iterations: 25, function evaluations: 80, CG iterations: 33,
optimality: 7.13e-07, constraint violation: 0.00e+00, execution time: 0.15 s.
Final Params: [8.00276479e-03 1.42800104e+01 2.29246361e-02]
Final Likelihood: 0.0
```

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Initial Params: [5.00000000e-01 1.77089435e+03 1.00000000e-01]
Initial Likelihood: -52153.55235631512
`xtol` termination condition is satisfied.
Number of iterations: 109, function evaluations: 448, CG iterations: 147,
optimality: 6.20e-04, constraint violation: 0.00e+00, execution time: 0.27 s.
Final Params: [ 0.68173566 29.98406946  0.99999976]
Final Likelihood: -51067.31712550258
Initial Params: [5.0000000e-01 1.7705557e+03 1.0000000e-01]
Initial Likelihood: 650.6386029095186
`gtol` termination condition is satisfied.
Number of iterations: 20, function evaluations: 56, CG iterations: 22,
optimality: 7.40e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [ 0.14702062 13.50124182  0.02720247]
Final Likelihood: 0.0
Initial Params: [5.0000000e-01 1.7705557e+03 1.0000000e-01]
Initial Likelihood: -52144.57420858272
`gtol` termination condition is satisfied.
Number of iterations: 26, function evaluations: 96, CG iterations: 37,
optimality: 3.47e-07, constraint violation: 0.00e+00, execution time: 0.066 s.
Final Params: [9.99997940e-01 1.91620269e-04 9.99989627e-01]
Final Likelihood: -51374.54385701068
Initial Params: [5.00000000e-01 1.77021704e+03 1.00000000e-01]
Initial Likelihood: 650.7203018692962
`gtol` termination condition is satisfied.
Number of iterations: 27, function evaluations: 88, CG iterations: 27,
optimality: 5.06e-07, constraint violation: 0.00e+00, execution time:  0.2 s.
Final Params: [3.78757364e-03 2.99841596e+01 1.34764962e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.77021704e+03 1.00000000e-01]
Initial Likelihood: -52133.94573758663
`gtol` termination condition is satisfied.
Number of iterations: 17, function evaluations: 60, CG iterations: 27,
optimality: 4.92e-07, constraint violation: 0.00e+00, execution time: 0.039 s.
Final Params: [0.99999133 0.00170957 0.99998649]
Final Likelihood: -51367.32408428384
Initial Params: [5.00000000e-01 1.76987839e+03 1.00000000e-01]
Initial Likelihood: 650.8017962234097
`gtol` termination condition is satisfied.
Number of iterations: 25, function evaluations: 72, CG iterations: 29,
optimality: 2.49e-07, constraint violation: 0.00e+00, execution time: 0.14 s.
Final Params: [1.25295706e-04 4.07505177e+00 8.99700793e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76987839e+03 1.00000000e-01]
Initial Likelihood: -52124.26871819998
`xtol` termination condition is satisfied.
Number of iterations: 69, function evaluations: 280, CG iterations: 97,
optimality: 1.07e-04, constraint violation: 0.00e+00, execution time: 0.17 s.
Final Params: [ 0.03487402 26.86046731  0.99999997]
```

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Final Likelihood: -51225.140584887486
Initial Params: [5.00000000e-01 1.76953973e+03 1.00000000e-01]
Initial Likelihood: 650.8830810323657
`xtol` termination condition is satisfied.
Number of iterations: 38, function evaluations: 172, CG iterations: 49,
optimality: 5.16e+00, constraint violation: 0.00e+00, execution time: 0.28 s.
Final Params: [0.00060898 0.29075995 0.06786432]
Final Likelihood: 108.10817627028115
Initial Params: [5.00000000e-01 1.76953973e+03 1.00000000e-01]
Initial Likelihood: -52110.267609551665
`xtol` termination condition is satisfied.
Number of iterations: 53, function evaluations: 256, CG iterations: 86,
optimality: 1.70e-03, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [0.03709622 3.60936631 0.99999915]
Final Likelihood: -51214.44076715378
Initial Params: [5.00000000e-01 1.76920108e+03 1.00000000e-01]
Initial Likelihood: 650.9641676878598
`gtol` termination condition is satisfied.
Number of iterations: 31, function evaluations: 96, CG iterations: 41,
optimality: 2.65e-07, constraint violation: 0.00e+00, execution time: 0.18 s.
Final Params: [9.19039112e-03 2.61505451e+01 4.02013602e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76920108e+03 1.00000000e-01]
Initial Likelihood: -52081.667362772336
`xtol` termination condition is satisfied.
Number of iterations: 52, function evaluations: 236, CG iterations: 73,
optimality: 6.12e-05, constraint violation: 0.00e+00, execution time: 0.14 s.
Final Params: [ 0.04007353 15.72298785  0.99999915]
Final Likelihood: -51204.637386065864
Initial Params: [5.00000000e-01 1.76886243e+03 1.00000000e-01]
Initial Likelihood: 651.045052979511
`gtol` termination condition is satisfied.
Number of iterations: 18, function evaluations: 52, CG iterations: 19,
optimality: 2.99e-07, constraint violation: 0.00e+00, execution time: 0.099 s.
Final Params: [2.41461149e-03 2.99984720e+01 9.53502195e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76886243e+03 1.00000000e-01]
Initial Likelihood: -52018.59604235305
`xtol` termination condition is satisfied.
Number of iterations: 74, function evaluations: 312, CG iterations: 105,
optimality: 2.83e-05, constraint violation: 0.00e+00, execution time: 0.19 s.
Final Params: [0.04252014 4.93680614 0.99999983]
Final Likelihood: -51195.74428909531
Initial Params: [5.00000000e-01 1.76852377e+03 1.00000000e-01]
Initial Likelihood: 651.1257334580536
`xtol` termination condition is satisfied.
Number of iterations: 40, function evaluations: 136, CG iterations: 45,
optimality: 5.42e+01, constraint violation: 0.00e+00, execution time: 0.23 s.
```

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Final Params: [1.42926482e-04 2.99808088e+01 7.41688071e-02]
Final Likelihood: 934.3362838119768
Initial Params: [5.00000000e-01 1.76852377e+03 1.00000000e-01]
Initial Likelihood: -51969.53506830246
`gtol` termination condition is satisfied.
Number of iterations: 25, function evaluations: 92, CG iterations: 36,
optimality: 4.64e-07, constraint violation: 0.00e+00, execution time: 0.062 s.
Final Params: [9.99998277e-01 2.22819174e-04 9.99987192e-01]
Final Likelihood: -51329.79021654696
Initial Params: [5.00000000e-01 1.76818512e+03 1.00000000e-01]
Initial Likelihood: 651.2062107988221
`gtol` termination condition is satisfied.
Number of iterations: 29, function evaluations: 92, CG iterations: 36,
optimality: 6.30e-07, constraint violation: 0.00e+00, execution time: 0.17 s.
Final Params: [1.17058584e-01 6.56742193e-02 1.00558727e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76818512e+03 1.00000000e-01]
Initial Likelihood: -51952.77481587144
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 80, CG iterations: 35,
optimality: 8.90e-08, constraint violation: 0.00e+00, execution time: 0.056 s.
Final Params: [9.99999524e-01 3.07092330e-04 9.99997751e-01]
Final Likelihood: -51323.85137071401
Initial Params: [5.00000000e-01 1.76784646e+03 1.00000000e-01]
Initial Likelihood: 651.2864815545048
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 68, CG iterations: 31,
optimality: 4.64e-07, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [0.00667477 0.07309583 0.04921529]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76784646e+03 1.00000000e-01]
Initial Likelihood: -51953.49388655857
`xtol` termination condition is satisfied.
Number of iterations: 65, function evaluations: 284, CG iterations: 88,
optimality: 1.13e-04, constraint violation: 0.00e+00, execution time: 0.16 s.
Final Params: [0.22964658 0.04838684 0.99999984]
Final Likelihood: -51149.37676156219
Initial Params: [5.00000000e-01 1.76750781e+03 1.00000000e-01]
Initial Likelihood: 651.3665526453269
`gtol` termination condition is satisfied.
Number of iterations: 16, function evaluations: 56, CG iterations: 21,
optimality: 5.12e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [2.27962841e-04 2.99999841e+01 1.02783213e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76750781e+03 1.00000000e-01]
Initial Likelihood: -51974.99890072542
`gtol` termination condition is satisfied.
Number of iterations: 27, function evaluations: 96, CG iterations: 48,
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optimality: 1.08e-08, constraint violation: 0.00e+00, execution time: 0.068 s.
Final Params: [9.99999655e-01 6.98108915e-04 9.99999056e-01]
Final Likelihood: -51299.947675134135
Initial Params: [5.00000000e-01 1.76716916e+03 1.00000000e-01]
Initial Likelihood: 651.4464157335481
`xtol` termination condition is satisfied.
Number of iterations: 35, function evaluations: 140, CG iterations: 41,
optimality: 2.39e+03, constraint violation: 1.74e+03, execution time: 0.23 s.
Final Params: [ 4.90036961e-01  1.76703501e+03 -7.42026102e-02]
Final Likelihood: 818.517415556596
Initial Params: [5.00000000e-01 1.76716916e+03 1.00000000e-01]
Initial Likelihood: -51942.18021893359
`gtol` termination condition is satisfied.
Number of iterations: 26, function evaluations: 92, CG iterations: 37,
optimality: 7.74e-07, constraint violation: 0.00e+00, execution time: 0.063 s.
Final Params: [0.99999991 0.00118672 0.99999865]
Final Likelihood: -51285.572031795135
Initial Params: [5.0000000e-01 1.7668305e+03 1.0000000e-01]
Initial Likelihood: 651.526076610814
`gtol` termination condition is satisfied.
Number of iterations: 25, function evaluations: 76, CG iterations: 25,
optimality: 4.21e-07, constraint violation: 0.00e+00, execution time: 0.15 s.
Final Params: [7.21530508e-02 2.99979797e+01 1.15675294e-04]
Final Likelihood: 0.0
Initial Params: [5.0000000e-01 1.7668305e+03 1.0000000e-01]
Initial Likelihood: -51913.16774345687
`gtol` termination condition is satisfied.
Number of iterations: 22, function evaluations: 80, CG iterations: 35,
optimality: 3.66e-07, constraint violation: 0.00e+00, execution time: 0.054 s.
Final Params: [0.99999852 0.0030348  0.99999446]
Final Likelihood: -51273.04195493298
Initial Params: [5.00000000e-01 1.76649185e+03 1.00000000e-01]
Initial Likelihood: 651.6055309868541
`gtol` termination condition is satisfied.
Number of iterations: 21, function evaluations: 64, CG iterations: 25,
optimality: 2.57e-07, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [0.00071276 0.00811915 0.00011657]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76649185e+03 1.00000000e-01]
Initial Likelihood: -51898.001935369786
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 80, CG iterations: 31,
optimality: 2.60e-07, constraint violation: 0.00e+00, execution time: 0.057 s.
Final Params: [9.99999157e-01 8.88017475e-04 9.99991886e-01]
Final Likelihood: -51262.33665266391
Initial Params: [5.0000000e-01 1.7661532e+03 1.0000000e-01]
Initial Likelihood: 651.6847875764685
`gtol` termination condition is satisfied.
```

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Number of iterations: 21, function evaluations: 56, CG iterations: 24,
optimality: 1.28e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.00001544e-04 2.98987577e+01 1.01441645e-03]
Final Likelihood: 0.0
Initial Params: [5.0000000e-01 1.7661532e+03 1.0000000e-01]
Initial Likelihood: -51896.682794670385
`xtol` termination condition is satisfied.
Number of iterations: 97, function evaluations: 404, CG iterations: 134,
optimality: 2.37e-05, constraint violation: 0.00e+00, execution time: 0.25 s.
Final Params: [0.12428222 0.02123044 0.99999977]
Final Likelihood: -51076.3467667257
Initial Params: [5.00000000e-01 1.76581454e+03 1.00000000e-01]
Initial Likelihood: 651.7638358347464
`gtol` termination condition is satisfied.
Number of iterations: 22, function evaluations: 64, CG iterations: 28,
optimality: 9.47e-07, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [8.56244939e-02 2.48476264e+01 2.55554886e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76581454e+03 1.00000000e-01]
Initial Likelihood: -51909.210321360835
`xtol` termination condition is satisfied.
Number of iterations: 59, function evaluations: 260, CG iterations: 74,
optimality: 2.90e-04, constraint violation: 0.00e+00, execution time: 0.15 s.
Final Params: [ 0.68177862 29.95823262  0.99999782]
Final Likelihood: -51160.750907575995
Initial Params: [5.00000000e-01 1.76547589e+03 1.00000000e-01]
Initial Likelihood: 651.8426806555733
`gtol` termination condition is satisfied.
Number of iterations: 18, function evaluations: 52, CG iterations: 21,
optimality: 5.18e-07, constraint violation: 0.00e+00, execution time: 0.097 s.
Final Params: [1.42923293e-02 2.99837983e+01 7.51960885e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76547589e+03 1.00000000e-01]
Initial Likelihood: -51935.58451543914
`xtol` termination condition is satisfied.
Number of iterations: 76, function evaluations: 292, CG iterations: 113,
optimality: 7.58e-04, constraint violation: 0.00e+00, execution time: 0.19 s.
Final Params: [0.12819964 4.45424518 0.9999998 ]
Final Likelihood: -51058.365013048366
Initial Params: [5.00000000e-01 1.76513723e+03 1.00000000e-01]
Initial Likelihood: 651.9213180525994
`xtol` termination condition is satisfied.
Number of iterations: 30, function evaluations: 120, CG iterations: 37,
optimality: 2.32e+03, constraint violation: 1.74e+03, execution time:  0.2 s.
Final Params: [ 4.93789060e-01  1.76500301e+03 -7.41602724e-02]
Final Likelihood: 817.4932073098356
Initial Params: [5.00000000e-01 1.76513723e+03 1.00000000e-01]
Initial Likelihood: -51958.23467469661
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`xtol` termination condition is satisfied.
Number of iterations: 148, function evaluations: 636, CG iterations: 225,
optimality: 1.62e-03, constraint violation: 0.00e+00, execution time: 0.41 s.
Final Params: [ 0.13954662 12.42820212  0.99999904]
Final Likelihood: -51048.40997465524
Initial Params: [5.00000000e-01 1.76479858e+03 1.00000000e-01]
Initial Likelihood: 651.9997542296322
`gtol` termination condition is satisfied.
Number of iterations: 14, function evaluations: 52, CG iterations: 19,
optimality: 4.08e-07, constraint violation: 0.00e+00, execution time:  0.1 s.
Final Params: [1.00334482e-04 2.99999796e+01 1.00217815e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76479858e+03 1.00000000e-01]
Initial Likelihood: -51984.19734399467
`xtol` termination condition is satisfied.
Number of iterations: 106, function evaluations: 444, CG iterations: 154,
optimality: 1.76e-03, constraint violation: 0.00e+00, execution time: 0.28 s.
Final Params: [ 0.14163243 18.98738897  0.9999991 ]
Final Likelihood: -51038.3946374806
Initial Params: [5.00000000e-01 1.76445993e+03 1.00000000e-01]
Initial Likelihood: 652.0779821627145
`gtol` termination condition is satisfied.
Number of iterations: 27, function evaluations: 88, CG iterations: 35,
optimality: 8.85e-07, constraint violation: 0.00e+00, execution time: 0.17 s.
Final Params: [8.66362072e-03 2.99722825e+01 2.64289265e-02]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76445993e+03 1.00000000e-01]
Initial Likelihood: -52020.24275836101
`xtol` termination condition is satisfied.
Number of iterations: 153, function evaluations: 640, CG iterations: 229,
optimality: 2.33e-03, constraint violation: 0.00e+00, execution time: 0.39 s.
Final Params: [0.20289692 0.35127077 0.99999896]
Final Likelihood: -51048.79071983221
Initial Params: [5.00000000e-01 1.76412127e+03 1.00000000e-01]
Initial Likelihood: 652.1560114508422
`gtol` termination condition is satisfied.
Number of iterations: 23, function evaluations: 64, CG iterations: 23,
optimality: 4.84e-07, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [2.68910048e-02 2.80555231e+01 1.06313064e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76412127e+03 1.00000000e-01]
Initial Likelihood: -52052.61576013919
`xtol` termination condition is satisfied.
Number of iterations: 62, function evaluations: 276, CG iterations: 79,
optimality: 1.58e-04, constraint violation: 0.00e+00, execution time: 0.16 s.
Final Params: [ 0.86285726 29.99521361  0.99999987]
Final Likelihood: -51358.70438978285
Initial Params: [5.00000000e-01 1.76378262e+03 1.00000000e-01]
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Initial Likelihood: 652.2338332812923
`gtol` termination condition is satisfied.
Number of iterations: 21, function evaluations: 56, CG iterations: 22,
optimality: 2.89e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.32001374e-04 2.64567064e+01 1.12825370e-02]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76378262e+03 1.00000000e-01]
Initial Likelihood: -52076.024794298864
`xtol` termination condition is satisfied.
Number of iterations: 64, function evaluations: 232, CG iterations: 101,
optimality: 1.45e-03, constraint violation: 0.00e+00, execution time: 0.15 s.
Final Params: [0.14494287 3.13756894 0.99999981]
Final Likelihood: -51016.2398169499
Initial Params: [5.00000000e-01 1.76344396e+03 1.00000000e-01]
Initial Likelihood: 652.3114464065147
`gtol` termination condition is satisfied.
Number of iterations: 24, function evaluations: 76, CG iterations: 30,
optimality: 3.19e-07, constraint violation: 0.00e+00, execution time: 0.14 s.
Final Params: [0.0004182  0.01008397 0.00011934]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76344396e+03 1.00000000e-01]
Initial Likelihood: -52067.37515025542
`xtol` termination condition is satisfied.
Number of iterations: 182, function evaluations: 752, CG iterations: 280,
optimality: 2.62e-03, constraint violation: 0.00e+00, execution time: 0.48 s.
Final Params: [0.15466265 1.47131732 0.99999919]
Final Likelihood: -51012.22296974568
Initial Params: [5.00000000e-01 1.76310531e+03 1.00000000e-01]
Initial Likelihood: 652.388856975086
`xtol` termination condition is satisfied.
Number of iterations: 36, function evaluations: 144, CG iterations: 44,
optimality: 2.27e+03, constraint violation: 1.73e+03, execution time: 0.23 s.
Final Params: [ 4.87458524e-01  1.76296766e+03 -7.42317236e-02]
Final Likelihood: 821.2754179507499
Initial Params: [5.00000000e-01 1.76310531e+03 1.00000000e-01]
Initial Likelihood: -52064.76342261469
`xtol` termination condition is satisfied.
Number of iterations: 78, function evaluations: 328, CG iterations: 122,
optimality: 9.74e-05, constraint violation: 0.00e+00, execution time:  0.2 s.
Final Params: [0.15571977 0.030024   0.99999983]
Final Likelihood: -51002.85184461324
Initial Params: [5.00000000e-01 1.76276666e+03 1.00000000e-01]
Initial Likelihood: 652.466061650583
`gtol` termination condition is satisfied.
Number of iterations: 20, function evaluations: 56, CG iterations: 23,
optimality: 3.80e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.01040615e-04 2.99402604e+01 1.01208434e-04]
Final Likelihood: 0.0
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Initial Params: [5.00000000e-01 1.76276666e+03 1.00000000e-01]
Initial Likelihood: -52068.47760216668
`xtol` termination condition is satisfied.
Number of iterations: 61, function evaluations: 228, CG iterations: 85,
optimality: 1.33e-04, constraint violation: 0.00e+00, execution time: 0.14 s.
Final Params: [0.15690768 0.04380482 0.99999994]
Final Likelihood: -50995.07084133467
Initial Params: [5.000000e-01 1.762428e+03 1.000000e-01]
Initial Likelihood: 652.5430622297773
`gtol` termination condition is satisfied.
Number of iterations: 22, function evaluations: 64, CG iterations: 25,
optimality: 6.02e-07, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [1.10528159e-01 1.77821397e+01 1.60719244e-03]
Final Likelihood: 0.0
Initial Params: [5.000000e-01 1.762428e+03 1.000000e-01]
Initial Likelihood: -52078.517688910346
The maximum number of function evaluations is exceeded.
Number of iterations: 250, function evaluations: 1044, CG iterations: 368,
optimality: 7.91e-03, constraint violation: 0.00e+00, execution time: 0.65 s.
Final Params: [0.15835622 2.89405202 0.99999914]
Final Likelihood: -50988.88674274257
Initial Params: [5.00000000e-01 1.76208935e+03 1.00000000e-01]
Initial Likelihood: 652.6198554391182
`gtol` termination condition is satisfied.
Number of iterations: 20, function evaluations: 64, CG iterations: 22,
optimality: 6.67e-07, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [1.02244357e-04 2.99958312e+01 1.01548350e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76208935e+03 1.00000000e-01]
Initial Likelihood: -52084.13812283387
`gtol` termination condition is satisfied.
Number of iterations: 27, function evaluations: 104, CG iterations: 33,
optimality: 2.32e-07, constraint violation: 0.00e+00, execution time: 0.068 s.
Final Params: [0.99999097 0.01535895 0.99997923]
Final Likelihood: -51286.79342942175
Initial Params: [5.00000000e-01 1.76175069e+03 1.00000000e-01]
Initial Likelihood: 652.6964440273
`xtol` termination condition is satisfied.
Number of iterations: 93, function evaluations: 428, CG iterations: 164,
optimality: 4.76e+01, constraint violation: 0.00e+00, execution time: 0.74 s.
Final Params: [1.59067088e-02 2.46013154e+01 7.41660503e-02]
Final Likelihood: 901.8071217386356
Initial Params: [5.00000000e-01 1.76175069e+03 1.00000000e-01]
Initial Likelihood: -52093.67220473801
`xtol` termination condition is satisfied.
Number of iterations: 55, function evaluations: 240, CG iterations: 70,
optimality: 2.33e-04, constraint violation: 0.00e+00, execution time: 0.14 s.
Final Params: [ 0.14722881 29.96329582  0.99999974]
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Final Likelihood: -50976.34359839605
Initial Params: [5.00000000e-01 1.76141204e+03 1.00000000e-01]
Initial Likelihood: 652.7728307391635
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 56, CG iterations: 19,
optimality: 2.46e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [3.20256200e-03 2.99923166e+01 3.39945059e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76141204e+03 1.00000000e-01]
Initial Likelihood: -52097.50218728212
`gtol` termination condition is satisfied.
Number of iterations: 51, function evaluations: 192, CG iterations: 63,
optimality: 7.02e-08, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [9.99999674e-01 4.25117753e-04 9.99998795e-01]
Final Likelihood: -51314.66298268062
Initial Params: [5.00000000e-01 1.76107339e+03 1.00000000e-01]
Initial Likelihood: 652.8490040753868
`xtol` termination condition is satisfied.
Number of iterations: 37, function evaluations: 172, CG iterations: 57,
optimality: 2.42e+01, constraint violation: 0.00e+00, execution time: 0.29 s.
Final Params: [3.34120980e-03 8.68471123e+00 7.03263427e-02]
Final Likelihood: 473.7050375149215
Initial Params: [5.00000000e-01 1.76107339e+03 1.00000000e-01]
Initial Likelihood: -52094.8679272711
`gtol` termination condition is satisfied.
Number of iterations: 24, function evaluations: 84, CG iterations: 35,
optimality: 4.47e-09, constraint violation: 0.00e+00, execution time: 0.061 s.
Final Params: [9.99999646e-01 5.38945520e-04 9.99999231e-01]
Final Likelihood: -51323.26405895039
Initial Params: [5.00000000e-01 1.76073473e+03 1.00000000e-01]
Initial Likelihood: 652.924977933553
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 56, CG iterations: 22,
optimality: 5.41e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.73077776e-02 2.99999885e+01 1.71655474e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76073473e+03 1.00000000e-01]
Initial Likelihood: -52101.83009857426
`xtol` termination condition is satisfied.
Number of iterations: 55, function evaluations: 180, CG iterations: 80,
optimality: 2.06e-04, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [ 0.1421121  29.99136949  0.9999997 ]
Final Likelihood: -50943.01844070328
Initial Params: [5.00000000e-01 1.76039608e+03 1.00000000e-01]
Initial Likelihood: 653.000743553742
`gtol` termination condition is satisfied.
Number of iterations: 21, function evaluations: 56, CG iterations: 22,
optimality: 2.82e-07, constraint violation: 0.00e+00, execution time:  0.1 s.
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Final Params: [0.07316362 3.63345773 0.03164013]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.76039608e+03 1.00000000e-01]
Initial Likelihood: -52089.29445222524
`xtol` termination condition is satisfied.
Number of iterations: 46, function evaluations: 132, CG iterations: 57,
optimality: 2.01e-04, constraint violation: 0.00e+00, execution time: 0.092 s.
Final Params: [ 0.14316671 29.98217206  0.99999984]
Final Likelihood: -50933.361383977935
Initial Params: [5.00000000e-01 1.76005742e+03 1.00000000e-01]
Initial Likelihood: 653.0763007027264
`xtol` termination condition is satisfied.
Number of iterations: 41, function evaluations: 140, CG iterations: 46,
optimality: 6.13e+01, constraint violation: 0.00e+00, execution time: 0.23 s.
Final Params: [1.43239132e-04 2.99799150e+01 8.31231360e-02]
Final Likelihood: 864.6015682141192
Initial Params: [5.00000000e-01 1.76005742e+03 1.00000000e-01]
Initial Likelihood: -52081.949609242416
`xtol` termination condition is satisfied.
Number of iterations: 39, function evaluations: 184, CG iterations: 54,
optimality: 1.38e-01, constraint violation: 0.00e+00, execution time:  0.1 s.
Final Params: [ 0.14412153 29.69663581  0.99999779]
Final Likelihood: -50924.965722367066
Initial Params: [5.00000000e-01 1.75971877e+03 1.00000000e-01]
Initial Likelihood: 653.151650875147
`gtol` termination condition is satisfied.
Number of iterations: 24, function evaluations: 72, CG iterations: 25,
optimality: 8.13e-07, constraint violation: 0.00e+00, execution time: 0.14 s.
Final Params: [0.17198078 7.98490554 0.05477554]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.75971877e+03 1.00000000e-01]
Initial Likelihood: -52058.04804716016
`xtol` termination condition is satisfied.
Number of iterations: 64, function evaluations: 284, CG iterations: 76,
optimality: 4.08e-04, constraint violation: 0.00e+00, execution time: 0.17 s.
Final Params: [ 0.67362079 29.99825016  0.99999986]
Final Likelihood: -51043.19682524272
Initial Params: [5.00000000e-01 1.75938012e+03 1.00000000e-01]
Initial Likelihood: 653.226796220654
`xtol` termination condition is satisfied.
Number of iterations: 34, function evaluations: 136, CG iterations: 44,
optimality: 2.32e+03, constraint violation: 1.73e+03, execution time: 0.23 s.
Final Params: [ 4.94552959e-01  1.75924562e+03 -7.41290332e-02]
Final Likelihood: 819.4127057474944
Initial Params: [5.00000000e-01 1.75938012e+03 1.00000000e-01]
Initial Likelihood: -52041.160034348184
`xtol` termination condition is satisfied.
Number of iterations: 58, function evaluations: 200, CG iterations: 98,
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optimality: 7.65e-04, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [0.02143951 5.72173674 0.99999982]
Final Likelihood: -51186.413485328594
Initial Params: [5.00000000e-01 1.75904146e+03 1.00000000e-01]
Initial Likelihood: 653.3017391288506
`gtol` termination condition is satisfied.
Number of iterations: 21, function evaluations: 60, CG iterations: 28,
optimality: 5.15e-07, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [0.03736584 0.08132443 0.00010044]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.75904146e+03 1.00000000e-01]
Initial Likelihood: -52034.24969187357
`xtol` termination condition is satisfied.
Number of iterations: 48, function evaluations: 224, CG iterations: 63,
optimality: 3.18e-05, constraint violation: 0.00e+00, execution time: 0.12 s.
Final Params: [ 0.14739255 29.99265627  0.99999982]
Final Likelihood: -50907.29277199554
Initial Params: [5.00000000e-01 1.75870281e+03 1.00000000e-01]
Initial Likelihood: 653.376471380715
`gtol` termination condition is satisfied.
Number of iterations: 19, function evaluations: 56, CG iterations: 20,
optimality: 5.07e-07, constraint violation: 0.00e+00, execution time: 0.13 s.
Final Params: [1.34007126e-04 2.99841366e+01 3.35857437e-03]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.75870281e+03 1.00000000e-01]
Initial Likelihood: -52037.31701973719
`xtol` termination condition is satisfied.
Number of iterations: 82, function evaluations: 324, CG iterations: 112,
optimality: 6.93e-05, constraint violation: 0.00e+00, execution time: 0.21 s.
Final Params: [ 0.13977795 29.98620784  0.99999996]
Final Likelihood: -50901.998309670744
Initial Params: [5.00000000e-01 1.75836415e+03 1.00000000e-01]
Initial Likelihood: 653.4509972108591
`gtol` termination condition is satisfied.
Number of iterations: 25, function evaluations: 76, CG iterations: 32,
optimality: 8.03e-07, constraint violation: 0.00e+00, execution time: 0.15 s.
Final Params: [0.03605124 0.12809814 0.02858672]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.75836415e+03 1.00000000e-01]
Initial Likelihood: -52050.3620179396
`xtol` termination condition is satisfied.
Number of iterations: 80, function evaluations: 284, CG iterations: 109,
optimality: 2.57e-04, constraint violation: 0.00e+00, execution time: 0.23 s.
Final Params: [ 0.1319649  29.89937842  0.9999991 ]
Final Likelihood: -50894.0844867111
Initial Params: [5.0000000e-01 1.7580255e+03 1.0000000e-01]
Initial Likelihood: 653.5253152505966
`gtol` termination condition is satisfied.
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Number of iterations: 19, function evaluations: 56, CG iterations: 19,
optimality: 2.53e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [2.04720559e-03 2.99920917e+01 2.90496177e-03]
Final Likelihood: 0.0
Initial Params: [5.0000000e-01 1.7580255e+03 1.0000000e-01]
Initial Likelihood: -52073.38468647913
`xtol` termination condition is satisfied.
Number of iterations: 74, function evaluations: 332, CG iterations: 107,
optimality: 1.81e-04, constraint violation: 0.00e+00, execution time: 0.19 s.
Final Params: [ 0.13280112 29.98001771  0.99999979]
Final Likelihood: -50886.90571336135
Initial Params: [5.00000000e-01 1.75768685e+03 1.00000000e-01]
Initial Likelihood: 653.5994265183738
`gtol` termination condition is satisfied.
Number of iterations: 20, function evaluations: 52, CG iterations: 21,
optimality: 5.46e-07, constraint violation: 0.00e+00, execution time: 0.099 s.
Final Params: [1.00258355e-04 2.94170986e+01 1.00327208e-04]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.75768685e+03 1.00000000e-01]
Initial Likelihood: -52084.059179042466
`xtol` termination condition is satisfied.
Number of iterations: 80, function evaluations: 348, CG iterations: 115,
optimality: 1.83e-04, constraint violation: 0.00e+00, execution time: 0.22 s.
Final Params: [ 0.13362812 29.93910684  0.99999981]
Final Likelihood: -50880.82554833588
Initial Params: [5.00000000e-01 1.75734819e+03 1.00000000e-01]
Initial Likelihood: 653.6733326309034
`xtol` termination condition is satisfied.
Number of iterations: 32, function evaluations: 128, CG iterations: 41,
optimality: 2.39e+03, constraint violation: 1.73e+03, execution time: 0.24 s.
Final Params: [ 4.92195964e-01  1.75721356e+03 -7.40669742e-02]
Final Likelihood: 821.4784143456634
Initial Params: [5.00000000e-01 1.75734819e+03 1.00000000e-01]
Initial Likelihood: -52092.85560430783
`xtol` termination condition is satisfied.
Number of iterations: 101, function evaluations: 436, CG iterations: 144,
optimality: 2.28e-04, constraint violation: 0.00e+00, execution time: 0.26 s.
Final Params: [ 0.13446176 29.97861125  0.99999982]
Final Likelihood: -50875.84174945368
Initial Params: [5.00000000e-01 1.75700954e+03 1.00000000e-01]
Initial Likelihood: 653.7470311445987
`xtol` termination condition is satisfied.
Number of iterations: 57, function evaluations: 212, CG iterations: 79,
optimality: 9.43e-02, constraint violation: 0.00e+00, execution time: 0.38 s.
Final Params: [0.0162573  9.63528875 0.06428469]
Final Likelihood: 2.8658316375145194
Initial Params: [5.00000000e-01 1.75700954e+03 1.00000000e-01]
Initial Likelihood: -52106.43889654445
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`xtol` termination condition is satisfied.
Number of iterations: 82, function evaluations: 344, CG iterations: 115,
optimality: 1.19e-04, constraint violation: 0.00e+00, execution time: 0.22 s.
Final Params: [ 0.13529076 29.95764758  0.99999981]
Final Likelihood: -50871.95557208848
Initial Params: [5.00000000e-01 1.75667088e+03 1.00000000e-01]
Initial Likelihood: 653.8205211116531
`gtol` termination condition is satisfied.
Number of iterations: 20, function evaluations: 56, CG iterations: 20,
optimality: 2.28e-07, constraint violation: 0.00e+00, execution time: 0.11 s.
Final Params: [1.41110200e-04 2.99801829e+01 3.72587748e-02]
Final Likelihood: 0.0
Initial Params: [5.00000000e-01 1.75667088e+03 1.00000000e-01]
Initial Likelihood: -52124.809055752936
`xtol` termination condition is satisfied.
Number of iterations: 61, function evaluations: 288, CG iterations: 76,
optimality: 2.08e-04, constraint violation: 0.00e+00, execution time: 0.16 s.
Final Params: [ 0.68115326 29.99523413  0.99999986]
Final Likelihood: -50964.26337443077
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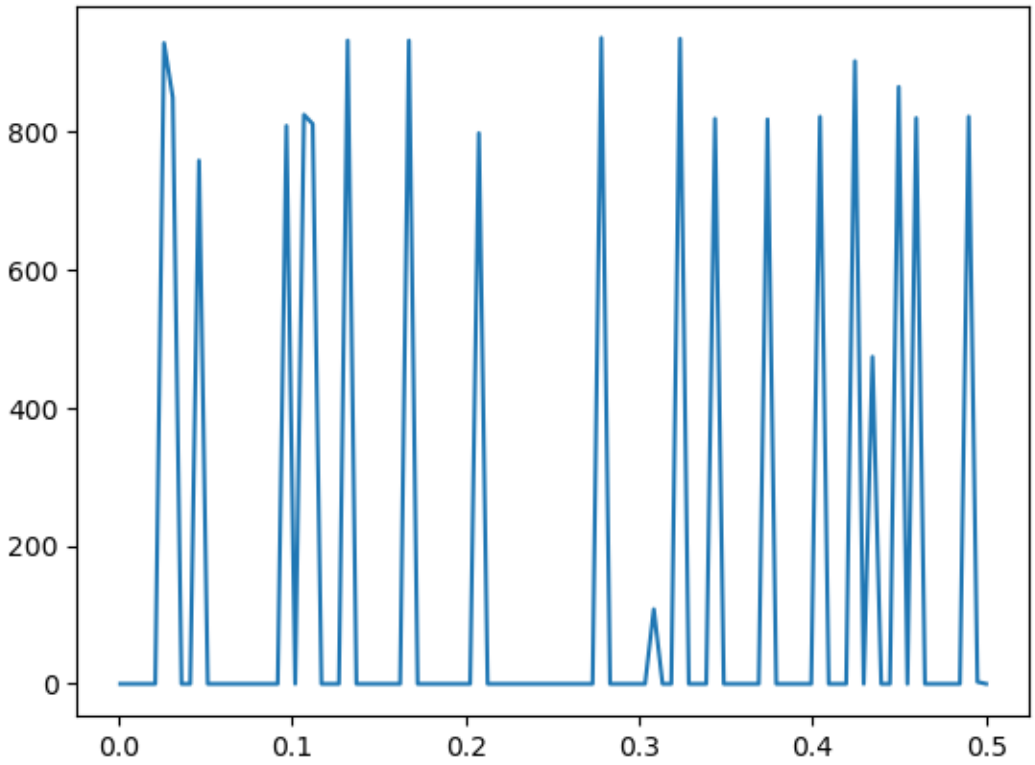
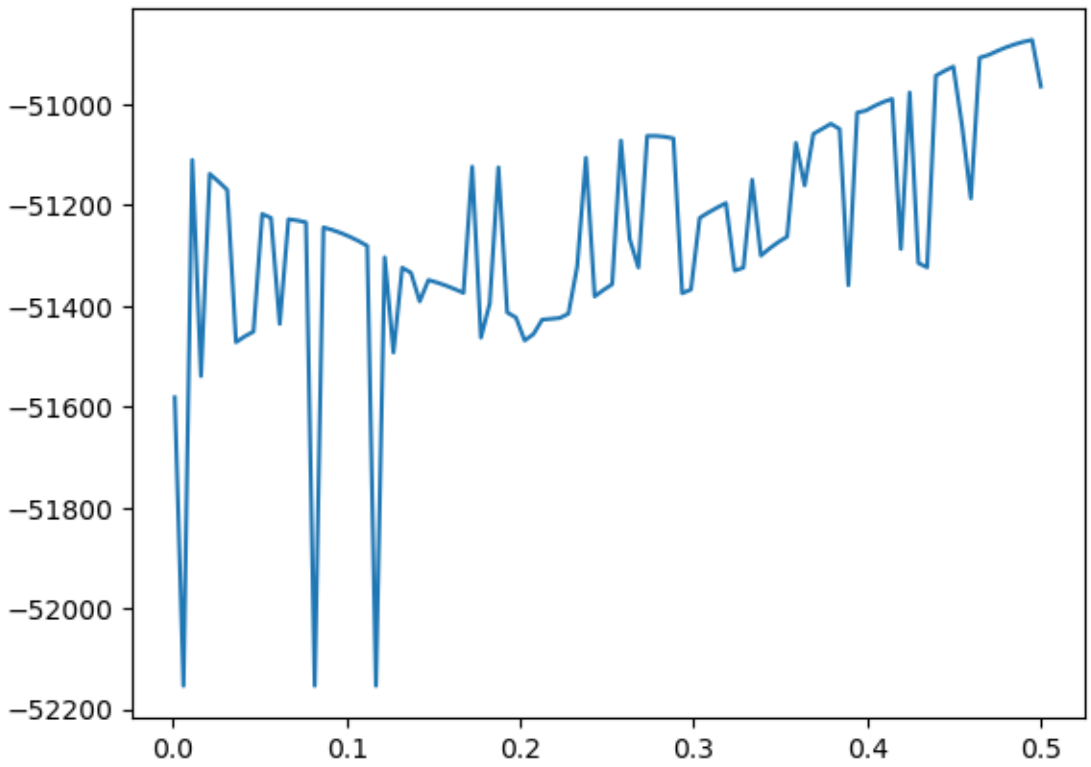
[ ]:
```
log_likelihoods = [goldCrudeModelsCIR[i].log_like for i in␣
 ↪range(len(goldCrudeModelsCIR))]
plt.plot(np.linspace(.001, 0.5, 100), log_likelihoods)
plt.show()
print("optimal parameters are: ", goldCrudeModelsCIR[np.
 ↪argmax(log_likelihoods)].params)
print("optimal B is: ", np.linspace(.001, 0.5, 100)[np.argmax(log_likelihoods)])

log_likelihoods = [goldCrudeModelsOU[i].log_like for i in␣
 ↪range(len(goldCrudeModelsOU))]
plt.plot(np.linspace(.001, 0.5, 100), log_likelihoods)
plt.show()
print("optimal parameters are: ", goldCrudeModelsOU[np.argmax(log_likelihoods)].
 ↪params)
print("optimal B is: ", np.linspace(.001, 0.5, 100)[np.argmax(log_likelihoods)])
```




```
optimal parameters are:  [1.07594908e-04 2.61563022e+01 7.42286488e-02]
optimal B is:  0.2782222222222222
```




```
optimal parameters are:  [ 0.13529076 29.95764758  0.99999981]
optimal B is:  0.494959595959596
```

A majority of time was spent playing around with different initial guesses. I tried to match the empirical results in [2] but I was unable to get the desired shape of the likelihood function. While the routine converged and I got parameter values, I was puzzled by the behavior of the graph. I expected a more "smooth" behavior but it appeared that different values of β produced widely different likelihoods. I had to look into the PyMLE library to figure out what was occuring and could not find any errors with the actual likelihood functions, similar to what I experienced when I wrote my function from scratch. I believe that the main issue relates to the possible bounds on the parameters (as well as initial guesses) that are causing the portfolio to have problematic values, thereby making the optimization routine less accurate. I played around with several different guesses and it led to different shapes for the graphs. I noticed that I had to change the bounds for the mean term because the actual mean for the time series was very large (due to the price of gold futures). Furthermore, I also had to lower my initial guess on the volatility parameter in order to ensure that the likelihood was not infinity. All in all, this was a part where I played around with data the most and depending on the time series, I would get very different results. For the scope of this report, I primarily focused on a portfolio consisting of crude oil and gold futures but this code can run for any combination of commodities futures.

With the MLE part finished, I then turned to implementing the optimal entry/exit thresholds as described on [1] page 82. The optimal stopping problem that we wish to solve is:

$$V^{\chi}(y) = \sup_{\tau \in \mathrm{T}} \mathbb{E}_y[e^{-r\tau}(Y_\tau - c_s)]$$

and:

$$J^{\chi}(y) = \sup_{v \in \mathrm{T}} \mathbb{E}_y[e^{-rv}(V^{\chi}(Y_v) - Y_v - c_b)]$$

We w.l.o.g assume that $c_s = c_b = c$ in our analysis

As part of the summary of analytical results, we consider the infinitesimal generator of a mean-reverting portfolio Y to be:

$$L^{\chi}(y) = \frac{\sigma^2 y}{2}\frac{d^2}{dy^2} + \mu(\theta - y)\frac{d}{dy}$$

and ODE:

$$L^{\chi}u(y) = ru(y)$$

The solutions to this ODE are of the form:

$$F^{\chi}(y) = M(\frac{r}{\mu}, \frac{2\mu\theta}{\sigma^2}; \frac{2\mu y}{\sigma^2}), G^{\chi}(y) = U(\frac{r}{\mu}, \frac{2\mu\theta}{\sigma^2}; \frac{2\mu y}{\sigma^2})$$

Where $M(a, b; z), U(a, b; z)$ are confluent hypergeometric functions of the first and second kind. With these, we state the following theorems:

Theorem 4.2: *The value function for the above optimal stopping problem is*

$$V^{\chi}(y) = \frac{b^{\chi^*} - c}{F^{\chi}(b^{\chi^*})}F^{\chi}(y)$$

if $y \in [0, b^{\chi^*})$ and

$$V^{\chi}(y) = y - c$$

otherwise

Theorem 4.4: *The optimal starting problem admits the solution*:

$$J^{\chi}(y) = V^{\chi}(y) - (y + c)$$

if $y \in [0, d^{\chi^*}]$ and:

$$J^{\chi}(y) = \frac{V^{\chi}(d^{\chi^*}) - (d^{\chi^*} + c)}{G^{\chi}(d^{\chi^*})}G^{\chi}(y)$$

otherwise

Implementation

With the theoretical result stated, I then turned to implementing the functions above that represent the optimal starting and exit signals. I referred to [2] for this part but with some changes. For one, [2] utilized a finite different scheme approach to approximate F'^{χ}, G'^{χ}. While this works, I wanted to be more precise since with a CIR process, these functions are easier to work with analytically so I found the actual derivatives of Kummer's function and Tricomi's function [4]:

$$\frac{d}{dz}M(a, b; z) = \frac{a}{b}M(a + 1, b + 1, z)$$

$$\frac{d}{dz}U(a, b; z) = -aU(a + 1, b + 1, z)$$

The challenging part for this code was translating the mathematical functions into code. I used the mpmath library which provides built-in hypergeometric functions and I used SciPy to derive the optimal b^*, d^*. A lot of time was spent on trying to understand the derivations for the formulas in [1] chapter 4, to which I may have struggled to fully internalize the concepts. This is where I believe some of the errors came from. I noted that I was able to follow Chapter 2 of the book fairly reasonably and use [2] as a reference. However, chapter 4 has more subtleties, when it comes to the bounds of the parameters and I feel as though I needed to take this into account more. I verified that the formulas I transcribed into code were correct but I need to play around with different inputs to get the desired behavior

[ ]:
```
import math
import scipy.optimize as sci
import numpy as np
import mpmath

# Theorem 4.2
def V_X(y, theta, mu, sigma, r,c):
    b_star = compute_b_star(mu,theta,sigma,r)
    if y >= 0 and y < b_star:
        return (b_star - c)/F_X(b_star,theta, mu, sigma, r,c)*F_X(y,theta, mu,␣
 ↪sigma, r,c)
    else:
        return y - c

def dV_X(y,theta, mu, sigma, r,c):
    b_star = compute_b_star(mu,theta,sigma,r)
    if y >= 0 and y < b_star:
        return (b_star - c)/F_X(b_star,theta, mu, sigma, r,c)*dF_X(y,theta, mu,␣
 ↪sigma, r,c)
    else:
        return 1

# Theorem 4.4
def J_X(y,theta, mu, sigma, r,c):
    d_star = compute_d_star(mu,theta,sigma,r)
    if y >= 0 and y < d_star:
        return V_X(y,theta, mu, sigma, r,c) - (y + c)
    else:
        return (V_X(d_star,theta, mu, sigma, r,c) - (d_star + c))/
 ↪G_X(d_star,theta, mu, sigma, r,c)*G_X(y,theta, mu, sigma, r,c)

# 4.9
def F_X(y,mu,theta,sigma,r):
    a,b,z = r/mu,(2*mu*theta)/sigma**2,(2*mu*y)/sigma**2
    return mpmath.hyp1f1(a,b,z)

def dF_X(y,mu,theta,sigma,r):
```

```
    a,b,z = r/mu,(2*mu*theta)/sigma**2,(2*mu*y)/sigma**2
    return a/b*mpmath.hyp1f1(a,b,z)

# 4.9
def G_X(y,mu,theta,sigma,r):
    a,b,z = r/mu,(2*mu*theta)/sigma**2,(2*mu*y)/sigma**2
    return mpmath.gamma(1 - b)/mpmath.gamma(a - b + 1)*mpmath.hyp1f1(a,b,z) +␣
 ↪mpmath.gamma(b - 1)/mpmath.gamma(a)*z**(1 - b)*mpmath.hyp1f1(a - b + 1,2 -␣
 ↪b,z)

def dG_X(y,mu,theta,sigma,r):
    a,b,z = r/mu,(2*mu*theta)/sigma**2,(2*mu*y)/sigma**2
    return -a*(mpmath.gamma(1 - b)/mpmath.gamma(a - b + 1)*mpmath.hyp1f1(a,b,z)␣
 ↪+ mpmath.gamma(b - 1)/mpmath.gamma(a)*z**(1 - b)*mpmath.hyp1f1(a - b + 1,2 -␣
 ↪b,z))


#def y_b(y):
#    return (self.mu*self.theta - self.r*self.c_b)/(self.mu + self.r)

#def y_s(y):
#    return (self.mu*self.theta + self.r*self.c_s)/(self.mu + self.r)

def compute_b_star(mu,theta,sigma,r):
    def f(y):
        return F_X(y,mu,theta,sigma,r) - (y)*dF_X(y,mu,theta,sigma,r)
    return sci.brentq(f, 0, 1)

def compute_d_star(mu,theta,sigma,r):
    def g(y):
        return G_X(y,mu,theta,sigma,r)(dV_X(y,mu,theta,sigma,r,r) - 1) -␣
 ↪dG_X(y,mu,theta,sigma,r)(V_X(y,mu,theta,sigma,r,r) - y)
    return sci.brentq(g, 0, 1)
```

There appears to be errors with my usage of the brentq method from SciPy. This is an error that I intend to fix in the near future

Miscellaneous: Time series forecast for gold futures

[ ]:
```
import pandas as pd
import numpy as np
import tensorflow as tf
from sklearn.preprocessing import MinMaxScaler

scaler = MinMaxScaler()
gld['Close'] = scaler.fit_transform(gld['Close'].values.reshape(-1, 1))
```

```
train_size = int(len(gld) * 0.8)
train_data = gld[:train_size]['Close'].values
test_data = gld[train_size:]['Close'].values


def create_sequences(data, seq_length):
    sequences = []
    for i in range(len(data) - seq_length):
        sequences.append(data[i:i+seq_length])
    return np.array(sequences)

seq_length = 10
train_sequences = create_sequences(train_data, seq_length)
test_sequences = create_sequences(test_data, seq_length)

model = tf.keras.Sequential([
    tf.keras.layers.LSTM(64, input_shape=(seq_length, 1)),
    tf.keras.layers.Dense(1)
])


model.compile(optimizer='adam', loss='mean_squared_error')
model.fit(train_sequences, train_data[seq_length:], epochs=10, batch_size=32)
predictions = model.predict(test_sequences)
predictions = scaler.inverse_transform(predictions)

print(predictions)
```

```
Epoch 1/10
19/19 [==============================] - 1s 3ms/step - loss: 0.1006
Epoch 2/10
19/19 [==============================] - 0s 2ms/step - loss: 0.0095
Epoch 3/10
19/19 [==============================] - 0s 2ms/step - loss: 0.0048
Epoch 4/10
19/19 [==============================] - 0s 2ms/step - loss: 0.0041
Epoch 5/10
19/19 [==============================] - 0s 2ms/step - loss: 0.0039
Epoch 6/10
19/19 [==============================] - 0s 2ms/step - loss: 0.0038
Epoch 7/10
19/19 [==============================] - 0s 2ms/step - loss: 0.0037
Epoch 8/10
19/19 [==============================] - 0s 2ms/step - loss: 0.0036
Epoch 9/10
19/19 [==============================] - 0s 2ms/step - loss: 0.0036
Epoch 10/10
19/19 [==============================] - 0s 2ms/step - loss: 0.0036
```

```
5/5 [==============================] - 0s 2ms/step
[[1844.175 ]
 [1844.3502]
 [1846.415 ]
 [1845.4471]
 [1841.7194]
 [1837.2373]
 [1834.6796]
 [1833.3654]
 [1832.4747]
 [1831.585 ]
 [1830.1086]
 [1828.7261]
 [1826.3945]
 [1825.0569]
 [1823.6387]
 [1820.9187]
 [1816.6061]
 [1809.4767]
 [1799.0314]
 [1788.0154]
 [1777.9812]
 [1768.1223]
 [1758.8096]
 [1751.5482]
 [1743.469 ]
 [1735.7977]
 [1729.58  ]
 [1725.5327]
 [1722.0455]
 [1719.745 ]
 [1719.451 ]
 [1719.8424]
 [1720.5088]
 [1721.0398]
 [1725.0858]
 [1731.1029]
 [1737.9602]
 [1745.0332]
 [1750.8267]
 [1757.602 ]
 [1762.7634]
 [1768.3364]
 [1774.1179]
 [1779.5382]
 [1783.0106]
 [1786.2314]
 [1787.6241]
```

[1787.3129]
[1785.3942]
[1781.285]
[1776.6573]
[1770.2727]
[1764.6958]
[1760.1409]
[1757.5411]
[1754.0138]
[1751.083]
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[1719.0615]
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[1713.4109]
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[1697.3356]
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[1692.2178]
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[1680.765]
[1673.9001]
[1667.4066]
[1663.6163]

```
[1661.8813]
[1660.882 ]
[1661.76  ]
[1663.1327]
[1663.6306]
[1662.5411]
[1661.9354]
[1661.9695]
[1660.2782]
[1661.257 ]
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[1779.6871]
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[1786.9739]
[1791.4326]
[1791.4382]
[1791.5435]
[1792.0901]
[1794.579 ]
[1797.4069]
[1797.9148]
[1797.8682]
[1799.8011]
[1800.7173]
[1802.8674]]
```

[ ]:
```
gld_tensor = tf.convert_to_tensor(gld['Close'].values)
```